MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Research Report CCS 478

THE DUALITY BETWEEN EXPECTED UTILITY AND PENALTY IN STOCHASTIC LINEAR PROGRAMMING

by

A. Ben-Tal*
M. Teboulle*

CENTER FOR CYBERNETIC STUDIES

The University of Texas
Austin, Texas 78712

MAY 2 4 1984

A

05 21 148

Research Report CCS 478

THE DUALITY BETWEEN EXPECTED UTILITY
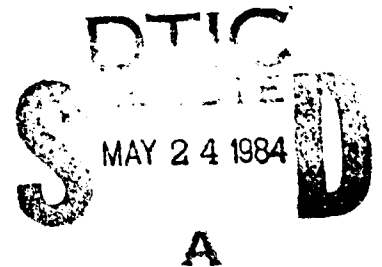## AND PENALTY IN STOCHASTIC
LINEAR PROGRAMMING

by

A. Ben-Tal*
M. Teboulle*

December 1983

*Faculty of Industrial Engineering & Management, Technion-Israel Institute of
Technology, Haifa, Israel

CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director
Graduate School of Business 4.138
The University of Texas at Austin
Austin, Texas 78712
(512) 471-1821

MAY 2 4 1984

A

ABSTRACT

We study the dual problem corresponding to a linear program in which the stochastic objective function is replaced by its expected utility, and discuss its relevance as a penalty method to a stochastically constrained dual linear program.

KEY WORDS

Stochastic Linear Programming

Expected Utility

Penalty Methods

1. INTRODUCTION

Consider a linear programming problem with a stochastic objective function:

$$\text{(SO-P)} \qquad \sup\{c^t x : Ax \le b , x \ge 0\}$$

where $x \in R^n$ is the decision vector, A is a given fixed matrix of size $m \times n$, b is a given fixed vector in R^m, and c is a random vector with a known distribution function $F_c(\cdot)$, and mean $E(c) = \mu \in R^n$.

A fundamental decision-theoretic approach to solve problem (SO-P) is the well known <u>expected utility principle</u> (see for example [9], [5]) which reduces problem (SO-P) to the deterministic nonlinear programming problem:

$$\text{(EU-P)} \qquad \text{Sup}\{Eu(c^t x) : Ax \le b , x \ge 0\}$$

where E denotes the mathematical expectation with respect to the random vector c, and u is the decision maker utility function, which is strictly increasing and concave. The latter property reflects a risk-aversion attitude. (See e.g. the papers of Arrow [1] and Pratt [6]). Under these conditions problem (EU-P) becomes a concave nonlinear programming problem.

The purpose of this paper is to investigate the dual problem of (EU-P) and to discuss its relations to the dual linear program of (SO-P):

$$\text{(SC-D)} \qquad \min\{b^t y : A^t y \ge c , y \ge 0\} .$$

which is a linear program with <u>stochastic constraints</u>. In fact we find it appropriate to deal, instead of problem (EU-P), with an equivalent problem

$$\text{(CE-P)} \qquad \sup\{u^{-1} Eu(c^t x) : Ax \le b , x \ge 0\}$$

where u^{-1} is the inverse of u. Note that for a random variable T

$$C(T) \triangleq u^{-1} Eu(T)$$

is the so-called certainty equivalent of T: it is the sure amount which leaves the decision maker indifferent to a gamble yielding T, since

$$u(C(T) = Eu(T).$$

There are few reasons to prefer the formulation (CE-P) to (EU-P):

(i) In the deterministic case (c being a degenerate random vector) problem (CE-P) reduces exactly to the original problem (SO-P), and so the linear structure of it (and its dual) are recovered.

(ii) According to the Von-Neumann-Morgenstern theory [8], the utility function is unique up to a monotone increasing affine transformation. If the (EU-P) formulation is used, the dual problem will depend on the particular choice of u . This is not the case with the (CE-P) formulation since C(T) is invariant to affine transformation in u.

