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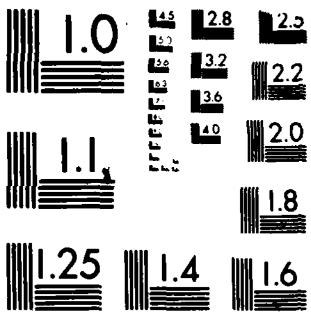

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MRC Technical Summary Report #2654

ASYMPTOTIC ESTIMATES FOR RESOLVENTS OF SOME NONINTEGRABLE VOLTERRA KERNELS

Stig-Olof Londen

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

March 1984

(Received November 9, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Let $a \in L^1_{loc}(R^+)$, $a \notin L^1(R^+)$, $a(t) = b(t)c(t)$, $0 < t < \infty$, where $b(t)$ is completely monotone and $c(t)$ is of positive type. Thus $b(t) = \int_{R^+} e^{-st}d\mu(s)$, $c(t) = \lim_{T\to\infty} \int_0^T \left(1 - \frac{s}{T}\right)\cos(st)dv(s)$ where $\mu(s)$, $v(s)$ are nondecreasing functions, defined on R^+, and locally of bounded variation. The resolvent r of a is the solution for $t > 0$ of $r(t) + (r*a)(t) = a(t)$. Extend v to R^- as odd and let $\hat{}$ denote Fourier-transforms. Then $\hat{a}(z) = \frac{1}{2}\int_R \hat{b}(z - ix)dv(x)$ for $\mathrm{Re}\, z$ sufficiently large. Using this formula we give conditions on μ, v which imply that $\hat{r}$ is locally sufficiently smooth to satisfy $r \in L^1(R^+)$. These conditions are shown to differ depending on the size of b at infinity.

AMS (MOS) Subject Classifications: 45D05, 45M05, 42A38

Key Words: Volterra equations, resolvent theory, Fourier transforms

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

The standard (real, linear, scalar) Volterra equation

$$x(t) + \int_0^t x(t - s)a(s)ds = f(t), \quad t > 0 ,$$

may easily be solved formally by the formula

$$(*) \qquad x(t) = f(t) - \int_0^t r(t - s)f(s)ds, \quad t > 0 ,$$

where the so called resolvent $r(t)$ satisfies

$$r(t) + \int_0^t r(t - s)a(s)ds = a(t), \quad t > 0 .$$

Equivalently

$$\hat{r}(z) = [1 + \hat{a}(z)]^{-1}\hat{a}(z), \quad \hat{a}(z) \overset{\text{def}}{=} \int_{R^+} e^{-zt}a(t)dt ,$$

for $\mathrm{Re}\, z$ sufficiently large. The usefulness of (*) does to a large extent depend on whether the size of r can be determined. In particular, if $r \in L^1(R^+)$ then properties of f like boundedness, integrability or convergence at infinity induce the same behavior in x. Consequently a key problem in Volterra equations is to obtain conditions on $a(t)$ which insure $r(t) \in L^1(R^+)$. This problem was completely solved by Paley and Wiener in the case when $a \in L^1(R^+)$. For the case when $a \notin L^1(R^+)$ very few results do however exist.

In this report we analyze the case when $a(t) = b(t)c(t)$, $t > 0$, $a \notin L^1(R^+)$, $b(t)$ completely monotone and $c(t)$ of positive type. Conditions are given under which $r \in L^1(R^+)$. The hypotheses differ depending on the size of b at infinity.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ASYMPTOTIC ESTIMATES FOR RESOLVENTS OF SOME NONINTEGRABLE VOLTERRA KERNELS

Stig-Olof Londen

1. INTRODUCTION

Consider the scalar, real, linear Volterra equation

$$x(t) + (x*a)(t) = f(t), \quad t > 0 ,$$

where a, f are given functions, $*$ denotes convolution, i.e. $(x*a)(t) = \int_0^t x(t-s)a(s)ds$, and x is to be determined. As is well-known this equation can be explicitly solved; under quite weak conditions on a and f one does in fact have

$$(F_1) \qquad x(t) = f(t) - (r*f)(t), \quad t > 0 ,$$

where the resolvent r satisfies $r(t) + (r*a)(t) = a(t)$, $t > 0$, or

$$\hat{r}(z) = [1 + \hat{a}(z)]^{-1}\hat{a}(z), \quad \hat{a}(z) \overset{\text{def}}{=} \int_{R^+} e^{-zt}a(t)dt ,$$

for $\text{Re } z$ sufficiently large. The representation (F_1) is obviously very useful for studying the asymptotic behavior of x, in particular provided $r \in L^1(R^+)$. In this case properties of f like boundedness, integrability, or convergence at infinity induce the same behavior in x. Recent results show that even for nonlinear equations the integrability of the resolvent is quite crucial in determining the asymptotic behavior of the solution, [4].

It is well known, [6], that if $a \in L^1(R^+)$ then $r \in L^1(R^+)$ iff

$$1 + \hat{a}(z) \neq 0, \quad \text{Re } z > 0 .$$

In applications however many kernels decay as $t^{-\alpha}$, $\alpha \in (0,1)$, and hence this condition cannot always be applied. Thus one is led to the question: When do nonintegrable kernels have integrable resolvents?

Due to the difficulty of the problem there exist comparatively few specific results giving explicit answers to this question. In [7] it was shown that

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$a \in L^1(0,1)$, a nonnegative, nonincreasing and convex on $(0,\infty)$ suffices to give $r \in L^1(R^+)$. Somewhat later an example of a locally integrable, nonnegative, nonincreasing kernel a was constructed [1] for which $r \notin L^1(R^+)$. This of course shows the importance of the convexity assumption. The results in [3] give more insight in the problem and reduce the question of integrability to one of extended local analyticity.

Large classes of nonintegrable nonconvex kernels do however have integrable resolvents. The kernels

$$a(t) = t^{-\alpha}\cos t, \quad \alpha \in [0,1), \quad a(t) = t^{-\alpha}J_0(t), \quad \alpha \in [0, \tfrac{1}{2}] ,$$

provide two particularly simple examples. Common to these examples is that they are products of a completely monotone function $b(t)$ $(=t^{-\alpha})$ and a function of positive type $c(t)$ $(=\cos t,\ J_0(t))$. Although this fact alone does not guarantee $r \in L^1(R^+)$ (a counterexample is $a(t) = t^{-\varepsilon}t^{-1/2}J_1(t^{1/2})$ where $\varepsilon \in (0, \tfrac{1}{4}])$ it may still be surmised that this is a class of functions worth investigating more thoroughly. This is also motivated by the convenient representation formulas available for a completely monotone function $b(t)$ and for a function $c(t)$ of positive type:

$$b(t) = \int_{R^+} e^{-xt}d\mu(x), \quad c(t) = \int_{R^+} \cos(xt)d\nu(x), \quad t > 0 .$$

Here μ, ν are nondecreasing and locally of bounded variation. These formulas provide a means of analyzing the behavior of $\hat{a}(z)$. In this report we analyze the question only locally, i.e. we let $\hat{a}(z)$ blow up at a single point - which for simplicity we take equal to the origin - and pay little or no attention to the behavior of $\hat{a}$ elsewhere. Global statements will appear in a forthcoming report. The same comment applies to a more detailed study of some examples and to a weakening of the size and continuity hypotheses (1.2), (1.3).

It was observed in [7] that a consequence of a result by Hardy and Littlewood is that if

$$\begin{cases} f \in H^1(\pi),\ f \in (C \cap L^\infty)(\{z | \operatorname{Re} z > 0\}) \\ f(i\omega) \text{ is locally absolutely continuous with } \frac{df}{d\omega} \in L^1(R) \end{cases}$$

(here $\pi \overset{\text{def}}{=} \{z | \text{Re } z > 0\}$) then there exists $g \in L^1(R^+)$ such that

$$f(z) = \hat{g}(z), \ \text{Re } z > 0; \ \|g\|_{L^1(R^+)} \le \frac{1}{2} \|\frac{df}{d\omega}\|_{L^1(R)} .$$

We let $f = \hat{r}$, define $\tilde{a}(\omega) = \lim_{\sigma \downarrow 0} \hat{a}(\sigma + i\omega)$ and conclude that the crucial thing which we need in order to apply this consequence and to obtain $[1 + \tilde{a}(\omega)]^{-1}\tilde{a}(\omega) \in \hat{L}^1$ is

$$[\frac{d\tilde{a}}{d\omega}][1 + \tilde{a}(\omega)]^{-2} \in L^1(R) ,$$

and in particular

$$(F_2) \qquad [\frac{d\tilde{a}}{d\omega}][\tilde{a}(\omega)]^{-2} \in L^1(-1,1) .$$

The attainment of (F_2) thus constitutes our primary goal.

Below we formulate our results which state hypotheses on the functions μ, ν under which (F_2) follows. In addition a few comments are given. In Sections 2, 3 and 4 we prove Theorems 1, 2 and 3 respectively. Finally in Section 5 we prove an auxiliary result needed in the proof of Theorem 1.

A preliminary version of Theorem 1 was given in [5].

THEOREM 1. Let $\mu(x)$ be a nondecreasing function defined for $x \ge 0$ with $\mu(0) = 0$ and such that

$$(1.1) \qquad \omega^{-1} \int_0^\omega \mu(x)dx \le \gamma\mu(\omega), \quad 0 < \omega \le 1 ,$$

for some constant $\gamma \in (0,1)$. Assume that the (completely monotone) function $b(t) \overset{\text{def}}{=} \int_{R^+} e^{-xt}d\mu(x)$ is well-defined for $t > 0$ and satisfies $b \in L^1(0,1)$.

Let $\nu(x)$ be an odd nondecreasing function defined for $x \in R$ and such that

$$(1.2) \qquad \nu(\infty) < \infty .$$

Suppose that

$$(1.3) \qquad \nu \in C^2(0,1] ,$$

with

$$(1.4) \qquad \int_0^1 \frac{\omega^2}{\nu^2(\omega)\mu(\omega)} d\nu_+'(\omega) < \infty ,$$

and let for some $\rho \in (0,\infty)$

(1.5) $$v'(\omega) \le \rho\omega^{-1}v(\omega) \quad \text{on} \quad (0,1) .$$

__Finally assume that__

(1.6) $$[v(\omega)\mu(\omega)]^{-1} \in L^1(0,1) .$$

__Define__

(1.7) $$\hat{b}(s) = \int_{R^+} \frac{d\mu(x)}{s + x}, \quad \text{Re } s > 0, \quad s \neq 0 .$$

__Then__ $(z = \sigma + i\omega)$

$$\tilde{a}(\omega) \overset{\text{dgf}}{=} \lim_{\sigma \downarrow 0} \hat{a}(z) = \lim_{\sigma \downarrow 0} \int_R \hat{b}(z - ix)dv(x)$$

__is well defined and continuously differentiable for__ $\omega \neq 0$ __and sufficiently small and__

(1.8) $$\lim_{\substack{|\sigma+i\omega| \to 0 \\ \sigma \geq 0, |\sigma+i\omega| \neq 0}} |\hat{a}(\sigma + i\omega)| = \infty, \quad \left\| \frac{d}{d\omega} \left(\frac{1}{\tilde{a}(\omega)}\right)\right\|_{L^1(-\epsilon_0,\epsilon_0)} \le c_0$$

__for some positive numbers__ ϵ_0, c_0 __which depend on the local behavior of__ μ, v __only through__ γ, ρ __and the__ L^1__-norms in (1.4) and (1.6).__

To clarify the notation in (1.4) observe that as $v' \in BV_{loc}(0,1]$ then $v'(\omega) = v'_+(\omega) + v'_-(\omega)$, $\omega > 0$, where v'_+ is nondecreasing and v'_- is nonincreasing.

Note that condition (1.1) implies $\mu(0+) = 0$ and that by (1.6) $\mu(x)v(x) > 0$, $x > 0$. But $v(0+) > 0$ is not excluded.

The condition (1.1) is crucial to the proof. In particular it is needed to obtain Lemma 1 (see (2.54)), that is to obtain an appropriate estimate of $\int_0^\omega f(s)ds$ in terms of $f(\omega)$. Here $f(\omega) \overset{\text{dgf}}{=} \text{Re } \hat{b}(i\omega)$, $\omega \neq 0$. In general Lemma 1 ceases to be true if (1.1) does not hold. To see that let for example $\mu(\omega) = [\log \frac{1}{\omega}]^{-\alpha}$, $\alpha > 0$; then $f(\omega) \sim \text{const } \omega^{-1}(\log(\frac{1}{\omega}))^{-1-\alpha}$ and (2.54) is not satisfied as a straightforward computation shows.

