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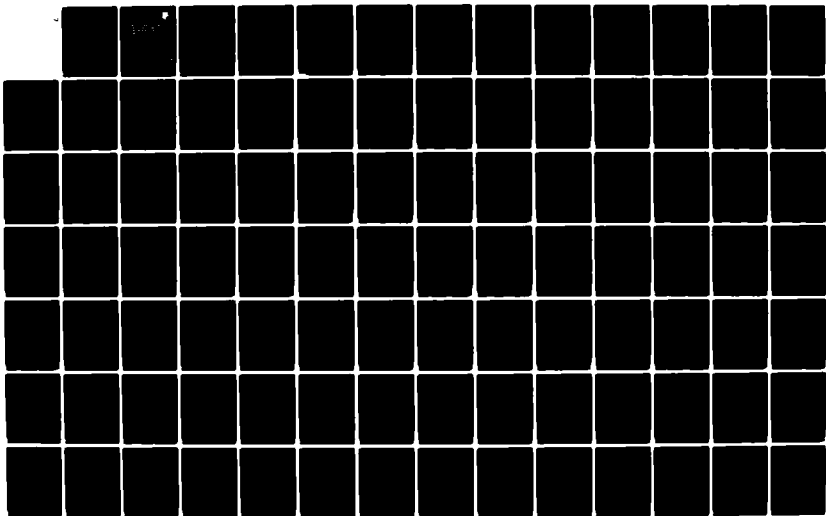
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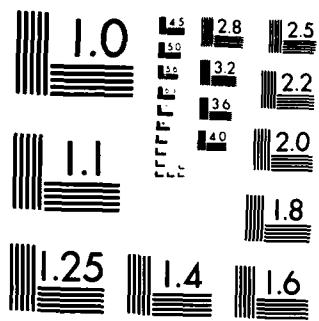
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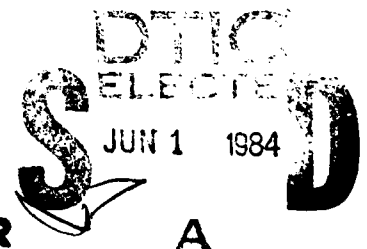
***ON THE SCATTERING OF ELECTROMAGNETIC
WAVES BY PERFECTLY CONDUCTING
BODIES MOVING IN VACUUM; Kinematic
Single Layer Potentials***

University of Delaware

Allan G. Dallas

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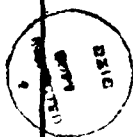
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exterior limiting values of these partial derivatives at the boundary are derived.



Section For	
1	2
3	4
5	6
7	8
9	10
11	12
13	14
15	16
17	18
19	20
21	22
23	24
25	26
27	28
29	30
31	32
33	34
35	36
37	38
39	40
41	42
43	44
45	46
47	48
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51	52
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79	80
81	82
83	84
85	86
87	88
89	90
91	92
93	94
95	96
97	98
99	100

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ORIENTATION

This is Part IV of a six-part report on the results of an investigation into the problem of determining the scattered field resulting from the interaction of a given electromagnetic incident wave with a perfectly conducting body executing specified motion and deformation in vacuum. Part I presents the principal results of the study of the case of a general motion, while Part II contains the specialization and completion of the general reasoning in the situation in which the scattering body is stationary. Part III is devoted to the derivation of a boundary-integral-type representation for the scattered field, in a form involving scalar and vector potentials. Parts IV, V, and VI are of the nature of appendices, containing the proofs of numerous auxiliary technical assertions utilized in the first three parts. Certain of the chapters of Part I are sufficient preparation for studying each of Parts III through VI. Specifically, the entire report is organized as follows:

- Part I. Formulation and Reformulation of the Scattering Problem
 - Chapter 1. Introduction
 - Chapter 2. Manifolds in Euclidean Spaces.
Regularity Properties of Domains
[Summary of Part VI]
 - Chapter 3. Motion and Retardation
[Summary of Part V]

- Chapter 4. Formulation of the Scattering Problem.
Theorems of Uniqueness
- Chapter 5. Kinematic Single Layer Potentials
[Summary of Part IV]
- Chapter 6. Reformulation of the Scattering Problem

- Part II. Scattering by Stationary Perfect Conductors
[Prerequisites: Part I]

- Part III. Representations of Sufficiently Smooth Solutions
of Maxwell's Equations and of the Scattering
Problem
[Prerequisites: Section [I.1.4], Chapters [I.2
and 3], Sections [I.4.1] and [I.5.1-10]]

- Part IV. Kinematic Single Layer Potentials
[Prerequisites: Section [I.1.4], Chapters [I.2
and 3]]

- Part V. A Description of Motion and Deformation. Retardation
of Sets and Functions
[Prerequisites: Section [I.1.4], Chapter [I.2]]

- Part VI. Manifolds in Euclidean Spaces. Regularity
Properties of Domains
[Prerequisite: Section [I.1.4]]

The section- and equation-numbering scheme is fairly self-explanatory. For example, "[I.5.4]" designates the fourth section of Chapter 5 of Part I, while "(I.5.4.1)" refers to the equation numbered (1) in that section; when the reference is made within Part I, however, these are shortened to "[5.4]" and "(5.4.1)," respectively. Note that Parts II-VI contain no chapter-subdivisions. "[IV.14]" indicates the fourteenth section of Part IV, "(IV.14.6)" the equation numbered (6) within that section; the Roman-numeral designations are never dropped in Parts II-VI.

A more detailed outline of the contents of the entire report appears in [I.1.2]. An index of notations and the bibliography are also to be found in Part I. References to the bibliography are made by citing, for example, "Mikhlin [34]." Finally, it should be pointed out that notations connected with the more common mathematical concepts are standardized for all parts of the report in [I.1.4].

PART IV

KINEMATIC SINGLE LAYER POTENTIALS

[IV.1] O R I E N T A T I O N. Motivated by the form of a certain representation of any (sufficiently smooth) solution of an appropriate scattering problem (cf., [I.4.1]), which is described in detail in Part III, we wish to introduce and study a class of functions constructed from the intrinsic objects associated with a smooth motion. Since these functions possess a form and properties which are, in many respects, similar to those of the classical and generalized single-layer potentials arising in the study of the Laplace and more general elliptic operators (cf., Günter [19], Mikhlin [34], and Miranda [38]), we term them "kinematic single layer potentials." Essentially, a kinematic single layer potential is a function tailored for a given smooth motion. In Part I, we attempt to construct from such functions a solution of the scattering problem connected with that motion, in the same way in which one can employ the classical single layer potential in generating an *ansatz* leading to a solution of the Neumann problem for the Laplace equation.