There is however some difficulty associated with the objective function

$$V(x) = C(c^t x) = u^{-1} Eu(c^t x)$$

of (CE-P); since u^{-1} is convex, it is not guaranteed that $V(x)$ is a concave function. We show however, in the next section, that for a large class of utility functions, including the important class of HARA utilities (see [2], [3], [5], [9]) $V(x)$ is concave regardless of the distribution of the random vector c .

Properties of the dual problem of (CE-P) are derived in section 3.

Section 4 outlines the general approach to treat the linear program with stochastic constraints (SC-D). Approximations are given in section 5.

2. CONCAVITY OF THE CERTAINTY EQUIVALENCE FUNCTIONAL

We assume now and henceforth in this paper that

(i) The random vector $c = (c_1, c_2, \ldots, c_n)$ is non-degenerate, i.e. $\forall x \neq 0$ $c^t x$ is not a degenerate univariate random variable.

(ii) $E(c^t x) < +\infty \ \forall \ x \in R^n$

We denote by $[\underline{c}_k, \bar{c}_k]$ the support of the random variable, by μ_k its mean and by σ_k its standard deviation. Let $\mathcal{U}$ be the class of twice differentiable concave utility functions with $u' > 0$ and $u'' \leqslant 0$.

For a given $u \in \mathcal{U}$, the inverse function u^{-1} exists, and it is a strictly monotone increasing convex function. Thus problem (EU-P) can be transformed into an equivalent one:

$$(\text{CE-P}) \qquad \sup\{V(x) \triangleq u^{-1} Eu(c^T x) : Ax \leq b , x \geq 0\}$$

Problem (CE-P) is called the <u>certainty equivalent problem</u>.

Note that $V(x)$ is a convex increasing transformation (u^{-1}) of a concave function $Eu(c^t x)$, thus in general $V(\cdot)$ is not necessarily a concave function. To be able to use the powerful duality theory of convex programming, it is then of major importance to study conditions under which $V(x)$ is concave.

The next result furnish a complete answer to this question. Surprisingly, the concavity of V(for arbitrary distribution of the random vector c) is fully characterized in term of the so-called <u>Arrow-Pratt local risk aversion indicator</u>:

$$r(t) \triangleq - \frac{u''(t)}{u'(t)} \qquad\qquad (2.1)$$

<u>Theorem 2.1</u>. Let u be a utility function in u. Then the function:

$$V(x) = u^{-1} Eu(c^T x)$$

is concave <u>for any random vector c</u> if and only if $\dfrac{1}{r(t)}$ (risk tolerance indicator) is concave.

Proof: (The details will appear elsewhere in a future paper and thus are omitted).

[outline]: (i) The concavity of V is established by showing that it satisfies the gradient inequality (see [7]). Denoting by $\varphi(t) \triangleq u^{-1}(t)$, and using the fact that $\varphi' > 0$, the gradient inequality here is:

$$\forall x, y \in R^n \; ; \; \frac{\varphi Eu(c^T y) - \varphi Eu(c^T x)}{\varphi'(Eu(c^T x))} \leq E[\frac{c^T(y-x)}{\varphi'[u(c^T x)]}] \qquad (2.2)$$

(ii) Given $u \in \mathcal{U}$ we define $h : R^2 \to R$ by

$$h(t_1, t_2) \triangleq \frac{\varphi(t_1) - \varphi(t_2)}{\varphi'(t_2)}$$

and show that h is a convex function if and only if $\frac{1}{r(t)}$ is concave.

(iii) Applying Jensen inequality to $h(t_1, t_2)$ with $t_1 = u(c^t y)$ and $t_2 = u(c^t x)$ we get the inequality (2.2).

□

Remark 2.1. The two parameter class of utility functions called <u>hyperbolic absolute risk aversion</u> (HARA), characterized by $r(t) = \frac{1}{at+b}$, which is widely used in economics ([2], [5], [9]) satisfies trivially the condition that $\frac{1}{r(t)}$ is concave.

The HARA family consists of the following utilities (defined for $t > -b/a$)

$$u(t) = \begin{cases} -e^{-t/b} & \text{if} \quad a = 0, \; b \neq 0 \\ \log(b+t) & \text{if} \quad a = 1 \\ (at+b)^{(a-1)/a} & \text{if} \quad a \neq 0, \; a \neq 1 \end{cases}$$

A utility function satisfying the condition that $1/r(t)$ is concave will be termed a <u>CRT-utility</u> (concave risk tolerance).