More generally one has that if $\mu(x)$ is slowly varying at the origin in the sense of Karamata then (1.1) does not hold. (This is not difficult to prove and we leave it to the reader). Conversely, suppose $\mu(x)$ is not slowly varying at the origin. It then follows that for some $\gamma < 1$ and some sequence $\omega_n \downarrow 0$ one has

$$\gamma\mu(\omega_n) \geq \omega_n^{-1} \int_0^{\omega_n} \mu(s)ds \ .$$

If in addition the sequence $\{\omega_n\}$ satisfies $\liminf_{n\to\infty} \frac{\omega_{n-1}}{\omega_n} > 0$ then one can show that (1.1) follows.

The conditions (1.2), (1.3) are of a technical nature and will be weakened in forthcoming work.

The multiplication of $c(t)$ by the completely monotone function $b(t)$ should have two consequences effecting a comparison between $\frac{d}{d\omega}[\tilde{c}(\omega)]^{-1}$ and $\frac{d}{d\omega}[\tilde{a}(\omega)]^{-1}$ in opposite directions. The first consequence is that as a at infinity will be less than b at infinity then - in general - the singularity of $\tilde{a}$ at the origin will be weaker than that of $\tilde{b}$, i.e. $|\tilde{a}(\omega)| < |\tilde{c}(\omega)|$. The second consequence is that as $\tilde{a} = \tilde{b}*v'$ then some local smoothing is to be expected; $\frac{d\tilde{a}}{d\omega}$ will behave in a nicer fashion than $\frac{d\tilde{c}}{d\omega}$. The hypothesis (1.4) brings out both these effects. To see this note that if v is absolutely continuous than $v'(\omega)$ may be interpreted as $\operatorname{Re} \tilde{c}(\omega)$ and hence (ignoring $\operatorname{Im} \tilde{a}(\omega)$) in order to have $\int_0^1 |\tilde{c}(\omega)|^{-2} |\frac{d\tilde{c}}{d\omega}| d\omega < \infty$ we need

(1.9) $\qquad v'$ absolutely continuous and $\int_0^1 (v')^{-2} |v''| d\omega < \infty$.

However, by (1.5) we have (1.4) provided

(1.10) $\qquad v' \in BV_{loc}(0,1]$ and $\int_0^1 \frac{dv'_+(\omega)}{v'^2(\omega)\mu(\omega)} < \infty$.

The factor μ in the denominator obviously accounts for the weakening of the singularity. On the other hand, the weakening of $|v''(\omega)|d\omega$ to $dv'_+(\omega)$ shows that much less smoothness is now required on v. (In this context the relation (1.3) is irrelevant). In addition note the important difference that (1.9) is a two-sided bound whereas (1.10) restricts only the increase of v' for ω increasing. Observe that as typically $\lim_{\omega\downarrow 0} v'(\omega) = \infty$ then (1.4) need not imply $\int_0^1 \frac{\omega^2}{v^2(\omega)\mu(\omega)} dv'_-(\omega) > -\infty$.

The technical condition (1.5) is primarily needed to obtain (see (2.48))

$$v(\omega + y) - v(\omega - y) \leq \text{const. } y\omega^{-1}v(\omega), \quad \omega > 0, \quad 0 < y \leq 2^{-1}\omega \ .$$

Note that also this condition is one-sided and that frequently, in specific examples, the condition (1.4) can be shown to imply (1.5).

The condition (1.6) basically requires the functions b,c to be sufficiently large at infinity. To see this note that under the assumption of Theorem 1 and assuming in addition

$$\int_\omega^\infty \frac{d\mu(x)}{x} \leq c\,\frac{\mu(\omega)}{\omega}, \quad \omega > 0 \ ,$$

for some constant c then

$$c_1 \frac{\mu(\omega)}{\omega} \leq |\tilde{b}(\omega)| \leq c_2 \frac{\mu(\omega)}{\omega}, \quad \omega > 0 \ ,$$

for some constants c_1, c_2. But by Lemma 1 of [7] we have

$$c_3|\tilde{b}(\omega)| \leq \int_0^{1/\omega} b(t)dt \leq c_4|\tilde{b}(\omega)|, \quad \omega > 0 \ ,$$

for some constants c_3, c_4. For simplicity let finally $c(t)$ be positive, nonincreasing; then $v(\omega) \geq c\cdot\omega \int_0^{1/\omega} c(t)dt$. Under the above conditions one thus has that (1.6) is implied by

$$\int_1^\infty \frac{dt}{\left(\int_0^t c(s)ds\right)\left(\int_0^t b(s)ds\right)} < \infty \ .$$

Note however that although being close to, the relation (1.6) is not equivalent to the requirement $a \notin L^1(R^+)$. To see this take the following simple example constructed in [2]:

$$c(t) \equiv 1, \quad b(t) = t^{-1}[1 - e^{-t}], \quad t > 0$$

Then $v(\omega) \equiv 1$, $\mu(\omega) = \omega$, $0 \leq \omega \leq 1$; $\mu(\omega) = 1$, $\omega \geq 1$, and so $\int_0^1 [v(\omega)\mu(\omega)]^{-1}d\omega = \infty$ although $a(t) = b(t) \notin L^1(R^+)$.

In the next result the condition (1.1) is weakened.

THEOREM 2. __Let__ $\mu(x)$ __be a nondecreasing function defined for__ $x \geq 0$ __with__ $\mu(0) = \mu(0+) = 0$. __Assume that the (completely monotone function__ $b(t) \overset{\text{def}}{=} \int_{R^+} e^{-xt}d\mu(x)$ __is well defined for__ $t > 0$ __and satisfies__ $b \in L^1(0,1)$.

Let $v(x)$ *be an odd, nondecreasing function defined for* $x \in R$ *and such that* (1.2), (1.3), (1.5) *hold. Assume that*

$$\int_0^1 \frac{\lambda^2(\omega)}{v(\omega)\mu(\omega)}\, d\omega < \infty \tag{1.6$_2$}$$

$$\int_0^1 \frac{\omega^2\lambda^2(\omega)}{v^2(\omega)\mu(\omega)}\, d|v'|(\omega) < \infty \tag{1.11}$$

and let for some constant c

$$\omega^3 \int_\omega^\infty \frac{d\mu(x)}{x^3} \le c\mu(\omega), \quad 0 < \omega \le 1 . \tag{1.12}$$

Define $\hat{b}(s)$ *as in (1.7). Then*

$$\operatorname{Re} \tilde{a}(\omega) \overset{\text{def}}{=} \lim_{\sigma \downarrow 0} \operatorname{Re} \hat{a}(z) = \lim_{\sigma \downarrow 0} \int_R \operatorname{Re} \hat{b}(z - ix)\, dv(x) \tag{1.13}$$

is well defined and continuously differentiable for $\omega \neq 0$ *and sufficiently small and*

$$\lim_{\substack{|\sigma+i\omega| \to 0 \\ \sigma > 0,\, |\sigma+i\omega| \neq 0}} [\operatorname{Re} \hat{a}(\sigma + i\omega)] = \infty, \quad \left\| \frac{d}{d\omega} \left(\frac{1}{\operatorname{Re} \tilde{a}(\omega)}\right) \right\|_{L^1(-\varepsilon_0,\varepsilon_0)} \le c_0 \tag{1.14}$$

for some positive numbers ε_0, c *which depend on the local behavior of* μ, v *only through* ρ, c *and the* L^1*-norms in* (1.6$_2$) *and* (1.11).

If in addition

$$\omega^2 \int_\omega^\infty \frac{d\mu(x)}{x^2} \le c\mu(\omega), \quad 0 < \omega \le 1 , \tag{1.15}$$

$$\int_0^1 \frac{\omega^2\lambda^2(\omega)}{v^2(\omega)\mu^2(\omega)}\, d|v'|(\omega) < \infty \tag{1.16}$$

$$\frac{\mu(y)}{y} \in L^1(0,1) \tag{1.17}$$

then $\operatorname{Im} \tilde{a}(\omega)$ *is well defined and continuously differentiable for* $\omega \neq 0$ *and sufficiently* small and

$$(1.18)\qquad \left\| \frac{d}{d\omega} (\mathrm{Im}\ \tilde{a}(\omega))^{-1} \right\|_{L^1(-\epsilon_0,\epsilon_0)} < c_0 ,$$

where ϵ_0, c_0 depend on the local behavior of μ, ν only through ρ, c and the L^1-norms in (1.6_2), (1.16), (1.17).

Above the assumption (1.1) has been weakened to $\omega^{-1}\mu(\omega) \in L^1(0,1)$. (Observe that even this is needed only for handling $\mathrm{Im}\ \tilde{a}(\omega)$). Thus we do allow a large class of slowly varying functions μ; for example

$$\mu(\omega) = \left[\log \frac{1}{\omega}\right]^{-\alpha},\ \alpha > 1 .$$

In this case $b(t) \sim [\log t]^{-\alpha},\ t \to \infty$.

Note that (1.17) implies $\tilde{b}(i\omega) \in L^1(0,1)$.

The key additional hypothesis compared to Theorem 1 that is now being made is th ne restriction on the variation of ν' is two-sided (see (1.11), (1.16)). The smoothnes assumptions on ν do however remain the same; i.e. in (1.11), (1.16) ν' need of course not be continuous.

The following technical complication also arises out of the fact that (1.1) does not hold. A factor $\lambda^2(\omega)$ appears in some of the integrands of Theorems 2 and 3. This factor is defined by

$$(1.19)\qquad \lambda(\omega) = \frac{\mu(\omega)}{\mu(\omega) - \frac{1}{\omega}\int_0^\omega \mu(s)ds} = \frac{\omega\mu(\omega)}{\int_0^\omega s d\mu(s)},\quad \omega > 0 .$$

Clearly $1 < \lambda(\omega) < \infty$ for $\omega > 0$. Note that (1.1) may be interpreted as requiring $\sup_{0<\omega<1} \lambda(\omega) < \infty$. In most cases of interest $\lambda(\omega)$ does however grow only logarithmically; if for example

$$(1.20)\qquad \mu(\omega) = \left[\log\left(\frac{1}{\omega}\right)\right]^{-\alpha},\quad \alpha > 0 ,$$

then

$$\limsup_{\omega \downarrow 0} \frac{\lambda(\omega)}{\alpha \log\left(\frac{1}{\omega}\right)} < 1 .$$

More generally, suppose $d\mu(s) = \mu'(s)ds$ with $\mu'(s)$ nonincreasing. Then by (1.19),

$\lambda(\omega) < 2\mu(\omega)[\omega\mu'(\omega)]^{-1}$. Hence the appearance of λ does not significantly increase the size of the integrands in (1.6_2), (1.11) and (1.16).

We also observe that the examples (1.20) do satisfy the fairly weak conditions (1.12), (1.15). Of course, (1.15) implies (1.12).

Our next result concerns the case when (1.17) does not hold; thus there is no size restriction on $b(t)$ at infinity except that $\lim_{t\to\infty} b(t) = 0$. Note that $\tilde{b}(\omega) \in L^1(0,1)$ need not be satisfied any more.

THEOREM 3. *Let the assumptions of Theorem 2 except (1.16), (1.17) be satisfied. In addition let*

$$(1.21) \qquad \left|\int_{\omega}^{\omega+y} \{v''(s) - v''(s - y)\}ds\right| < \frac{y}{\log^{\alpha}(\frac{1}{y})} F(\omega), \quad 0 < \omega < 1, \quad 0 < y < \omega/2 ,$$

for some $\alpha > 1$ *and some* F *satisfying*

$$(1.22) \qquad \int_0^1 \frac{\omega^2\lambda^2(\omega)F(\omega)}{v^2(\omega)\mu(\omega)}\, d\omega < \infty .$$

Define for $\varepsilon > 0$, $\omega \neq 0$, $\tilde{a}_{\varepsilon}(\omega) = \int_{|x-\omega|>\varepsilon\omega} \tilde{b}(\omega - x)dv(x)$. *Then* $\tilde{a}_{\varepsilon}(\omega)$ *is continuously differentiable for* $\omega \neq 0$ *and sufficiently small and there exist positive constants* ε_0, c_0 *such that*

$$(1.23) \qquad \sup_{0<\varepsilon<\varepsilon_0} \left\|\frac{d}{d\omega}\left(\frac{1}{\tilde{a}_{\varepsilon}(\omega)}\right)\right\|_{L^1(-\varepsilon_0,\varepsilon_0)} < c_0 .$$

The constants ε_0, c_0 *depend on the local behavior of* μ, v *only through* ρ, c *of (1.15) and the* L^1*-norms of* (1.6_2), (1.11), (1.22).

The above condition (1.21) does require v' to be absolutely continuous and is thus stronger as to the smoothness of v than (1.4), (1.11), (1.16). Typically $F = v''$ in which case (1.22) is seen to reduce to (1.11) - modulo the added smoothness.