Of course, one must beware: there are important differences between the properties of these kinematic single layers and those of the elliptic case, perhaps most notably in their differentiability properties, as we shall discover.

We aim to arrange our analysis of kinematic single layer potentials in such a way that we can employ variants of the arguments set forth in the works of Günter [19], Mikhlin [34], and Pogorzelski [42], which are concerned with the classical potentials.

[IV.2] D E F I N I T I O N S. Let M be a motion in $M(1)$, and suppose that $\mu: \partial B \rightarrow K$. For $(X,t) \in B^0 \cup \Omega^\sigma$ such that $[\mu]_{[X,t]} \in L_1(\partial B(X,t))^\dagger$ we define

$$V\{\mu\}(X,t) := \frac{1}{4\pi} \int_{\partial B(X,t)} \frac{[\mu]_{[X,t]}}{r_X \cdot ([v]_{[X,t]} + [v^c]_{[X,t]}) \text{grad } r_X|_3} d\lambda_{\partial B(X,t)}, \quad (1)$$

and (supposing that the integral exists for some (X,t)) call the resultant function a *kinematic single layer potential associated with the motion M* , or, for brevity, a KSLM. If $V\{\mu\}$ is defined at least on B^0 , we define

$$V^I\{\mu\} := V\{\mu\}|_{B^0}; \quad (2)$$

in case $V\{\mu\}$ is defined at least on Ω^σ , we set

$$V^0\{\mu\} := V\{\mu\}|_{\Omega^\sigma}. \quad \blacksquare. \quad (3)$$

[IV.3] R E M A R K S. Retain the setting and notation of [IV.2].

(a) Since $M \in M(1)$, we may invoke the conclusions of

[†] We shall use such abbreviated notation for the various Lebesgue spaces, wherever convenient.

Theorem [I.3.27], and recall the notations established there. In particular, if $(X,t) \in \mathcal{B}^0 \cup \Omega^\sigma$, then $\mathcal{B}(X,t)^0$ is a 1-regular domain which is also regularly open, by [I.3.13.i]; the latter also says that $\mathcal{B}(X,t)$ is compact, so $\partial\mathcal{B}(X,t) = \partial\{\mathcal{B}(X,t)^0\}$ is a compact (2,3;1)-manifold. Recall that we indicated how these statements fail, in general, to be true in case $(X,t) \in \partial\mathcal{B}$, whence the restriction placed on (X,t) in [IV.2], since we wish to be sure that we can integrate over $\partial\mathcal{B}(X,t)$.

(b) For a given density μ , it is clear from (1) that $V\{\mu\}$ is intrinsic to M .

(c) Let $(X,t) \in \mathcal{B}^0 \cup \Omega^\sigma$: we have seen, in [I.3.27.iv], that $v \in C(\partial\mathcal{B}; \mathbb{R}^3)$ and $v \in C(\partial\mathcal{B})$, so the inclusions $[v]_{[X,t]} \in C(\partial\mathcal{B}(X,t); \mathbb{R}^3)$ and $[v]_{[X,t]} \in C(\partial\mathcal{B}(X,t))$ follow, as in Remark [I.3.18.d]. Since $(X,t) \notin \partial\mathcal{B}$, $X \notin \partial\mathcal{B}(X,t)$, and so $r_X(Z) > 0$ for each $Z \in \partial\mathcal{B}(X,t)$, whence $\text{grad } r_X$ is defined and continuous on $\partial\mathcal{B}(X,t)$. It is easy to see, as in the proof of [I.3.27.vi], that $[v]_{[X,t]}(Z) + [v^c]_{[X,t]}(Z) \text{ grad } r_X(Z) \neq 0$ (and is in fact an exterior normal for $\partial\mathcal{B}(X,t)$ at Z), for each $Z \in \partial\mathcal{B}(X,t)$. Consequently, $\{r_X \cdot ([v]_{[X,t]} + [v^c]_{[X,t]} \text{ grad } r_X|_3)^{-1} \in C(\partial\mathcal{B}(X,t))$; hence this kernel is measurable and bounded (recall that $\partial\mathcal{B}(X,t)$ is compact), from which it follows that $V\{\mu\}(X,t)$ exists iff the retardation $[\mu]_{[X,t]} \in L_1(\partial\mathcal{B}(X,t))$, as required in the definition. In particular, if $\mu \in C(\partial\mathcal{B})$, then $[\mu]_{[X,t]} \in C(\partial\mathcal{B}(X,t)) \subset L_1(\partial\mathcal{B}(X,t))$, the latter inclusion following from the compactness of $\partial\mathcal{B}(X,t)$, showing that $V\{\mu\}$ is defined on $\mathcal{B}^0 \cup \Omega^\sigma$, in this case.

(d) Let (R, X) be a reference pair for M , possessing the properties listed in Definition [I.3.25]. Using this reference pair, we can recast the integral appearing in (1) into a form involving integration over the $(2,3;1)$ -manifold ∂R , which is more convenient for analysis of the function $V\{\mu\}$. Then, choose $(X, t) \in B^0 \cup \Omega^0$, and suppose that $[\mu]_{[X, t]} \in L_1(\partial B(X, t))$, so that, as pointed out, the integrand on the right of (1) is also in this space. By Theorem [I.3.27.vi.1], $[X]_{(X, t)}: \partial R \rightarrow \mathbb{R}^3$ is a 1-embedding, taking ∂R onto $\partial B(X, t)$, so we can apply the transformation formula (I.2.25.1) to write