3. <u>THE DUAL PROBLEM: AN INDUCED u-PENALTY</u>

The dual problem of

(CE-P) $\sup\{V(x) \triangleq u^{-1} Eu(c^t x) : Ax \leq b, \; x \geq 0\}$

is derived via Lagrangian duality. We assume that u is a CRT-utility so (CE-P) is a concave program. The Lagrangian for problem (CE-P) is the function $L: R_+^n \times R_+^m \to R$ with values:

$$L(x,y) = V(x) + y^t(b-Ax) \tag{3.1}$$

The dual objective function is defined by:

$$D(y) = \sup_{x \geq 0}\{V(x) + y^t(b-Ax)\} \tag{3.2}$$

and thus the dual problem for (CE-P) is given by:

$$\text{(CE-D)} \quad \inf_{y \geq 0}\{y^t b + \sup_{x \geq 0}\{V(x) - y^t Ax\}\} \tag{3.3}$$

Let

$$P_u(y) \overset{\Delta}{=} \sup_{x \geq 0}\{V(x) - y^t Ax\} \tag{3.4}$$

Then problem (CE-D) becomes:

$$\text{(CE-D)} \quad \inf_{y \geq 0}\{y^t b + P_u(y)\} \tag{3.5}$$

The first term, $y^t b$ is just the objective function of the dual problem of (SO-P):

$$\text{(SC-D)} \quad \inf_{y \geq 0}\{b^t y : A^t y \geq c\} \tag{3.6}$$

We show below that the second term, $P_u(y)$ plays the role of a penalty function for the stochastic constraints in (SC-D), P_u will be called accordingly u-penalty.

Theorem 3.1. The u-penalty function (3.5) is convex and satisfies:

$$\text{(i)} \quad P_u(y) = 0 \quad \text{if} \quad A^t y \geq \mu \tag{3.7}$$

$$\text{(ii)} \quad P_u(y) > 0 \quad \text{if} \quad A^t y \ngeq \mu \tag{3.8}$$

Proof: As a pointwise supremum of affine functions, $P_u(y)$ is convex.

(i) Let $Q(y;x) \triangleq V(x) - y^t A x$ then

$$P_u(y) = \sup_{x \geq 0} Q(y;x) \geq Q(y;0) = 0 \qquad (3.9)$$

since $u \in U$, by Jensen inequality $v(x) = u^{-1} E u(c^t x) \leq \mu^t x$, where equality holds only for $x = 0$, and so

$$P_u(y) = \sup_{x \geq 0} Q(y;x) \leq \sup x^t(\mu - A^t y) \qquad (3.10)$$

Now if $A^t y \geq \mu$, the inequality (3.10) shows that $P_u(y) \leq 0$ which together with (3.9) proves (3.7).

(ii) Assume that for some $k(k=1,\ldots,n)$, $A^t_k y < \mu_k$ where A^t_k is the k-th row of A.

Let $Q_k(y;x_k) \triangleq Q(y;0,0,\ldots,x_k,\ldots 0) = V(x_k) - x_k A^T_k y$

Then we have $Q_k(y;0) = 0$ and $\dfrac{d}{dx_k} Q_k(y;0) = \mu_k - A^T_k y > 0$.

Hence there exists $\hat{x}_k > 0$ such that:

$$Q_k(y;\hat{x}_k) > Q_k(y;0) = 0 \qquad (3.11)$$

Noting that $P_u(y) \overset{\geq}{=} \sup_{R^n \ni x_k \geq 0} Q_k(y;x_k) \qquad (3.12)$

and using (3.11), it follows that $P_u(y) > 0$.

$\square$

The theorem demonstrates that $P_u(y)$ penalizes solutions of (SC-D) which are not feasible in the mean.

We say that y_1 __is less feasible than__ y_2 for the k-th constraint (in the mean) if:

$$A_k^t y_2 > A_k^t y_1 \qquad (3.13)$$

and we write it $y_1 \succ y_2$.