Our final result, Theorem 4, also considers the case when $\tilde{b} \notin L^1_{loc}$. We state it without proof; the proof will be given in a forthcoming report.

THEOREM 4. *Let the assumptions of Theorem 2 except (1.16), (1.17) be satisfied. In addition let* μ *be absolutely continuous with*

(1.24) $$\mu'(\omega) \le c\,\frac{\mu(\omega)}{\omega}, \quad 0 < \omega \le 1,$$

<u>for some constant</u> c. <u>Define</u> $h(\omega) = \omega^2\lambda^2(\omega)\nu''(\omega)[\nu^2(\omega)\mu(\omega)]^{-1}$ <u>and</u>

$\omega_h(y) = \|h(\omega + y) - h(\omega)\|_{L^1(0,1)}$. <u>(By (1.11)</u> $h \in L^1(0,1)$). <u>Assume that</u> h <u>satisfies</u> <u>the integrated Dini-condition</u>

(1.25) $$\int_0^1 \frac{\omega_h(y)}{y}\,dy < \infty .$$

<u>Define</u> $\tilde{a}_\varepsilon$ <u>as in Theorem 3. Then (1.23) holds with the constants</u> ε_0, c_0 <u>depending on the local behavior of</u> μ, ν <u>only through</u> ρ, c <u>of (1.15), (1.24), and the</u> L^1<u>-norms of</u> $h(y)$, $y^{-1}\omega_h(y)$ <u>and</u> $\lambda^2[\nu\mu]^{-1}$.

Note that at the expense of more smoothness on μ and the Dini-condition we are able to do without $\mu(y)y^{-1} \in L^1_{loc}$.

The result (1.23) of Theorems 3 and 4 is of course intermediate in nature. Although it does constitute the essential step towards obtaining $r \in L^1(R^+)$ it is not entirely routine how to accomplish the remaining steps. As we intend to return to this topic in a forthcoming report we only sketch the procedure here.

The root of the difficulty lies of course in the fact that under the conditions of Theorems 3 and 4 one does not have $\tilde{b}(\omega) \in L^1_{loc}(R)$. A consequence is that the analysis yielding (2.29) and in particular (2.41) is not valid. But note that we always have $\mathrm{Re}\,\tilde{b}(\omega) \in L^1_{loc}(R)$ and thus the difficulty appears only when the imaginary part is considered. However, as $\mathrm{Im}\,\hat{b}(\sigma + i\omega)$ is odd with respect to ω for $\sigma > 0$ one may by assuming, with $i = 1$,

(1.26) $$\begin{cases} |\nu^{(i)}(\omega) - \nu^{(i)}(x)| \le c(\omega)|\omega - x|^\alpha, & \omega \ne 0,\ 0 < |x| \le \omega/2, \\ \text{for some } \alpha > 0,\ c(\omega) \in L^\infty[\delta,\infty), & \delta > 0, \end{cases}$$

still obtain (2.29) as a principal value integral. Suppose now that ν_n satisfy (1.26) with $i = 2$, ($c(\omega)$, α may depend on n) and that (for example)

$$(1.27)\qquad \begin{cases} \sup_n \left\{ \|h_n\|_{L^1(0,1)} + \left\|\frac{\omega_{h_n}(y)}{y}\right\|_{L^1(0,1)} + \left\|\frac{\lambda^2}{v_n^2\mu^2}\right\|_{L^1(0,1)} \right\} < \infty , \\ \sup_n \rho_n < \infty , \end{cases}$$

where h_n, ω_{h_n} of course correspond to v_n in the manner h, ω_h correspond to v and ρ_n is such that $v_n^1 < \rho_n \omega^{-1} v_n$. Denote $\tilde{a}_n = \tilde{b} * v_n'$, $\tilde{a}_{\varepsilon n}(\omega) = \int_{|\omega - x| > \varepsilon\omega} \tilde{b}(\omega - x) v_n'(x)dx$. Then $\tilde{a}_{\varepsilon n} \to \tilde{a}_n$ as $\varepsilon \downarrow 0$ in the sense of (2.41) and by (1.27) and the last sentence of Theorem 4,

$$(1.28)\qquad \sup_n \int_0^{\varepsilon_0} |\tilde{a}_n|^2 \left|\frac{d\tilde{a}_n}{d\omega}\right| d\omega < \infty .$$

Suppose that v_n can be made to approximate v in the sense that for some $q \in (1,2]$ and any $\delta > 0$

$$\|\tilde{a}_n - \tilde{a}\|_{L^q((-\infty,-\delta)\cup(\delta,\infty))} \to 0 \quad \text{for} \quad n \to \infty .$$

Then $\|\tilde{r}_n - \tilde{r}\|_{L^q(R)} \to 0$, $\tilde{r}_n \overset{\text{def}}{=} \tilde{a}_n[1 + \tilde{a}_n]^{-1}$, and consequently $\|r_n - r\|_{L^{q'}(R^+)} \to 0$, as $n \to \infty$. But by (1.28) $r_n \in L^1(R^+)$ with $\sup_n \|r_n\|_{L^1(R^+)} < \infty$ and so $r \in L^1(R^+)$.

2. PROOF OF THEOREM 1

We begin by deducing some properties of $\tilde{b}(\omega) \overset{\text{def}}{=} \hat{b}(i\omega)$. Write

$$(2.1) \qquad f(\omega) \overset{\text{def}}{=} \operatorname{Re} \tilde{b}(\omega) = \int_{R^+} \frac{x}{x^2+\omega^2}\, d\mu(x), \quad \omega \neq 0 ,$$

$$(2.2) \qquad g(\omega) \overset{\text{def}}{=} \operatorname{Im} \tilde{b}(\omega) = -\int_{R^+} \frac{\omega d\mu(x)}{x^2+\omega^2}, \quad \omega \neq 0 .$$

Note that both f and g are locally absolutely continuous on $R\backslash\{0\}$, and that

$$(2.3) \qquad f \text{ is even with } f(\omega) > 0, \quad \omega > 0 ,$$

$$(2.4) \qquad g \text{ is odd with } g(\omega) < 0, \quad \omega > 0 ,$$

$$(2.5) \qquad \frac{df}{d\omega} = -\int_{R^+} \frac{2x\omega\, d\mu(x)}{(x^2+\omega^2)^2}, \quad \omega \neq 0 ,$$

$$(2.6) \qquad \frac{dg}{d\omega} = \int_{R^+} \frac{\omega^2 - x^2}{(\omega^2+x^2)^2}\, d\mu(x), \quad \omega \neq 0 ,$$

$$(2.7) \qquad \int_0^\omega f(s)ds = \int_{R^+} tg^{-1}\left(\frac{\omega}{x}\right) d\mu(x), \quad \omega > 0 .$$

Thus in particular $f \in L^1(0,1)$. From (1.1) follows

$$(2.8) \qquad [1-\gamma]\mu(\omega) < \mu(\omega) - \omega^{-1}\int_0^\omega \mu(s)ds = \omega^{-1}\left[\int_0^\omega x d\mu(x)\right], \quad 0 < \omega < 1 ,$$

and so $(\lambda \overset{\text{def}}{=} [1-\gamma]^{-1})$

$$(2.9) \qquad 0 < \frac{\mu(\omega)}{\omega} < \lambda\omega^{-2}\int_0^\omega x d\mu(x), \quad 0 < \omega < 1 .$$

But a straightforward integration shows that $\omega^{-2}\int_0^\omega x d\mu(x) \in L^1(0,1)$ and therefore

$$(2.10) \qquad \omega^{-1}\mu(\omega) \in L^1(0,1) .$$

It is not difficult to show that (2.2), (2.10) imply $g \in L^1(0,1)$ (we leave the verification of this to the reader) and thus

(2.11) $$\int_{|\omega|<1} |\tilde{b}(\omega)| d\omega < \infty .$$

From the monotonicity of f we immediately have

(2.12) $$0 < f(\omega) < \frac{1}{|\omega|} \int_0^{|\omega|} f(s)ds, \quad \omega \neq 0 .$$

We wish to estimate $\frac{df}{d\omega}$, g, $\frac{dg}{d\omega}$ in terms of f. For $\omega \neq 0$ we have

(2.13) $$\left|\frac{df}{d\omega}\right| < \frac{2}{|\omega|} \int_{R^+} \frac{x d\mu(x)}{x^2 + \omega^2} = \frac{2}{|\omega|} f(\omega) .$$

Then note that from (2.7) follows

(2.14) $$\int_0^{\omega} \frac{\omega d\mu(x)}{x^2 + \omega^2} < \frac{\mu(\omega)}{\omega} < \frac{4}{\pi\omega} \int_0^{\omega} tg^{-1}\left(\frac{\omega}{x}\right) d\mu(x) < \frac{4}{\pi\omega} \int_0^{\omega} f(s)ds$$

and as obviously

(2.15) $$\int_{\omega}^{\infty} \frac{\omega d\mu(x)}{x^2 + \omega^2} < f(\omega)$$

then

(2.16) $$|g(\omega)| < \left[1 + \frac{4}{\pi}\right] |\omega|^{-1} \int_0^{|\omega|} f(s)ds, \quad \omega \neq 0 .$$

Observe that (2.12), (2.16) yield

(2.17) $$|\omega\tilde{b}(\omega)| \to 0 \quad \text{for} \quad \omega \to 0 .$$

Continuing the relation (2.9) we have

(2.18) $$\frac{\mu(\omega)}{\omega} < 2\lambda \int_0^{\omega} \frac{x d\mu(x)}{x^2 + \omega^2} < 2\lambda f(\omega) .$$

Thus

(2.19) $$\int_0^{\omega} \frac{\omega^2 - x^2}{(\omega^2 + x^2)^2} d\mu((x) < \frac{\mu(\omega)}{\omega^2} < \frac{2\lambda}{\omega} f(\omega), \quad \omega > 0 .$$

Let $\omega > 0$. If $x > \omega$ then $\frac{x^2 - \omega^2}{x^2 + \omega^2} < \frac{x}{\omega}$ and so

(2.20) $$\left|\int_\omega^\infty \frac{\omega^2 - x^2}{(\omega^2 + x^2)^2}\, d\mu(x)\right| < \frac{1}{\omega}\int_\omega^\infty \frac{x\,d\mu(x)}{x^2 + \omega^2} < \frac{f(\omega)}{\omega} .$$

From (2.6), (2.19), (2.20)

(2.21) $$\left|\frac{dq}{d\omega}\right| < [1 + 2\lambda]\, \frac{1}{|\omega|}\, f(\omega), \quad \omega \neq 0 .$$

Define c by

(2.22) $$c(t) = \frac{1}{2}\int_R e^{itx} dv(x), \quad 0 < t < \infty$$

For $\operatorname{Re} z > 0$ we then have

(2.23) $$\int_{R^+} e^{-zt} b(t)c(t)dt = \frac{1}{2}\int_R \left\{\int_{R^+} e^{-zt}e^{itx} b(t)dt\right\}dv(x) = \frac{1}{2}\int_R \hat{b}(z - ix)dv(x) = \hat{a}(z)$$

where the use of Fubinis theorem is allowed as

(2.24) $$dv > 0, \qquad v(\infty) < \infty ,$$

and by the fact that $e^{-zt}b(t) \in L^1(R^+)$ for $\operatorname{Re} z > 0$. Write $z = \sigma + i\omega$; fix ε, ω such that $0 < \varepsilon < |\omega| < \frac{1}{2}$ and split the integral defining $\hat{a}$:

(2.25) $$2\hat{a}(\sigma + i\omega) = \int_{|\omega - x| > \varepsilon} + \int_{|\omega - x| < \varepsilon} \{\hat{b}(\sigma + i\omega - ix)dv(x)\} .$$

Provided $\omega \neq x$ then $\lim_{\sigma \downarrow 0} \hat{b}(\sigma + i\omega - ix)$ is well-defined. Also, by (2.24) and as $\sup_{\substack{\sigma > 0\\ |\omega - x| > \varepsilon}} |\hat{b}(\sigma + i\omega - ix)| < \infty$ we may use the dominated convergence theorem to get

(2.26) $$\lim_{\sigma \downarrow 0} \int_{|\omega - x| > \varepsilon} \hat{b}(\sigma + i\omega - ix)dv(x) = \int_{|\omega - x| > \varepsilon} \tilde{b}(\omega - x)dv(x) .$$

By (1.3) $\int_{|\omega - x| < \varepsilon} v'(x) < \infty$. Also note that if $b_\sigma(t) \overset{\text{def}}{=} e^{-\sigma t}b(t)$, $\sigma > 0$; then $b_\sigma(t)$ is convex and so by Lemma 1 of [7], $\omega \neq 0$,

(2.27) $$|\hat{b}(\sigma + i\omega)| = |\tilde{b}_\sigma(\omega)| < c_1 \int_0^{1/|\omega|} e^{-\sigma t} b(t)dt < c_1 \int_0^{1/|\omega|} b(t)dt < c_2\, |\tilde{b}(\omega)| ,$$

for some constants c_1, c_2. If $b(0+) = \infty$ or if $|b^1(0+)| = \infty$, define

$b_n(t) = b(t + n^{-1})$, obtain (2.27) for $b_n(t)$, let $n \to \infty$ and apply the monotone convergence theorem to get (2.27) for b). Thus an application of the dominated convergence theorem yields

$$\text{(2.28)} \qquad \lim_{\sigma\downarrow 0} \int_{|\omega-x|<\varepsilon} \hat{b}(\sigma + i\omega - ix)dv(x) = \int_{|\omega-x|<\varepsilon} \tilde{b}(\omega - x)v'(x)dx, \quad 0 < \varepsilon < |\omega| < \frac{1}{2} .$$

Hence, by (2.25), (2.26), (2.28) $\tilde{a}(\omega) = \lim_{\sigma\downarrow 0} \hat{a}(\sigma + i\omega)$ exists and satisfies

$$\text{(2.29)} \qquad \tilde{a}(\omega) = \frac{1}{2}\int_R \tilde{b}(\omega - x)dv(x), \quad 0 < |\omega| < \frac{1}{2} .$$

Also observe that the uniform continuity of $\tilde{b}(\omega)$ on $0 < \sigma < |\omega|$, (1.3) and (2.24) yield $\tilde{a} \in C\left(0 < |\omega| < \frac{1}{2}\right)$.