$$\begin{aligned} V\{\mu\}(X, t) &= \frac{1}{4\pi} \int_{\partial R} \left\{ \frac{[\mu]_{[X, t]}}{r_X \cdot |[\nu]_{[X, t]} + [\nu^c]_{[X, t]} \cdot \text{grad } r_X|_3} \right\} \circ [X]_{(X, t)} \cdot J[X]_{(X, t)} d\lambda_{\partial R} \\ &= \frac{1}{4\pi} \int_{\partial R} \frac{[\mu^0]_{(X, t)} \cdot J[X]_{(X, t)}}{r_X^0[X]_{(X, t)} \cdot |[\nu^0]_{(X, t)} + [\nu^{0c}]_{(X, t)} \cdot \text{grad } r_X^0[X]_{(X, t)}|_3} d\lambda_{\partial R}, \end{aligned} \quad (4)$$

the second equality holding in view of (I.3.24.2). We have developed the expression (I.3.27.7), giving $J[X]_{(X, t)}$ on ∂R ; using this, (4) becomes

$$\begin{aligned} V\{\mu\}(X, t) &= \frac{1}{4\pi} \int_{\partial R} \frac{1 - \tau_4(\cdot; X, t)}{r_X^0[X]_{(X, t)}} \cdot [\mu^0 \cdot \hat{J}X]_{(X, t)} d\lambda_{\partial R} \\ &= \frac{1}{4\pi} \int_{\partial R} \kappa(\cdot; X, t) \cdot [\mu^0 \cdot \hat{J}X]_{(X, t)} d\lambda_{\partial R}, \end{aligned} \quad (5)$$

where $\kappa: \{(P, Y, s) \mid P \in \partial R, Y \in \mathbb{R}^3, s \in \mathbb{R}, Y \neq X(P, s)\} \rightarrow (0, \infty)$ is given by

$$\kappa(P;Y,s) := \frac{1-\tau;_4(P;Y,s)}{c\tau(P;Y,s)}, \quad (6)$$

whenever $P \in \partial R$, $Y \in \mathbb{R}^3$, $s \in \mathbb{R}$, $Y \neq \chi(P,s)$.

If (P,Y,s) satisfies the conditions of (6), observe that $\tau(P;Y,s) > 0$, $\tau;_4(P;Y,s)$ exists by [I.3.22.i], and $1-\tau;_4(P;Y,s) > 0$ clearly follows from (I.3.22.2). Thus, κ is well-defined and positive on the indicated set. If $\mu \in C(\partial B)$, the reasoning of (c), *supra*, shows that (5) holds for each $(X,t) \in B^0 \cup \mathcal{N}^\sigma$, since $[\mu]_{[X,t]} \in L_1(\partial B(X,t))$ in that case.

(e) Continuing with the setting introduced in (d), consider further the properties of the positive function κ , defined by (6)⁺ on the set $\{(P,Y,S) \mid P \in \partial R, Y \in \mathbb{R}^3, s \in \mathbb{R}, Y \neq \chi(P,s)\}$, at each point of which $\tau;_4$ is defined and τ is positive. More explicitly, at each point of this set, we have, using (I.3.22.2),

$$\kappa(P;X,t) = r_X^{-1}([X]_{(X,t)}(P)) \cdot \{1 + r_{X,\ell}([X]_{(X,t)}(P)) \cdot [X;_4^{\ell,c}]_{(X,t)}(P)\}^{-1}. \quad (7)$$

Now, if $P \in \partial R$, then $\kappa(P;\cdot,\cdot)$ is defined on the open set $\{(Y,s) \in \mathbb{R}^4 \mid Y \neq \chi(P,s)\} = \mathbb{R}^4 \cap \{\chi^*(\{P\} \times \mathbb{R})\}'$, and [I.3.22.i] leads to a description of the regularity properties of this function: if $\chi(P,\cdot) \in C^k(\mathbb{R};\mathbb{R}^3)$ for some $k \in \mathbb{N}$, then clearly $\kappa(P;\cdot,\cdot) \in C^{k-1}(\mathbb{R}^4 \cap \{\chi^*(\{P\} \times \mathbb{R})\}')$, and we can compute, for $k \geq 2$, the partial derivatives of this function using either (6) or (7), in conjunction with the results given in (I.3.22.1-4). Even though the calculations are entirely trivial, we shall briefly indicate their sequence and display

⁺ Although κ depends upon the particular reference pair chosen for the motion, we omit any indication of this dependence (as for τ).

the final expressions. Supposing, then, that $P \in \partial R$ and $x(P, \cdot) \in C^2(R; R^3)$, set, for $i = 1, 2, 3$,

$$r^i(P; X, t) := r_{X,i}([x]_{(X,t)}(P)) = \frac{1}{c\tau(P; X, t)} \{x^i(P, t-\tau(P; X, t)) - x^i\},$$

$$\text{for } (X, t) \in R^4 \cap \{X^*((P) \times R)\}', \quad (8)$$

so that, by (I.3.22.2),

$$1-\tau;_4(P; X, t) = \{1+r^j(P; X, t) \cdot x^j;_4(P, t-\tau(P; X, t))\}^{-1},$$

$$\text{for } (X, t) \in R^4 \cap \{X^*((P) \times R)\}'. \quad (9)$$

Choose $(X, t) \in R^4 \cap \{X^*((P) \times R)\}'$. From (8), we find, for $i, k = 1, 2, 3$,

$$r^i;_k(P; X, t) = \frac{1}{c\tau(P; X, t)} \{ \{1-\tau;_4(P; X, t)\} \\ \cdot \{r_{X,i}([x]_{(X,t)}(P)) \cdot r_{X,k}([x]_{(X,t)}(P)) \\ + [x^i;_4]_{(X,t)}(P) \cdot r_{X,k}([x]_{(X,t)}(P))\} - \delta_{ik} \},$$

$$(10)$$

having also used (I.3.22.4), and

$$r^i;_4(P; X, t) = \frac{1}{\tau(P; X, t)} \{ -\tau;_4(P; X, t) \cdot r_{X,i}([x]_{(X,t)}(P)) \\ + \{1-\tau;_4(P; X, t)\} \cdot [x^i;_4]_{(X,t)}(P) \}.$$