The next result shows other desirable properties of the u-penalty. For the k-th constraint we define:

$$P_u^k(y) = \sup_{x_k \geq 0}\{V(x_k) - x_k A_k^t y\} \qquad (3.14)$$

Theorem 3.2 The u-penalty function satisfies

(i)   if $y_1 \succ y_2$ then $P_u^k(y_1) > P_u^k(y_2)$ (3.15)

(ii)   if for some k: $A_k^t y < c_{-k}$ then $P_u(y) = \infty$ (3.16)

Proof:

(i)   For $y_1 > y_2$ we have by (3.13):

$$V(x_k) - x_k A_k' y_k > V(x_k) - x_k A_k' y_2$$

and then using (3.14) the inequality (3.15) holds.

(ii)   Let $A_k^T y < \underline{c}_k$. By (3.12) we have:

$$P_u(y) \geq \sup_{x_k \geq 0}\{V(x_k) - x_k A_k' y\} \qquad (3.17)$$

since u is increasing and $c_k \geq \underline{c}_k$, (3.17) implies:

$$P_u(y) \geq \sup_{x_k \geq 0}\{u^{-1}Eu(\underline{c}_k x_k) - x_k A_k^T y\} = \sup_{x_k \geq 0}\{(\underline{c}_k - A_k^T y)x_k\} \qquad (3.18)$$

and hence (3.16) follows.   □

Theorem 3.2 demonstrates that the u-penalty is <u>monotone</u> in the sense defined in (3.13)

The second property (3.16) shows that the u-penalty automatically excludes solutions which are <u>not feasible</u> for (SC-D) <u>in probability 1</u>, since for those $P_u(y) = \infty$.

4. LINEAR PROGRAMMING WITH STOCHASTIC CONSTRAINTS: A GENERAL APPROACH

In this section we use the properties of the u-penalty studied previously , outline a general approach for linear programming problem with stochastic constra... :

$$\inf \ y^t b$$

$$\text{(SC)} \qquad A_k^t y \geq c_k \qquad k = 1,\ldots,n$$

$$y \geq 0$$

Suppose that the decision-maker accepts a solution y as being "feasible" for the constraint $A_k^t y \geq c_k$ if $A^t y$ is "large enough", and rejects solutions y for which $A^t y$ is "too small", i.e. he can choose two positive numbers k_1, k_2 such that:

$$y \text{ is feasible if } A_k^t y \geq \mu_k + k_1 \sigma_k \qquad ^{(*)}$$

$$y \text{ is infeasible if } A_k^t < \mu_k - k_2 \sigma_k$$

all other solutions , i.e. y's such that:

$$\mu_k - k_2 \sigma \leq A_k^t y < \mu_k + k_1 \sigma_k$$

are "semi-feasible".

(*) If "Feasibility" is modeled by chance constraint [4], i.e. $\Pr(A_k^t y \geq c_k) \geq \alpha_k$, then $k_1 = F_{z_k}^{-1}(\alpha_k)$ where F_{z_k} is the distribution function of the random variable $z_k = \dfrac{c_k - \mu_k}{\sigma_k}$, and $\alpha_k \in [0,1]$ are prescribed probabilities levels.

We will now build a penalty function $P_u^k(y)$ for the k-th constraint of (SC) which will reflect the above subjective attitude for the decision-maker.

Let u be a CRT-utility. For fixed k define:

$$\alpha_k = (k_1 + k_2) \cdot \frac{\sigma_k}{\mu_k - c_{-k}} \tag{4.1}$$

$$\beta_k = \mu_k - (k_1 c_{-k} + k_2 \mu_k) \cdot \frac{\sigma_k}{\mu_k - c_{-k}} \tag{4.2}$$

and let c_k^* be the random variable:

$$c_k^* = \alpha_k c_k + \beta_k$$

The penalty function is given by:

$$P_u^k(y) = \sup_{x_k \geq 0} \{u^{-1} Eu(c_k^* x_k) - x_k A_k^t y\} \tag{4.3}$$