We next demonstrate that the first part of (1.8) holds. In particular we show that

$$\text{(2.30)} \qquad \lim_{\substack{\sigma+i\omega\to 0\\ \sigma\geq 0,\,\sigma+i\omega\neq 0}} \int_{-1/2}^{1/2} \operatorname{Re} \hat{b}(\sigma + i\omega - ix)dv(x) = \infty .$$

The first part of (1.8) follows from (2.30) as $dv \geq 0$ and as $\operatorname{Re} \hat{b}(z) \geq 0$ for $\operatorname{Re} z \geq 0$, $z \neq 0$. For $\sigma \geq 0$, $\omega \in R$, $s > 0$ one has

$$\frac{\sigma + s}{(\sigma + s)^2 + \omega^2} \geq \frac{1}{2}\,\frac{s}{(\sigma + |\omega|)^2 + s^2}$$

and therefore, as μ is nondecreasing,

$$\text{(2.31)} \qquad \operatorname{Re} \hat{b}(\sigma + i\omega) = \int_{R^+} \frac{(\sigma + s)d\mu(s)}{(\sigma + s)^2 + \omega^2} \geq \frac{1}{2}\int_{R^+} \frac{s\,d\mu(s)}{(\sigma + |\omega|)^2 + s^2} = \frac{1}{2} f(\sigma + |\omega|)$$

We assert that (2.33) holds. Observe that from (2.18) and by (1.6)

$$\text{(2.32)} \qquad \frac{1}{sf(s)v(s)} \leq \frac{2\lambda}{\mu(s)v(s)} \in L^1(0,1) .$$

Therefore, if $s \frac{d}{ds}\left[\frac{1}{sf(s)v(s)}\right] \in L^1(0,1)$ then

$$\text{(2.33)} \qquad \lim_{s\downarrow 0} [f(s)v(s)]^{-1} = 0 .$$

But from (1.5), (2.13), (2.32)

$$\left|s\,\frac{d}{ds}\left[\frac{1}{sf(s)v(s)}\right]\right| < [3+\rho][sf(s)v(s)]^{-1} \in L^1(0,1)$$

and so (2.33) is true.

For $\operatorname{Re} z > 0$, $z \neq 0$, one has - as pointed out above - $\operatorname{Re}\hat{b}(z) > 0$ and also $\omega \frac{d}{d\omega}\operatorname{Re}\hat{b}(\sigma + i\omega) < 0$, $\omega \neq 0$. Therefore, for $0 < T < 2^{-1}$ and by (2.31)

$$(2.34)\quad \begin{cases} \int_{-1/2}^{1/2} \operatorname{Re}\hat{b}(\sigma + i\omega - ix)dv(x) > \int_{|x|<T} \operatorname{Re}\hat{b}(\sigma + i\omega - ix)dv(x) > \\ 2\operatorname{Re}\hat{b}(\sigma + i|\omega + T|)v(T) > f(\sigma + |\omega + T|)v(T) \end{cases}$$

If $0 < \sigma < |\omega| < \frac{1}{2}$ let $T = |\omega|$. Then by (2.13), (2.34)

$$(2.35)\quad \int_{-1/2}^{1/2} \operatorname{Re}\hat{b}(\sigma + i\omega - ix)dv(x) > f(3\omega)v(|\omega|) > \frac{1}{9}f(\omega)v(|\omega|)\ .$$

If $0 < |\omega| < \sigma < \frac{1}{2}$ let $T = \tau$. Again by (2.13), (2.34)

$$(2.36)\quad \int_{-1/2}^{1/2} \operatorname{Re}\hat{b}(\sigma + i\omega - ix)dv(x) > \frac{1}{9}f(\sigma)v(\sigma)$$

From (2.33), (2.35), (2.36) we have (2.30)

Our next aim is to approximate $\tilde{a}(\omega)$. Take $\varepsilon \in \left(0, \frac{1}{4}\right]$ and define $\tilde{a}_\varepsilon$ by

$$2\tilde{a}_\varepsilon(\omega) = \int_{|\omega-x|>\varepsilon\omega} \tilde{b}(\omega - x)dv(x),\quad 0 < |\omega| < \frac{1}{4}\ .$$

Then $\tilde{a}_\varepsilon \in C\left(0 < |\omega| < \frac{1}{4}\right)$, $\operatorname{Re}\tilde{a}_\varepsilon$ is even, $\operatorname{Im}\tilde{a}_\varepsilon$ is odd and obviously

$$(2.37)\quad 2(\tilde{a} - \tilde{a}_\varepsilon)(\omega) = \int_{|\omega-x|<\varepsilon|\omega|} \tilde{b}(\omega - x)dv(x),\quad 0 < |\omega| < \frac{1}{4}\ .$$

Also note that

$$(2.38)\quad \begin{cases} \dfrac{d\tilde{a}_\varepsilon}{d\omega} \in C\left(0 < |\omega| < \frac{1}{4}\right)\ , \\ 2\,\dfrac{d}{d\omega}(\tilde{a} - \tilde{a}_\varepsilon)(\omega) = \varepsilon\tilde{b}(\varepsilon\omega)v'(\omega - \varepsilon\omega) + \varepsilon\tilde{b}(-\varepsilon\omega)v'(\omega + \varepsilon\omega) + \int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} \tilde{b}(\omega-s)v''(s)ds\ , \end{cases}$$

and hence

$$\frac{d\tilde{a}}{d\omega} \in C\left(0 < |\omega| < \frac{1}{4}\right) \tag{2.39}$$

follows from (1.3), (2.24) and from the fact that $\frac{d\tilde{b}}{d\omega}$ is uniformly continuous on $[\tau,\infty)$, $\tau > 0$. We claim that for any fixed $s \in \left(0, \frac{1}{4}\right)$ one has

$$\lim_{\varepsilon \downarrow 0} \sup_{s<|\omega|<\frac{1}{4}} \left[|\tilde{a}(\omega) - \tilde{a}_\varepsilon(\omega)| + \left|\frac{d\tilde{a}(\omega)}{d\omega} - \frac{d\tilde{a}_\varepsilon(\omega)}{d\omega}\right| \right] = 0 . \tag{2.40}$$

To obtain the first part of (2.40) it suffices to use (1.3) in (2.37). The second part of (2.40) follows provided one employs (2.17) and (1.3) in the second part of (2.38).

From (2.30) follows that $|\tilde{a}|$ is certainly bounded away from zero on $0 < |\omega| < \frac{1}{4}$. Consequently by (2.40) and the continuity of all functions involved,

$$\lim_{\varepsilon \downarrow 0} \int_{\tau_1<|\omega|<\tau_2} |\tilde{a}_\varepsilon(\omega)|^{-2}\left|\frac{d\tilde{a}_\varepsilon}{d\omega}\right| d\omega = \int_{\tau_1<|\omega|<\tau_2} |\tilde{a}(\omega)|^{-2}\left|\frac{d\tilde{a}}{d\omega}\right| d\omega < \infty \tag{2.41}$$

for any $0 < \tau_1 < \tau_2 < \frac{1}{4}$. Suppose there exists $\varepsilon_0 \in \left(0, \frac{1}{4}\right]$ such that

$$C_0 \overset{\text{def}}{=} \sup_{0<\varepsilon<\varepsilon_0} \int_{|\omega|<\varepsilon_0} |\tilde{a}_\varepsilon(\omega)|^{-2}\left|\frac{d\tilde{a}_\varepsilon}{d\omega}\right| d\omega < \infty . \tag{2.42}$$

Then, by (2.41), we would have the second part of (1.8). Thus it suffices to prove (2.42). *Before attempting to prove this relation we make an additional simplification of* the problem.

For $0 < |\omega| < \frac{1}{4}$, $0 < \varepsilon < \frac{1}{4}$ define $\tilde{\alpha}_\varepsilon(\omega)$ by

$$2\tilde{\alpha}_\varepsilon(\omega) = \int_{\omega+\varepsilon\omega}^{1/2} + \int_{-1/2}^{\omega-\varepsilon\omega} \tilde{b}(\omega - s)dv(s) .$$

As $\mathrm{Re}(\tilde{a}_\varepsilon - \tilde{\alpha}_\varepsilon)(\omega) > 0$, $\mathrm{Re}\,\tilde{\alpha}_\varepsilon(\omega) > 0$ it follows that

(2.43) $$|\tilde{a}_\varepsilon|^{-2}\left|\frac{d\tilde{a}_\varepsilon}{d\omega}\right| \le [\mathrm{Re}\ \tilde{\alpha}_\varepsilon]^{-2}\left|\frac{d\tilde{\alpha}_\varepsilon}{d\omega}\right| + [\mathrm{Re}\ \tilde{\alpha}_\varepsilon]^{-2}\left|\frac{d(\tilde{a}_\varepsilon - \tilde{\alpha}_\varepsilon)}{d\omega}\right| .$$

Straightforward estimates yield

(2.44) $$\sup_{0<\varepsilon\le\frac{1}{4}} \int_{-1/4}^{1/4} [\mathrm{Re}\ \tilde{\alpha}_\varepsilon]^{-2}\left|\frac{d(\tilde{a}_\varepsilon - \tilde{\alpha}_\varepsilon)}{d\omega}\right| d\omega \le c_0$$

where the constant c_0 depends on v only through $v(\frac{1}{2})^{-2} \int_{1/2}^{\infty} \frac{dv(x)}{x}$. A consequence of (2.43), (2.44) is of course that (2.42) holds provided we show that there exists $\varepsilon_0 \in (0, \frac{1}{4}]$, $c_0 < \infty$ such that

(2.45) $$\int_{|\omega|\le\varepsilon_0} \alpha_\varepsilon^{-2}\left|\frac{d\alpha_\varepsilon}{d\omega}\right| d\omega \le c_0, \quad 0 < \varepsilon \le \varepsilon_0 ,$$

(2.46) $$\int_{|\omega|\le\varepsilon_0} \alpha_\varepsilon^{-2}\left|\frac{d\beta_\varepsilon}{d\omega}\right| d\omega \le c_0, \quad 0 < \varepsilon \le \varepsilon_0 ,$$

where we have defined $\alpha_\varepsilon = \mathrm{Re}\ \tilde{\alpha}_\varepsilon$, $\beta_\varepsilon = \mathrm{Im}\ \tilde{\alpha}_\varepsilon$.