$$(11)$$

Using (10), (11), and (I.3.22.4) with (9) leads to

$$\begin{aligned}
 \tau_{;4k}(P;X,t) = & -\{1-\tau_{;4}(P;X,t)\}^2 \cdot \left\{ \frac{1}{c} \{1-\tau_{;4}(P;X,t)\} \cdot r_{X,k}([X]_{(X,t)}(P)) \right. \\
 & \cdot r_{X,l}([X]_{(X,t)}(P)) \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) \\
 & + \frac{\{1-\tau_{;4}(P;X,t)\}}{c\tau(P;X,t)} \cdot \{r_{X,l}([X]_{(X,t)}(P)) + [X,_{44}]_{(X,t)}^{\ell c}(P)\} \quad (12) \\
 & \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) \cdot r_{X,k}([X]_{(X,t)}(P)) \\
 & \left. - \frac{1}{c\tau(P;X,t)} \cdot [X,_{44}]_{(X,t)}^{kc}(P) \right\}, \quad \text{for } k = 1, 2, 3,
 \end{aligned}$$

and

$$\begin{aligned}
 \tau_{;44}(P;X,t) = & -\{1-\tau_{;4}(P;X,t)\}^2 \cdot \{ \{1-\tau_{;4}(P;X,t)\} \cdot r_{X,l}([X]_{(X,t)}(P)) \\
 & \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) - \frac{\tau_{;4}(P;X,t)}{\tau(P;X,t)} \cdot r_{X,l}([X]_{(X,t)}(P)) \\
 & \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) + \frac{\{1-\tau_{;4}(P;X,t)\}}{\tau(P;X,t)} \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) \\
 & \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) \}. \quad (13)
 \end{aligned}$$

Consequently, computing from (6) and using (12) and (13) to replace the second-order partial derivatives of $\tau(P; \cdot, \cdot)$ which appear, we can show that the following are correct:

$$\begin{aligned}
 \kappa_{;j}(P;X,t) = & - \frac{\{1-\tau_{;4}(P;X,t)\}^3}{c^2 \tau(P;X,t)} \cdot \{ r_{X,l}([X]_{(X,t)}(P)) \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) \} \\
 & \cdot r_{X,j}([X]_{(X,t)}(P)) + \frac{\{1-\tau_{;4}(P;X,t)\}^3}{c^2 \tau^2(P;X,t)} \\
 & \cdot \{ 1 - [X,_{44}]_{(X,t)}^{\ell c}(P) \cdot [X,_{44}]_{(X,t)}^{\ell c}(P) \} \cdot r_{X,j}([X]_{(X,t)}(P)) \\
 & + \frac{\{1-\tau_{;4}(P;X,t)\}^2}{c^2 \tau^2(P;X,t)} \cdot [X,_{44}]_{(X,t)}^{jc}(P), \quad \text{for } j = 1, 2, 3, \quad (14)
 \end{aligned}$$

and

$$\begin{aligned} \kappa_{;4}(P;X,t) = & - \frac{\{1-\tau_{;4}(P;X,t)\}^3}{c\tau(P;X,t)} \cdot r_{X,4}([X]_{(X,t)}(P)) \cdot [X_{,44}^c]_{(X,t)}(P) \\ & - \frac{\{1-\tau_{;4}(P;X,t)\}^3}{c\tau^2(P;X,t)} \cdot \{r_{X,4}([X]_{(X,t)}(P)) + [X_{,4}^c]_{(X,t)}(P)\} \\ & \cdot [X_{,4}^c]_{(X,t)}(P), \end{aligned} \quad (15)$$

for each $(X,t) \in \mathbb{R}^4 \cap \{X^*(\{P\} \times \mathbb{R})\}'$, whenever $P \in \partial R$ and $X(P, \cdot) \in C^2(\mathbb{R}; \mathbb{R}^3)$. Further, suppose that $P \in \partial R$ and $X(P, \cdot) \in C^3(\mathbb{R}; \mathbb{R}^3)$: it turns out then that

$$\begin{aligned} \square_c \kappa(P; \cdot, \cdot) &:= \kappa_{;11}(P; \cdot, \cdot) - \frac{1}{c^2} \kappa_{;44}(P; \cdot, \cdot) = 0, \\ &\text{in } \mathbb{R}^4 \cap \{X^*(\{P\} \times \mathbb{R})\}'. \end{aligned} \quad (16)$$

The verification of (16) is a tedious routine exercise, starting from (14) and (15), and employing (12), (13), and [I.3.22]; we omit the details. Now, having (16), we can prove a bit more, viz., that if $f: \partial R \times \mathbb{R} \rightarrow K$ and, for some $P \in \partial R$, $f(P, \cdot) \in C^2(\mathbb{R})$ and $X(P, \cdot) \in C^3(\mathbb{R}; \mathbb{R}^3)$, then[†]

$$\square_c (\kappa[f])(P; \cdot, \cdot) = 0 \quad \text{in } \mathbb{R}^4 \cap \{X^*(\{P\} \times \mathbb{R})\}'. \quad (17)$$

(17) is the result of a short computation, using relations already established: letting $(X,t) \in \mathbb{R}^4 \cap \{X^*(\{P\} \times \mathbb{R})\}'$, we find, first,

[†] Recall the alternate notation

$[f](P;X,t) := [f]_{(X,t)}(P), \quad \text{for } P \in \partial R, \quad (X,t) \in \mathbb{R}^4.$

$$\begin{aligned}
 (\kappa[f]);_1(P;X,t) &= \kappa;_1(P;X,t) \cdot [f](P;X,t) \\
 &\quad - \kappa(P;X,t) \cdot \tau;_1(P;X,t) \cdot [f,{}_4](P;X,t) \\
 &= \kappa;_1(P;X,t) \cdot [f](P;X,t) \\
 &\quad + \frac{1}{c} \kappa(P;X,t) \cdot \{1-\tau;_4(P;X,t)\} \cdot r_{X,1}([X]_{(X,t)}(P)) \\
 &\quad \cdot [f,{}_4](P;X,t),
 \end{aligned}$$

the second equality holding *via* (I.3.22.4), and

$$\begin{aligned}
 (\kappa[f]);_4(P;X,t) &= \kappa;_4(P;X,t) \cdot [f](P;X,t) \\
 &\quad + \kappa(P;X,t) \cdot \{1-\tau;_4(P;X,t)\} \cdot [f,{}_4](P;X,t).
 \end{aligned}$$