The coefficients α_k, β_k were chosen so that:

$$E(c_k^*) = \mu_k + k_1 \sigma_k$$

$$c_{-k}^* = \mu_k - k_2 \sigma_k,$$

hence by Theorem 3.1 and Theorem 3.2 (ii) we have:

$$P_u^k(y) = \begin{cases} 0 & \text{if } A_k^t y \geq \mu_k + k_1 \sigma_k \qquad (4.4) \\ \infty & \text{if } A_k^t y < \mu_k - k_2 \sigma_k \qquad (4.5) \\ \text{positive} & \text{otherwise} \qquad\qquad\quad (4.6) \end{cases}$$

Thus semi-feasible solutions cause a positive, but finite penalty. Moreover, by Theorem 3.2 (i), the more they violate feasibility, the more they are penalized. Thus,

the approach we advocate for treating the stochastic linear program (SC) is to use the surrogate deterministic program (P):

$$\text{(P)} \qquad \inf_{y \geq 0} \{ y^t b + \sum_{k=1}^{n} P_u^k(y) \} \qquad\qquad (4.7)$$

In comparison with problem (CE-D) in (3.5), we use here an additive penalty (sum of penalties for individual constraints) rather than a joint constraints penalty. The additive form is clearly advantageous from the computational viewpoint. It is applicable whenever the decision maker <u>can</u> treat the constraints individually. However there is one choice of the utility function under which the joint constraints penalty <u>is</u> additive:

<u>Theorem 4.1</u>. Let u be an exponential utility function:

$$u(t) = a - be^{-t/p} \qquad (p > 0 \ , \ b > 0 \ , \ a \in R)$$

If the random variables $c_1, c_2, \ldots, c_n$ are independents then:

$$P_u(y) = \sum_{k=1}^{n} P_u^k(y)$$

where here: $\quad P_u^k(y) = p \cdot \sup_{x_k \geq 0} \{ -\log E(e^{-c_k x_k}) - x_k A_k^t y \}$

<u>Proof</u>: The result follows immediately from the fact that in the case of exponential utility the certainty equivalent, in terms of which P_u is defined, is additive. (See, [2], Theorem 4). □

We close this section by a simple illustrative example. Consider the one dimensional inventory problem:

$$\text{(SC)} \qquad \min\{hy : y \geq d \ , \ y \geq 0\}$$

where h is the unit holding cost and d is the demand. Assume that $d \sim \exp(\lambda)$ with mean $\mu = \frac{1}{\lambda}$ and $\sigma = \frac{1}{\lambda}$ $(\lambda > 0)$. Let $u(t) = 1 - e^{-t/p}$ $(p>0)$; i.e. the risk-aversion indicator is $r(t) = \frac{1}{p}$. Then by (4.3) the penalty function is:

$$P_u(y) = \sup_{x \geq 0}\{-p \log\frac{\lambda}{\lambda+x} - (\frac{y-\beta}{\alpha})\}$$

where here, by (4.1) - (4.2) : $\alpha = k_1 + k_2$; $\beta = \mu(1-k_2)$.

By simple calculus we obtain:

$$P_u(y) = \begin{cases} 0 & \text{if } y \geq (1+k_1)\mu \\[2ex] p\{\log \mu \frac{k_1 + k_2}{y+\mu(k_2-1)} + \frac{y-\mu(k_2+1)}{\mu(k_1 + k_2)}\} & \text{if } \mu(1-k_2) \leq y < \mu(1+k_2) \\[2ex] +\infty & \text{if } y < \mu(1-k_2) \end{cases}$$

Note that p plays here the role of a penalty-parameter. Subsittuting P_u in (4.7) and solving problem (P) we obtain

$$y^* = \mu(1-k_2) + \mu \frac{k_1 + k_2}{1 + \mu(k_1+k_2)\frac{h}{p}}$$

The optimal inventory y^* is a monotone decreasing function of both the holding cost h and the risk-aversion $1/p$. If either $h = 0$ or $1/p \to 0$ (risk neutrality) then y^* equals to the highest value $\mu(1+k_1)$ while if $h \to \infty$ or $1/p \to \infty$ (extreme risk aversion) y^* equals to lowest value $\mu(1-k_2)$.