For the proof of (2.45), (2.46) we need the *following easy consequences* of (1.5). Without loss of generality let $\rho \ge 1$ and assume $\varepsilon_0 > 0$ satisfies $2\rho\varepsilon_0 \le 1$. Then (1.5) yields, provided $0 < \omega \le \frac{1}{4}$, $0 < y \le \varepsilon_0\omega$,

(2.47) $$v(\omega + \varepsilon_0\omega) \le 2v(\omega) ,$$

(2.48) $$v(\omega + y) - v(\omega - y) \le c_\rho y\omega^{-1}v(\omega), \quad c_\rho = 2^3\rho .$$

A consequence of $dv \ge 0$, (2.47) and of the fact that f is positive, decreasing for $\omega > 0$ is (provided $\varepsilon \le \varepsilon_0$)

(2.49) $$\begin{aligned} \alpha_\varepsilon(\omega) &\ge \int_0^{\omega-\varepsilon\omega} f(\omega - s)dv(s) \ge f(\omega)v(\omega - \varepsilon\omega) \ge \frac{1}{2}\, v(\omega)f(\omega) \\ &\ge (4\lambda\omega)^{-1}v(\omega)\mu(\omega), \quad 0 < \omega \le \frac{1}{4} , \end{aligned}$$

where the last inequality follows from (2.19). Therefore, recall (1.6),

$$(2.50) \qquad \sup_{0<\varepsilon<\varepsilon_0} \int_0^{1/4} \frac{d\omega}{\omega\alpha_\varepsilon(\omega)} < 4\lambda \int_0^{1/4} [v(\omega)\mu(\omega)]^{-1} d\omega < \infty .$$

For $0 < \omega < \frac{1}{4}$, $0 < \varepsilon < \varepsilon_0$, we have

$$(2.51) \qquad \frac{d\alpha_\varepsilon}{d\omega} = \sum_{i=1}^{3} I_i , \qquad \frac{d\beta_\varepsilon}{d\omega} = \sum_{i=4}^{6} I_i ,$$

where $\left(h \overset{\mathrm{def}}{=} |g|\right)$

$$(2.52) \qquad \begin{aligned}
I_1 &= \int_{\omega+\varepsilon_0\omega}^{1/2} + \int_{-1/2}^{\omega-\varepsilon_0\omega} \{f'(\omega - s)v'(s)\}ds \\
I_2 &= \int_{\omega+\varepsilon\omega}^{\omega+\varepsilon_0\omega} + \int_{\omega-\varepsilon_0\omega}^{\omega-\varepsilon\omega} \{f'(\omega - s)v'(s)\}ds \\
I_3 &= f(\varepsilon\omega)[(1 - \varepsilon)v'(\omega - \varepsilon\omega) - (1 + \varepsilon)v'(\omega + \varepsilon\omega)] \\
I_4 &= \int_{\omega+\varepsilon_0\omega}^{1/2} \int_{-1/2}^{\omega-\varepsilon_0\omega} \{h'(\omega - s)v'(s)\}ds \\
I_5 &= h(\varepsilon\omega)v'(\omega + \varepsilon\omega)[-1 - \varepsilon] + \int_{\omega+\varepsilon\omega}^{\omega+\varepsilon_0\omega} h'(\omega - s)v'(s)ds \\
I_6 &= -h(\varepsilon\omega)v'(\omega - \varepsilon\omega)[1 - \varepsilon] - \int_{\omega-\varepsilon_0\omega}^{\omega-\varepsilon\omega} h'(\omega - s)v'(s)ds
\end{aligned}$$

Denote

$$(2.53) \qquad S_i = \int_0^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)|I_i|d\omega; \quad i = 1,\dots,6 .$$

To proceed with the estimation of the quantities S_i we need the following lemma which crucially depends on (1.1).

LEMMA 1. For any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\int_0^{\varepsilon_0\omega} f(s)ds < \delta\omega f(\omega), \quad 0 < \omega < \varepsilon_0 . \tag{2.54}$$

The proof of this lemma is given in Section 5.

Choose $\delta > 0$ sufficiently small then fix $\varepsilon_0 < (2\rho)^{-1}$ such that (2.54) holds. Let $\varepsilon \in (0,\varepsilon_0]$ thus

$$\int_0^{\varepsilon\omega} f(s)ds < \int_0^{\varepsilon_0\omega} f(s)ds < \delta\omega f(\omega), \quad 0 < \omega < \varepsilon_0 . \tag{2.55}$$

From (2.13) and as f, $dv > 0$,

$$|I_1| < \frac{2}{\varepsilon_0\omega}\int_{\omega+\varepsilon_0\omega}^{1/2} + \int_{-1/2}^{\omega-\varepsilon_0\omega} \{f(\omega - s)v'(s)ds\} < 2[\varepsilon_0\omega]^{-1}\alpha_\varepsilon(\omega)$$

Consequently, by (2.49)

$$S_1 < 2^3\lambda\varepsilon_0^{-1}\int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} . \tag{2.56}$$

To avoid any convergence problem in the estimation of S_2 (which will occupy us until (2.71)) we conider $S_{2\eta} \overset{\text{def}}{=} \int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}|I_2|d\omega$, $0 < \eta < \varepsilon_0$, and obtain bounds independent of η. The fact that f is even gives us

$$\begin{aligned} I_2 &= \int_{\varepsilon\omega}^{\varepsilon_0\omega} f'(y)[v'(\omega - y) - v'(\omega + y)]dy \\ &= -\int_{\varepsilon\omega}^{\varepsilon_0\omega} f'(y)\Big[\int_{\omega-y}^{\omega+y} dv_+'(s)\Big]dy + \int_{\varepsilon\omega}^{\varepsilon_0\omega} f'(y)\Big[v'(\omega - y) - v'(\omega + y) + \int_{\omega-y}^{\omega+y} dv_+'(s)\Big]dy \end{aligned}$$

where an additional term has been added and subtracted. As

$$v'(\omega - y) - v'(\omega + y) + \int_{\omega-y}^{\omega+y} dv_+'(s) = -\int_{\omega-y}^{\omega+y} dv_-'(s) > 0$$

we obtain

(2.57) $$S_{2\eta} \le 2R_{1\eta} + R_{2\eta} ,$$

where

(2.58) $$R_{1\eta} = \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\Big[\int_{\varepsilon\omega}^{\varepsilon_0\omega} |f'(y)|\{\int_{\omega-y}^{\omega+y} dv_+'(s)\}dy\Big]d\omega ,$$

(2.59) $$R_{2\eta} = \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\Big[\int_{\varepsilon\omega}^{\varepsilon_0\omega} |f'(y)|\{v'(\omega - y) - v'(\omega + y)\}dy\Big]d\omega .$$

For the estimation of $R_{1\eta}$ it is convenient to approximate $\int_{\omega-y}^{\omega+y} dv_+'(s)$ by a step function. Take any sequence $\{\omega_i\}_{i=1}^{\infty}$ satisfying

(2.60) $$2^{-1} > \omega_1 > \omega_2 > \dots > 0; \quad 2\omega_i > \omega_{i-1}, \ \forall_i; \quad \omega_i \to 0 \ \text{ for } \ i \to \infty ,$$

and define $a_i = \int_{\omega_i}^{\omega_{i-1}} dv_+'(s) \ge 0$. (Formally $dv_+'(s) \approx \sum_i a_i \delta_{\omega_i}(s)$ where $\delta_{\omega_i}(s)$ is the Dirac δ-function with support at ω_i). Our task is of course now to estimate

(2.61) $$\Psi \overset{\text{def}}{=} \sum_i \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\Big[\int_{\varepsilon\omega}^{\varepsilon_0\omega} |f'(y)|\{\phi_i(\omega + y) - \phi_i(\omega - y)\}dy\Big]d\omega$$

where we have defined

(2.62) $$\phi_i(\omega) = -a_i h(\omega_i - \omega), \quad 0 < \omega < \tfrac{1}{2}, \quad h(\omega) = 1, \quad \omega > 0; \quad h(\omega) = 0, \quad \omega < 0 ,$$

and to obtain a bound independent of $\{\omega_i\}$, η. Note that for each fixed $\eta > 0$ the series in (2.61) contains only a finite number of terms.

For each i we have (c is an a priori constant depending on ρ, γ whose value varies from one place to another)

(2.63) $$\int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\Big[\int_{\varepsilon\omega}^{\varepsilon_0\omega} |f'(y)|\{\phi_i(\omega + y) - \phi_i(\omega - y)\}dy\Big]d\omega$$

$$\le a_i \int_{\omega_i[1+\varepsilon_0]^{-1}}^{\omega_i[1-\varepsilon_0]^{-1}} \alpha_\varepsilon^{-2}(\omega)\Big[\int_{|\omega-\omega_i|}^{\varepsilon_0\omega} |f'(y)|dy\Big]d\omega$$

$$< \frac{4a_i}{v^2(\omega_i[1+\varepsilon_0]^{-1})f^2(\omega_i[1-\varepsilon_0]^{-1})} \int_{\omega_i[1+\varepsilon_0]^{-1}}^{\omega_i[1-\varepsilon_0]^{-1}} f(|\omega - \omega_i|)d\omega$$

$$< \frac{ca_i}{v^2(\omega_{i-1})f^2(\omega_i)} \int_0^{2\varepsilon_0\omega_i} f(y)dy < \frac{[16\lambda]^{-1}a_i\omega_i}{v^2(\omega_{i-1})f(\omega_i)} < \frac{8^{-1}a_i\omega_i^2}{v^2(\omega_{i-1})\mu(\omega_{i-1})}$$

where the first inequality follows by the definition of ϕ_i; the second uses the monotonicity of f, v and (2.49); the third follows by (1.5), (2.13) and $2\omega_i > \omega_{i-1}$; the fourth follows from (2.55) after taking δ sufficiently small and the last uses the monotonicity of f, (2.49) and again the fact that $2\omega_i > \omega_{i-1}$. Thus, by the monotonicity of ω, v, μ,

$$\Psi < \frac{1}{8}\sum_i \frac{a_i\omega_i^2}{v^2(\omega_{i-1})\mu(\omega_{i-1})} < \frac{1}{8}\int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)}$$

and one concludes that

$$2R_{1\eta} < \frac{1}{4}\int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)} . \tag{2.64}$$

We proceed to estimate $R_{2\eta}$. An interchange of the order of integration produces

$$R_{2\eta} = \int_{\varepsilon\varepsilon_0}^{\varepsilon_0^2}\int_z^{\varepsilon_0} + \int_{\varepsilon_0\eta}^{\varepsilon\varepsilon_0}\int_z^{y/\varepsilon} + \int_{\varepsilon\eta}^{\varepsilon_0\eta}\int_\eta^{y/\varepsilon} |f'(y)|\alpha_\varepsilon^{-2}(\omega)[v'(\omega - y) - v'(\omega + y)]d\omega dy \tag{2.65}$$

where $z \overset{\text{def}}{=} \frac{y}{\varepsilon_0}$. Consider the inner integral in the first term. An integration by parts gives

$$\begin{aligned}\int_z^{\varepsilon_0} \alpha_\varepsilon^{-2}[v'(\omega - y) - v'(\omega + y)]d\omega &= \Big|_z^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)[v(\omega - y) - v(\omega + y)] \\ &+ 2\int_z^{\varepsilon_0} \alpha_\varepsilon^{-3}(\omega)\frac{d\alpha_\varepsilon(\omega)}{d\omega}[v(\omega - y) - v(\omega + y)]dy .\end{aligned} \tag{2.66}$$

As $y > 0$ then $v(\omega - y) - v(\omega + y) < 0$. This fact coupled with $\alpha_\varepsilon > 0$ implies that the substitution term from the upper limit ε_0 may be dropped. Also note that as y is sufficiently small compared to z then (2.48) may be used to estimate $v(z + y) - v(z - y)$. Therefore the substitution term on the right side of (2.66) has the upper bound $c_\rho y z^{-1}\alpha_\varepsilon^{-2}(z)v(z)$.