Generating the combination $(\kappa[f]);_{11}(P;X,t) - \frac{1}{c} (\kappa[f]);_{44}(P;X,t)$ from the latter results, using (I.3.22.4), accounting for (16), and noting from (10) that, in view of (I.3.22.2),

$$\begin{aligned}
 r;_1^1(P;X,t) &= \frac{1}{c\tau(P;X,t)} \{ \{1-\tau;_4(P;X,t)\} \\
 &\quad \cdot \{1+r_{X,1}([X]_{(X,t)}(P)) \cdot [X,{}_4^1]_{(X,t)}(P)\} - 3 \} \\
 &= \frac{-2}{c\tau(P;X,t)},
 \end{aligned}$$

yields, following some simple manipulations,

$$\begin{aligned} \square_c(\kappa[f])(P;X,t) &= \frac{1}{c} \left\{ -2\kappa^2(P;X,t) + 2\{1-\tau;_4(P;X,t)\} \cdot \{\kappa;_1(P;X,t) \right. \\ &\quad \cdot r_{X,1}([X]_{(X,t)}(P)) - \frac{1}{c} \kappa;_4(P;X,t) \} \\ &\quad + \kappa(P;X,t) \cdot \left\{ \frac{1}{c} \tau;_{44}(P;X,t) - \tau;_{41}(P;X,t) \right. \\ &\quad \cdot r_{X,1}([X]_{(X,t)}(P)) \} \left. \right\} [f,{}_4](P;X,t). \end{aligned}$$

Finally, it is easy to check, from (12) and (13), that

$$\frac{1}{c} \tau;_{44}(P;X,t) - \tau;_{41}(P;X,t) \cdot r_{X,1}([X]_{(X,t)}(P)) = 0,$$

while (14) and (15) imply that

$$\begin{aligned} &\kappa;_1(P;X,t) \cdot r_{X,1}([X]_{(X,t)}(P)) - \frac{1}{c} \kappa;_4(P;X,t) \\ &= \kappa^2(P;X,t) \cdot \{1 + r_{X,2}([X]_{(X,t)}(P)) \cdot [X,{}_4^c]_{(X,t)}(P)\} \\ &\quad - \kappa^2(P;X,t) \{ \tau;_4(P;X,t) - \{1-\tau;_4(P;X,t)\} \cdot r_{X,2}([X]_{(X,t)}(P)) \\ &\quad \cdot [X,{}_4^c]_{(X,t)}(P) \} \\ &= \kappa^2(P;X,t) \cdot \{1-\tau;_4(P;X,t)\}^{-1}, \end{aligned}$$

because of (I.3.22.2) and (I.3.22.3). Now (17) results from the latter three equalities.

Continuing with our discussion of κ , let us observe that

$$\partial R \times \{B^0 \cup \Omega^0\} \subset \{(P,Y,s) \mid P \in \partial R, \quad Y \in \mathbb{R}^3, \quad s \in \mathbb{R}, \quad Y \neq X(P,s)\},$$

because $(X(P,s),s) \in \partial B$ whenever $P \in \partial R$ and $s \in \mathbb{R}$, so we can consider the function $\kappa \mid \partial R \times \{B^0 \cup \Omega^0\}$; in view of the definition (6),

the properties of this restriction follow from those of $\tau|_{\partial R \times \{B^0 \cup \Omega^0\}}$, which are described in [I.3.16] and [I.3.22.ii]. Thus, we know that τ is continuous on $\partial R \times \mathbb{R}^4$ and positive on $\partial R \times \{B^0 \cup \Omega^0\}$, while $\tau|_4|_{\partial R \times \{B^0 \cup \Omega^0\}} \in C(\partial R \times \{B^0 \cup \Omega^0\})$, since we know that $D_4 \chi \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$ by Remark [I.3.26.c], whence the inclusion $\kappa|_{\partial R \times \{B^0 \cup \Omega^0\}} \in C(\partial R \times \{B^0 \cup \Omega^0\})$ is an immediate consequence of (6). Let us suppose further that, for some $q \in \mathbb{N} \cup \{\infty\}$, $q \geq 2$, $D_4^k \chi \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$ for each $k \leq q$ if $q \in \mathbb{N}$, or each $k \in \mathbb{N}$, if $q = \infty$; we know that this is the case if, for example, $M \in M(q)$ and (R, X) possesses the properties of Definition [I.3.25], as we pointed out in [I.3.26.c]. At any rate, with this condition fulfilled, [I.3.22.ii] and the positivity of τ on $\partial R \times \{B^0 \cup \Omega^0\}$ clearly imply that $\kappa|_{\alpha}|_{\partial R \times \{B^0 \cup \Omega^0\}} \in C(\partial R \times \{B^0 \cup \Omega^0\})$ whenever the degree of the multi-index $\alpha = (\alpha_1, \dots, \alpha_4)$ is no greater than $q-1$ if $q \in \mathbb{N}$ or for each such multi-index if $q = \infty$. In particular, it is easy to see that (14) and (15) are valid for each $(P, X, t) \in \partial R \times \{B^0 \cup \Omega^0\}$, while, if $q \geq 3$, then

$$\square_c(\kappa|_{\partial R \times \{B^0 \cup \Omega^0\}}) = 0 \quad (18)$$

and

$$\square_c((\kappa[f])|_{\partial R \times \{B^0 \cup \Omega^0\}}) = 0, \quad (19)$$

from (16) and (17), respectively, where the wave operators "act in the second group of variables," and $f: \partial R \times \mathbb{R} \rightarrow K$, with $f(P, \cdot) \in C^2(\mathbb{R})$ for each $P \in \partial R$.

Finally, consider the case in which $(X, t) \in \partial B$: then $X \in \partial \delta_c$.