5. MEAN VARIANCE APPROXIMATIONS

In this section we derive a quadratic approximation for the u-penalty function $P_u(y)$.

Denote the mean vector of c by μ, and by V its variance-covariance matrix (positive definite).

It can be shown by direct differentiation of the certainty equivalent functional

$$V(x) \triangleq u^{-1} E u(c^t x)$$

that:

$$V(0) = 0$$

$$\nabla V(0) = \mu$$

$$\nabla^2 V(0) = -r_0 V \quad \text{with} \quad r_0 \triangleq -\frac{u''(0)}{u'(0)} \geq 0$$

Using this in (3.4) we obtain:

Proposition 5.1. A second order approximation of $P_u(y)$ is:

$$\hat{P}_u(y) = \sup_{x \geq 0} \{ x^t (A^t y - \mu) - \frac{1}{2} r_0 x^t V x \} \tag{5.1}$$

□

Thus the approximate u-penalty $\hat{P}_u(y)$ is given in term of a concave quadratic program. A direct compution shows that the approximation is exact for the case of exponential utility and c being jointly normal.

For additive penalties, even a more explicit approximation is possible:

Proposition 5.2. A second order approximation $\hat{P}_u(y)$ of $P_u(y) = \sum_{k=1}^{n} P_u^k(y)$ is:

$$\hat{P}_u(y) = \frac{1}{2r_0} \sum_{k=1}^{n} \frac{1}{\sigma_k^2} [(\mu_k - A_k^T y)_+]^2 \tag{5.2}$$

where $a_+ = \max(0, a)$.

Proof: We have to show that:

$$P_u^k(y) = \frac{1}{2r_0 \sigma_k^2} [(\mu_k - A_k^T y)_+]^2 \tag{5.3}$$

Now from Proposition 5.1 we have for the k-th constraint:

$$P_u^k(y) = \sup_{x_k \geq 0} \{x_k(\mu_k - A_k^t y) - r_o \sigma_k^2 x_k^2\}$$

which by simple calculus gives (5.3).

□

REFERENCES

[1] Arrow, K.J.: Essays on the Theory of Risk-bearing, Markham Publishing, Chicago, 1975.

[2] Bamberg, G. and Spremann, K.: "Implications of constant risk aversion", Zeit. für Operations Research, 25, 1981, pp. 205-224.

[3] Cass, D. and Stiglitz, J.: "The structure of investor preferences and asset returns and separability in portfolio allocation: A contribution to the pure theory of mutual funds". J. Eco. Th., 2, 1970, pp. 122-160.

[4] Charnes, A. and Cooper, W.W.: "Chance constrained programming", Manag.Sci., 5, 1959, pp. 73-79.

[5] Hammond, J.S.: "Simplifying the choice between uncertain prospects where preference is non-linear", Manag.Sci., 20, 1974, pp. 1047-1072.

[6] Pratt, J.W.: "Risk aversion in the small and in the large", Econometrica, 32, 1964, pp. 122-136.

[7] Roberts, A.W., Varberg, D.E., Convex Functions, Academic Press, New York, 1973.

[8] von Neumann, J. and Morgenstern, O.: Theory of Games and Economic Behavior, 2nd ed., Princeton University Press, Princeton, N.J., 1947.

[9] Wilson, R.:: "The theory of syndicates", Econometrica, 36, 1968, pp. 119-132.

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CCS 478	2. GOVT ACCESSION NO. AD-A141 393	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* The Duality Between Expected Utility and Penalty in Stochastic Linear Programming		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. Ben-Tal, M. Teboulle		8. CONTRACT OR GRANT NUMBER(s) N0001481-C-0236
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Cybernetic Studies The University of Texas at Austin Austin, Texas 78712		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research (Code 434) Washington, D.C.		12. REPORT DATE December 1983
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS *(If different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Stochastic linear programming, expected utility, penalty methods

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

We study the dual problem corresponding to a linear program in which the stochastic objective function is replaced by its expected utility, and discuss its relevance as a penalty method to a stochastically constrained dual linear program.

DD $_{1 \text{ JAN } 73}^{\text{FORM}}$ 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

END

FILMED