The inner integrals in the second and third term on the right side of (2.65) may be estimated in the same fashion. Substituting back into (2.65) we then obtain

(2.67)
$$R_{2\eta} < c_\rho \int_{\varepsilon_0\eta}^{\varepsilon_0^2} |f'(y)| y z^{-1}\alpha_\varepsilon^{-2}(z)v(z)dy + c_\rho \eta^{-1}\alpha_\varepsilon^{-2}(\eta)v(\eta)\int_{\varepsilon\eta}^{\varepsilon_0\eta} |f'(y)| y dy$$
$$+ 2\int_\eta^{\varepsilon_0} \frac{d\alpha_\varepsilon}{d\omega}[\alpha_\varepsilon(\omega)]^{-2}\left[\alpha_\varepsilon^{-1}(\omega)\int_{\varepsilon\omega}^{\varepsilon_0\omega} |f'(y)|\{v(\omega + y) - v(\omega - y)\}dy\right]d\omega$$

where we have combined the two first substitution terms and - after again interchanging the order of integration - combined the double integrals to a single term. Denote the three terms on the right side of (2.67) by $T_{i\eta}$; $i = 1,2,3$. For T_1 we have the estimate (use (2.12), (2.13), (2.49), (2.55)

(2.68)
$$T_1 < c\int_{\varepsilon_0\eta}^{\varepsilon_0^2} y|f'(y)|[zv(z)f^2(z)]^{-1}dy < \frac{(1-\gamma)}{2}\int_\eta^{\varepsilon_0} \varepsilon_0 f(\varepsilon_0 z)[zv(z)f^2(z)]^{-1}dz$$
$$< \int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)},$$

provided δ is taken sufficiently small. From (2.13), (2.49), (2.55) follows (with δ small enough)

(2.69)
$$T_2 < \frac{\eta}{4v(\eta)\mu(\eta)},$$

and from (2.13), (2.48), (2.49, (2.55) one has $\left(F_\varepsilon \overset{\text{def}}{=} \alpha_\varepsilon^{-2}\left|\frac{d\alpha_\varepsilon}{d\omega}\right|\right)$

$$T_3 < 2c_\rho \int_\eta^{\varepsilon_0} F_\varepsilon(\omega)[\omega\alpha_\varepsilon(\omega)]^{-1} v(\omega) \int_0^{\varepsilon_0\omega} f(y)dy\, d\omega$$

$$(2.70) \qquad < 4^{-1} \int_\eta^{\varepsilon_0} F_\varepsilon(\omega)d\omega$$

provided δ was chosen small enough. By (2.57), (2.64), (2.67)-(2.70),

$$(2.71) \qquad S_{2\eta} < \frac{1}{4}\int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)} + \int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} + \frac{\eta}{4v(\eta)\mu(\eta)} + \frac{1}{4}\int_\eta^{\varepsilon_0} F_\varepsilon(\omega)d\omega \ .$$

We proceed to examine $S_{3\eta} \overset{\text{def}}{=} \int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}|I_3|d\omega$. As $v' > 0$ we have

$$|(1-\varepsilon)v'(\omega-\varepsilon\omega) - (1+\varepsilon)v'(\omega+\varepsilon\omega)|$$

$$< |v'(\omega-\varepsilon\omega) - v'(\omega+\varepsilon\omega) + \int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s)|$$

$$+ \varepsilon v'(\omega-\varepsilon\omega) + \varepsilon v'(\omega+\varepsilon\omega) + \int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s)$$

$$(2.72) \qquad = v'(\omega-\varepsilon\omega) - v'(\omega+\varepsilon\omega) + 2\int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s) + \varepsilon v'(\omega-\varepsilon\omega) + \varepsilon v'(\omega+\varepsilon\omega)$$

$$= \frac{d}{d\omega}[v(\omega-\varepsilon\omega) - v(\omega+\varepsilon\omega)] + 2\int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s) + 2\varepsilon v'(\omega-\varepsilon\omega) + 2\varepsilon v'(\omega+\varepsilon\omega)$$

$$< \frac{d}{d\omega}[v(\omega-\varepsilon\omega) - v(\omega+\varepsilon\omega)] + 2\int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s) + \varepsilon c\omega^{-1}v(\omega) \ .$$

The contribution of the very last term of (2.72) to $S_{3\eta}$ is easily estimated (use (2.12), (2.49), (2.55) and take δ sufficiently small)

$$(2.73) \qquad \varepsilon c \int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega) f(\varepsilon\omega)\omega^{-1} v(\omega)d\omega < \frac{1}{2}\int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} \ .$$

We next wish to obtain the inequality

$$(2.74)\qquad 2\int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)f(\varepsilon\omega)\left\{\int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s)\right\}d\omega < \frac{1}{4}\int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)} .$$

To accomplish this we proceed as when $R_{1\eta}$ was estimated; i.e. choose $\{\omega_i\}_{i=1}^{\infty}$ as in (2.60) and approximate $\int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} dv_+'(s)$ by a stepfunction. For each i we now have the following contribution to the left side of (2.74) (use (2.12), (2.49), (2.55)),

$$(2.75)\qquad 2a_i \int_{\omega_i[1+\varepsilon]^{-1}}^{\omega_i[1-\varepsilon]^{-1}} \alpha_\varepsilon^{-2}(\omega)f(\varepsilon\omega)d\omega < \delta a_i \varepsilon^{-1} \int_{\omega_i[1+\varepsilon]^{-1}}^{\omega_i[1-\varepsilon]^{-1}} \frac{d\omega}{v^2(\omega)f(\omega)} < \frac{1}{4}\frac{a_i\omega_i^2}{v^2(\omega_{i-1})\mu(\omega_{i-1})} .$$

Hence, after summing with respect to i and using the monotonicity of ω, v, μ we have (2.74).

To obtain an upper bound for $S_{3\eta}$ it thus remains to obtain an upper bound for

$$(2.76)\qquad \Phi \overset{\text{def}}{=} \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)f(\varepsilon\omega)\left\{\frac{d}{d\omega}[v(\omega-\varepsilon\omega) - v(\omega+\varepsilon\omega)]\right\}d\omega .$$

We integrate Φ by parts and obtain

$$(2.77)\qquad \begin{aligned} \Phi = {} & \Big|_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)f(\varepsilon\omega)[v(\omega-\varepsilon\omega) - v(\omega+\varepsilon\omega)] \\ & + 2\int_{\eta}^{\varepsilon_0} F_\varepsilon(\omega)\alpha_\varepsilon^{-1}(\omega)f(\varepsilon\omega)[v(\omega-\varepsilon\omega) - v(\omega+\varepsilon\omega)]d\omega \\ & - \varepsilon\int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)f'(\varepsilon\omega)[v(\omega-\varepsilon\omega) - v(\omega+\varepsilon\omega)]d\omega \end{aligned}$$

Denote the integrals on the right side of (2.77) by T_4, T_5. The substitution term from ε_0 is negative and may be dropped. An estimation of the term given by the lower limit yields

$$(2.78)\qquad \alpha_\varepsilon^{-2}(\eta)f(\varepsilon\eta)[v(\eta+\varepsilon\eta) - v(\eta-\varepsilon\eta)] < \frac{\eta}{4v(\eta)\mu(\eta)}$$

where (2.12), (2.48), (2.49) and (2.55) were used and δ was taken sufficiently small. For T_4 one gets by analogous estimates

(2.79) $$T_4 < \frac{1}{4}\int_{\eta}^{\varepsilon_0} F_\varepsilon(\omega)d\omega \ .$$

Finally note that T_5 has the upper bound $\frac{1}{2}\int_0^1 [v(\omega)\mu(\omega)]^{-1}d\omega$. Hence

(2.80) $$S_{3\eta} < \int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} + \frac{1}{4}\int_0^1 \frac{\omega^2}{v^2(\omega)\mu(\omega)}\, dv_+'(\omega) + \frac{\eta}{4v(\eta)\mu(\eta)} + \frac{1}{4}\int_{\eta}^{\varepsilon_0} F_\varepsilon(\omega)d\omega \ .$$

But without loss of generality we may (recall (1.6)) choose $\eta \downarrow 0$ such that $\eta[v(\eta)\mu(\eta)]^{-1} \to 0$. Hence by (2.51), (2.56), (2.71), (2.80) we have $F_\varepsilon \in L^1(0,\varepsilon_0)$ and

(2.81) $$\sup_{0<\varepsilon<\varepsilon_0} \int_0^{\varepsilon_0} F_\varepsilon(\omega)d\omega < c\int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} + \int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)} < \infty \ ,$$

where c, ε_0 depend only on ρ, γ *and where we have used* (1.4), (1.6).

We proceed to show that $\sup\limits_{0<\varepsilon<\varepsilon_0} \sum\limits_{i=4}^{6} S_i < \infty$, and begin by examining the contribution of I_4. From (2.21) follows $|I_4| < [1 + 2\lambda][\varepsilon_0\omega]^{-1}\alpha_\varepsilon(\omega)$ and so by (2.49)

(2.82) $$S_4 < c\int_0^{\varepsilon_0} \frac{d\omega}{v(\omega)\mu(\omega)} \ .$$

Next note that as h is even we may write (add and subtract an extra term)

(2.83)
$$I_5 = -h(\varepsilon_0\omega)[1 + \varepsilon]v'(\omega + \varepsilon\omega) - \int_{\varepsilon\omega}^{\varepsilon_0\omega} h'(y)\int_{\omega+\varepsilon\omega}^{\omega+y} dv_+'(s)dy$$
$$+ \int_{\varepsilon\omega}^{\varepsilon_0\omega} h'(y)\Big[-v'(y + \omega) + (1 + \varepsilon)v'(\omega + \varepsilon\omega) + \int_{\omega+\varepsilon\omega}^{\omega+y} dv_+'(s)\Big]dy \ .$$

By (1.5), (2.16), (2.47), (2.49), (2.55) and provided δ is sufficiently small,

(2.84) $$\int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)h(\varepsilon_0\omega)(1 + \varepsilon)v'(\omega + \varepsilon\omega)d\omega < \varepsilon_0^{-1}\int_0^{\varepsilon_0} \frac{d\omega}{v(\omega)\mu(\omega)} \ .$$

As the quantity inside the square brackets in (2.83) is nonnegative we have that what remains to be estimated in (2.83) is bounded from above by

$$(2.85)\qquad 2\int_{\varepsilon\omega}^{\varepsilon_0\omega} |h'(y)|\Big\{\int_{\omega+\varepsilon\omega}^{\omega+y} dv_+'(s)\Big\}dy + \int_{\varepsilon\omega}^{\varepsilon_0\omega} |h'(y)|[-v'(y+\omega) + (1+\varepsilon)v'(\omega+\varepsilon\omega)]dy .$$

Approximate $\int_{\omega+\varepsilon\omega}^{\omega+y} dv_+'(s)$ as before. Then, for each i, by (2.21), (2.47), (2.49), (2.55),

$$2\int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\int_{\varepsilon\omega}^{\varepsilon_0\omega} |h'(y)|\Big[\int_{\omega+\varepsilon\omega}^{\omega+y} a_i\delta_{\omega_i}(s)\Big]dy\, d\omega$$

$$(2.86)\qquad \begin{aligned} &< 2a_i \int_{\omega_i[1+\varepsilon_0]^{-1}}^{\omega_i[1+\varepsilon]^{-1}} \alpha_\varepsilon^{-2}(\omega)\int_{\omega_i-\omega}^{\varepsilon_0\omega} |h'(y)|dy\, d\omega \\ &< ca_i[f(\omega_i)v(\omega_i)]^{-2}\int_{\omega_i[1+\varepsilon_0]^{-1}}^{\omega_i[1+\varepsilon]^{-1}}\int_{\omega_i-\omega}^{\varepsilon_0\omega} |h'(y)|dy\, d\omega \\ &< ca_i[f(\omega_i)v(\omega_i)]^{-2}\int_{\varepsilon\omega_i}^{\varepsilon_0\omega_i} y|h'(y)|dy < \frac{a_i\omega_i^2}{v^2(\omega_{i-1})\mu(\omega_{i-1})} . \end{aligned}$$

Thus

$$(2.87)\qquad 2\int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\int_{\varepsilon\omega}^{\varepsilon_0\omega} |h'(y)|\Big\{\int_{\omega-y}^{\omega+y} dv_+'(s)\Big\}dy\, d\omega < \int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)} .$$

Our final purpose is to estimate the contribution to S_5 by the second part of (2.85):

$$(2.88)\qquad \Phi_\eta \overset{\text{def}}{=} \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\Big\{\int_{\varepsilon\omega}^{\varepsilon_0\omega} |h'(y)\frac{d}{d\omega}[v(\omega+\varepsilon\omega) - v(\omega+y)]dy\Big\}d\omega .$$

To estimate Φ_η we proceed as when $R_{2\eta}$ was estimated. Interchange the order of integration and write Φ_η as the sum of three terms with the limits of integration identical to those in (2.65). Consider the inner integral in the first term. An integration by parts produces $(z = \varepsilon_0^{-1}y)$

$$\int_z^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\, \frac{d}{d\omega}\, [v(\omega + \varepsilon\omega) - v(\omega + y)]d\omega = \Big|_z^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)[v(\omega + \varepsilon\omega) - v(\omega + y)]$$

$$+ 2\int_z^{\varepsilon_0} \alpha_\varepsilon^{-1}(\omega)F_\varepsilon(\omega)[v(\omega + \varepsilon\omega) - v(\omega + y)]d\omega \ .$$

As $y > \varepsilon\omega$, $dv > 0$ and $\alpha_\varepsilon > 0$ then the upper limit ε_0 yields a negative term. Substitution of the lower limit gives the left side of (2.89). The first inequality in (2.89) follows by $dv > 0$ and the second from (2.48).

$$\alpha_\varepsilon^{-2}(z)[v(z + y) - v(z + \varepsilon z)] < \alpha_\varepsilon^{-2}(z)[v(z + y) - v(z - y)]$$

(2.89)

$$< c_\rho y z^{-1}\alpha_\varepsilon^{-2}(z)v(z) \ .$$

The inner integrals in the second and third term are estimated in the same fashion and the result is that Φ_η is dominated by the right side of (2.67) with f' replaced by h'. But by (2.21) $y|h'(y)| < cf(y)$ and so the estimation can be carried out entirely as when $R_{2\eta}$ was estimated. Thus

$$\text{(2.90)} \qquad \Phi_\eta < \int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} + \frac{\eta}{4v(\eta)\mu(\eta)} + \frac{1}{4}\int_\eta^{\varepsilon_0} F_\varepsilon(\omega)d\omega \ .$$

Consequently, by (2.81), (2.84), (2.87), (2.90), after letting $\eta \downarrow 0$.

$$S_5 < c\int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} + \frac{5}{4}\int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)} \ .$$

By similar computations one obtains the same bound for S_6. Recalling also (2.82) one finally has

$$\sum_{i=4}^{6} S_i < c \int_0^1 \frac{d\omega}{v(\omega)\mu(\omega)} + \frac{5}{2} \int_0^1 \frac{\omega^2 dv_+'(\omega)}{v^2(\omega)\mu(\omega)}$$

which completes the proof.