$x_t^{-1}(X) \in \partial R$, and $\kappa(\cdot; X, t)$ is defined by (6) only on $\partial R \cap \{x_t^{-1}(X)\}'$. Since $1 - \tau_4(\cdot; X, t)$ is bounded away from zero on $\partial R \cap \{x_t^{-1}(X)\}'$, and $\tau(\cdot; X, t)$ is continuous on ∂R and vanishes at $x_t^{-1}(X)$, it is clear that $\kappa(\cdot; X, t)$ is unbounded in any deleted ∂R -neighborhood of $x_t^{-1}(X)$. In fact, using the estimate $|D_4 x|_3 \leq c^*$ on $\partial R \times \mathbb{R}$, the inequalities

$$\left(1 + \frac{c^*}{c}\right)^{-1} r_X^{-1}([X]_{(X,t)}(P)) \leq \kappa(P; X, t) \leq \left(1 - \frac{c^*}{c}\right) r_X^{-1}([X]_{(X,t)}(P)) \quad (20)$$

follow easily from (7), for $P \in \partial R \cap \{x_t^{-1}(X)\}'$, while $[X]_{(X,t)}(x_t^{-1}(X)) = x(x_t^{-1}(X), t) = X$.

(f) Again let (R, X) be a reference pair for M as in [I.3.45], $(X, t) \in \mathbb{B} \cup \Omega^\sigma$, and $[\mu]_{(X,t)} \in L_1(\partial B(X, t))$, so that the representations (4) and (5) are valid. Recalling Remark [I.3.28.e], we know that $[X^*]_{(X,t)}: \partial R \rightarrow \mathbb{R}^4$ is a 1-imbedding carrying ∂R onto the $(2, 4; 1)$ -manifold $\partial B \cap C_-(X, t)$. Consequently, using (I.2.25.1) and the 1-imbedding $[X^*]_{(X,t)}^{-1}$, we can recast $V\{\mu\}(X, t)$ as an integral over $\partial B \cap C_-(X, t)$, by transformation of the integral appearing on the right-hand side of (5). In fact, we could have defined $V\{\mu\}(X, t)$ originally in terms of integration over $\partial B \cap C_-(X, t)$; one might consider such a definition as "more natural," if the intersection $\partial B \cap C_-(X, t)$ is regarded as a "more natural" intrinsic object than its projection $\partial B(X, t)$. However, aside from this somewhat academic consideration, the alternative form of definition might offer a more substantive advantage when one is engaged in a deeper study of functions $V\{\mu\}$ under hypotheses on the density μ more general than those of the classical

sort which we shall later impose. Indeed, the form of definition via integration over $\partial B \cap C_-(X,t)$ can be regarded as involving the trace of μ on the submanifold $\partial B \cap C_-(X,t)$ of ∂B , which exists in a well-defined sense if μ lies in an appropriate space of functions on ∂B , and the motion is sufficiently regular. We shall not pursue this observation further, being content to restrict our attention to the case in which μ is at least continuous on ∂B .

We proceed now to the study of the kinematic single layer potential under various assumptions concerning the motion and density function generating it. By far the easiest to discern are the simple properties of continuity and differentiability in the open set $B^0 \cup \Omega^0$; more delicate are those questions relating to behavior near the manifold ∂B . Accordingly, we shall deal first with the former, after pointing out a property of the support of a KSLM.

[IV.4] P R O P O S I T I O N. Let $M \in \mathcal{M}(1)$ and $\mu: \partial B \rightarrow \mathbb{K}$, with $[\mu]_{[X,t]} \in L_1(\partial B(X,t))$ for each $(X,t) \in B^0 \cup \Omega^0$. Suppose that there exists a $t_0 \in \mathbb{R}$ for which

$$\mu(Z,\zeta) = 0 \quad \text{whenever} \quad \zeta \leq t_0 \quad \text{and} \quad Z \in \partial B_\zeta. \quad (1)$$

Then $V\{\mu\}: B^0 \cup \Omega^0 \rightarrow \mathbb{K}$ vanishes on the set

$$\{(X,t) \in B^0 \cup \Omega^0 \mid t \leq t_0, \text{ or } t > t_0 \text{ and } \text{dist}(X, \partial B_{t_0}) \geq c(t-t_0)\}. \quad (2)$$

P R O O F. It is clear that $V\{\mu\}$ is defined by (IV.2.1) on $B^0 \cup \Omega^0$ (cf., [IV.3.c]). Choose $(X,t) \in B^0 \cup \Omega^0$. Suppose first that $t \leq t_0$:

then $t - \frac{1}{c} r_X(Z) \leq t_0$ whenever $Z \in \mathbb{R}^3$, so (1) implies that, if $Z \in \partial B(X, t)$, i.e., $Z \in \partial B_{t - \frac{1}{c} r_X(Z)}$, we have $[u]_{[X, t]}(Z) := u(Z, t - \frac{1}{c} r_X(Z)) = 0$. Thus, $[u]_{[X, t]} = 0$ if $t \leq t_0$, giving $V\{u\}(X, t) = 0$. Next, let $t > t_0$ and $\text{dist}(X, \partial B_{t_0}) \geq c(t - t_0)$, or $\inf \{r_X(Z) \mid Z \in \partial B_{t_0}\} \geq c(t - t_0)$. Choose any reference pair (R, χ) for M ; since χ_{t_0} is a bijection of ∂R onto ∂B_{t_0} , we see that

$$\inf \{r_X(\chi(P, t_0)) \mid P \in \partial R\} \geq c(t - t_0). \quad (3)$$

We claim that

$$r_X(\{ \chi \}_{(X, t)}(P)) \geq c(t - t_0) \quad \text{for each } P \in \partial R. \quad (4)$$

To see that this is so, choose $P \in \partial R$. By (3), $r_X(\chi(P, t_0)) \geq c(t - t_0)$, so whenever $t_0 < \zeta$,

$$\begin{aligned} r_X(\chi(P, \zeta)) &= r_X(\chi(P, t_0)) + \{r_X(\chi(P, \zeta)) - r_X(\chi(P, t_0))\} \\ &\geq c(t - t_0) - |r_X(\chi(P, \zeta)) - r_X(\chi(P, t_0))| \\ &\geq c(t - t_0) - |\chi(P, \zeta) - \chi(P, t_0)|_3 \\ &\geq c(t - t_0) - c^*|\zeta - t_0| = c(t - t_0) - c^*(\zeta - t_0) \\ &> c(t - t_0) - c(\zeta - t_0) = c(t - \zeta). \end{aligned}$$

Consequently, if we assume that $t_0 < t - \tau(P; X, t)$, where τ is, of course, the retardation function for (R, χ) , we should have

$$c\tau(P; X, t) = r_X(\chi(P, t - \tau(P; X, t))) > c\tau(P; X, t),$$

which is impossible. Therefore, $t_0 \geq t - \tau(P; X, t)$ holds, i.e.,
 $t_0 \geq t - \frac{1}{c} r_X(X(P, t - \tau(P; X, t)))$; this clearly implies that (4) is true.
 Recalling that $[X]_{(X, t)}$ is a bijection of ∂R onto $\partial B(X, t)$, (4)
 now shows that $r_X(Z) \geq c(t - t_0)$, or $t - \frac{1}{c} r_X(Z) \leq t_0$, whenever
 $Z \in \partial B(X, t)$. With (1), this obviously leads once again to $[\mu]_{[X, t]} = 0$
 on $\partial B(X, t)$, and thence $V\{\mu\}(X, t) = 0$ follows. $\square$.

We shall prepare certain classical statements describing
 properties of a function defined by integration over a manifold.

[IV.5] L E M M A. Let $D \subset \mathbb{R}^k$ be an open set, for some $k \in \mathbb{N}$, and
 M an $(r, n; q)$ -manifold. For each $y \in D$, let $f(\cdot; y)$ be a complex
 function defined λ_M -a.e. on M such that

- (i) for each $y \in D$, $f(\cdot; y)$ is M_{λ_M} -measurable;
- (ii) for λ_M -a.a. $x \in M$, $f(x; \cdot)$ is defined on D and
 in $C(D)$;
- (iii) for each $y \in D$, there exist a neighborhood $V_y \subset D$
 of y and a non-negative function $F_y \in L_1(M, M_{\lambda_M}, \lambda_M)$
 such that

$$|f(\cdot, \tilde{y})| \leq F_y \quad \lambda_M\text{-a.e. on } M, \quad \text{for each } \tilde{y} \in V_y. \quad (1)$$

Then the correspondence

$$y \mapsto \phi(y) := \int_M f(\cdot; y) d\lambda_M, \quad y \in D, \quad (2)$$

defines an element ϕ of $C(D)$.

P R O O F. It is clear, from (i) and (iii), that $f(\cdot; y) \in L_1(M, M_{\lambda_M}, \lambda_M)$ whenever $y \in \mathcal{D}$, since $\int_M |f(\cdot; y)| d\lambda_M \leq \int_M F_y d\lambda_M < \infty$ for $y \in \mathcal{D}$, by (1). Thus, (2) defines a function $\phi: \mathcal{D} \rightarrow \mathbb{K}$. To see that ϕ is continuous on $\mathcal{D}$, let $y \in \mathcal{D}$, and suppose that $(y_j)_{j=1}^\infty$ is any sequence in $\mathcal{D}$ converging to y . Then $(f(\cdot; y_j))_{j=1}^\infty$ is a sequence in $L_1(M, M_{\lambda_M}, \lambda_M)$ converging λ_M -a.e. on M to $f(\cdot; y)$, by (ii). There is an $n_0 \in \mathbb{N}$ such that $y_j \in V_y$ for all $j \geq n_0$, whence (1) gives $|f(\cdot; y_j)| \leq F_y$ λ_M -a.e. on M , for each $j \geq n_0$. Lebesgue's dominated convergence theorem then allows us to write

$$\begin{aligned} \lim_{j \rightarrow \infty} \phi(y_j) &= \lim_{j \rightarrow \infty} \int_M f(\cdot; y_j) d\lambda_M = \int_M \lim_{j \rightarrow \infty} f(\cdot; y_j) d\lambda_M \\ &= \int_M f(\cdot; y) d\lambda_M = \phi(y). \end{aligned}$$

Thus, ϕ is sequentially continuous, hence continuous, on $\mathcal{D}$. $\square$.

[IV.6] C O R O L L A R Y. Let $\mathcal{D} \subset \mathbb{R}^k$ be an open set, for some $k \in \mathbb{N}$, M a compact $(r, n; q)$ -manifold, and $f \in C(M \times \mathcal{D})$. Then (IV.5.2) defines an element ϕ of $C(\mathcal{D})$.

P R O O F. It is quite simple to check that the hypotheses of Lemma [IV.5] are fulfilled in this setting: for each $y \in \mathcal{D}$, $f(\cdot, y) \in C(M)$, hence it is M_{λ_M} -measurable (recall that M_{λ_M} contains the Borel sets of M), and $f(x, \cdot) \in C(\mathcal{D})$ for each $x \in M$. If $y \in \mathcal{D}$, choose $\delta_y > 0$ such that $B_{\delta_y}^k(y)^- \subset \mathcal{D}$. Taking $V_y := B_{\delta_y}^k(y)$ and, noting that $M \times V_y^-$ is compact in $M \times \mathcal{D}$,

$$F_y(x) := \max_{\substack{\tilde{y} \in V_y^- \\ z \in M}} |f(z, \tilde{y})|, \quad \text{for each } x \in M,$$

it is clear that [IV.5.iii] holds. $\square$.

[IV.7] L E M M A. Let $D \subset \mathbb{R}^k$ be an open set, for some $k \in \mathbb{N}$, and M an $(r, n; q)$ -manifold. For each $y \in D$, let $f(\cdot; y)$ be a complex function defined λ_M -a.e. on M such that

- (i) for each $y \in D$, $f(\cdot; y) \in L_1(M, M_{\lambda_M}, \lambda_M)$;
- (ii) for λ_M -a.a. $x \in M$, $f(x; \cdot)$ is defined on D , with $f_{;i}(x; \cdot)$ existing on D for each $i \in \{1, \dots, k\}$;
- (iii) for each $y \in D$, there exist a positive $\delta_y \in \mathbb{R}$ and a non-negative $F_y \in L_1(M, M_{\lambda_M}, \lambda_M)$ such that $B_{\delta_y}^k(y)^- \subset D$ and

$$|f_{;i}(\cdot; \tilde{y})| \leq F_y \quad \lambda_M\text{-a.e. on } M,$$
 for each $\tilde{y} \in B_{\delta_y}^k(y)$ and $i \in \{1, \dots, k\}$. (1)

Define $\phi: D \rightarrow \mathbb{K}$ by

$$\phi(y) := \int_M f(\cdot; y) d\lambda_M, \quad \text{for each } y \in D. \quad (2)$$

Then, for each $i \in \{1, \dots, k\}$, $\phi_{;i}$ exists on D , with

$$\phi_{;i}(y) = \int_M f_{;i}(\cdot; y) d\lambda_M \quad \text{for each } y \in D. \quad (3)$$

If, in addition,

(iv) for λ_M -a.a. $x \in M$, $f_{;i}(x; \cdot) \in C(D)$ for each $i \in \{1, \dots, k\}$,

then $\phi \in C^1(D)$.