3. PROOF OF THEOREM 2.

Choose $\varepsilon_0 \in (0, \frac{1}{4}]$ such that $2\rho\varepsilon_0 < 1$ and consider the proof of Theorem 1 up to and including (2.48). The only relations in this part which are now not valid are of course (2.8), (2.9), (2.18), (2.19), (2.21). (Note that (2.10) is assumed).

Define $\lambda(\omega)$ as in (1.19). Then (compare with (2.8), (2.9), (2.18)

$$\frac{\mu(\omega)}{\omega} = \frac{\lambda(\omega)}{\omega^2}\int_0^\omega x\,d\mu(x) < 2\lambda(\omega)\int_0^\omega \frac{x\,d\mu(x)}{x^2+\omega^2} < 2\lambda(\omega)f(\omega)\ , \tag{3.1}$$

and so, instead of (2.49) we have

$$\alpha_\varepsilon(\omega) > \frac{v(\omega)f(\omega)}{2} > [4\lambda(\omega)]^{-1}v(\omega)\mu(\omega)\ . \tag{3.2}$$

Let $\alpha_\varepsilon, \beta_\varepsilon, I_i, S_i$ be as in the proof of Theorem 1. For S_1 we now have the estimate

$$S_1 < 8\varepsilon_0^{-1}\int_0^1 \frac{\lambda(\omega)}{v(\omega)\mu(\omega)}\,d\omega\ . \tag{3.3}$$

To estimate S_2 we write

$$|I_2| = \left|\int_{\varepsilon\omega}^{\varepsilon_0\omega} f'(y)[v'(\omega-y) - v'(\omega+y)]dy\right| < \int_{\varepsilon\omega}^{\varepsilon_0\omega} |f'(y)|\left\{\int_{\omega-y}^{\omega+y} d|v'|(s)\right\}dy \tag{3.4}$$

and approximate $\int_{\omega-y}^{\omega+y} d|v'|(s)$ in the same fashion as $\int_{\omega-y}^{\omega+y} dv'_+(s)$ was approximated in the proof of Theorem 1. Note at first that straightforward calculations which use (1.12) give

$$\int_0^\omega \int_s^\omega |f'(y)|dy\,ds < c\mu(\omega)\ . \tag{3.5}$$

Then choose for each i, $\omega_{1i}, \omega_{2i} \in [\omega_i, \omega_{i-1}]$ so that

$$\begin{aligned} &\lambda(\omega_{1i}) < \lambda(\omega), && 2^{-1}\omega_i < \omega < 2\omega_i\ , \\ &\frac{\lambda(\omega_{2i})}{\mu(\omega_{2i})} < \frac{\lambda(\omega)}{\mu(\omega)}, && 2^{-1}\omega_i < \omega < 2\omega_i\ . \end{aligned} \tag{3.6}$$

We now have, for each i,

$$\int_{n}^{\epsilon_0} \alpha_\epsilon^{-2}(\omega) \int_{\epsilon\omega}^{\epsilon_0\omega} |f'(y)|[\phi_i(\omega + y) - \phi_i(\omega - y)]dy\, d\omega$$

$$< \frac{ca_i}{v^2(\omega_{i-1})f(\omega_{1i})f(\omega_{2i})} \int_0^{\frac{\omega i}{3}} \int_s^{\frac{\omega i}{3}} |f'(y)|dy\, ds$$

$$< \frac{ca_i\mu(\frac{\omega_i}{3})}{v^2(\omega_{i-1})f(\omega_{1i})f(\omega_{2i})} < \frac{ca_i\omega_i^2\lambda(\omega_{1i})\lambda(\omega_{2i})}{v^2(\omega_{i-1})\mu(\omega_{2i})} .$$

The first inequality follows by the definition of ϕ_i, by the first part of (3.2) and by the fact that the growth of v, f is regulated; the second uses (3.5) and the last follows as μ is nondecreasing and by the second part of (3.2). Therefore, use the monotonicity of ω, v and (3.6),

(3.8) $$S_2 < c \int_0^1 \frac{\omega^2\lambda^2(\omega)}{v^2(\omega)\mu(\omega)} d|v'|(\omega) .$$

For the estimation of S_3 we observe that straightforward calculations which use (1.11) give

$$f(\epsilon\omega) - f(\tilde{\omega}) < c\,\frac{\mu(\tilde{\omega})}{\epsilon\tilde{\omega}}, \quad \frac{\omega}{2} < \tilde{\omega} < 2\omega ,$$

which together with (3.2) yields

(3.9) $$\epsilon f(\epsilon\omega) < c\lambda(\tilde{\omega})f(\tilde{\omega}), \quad \frac{\omega}{2} < \tilde{\omega} < 2\omega .$$

Now note that

(3.10) $$|(1 - \epsilon)v'(\omega - \epsilon\omega) - (1 + \epsilon)v'(\omega + \epsilon\omega)| < \epsilon c\,\frac{v(\omega)}{\omega} + \int_{\omega-\epsilon\omega}^{\omega+\epsilon\omega} d|v'|(s) .$$

The first term on the right side of (3.10) gives

$$(3.11) \qquad \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\varepsilon f(\varepsilon\omega)\omega^{-1} v(\omega)d\omega < c\int_0^1 \frac{\lambda^2(\omega)d\omega}{v(\omega)\mu(\omega)}$$

where (3.9) with $\tilde{\omega} = \omega$, and (3.2) were used. To obtain our next goal:

$$(3.12) \qquad \int_{\eta}^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)f(\varepsilon\omega) \int_{\omega-\varepsilon\omega}^{\omega+\varepsilon\omega} d|v'|(s)d\omega < c\int_0^1 \frac{\omega^2\lambda^2(\omega)}{v^2(\omega)\mu(\omega)} d|v'|(\omega)$$

we proceed in the usual fashion. Let ω_{1i}, ω_{2i} be as in (3.6). By (3.2), (3.9) - where $\tilde{\omega} = \omega_{1i}$ - and as we can regulate the growth of f, v;

$$(3.13) \qquad a_i \int_{\omega_i[1+\varepsilon]^{-1}}^{\omega_i[1-\varepsilon]^{-1}} \alpha_\varepsilon^{-2}(\omega)f(\varepsilon\omega)d\omega < c\varepsilon^{-1}a_i\lambda(\omega_{1i})f(\omega_{1i}) \int_{\omega_i[1+\varepsilon]^{-1}}^{\omega_i[1-\varepsilon]^{-1}} \alpha_\varepsilon^{-2}(\omega)d\omega$$

$$< ca_i \frac{\omega_i^2\lambda(\omega_{1i})}{v^2(\omega_{i-1})} \frac{\lambda(\omega_{2i})}{\mu(\omega_{2i})}$$

and thus, by the monotonicity of ω, v and from (3.6) we have (3.12). From (3.10)-(3.12)

$$(3.14) \qquad S_3 < c\left[\int_0^1 \frac{\lambda^2(\omega)}{v(\omega)\mu(\omega)} d\omega + \int_0^1 \frac{\lambda^2(\omega)\omega^2}{v^2(\omega)\mu(\omega)} d|v'|(\omega)\right]$$

and so, by (3.3), (3.8), (3.14) and as $\lambda(\omega) > 1$,

$$(3.15) \qquad \int_0^{\varepsilon_0} \left|\frac{d\alpha_\varepsilon}{d\omega}\right| \alpha_\varepsilon^{-2}d\omega < c\left[\int_0^1 \frac{\lambda^2(\omega)}{v(\omega)\mu(\omega)} d\omega + \int_0^1 \frac{\lambda^2(\omega)\omega^2}{v^2(\omega)\mu(\omega)} d|v'|(\omega)\right]$$

We next estimate $\int_0^{\varepsilon_0} \left|\frac{d\beta_\varepsilon}{d\omega}\right| \alpha_\varepsilon^{-2}d\omega$. From the definition of $\lambda(\omega)$ and as $d\mu > 0$ follows after some calculations that $\omega^{-1}\lambda(\omega)$ is nonincreasing. Then note that now (compare with (2.21) and use $\lambda(\omega) > 1$)

$$(3.16) \qquad \left|\frac{dh}{d\omega}\right| < 3\lambda(\omega)\frac{f(\omega)}{\omega}, \quad \omega > 0 .$$

Consequently

$$|I_4| \leq \int_{-1/2}^{\omega-\varepsilon_0\omega} + \int_{\omega+\varepsilon_0\omega}^{-1/2} |h'(\omega - s)|v'(s)ds \leq \frac{\lambda(\varepsilon_0\omega)}{\varepsilon_0\omega}\, \alpha_\varepsilon(\omega) .$$

Note that $\lambda(\omega) \leq \varepsilon_0^{-1}\lambda(\varepsilon_0\omega)$. This fact together with (3.2) and as v, μ are nondecreasing implies

$$\int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}|I_4|d\omega \leq \varepsilon_0^{-1}\int_\eta^{\varepsilon_0} \frac{\lambda(\varepsilon_0\omega)}{\omega\alpha_\varepsilon(\omega)} d\omega \leq c\int_\eta^{\varepsilon_0} \frac{\lambda(\varepsilon_0\omega)\lambda(\omega)}{v(\omega)\mu(\omega)} d\omega$$

(3.17)

$$\leq c\int_\eta^{\varepsilon_0} \frac{\lambda^2(\varepsilon_0\omega)}{v(\varepsilon_0\omega)\mu(\varepsilon_0\omega)} d\omega \leq c\int_0^1 \frac{\lambda^2(s)}{v(s)\mu(s)} ds .$$

Write I_5 as in (2.83) and observe that from (1.15) and (3.2) follows $h(\omega) \leq c\omega^{-1}\mu(\omega) \leq c\lambda(\omega)f(\omega)$ and thus

$$h(\varepsilon_0\omega) \leq c\lambda(\varepsilon_0\omega)f(\varepsilon_0\omega) . \tag{3.18}$$

By (1.5), (2.47), (3.2), (3.18), as $f(\varepsilon_0\omega) \leq \varepsilon_0^{-2}f(\omega)$, as v, μ are monotone, and by the fact that $\lambda(\omega) \leq \varepsilon_0^{-1}\lambda(\varepsilon_0\omega)$ we have (note that c varies from place to place)

$$\int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)h(\varepsilon_0\omega)[1 + \varepsilon]v'(\omega + \varepsilon\omega)d\omega \leq 2\int_\eta^{\varepsilon_0} \frac{h(\varepsilon_0\omega)v(\omega)}{\omega\alpha_\varepsilon^2(\omega)} d\omega$$

(3.19)

$$\leq c\int_\eta^{\varepsilon_0} \frac{\lambda(\varepsilon_0\omega)f(\varepsilon_0\omega)}{\omega v(\omega)f^2(\omega)} \leq c\int_\eta^{\varepsilon_0} \frac{\lambda(\varepsilon_0\omega)}{\omega v(\omega)f(\omega)} d\omega$$

$$\leq c\int_\eta^{\varepsilon_0} \frac{\lambda(\varepsilon_0\omega)\lambda(\omega)}{v(\varepsilon_0\omega)\mu(\varepsilon_0\omega)} d\omega \leq c\int_0^1 \frac{\lambda^2(s)}{v(s)\mu(s)} ds$$

Part of the second term on the right side of (2.83) is estimated by

$$\epsilon \int_{\epsilon\omega}^{\epsilon_0\omega} |h'(y)|v'(\omega+\epsilon\omega)dy \le \frac{2\epsilon v(\omega)}{\omega} \int_{\epsilon\omega}^{\epsilon_0\omega} |h'(y)|dy \tag{3.20}$$

$$\le c\,\frac{\epsilon v(\omega)}{\omega} \int_{\epsilon\omega}^{\epsilon_0\omega} \frac{\mu(s)}{s^2}\,ds \le c\,\frac{v(\omega)\mu(\omega)}{\omega^2} ,$$

where we used (recall (1.15))

$$|h'(\omega)| \le \int_\omega^\infty \frac{d\mu(x)}{x^2} + \frac{\mu(\omega)}{\omega^2} \le c\,\frac{\mu(\omega)}{\omega^2} . \tag{3.21}$$

Hence, by (3.2), (3.20)

$$\epsilon \int_n^{\epsilon_0} \alpha_\epsilon^{-2}(\omega) \int_{\epsilon\omega}^{\epsilon_0\omega} |h'(y)|v'(\omega+\epsilon\omega)dy\,d\omega \le c \int_0^1 \frac{\lambda^2(\omega)}{v(\omega)\mu(\omega)}\,d\omega , \tag{3.22}$$

and it only remains to obtain

$$\int_n^{\epsilon_0} \alpha_\epsilon^{-2}(\omega) \int_{\epsilon\omega}^{\epsilon_0\omega} |h'(y)|\left\{\int_{\omega+\epsilon\omega}^{\omega+y} d|v'|(s)\right\}dy\,d\omega \le c \int_0^1 \frac{\omega^2\lambda^2(\omega)}{v^2(\omega)\mu^2(\omega)}\,d|v'|(\omega) . \tag{3.23}$$

We proceed as before and remark that from (3.2), (3.21) and by (1.17),

$$a_i \int_n^{\epsilon_0} \alpha_\epsilon^{-2}(\omega) \int_{\epsilon\omega}^{\epsilon_0\omega} |h'(y)|\left\{\int_{\omega+\epsilon\omega}^{\omega+y} \delta_{\omega_i}(s)\right\}dy\,d\omega \le \tag{3.24}$$

$$\le a_i \int_{\omega_i[1+\epsilon_0]^{-1}}^{\omega_i[1+\epsilon]^{-1}} \alpha_\epsilon^{-2}(\omega) \int_{\omega_i-\omega}^{\epsilon_0\omega} |h'(y)|dy\,d\omega$$

$$\le a_i c \int_{\omega_i[1+\epsilon_0]^{-1}}^{\omega_i[1+\epsilon]^{-1}} \alpha_\epsilon^{-2}(\omega) \int_{\omega_i-\omega}^{\epsilon_0\omega} \frac{\mu(y)}{y^2}\,dy\,d\omega$$

$$< ca_i \frac{1}{v^2(\omega_i)f^2(\omega_i)} \int_{\varepsilon\omega_i}^{\varepsilon_0\omega_i} \int_{s}^{\varepsilon_0\omega_i} \frac{\mu(y)}{y^2}\, dy\, ds$$

$$< \frac{ca_i}{v^2(\omega_i)f^2(\omega_i)} \int_0^{\varepsilon_0\omega_i} \frac{\mu(y)}{y}\, dy < \frac{ca_i}{v^2(\omega_i)f^2(\omega_i)}$$

and so (3.23) is satisfied. By (3.17), (3.19), (3.22), (3.23) and as S_6 is estimated as S_5 we obtain using (1.6_2) and (1.16)

$$(3.25) \qquad \int_0^{\varepsilon_0} \left|\frac{d\beta_\varepsilon}{d\omega}\right| \alpha_\varepsilon^{-2}(\omega)d\omega < c\left[\int_0^1 \frac{\lambda^2(s)}{v(s)\mu(s)}\, ds + \int_0^1 \frac{\omega^2\lambda^2(\omega)}{v^2(\omega)\mu^2(\omega)}\, d|v'|(\omega)\right] < \infty$$

which completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

By Theorem 2 we obviously have the relation (1.14) and it only remains to establish

$$\sup_{0<\varepsilon<\varepsilon_0} \int_0^{\varepsilon_0} \left|\frac{d\beta_\varepsilon}{d\omega}\right| \alpha_\varepsilon^{-2}(\omega)d\omega < \infty . \tag{4.1}$$

The estimate (3.17) for S_4 is still valid. To estimate S_5, S_6 we combine I_5, I_6 as

$$\begin{aligned} I_5 + I_6 &= -h(\varepsilon_0\omega)[v'(\omega+\varepsilon\omega) + v'(\omega-\varepsilon\omega)] + \varepsilon h(\varepsilon\omega)[v'(\omega-\varepsilon\omega) - v'(\omega+\varepsilon\omega)] \\ &+ \int_{\omega+\varepsilon\omega}^{\omega+\varepsilon_0\omega} h'(\omega-s)[v'(s) - v'(\omega+\varepsilon\omega) + v'(2\omega-s) - v'(\omega-\varepsilon\omega)]ds \end{aligned} \tag{4.2}$$

The contribution to $S_5 + S_6$ by the first term on the right side of (4.2) is estimated as in (3.19). (Note that (1.16), (1.17) are not needed for this estimate to hold). The contribution by the second term is estimated by (use $h(\omega) < c\omega^{-1}\mu(\omega)$ and (3.2))

$$\int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)\varepsilon h(\varepsilon\omega)|v'(\omega-\varepsilon\omega) - v'(\omega+\varepsilon\omega)|d\omega < c\int_\eta^{\varepsilon_0} \frac{\lambda^2(\omega)}{v(\omega)\mu(\omega)} d\omega < \infty . \tag{4.3}$$

Finally we write the last term as

$$L \overset{\text{def}}{=} \int_{\varepsilon\omega}^{\varepsilon_0\omega} h'(y)\left[\int_\omega^{\omega+y} [v''(s-y) - v''(s)]ds + \int_\omega^{\omega+\varepsilon\omega} [v''(s) - v''(s-\varepsilon\omega)]ds\right]dy .$$

By (1.21), (3.21), as μ is nondecreasing and $y > \varepsilon\omega$,

$$|L| < \mu(\omega)\int_{\varepsilon\omega}^{\varepsilon_0\omega} y^{-2}\left\{\frac{2y}{\log^\alpha(\frac{1}{y})}\right\} F(\omega)dy < c\mu(\omega)F(\omega) . \tag{4.4}$$

Thus, from (1.22),

$$\int_\eta^{\varepsilon_0} \alpha_\varepsilon^{-2}(\omega)|L(\omega)|d\omega < c\int_0^1 \frac{\omega^2\lambda^2(\omega)F(\omega)}{v^2(\omega)\mu(\omega)} d\omega < \infty . \tag{4.5}$$

This completes the proof.

5. PROOF OF LEMMA 1

For any $\varepsilon_0 > 0$ the left side of (2.54) may be estimated as

$$\int_0^{\varepsilon_0\omega} f(s)ds = \int_0^{\infty} tg^{-1}\left(\frac{\varepsilon_0\omega}{x}\right)d\mu(x) \leq \frac{\pi}{2}\mu(\varepsilon_0\omega) + \varepsilon_0\omega\int_{\varepsilon_0\omega}^{\infty}\frac{d\mu(x)}{x} . \tag{5.1}$$

For the right side of (2.54) we have (see (2.18))

$$f(\omega) = \int_0^{\omega} + \int_{\omega}^{\infty}\left\{\frac{x d\mu(x)}{x^2+\omega^2}\right\} \geq \frac{(1-\gamma)\mu(\omega)}{2\omega} + \frac{1}{2}\int_{\omega}^{\infty}\frac{d\mu(x)}{x}, \quad 0 < \omega \leq 1 . \tag{5.2}$$

From (5.1), (5.2) follows that (2.54) certainly holds provided for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\mu(\varepsilon_0\omega) + \varepsilon_0\omega\int_{\varepsilon_0\omega}^{\infty}\frac{d\mu(x)}{x} \leq \frac{\delta(1-\gamma)}{\pi}\left[\mu(\omega) + \omega\int_{\omega}^{\infty}\frac{d\mu(x)}{x}\right] , \tag{5.3}$$

for $0 < \omega \leq \varepsilon_0$. Note however that if $G(\omega) \overset{\text{def}}{=} \mu(\omega) + \omega\int_{\omega}^{\infty}\frac{d\mu(s)}{s}$ then (5.3) can be written

$$G(\varepsilon_0\omega) \leq \frac{\delta}{\pi}(1-\gamma)G(\omega), \quad 0 < \omega \leq \varepsilon_0 . \tag{5.4}$$

To show that (5.4) holds we begin by demonstrating that

$$\omega^{-1}\int_0^{\omega} G(s)ds \leq \tilde{\gamma}G(\omega), \quad 0 < \omega \leq 1 , \tag{5.5}$$

where $\tilde{\gamma} \overset{\text{def}}{=} 2^{-1}[\gamma + 1] \in (0,1)$. Immediately one has $G \in L^1(0,1)$ and

$$\mu(\omega) = \omega\mu(\omega)\int_{\omega}^{\infty} s^{-2}ds \leq \omega\int_{\omega}^{\infty}\mu(s)s^{-2}ds = G(\omega), \quad \omega > 0 \tag{5.6}$$

From (1.1), (5.6) one has

$$\omega^{-1}\int_0^{\omega}\mu(s)ds \leq \gamma G(\omega), \quad 0 < \omega \leq 1 .$$

Thus

$$(5.7) \qquad (2\omega)^{-1}\int_0^{\omega}\mu(s)ds + 2^{-1}G(\omega) \le \tilde{\gamma}G(\omega), \qquad 0 < \omega \le 1 .$$

But straightforward calculations show that the left side of (5.7) equals $\omega^{-1}\int_0^{\omega} G(s)ds$ and so (5.5) follows. In addition note that

$$(5.8) \qquad G \text{ is nonnegative and nondecreasing, } \omega > 0 .$$

To complete the proof of (5.4) we use the following simple claim.

CLAIM. *Let a function* $G(\omega)$ *be defined on* $0 < \omega \le 1$, *let* G *be nonnegative, nondecreasing and suppose there exists* $r \in (0,1)$ *such that*

$$(5.9) \qquad \omega^{-1}\int_0^{\omega} G(s)ds \le rG(\omega), \qquad 0 < \omega \le 1 .$$

Then given any $\hat{\delta} > 0$ *there exists* $\varepsilon_0 > 0$ *such that*

$$(5.10) \qquad G(\varepsilon_0\omega) \le \hat{\delta}G(\omega), \qquad 0 < \omega \le \varepsilon_0 .$$

Clearly this Claim together with (5.5), (5.8) gives (5.4).

To prove the Claim take any $\omega \in (0,1]$ and suppose ω is such that $G((1 - r^{1/2})\omega) > r^{1/2}G(\omega)$. Then, as G is nonnegative and nondecreasing,

$$\omega^{-1}\int_0^{\omega} G(s)ds \ge \omega^{-1}\int_{(1-r^{1/2})\omega}^{\omega} G(s)ds > \omega^{-1}r^{1/2}G(\omega)[\omega - (1 - r^{1/2})\omega] = rG(\omega) ,$$

which violates (5.9). Thus, we must have $G((1 - r^{1/2})\omega) \le r^{1/2}G(\omega)$ for $0 < \omega \le 1$, and so, by induction, for any positive integer n,

$$G((1 - r^{1/2})^n\omega) \le r^{n/2}G(\omega), \qquad 0 < \omega \le 1 .$$

Fix $\hat{\delta} > 0$ and take n sufficiently large so that $r^{n/2} \le \hat{\delta}$. Define $\varepsilon_0 = (1 - r^{1/2})^n$. Then $G(\varepsilon_0\omega) \le \hat{\delta}G(\omega)$, $0 < \omega \le 1$, which contains (5.10).

Acknowledgement. This work was initiated together with D. F. Shea and the author is indebted to him for several fruitful discussions.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2654	2. GOVT ACCESSION NO. AD-A141662	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ASYMPTOTIC ESTIMATES FOR RESOLVENTS OF SOME NONINTEGRABLE VOLTERRA KERNELS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Stig-Olof Londen		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE March 1984
		13. NUMBER OF PAGES 39
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Volterra equations, resolvent theory, Fourier transforms

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $a \in L^1_{loc}(R^+)$, $a \notin L^1(R^+)$, $a(t) = b(t)c(t)$, $0 < t < \infty$, where $b(t)$ is completely monotone and $c(t)$ is of positive type. Thus $b(t) = \int_{R^+} e^{-st} d\mu(s)$, $c(t) = \lim_{T\to\infty} \int_0^T (1 - \frac{s}{T})\cos(st)d\nu(s)$ where $\mu(s)$, $\nu(s)$ are nondecreasing functions, defined on R^+, and locally of bounded

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ABSTRACT (cont.)

variation. The resolvent r of a is the solution for $t > 0$ of $r(t) + (r*a)(t) = a(t)$. Extend ν to R^- as odd and let $\hat{}$ denote Fourier-transforms. Then $\hat{a}(z) = \frac{1}{2}\int_R \hat{b}(z - ix)d\nu(x)$ for Re z sufficiently large. Using this formula we give conditions on μ, ν which imply that $\hat{r}$ is locally sufficiently smooth to satisfy $r \in L^1(R^+)$. These conditions are shown to differ depending on the size of b at infinity.