P R O O F. Note that (i) ensures that ϕ is defined by (2) on D . According to (ii), there exists a set $M_0 \in M_{\lambda_M}$ with $\lambda_M(M_0) = 0$, such that $f(x; \cdot)$ is defined on D and $f_{;i}(x; \cdot)$ exists on D for each $i \in \{1, \dots, k\}$, for each $x \in M_0$. Choose $y \in D$ and $i \in \{1, \dots, k\}$. Let $(\sigma_j)_{j=1}^\infty$ be a sequence with $0 < |\sigma_j| < \delta_y$ for each $j \in \mathbb{N}$ and converging to zero. For each $j \in \mathbb{N}$, define $\sigma_{ij} \in \mathbb{R}^k$ by setting $\sigma_{ij}^m := \sigma_j \delta_1^m$ for $m \in \{1, \dots, k\}$. Then

$$f_{;i}(x; y) = \lim_{j \rightarrow \infty} \frac{1}{\sigma_j} \{f(x; y + \sigma_{ij}) - f(x; y)\} \quad \text{for each } x \in M_0,$$

so $f_{;i}(\cdot; y)$ is the pointwise limit of the sequence $\left\{ \frac{1}{\sigma_j} \{f(\cdot; y + \sigma_{ij}) - f(\cdot; y)\} \right\}_{j=1}^\infty$ in $L_1(M, M_{\lambda_M}, \lambda_M)$, λ_M -a.e. on M . For each $x \in M_0$,

let us apply the mean-value theorem to the function on $(-\delta_y, \delta_y)$ given by $s \mapsto f(x; y + s_1)$, where $s_1^m := s \delta_1^m$ for $m \in \{1, \dots, k\}$: we conclude that, for each $j \in \mathbb{N}$, there exists $\zeta_j(x) \in \mathbb{R}$, lying between 0 and σ_j , hence in $(-\delta_y, \delta_y)$, for which

$$\left| \frac{1}{\sigma_j} \{f(x; y + \sigma_{ij}) - f(x; y)\} \right| = |f_{;i}(x; y + \zeta_{ij}(x))|, \quad (4)$$

where $\zeta_{ij}(x)^m := \zeta_j(x) \delta_1^m$ for $m \in \{1, \dots, k\}$. Since $y + \zeta_{ij}(x) \in B_{\delta_y}^k(y)$, (1) and (4) combine to give

$$\left| \frac{1}{\sigma_j} \{f(\cdot; y + \sigma_{ij}) - f(\cdot; y)\} \right| \leq F_y \quad \lambda_M\text{-a.e. on } M, \quad \text{for each } j \in \mathbb{N}.$$

Therefore, we may apply the dominated convergence theorem of Lebesgue to conclude that $f_{;i}(\cdot; y) \in L_1(M, \mathcal{M}_{\lambda_M}, \lambda_M)$ and

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{\sigma_j} \{\phi(y + \sigma_{ij}) - \phi(y)\} &= \lim_{j \rightarrow \infty} \int_M \frac{1}{\sigma_j} \{f(\cdot; y + \sigma_{ij}) - f(\cdot; y)\} d\lambda_M \\ &= \int_M \lim_{j \rightarrow \infty} \frac{1}{\sigma_j} \{f(\cdot; y + \sigma_{ij}) - f(\cdot; y)\} d\lambda_M \\ &= \int_M f_{;i}(\cdot; y) d\lambda_M, \end{aligned}$$

whence it is clear that $\phi_{;i}$ exists on $\mathcal{D}$, and (3) holds.

Finally, if (iv) holds, then, for each $i \in \{1, \dots, k\}$, $f_{;i}(\cdot; y)$ is defined λ_M -a.e. on M for each $y \in \mathcal{D}$, and hypotheses (i)-(iii) of Lemma [IV.5] are clearly fulfilled by $f_{;i}$ (the first part of the proof showed that $f_{;i}(\cdot; y) \in L_1(M, \mathcal{M}_{\lambda_M}, \lambda_M)$, so is certainly $\mathcal{M}_{\lambda_M}$ -measurable). Because of (3), Lemma [IV.5] then says that $\phi_{;i} \in C(\mathcal{D})$. $\square$.

[IV.8] C O R O L L A R Y. Let $\mathcal{D} \subset \mathbb{R}^k$ be an open set, for some $k \in \mathbb{N}$, and M a compact $(r, n; q)$ -manifold. Suppose that $(x, y) \mapsto f(x; y)$ denotes an element of $C(M \times \mathcal{D})$ such that $f_{;i} \in C(M \times \mathcal{D})$ for each $i \in \{1, \dots, k\}$. Then (IV.7.2) defines a function $\phi \in C^1(\mathcal{D})$, for which (IV.7.3) holds.

P R O O F. We need only check that the hypotheses of [IV.7] are

satisfied: if $y \in \mathcal{D}$, $f(\cdot; y) \in C(M) \subset L_1(M, M_{\lambda_M}, \lambda_M)$. [IV.17.iv] is clearly true here. Finally, for $y \in \mathcal{D}$, choose any $\delta_y > 0$ such that $B_{\delta_y}^k(y)^- \subset \mathcal{D}$, observe that $M \times B_{\delta_y}^k(y)^-$ is compact, and simply take

$$F_y(x) := \max_{1 \leq i \leq k} \max_{\substack{z \in M \\ \tilde{y} \in B_{\delta_y}^k(y)^-}} |f_i(z; \tilde{y})| \quad \text{for each } x \in M.$$

Then [IV.7.iii] holds. $\square$.

Returning to the study of the kinematic single layer potential, let us establish the following simple statement:

[IV.9] P R O P O S I T I O N. Let $M \in \mathcal{M}(1)$ and $\mu \in C(\partial B)$. Then the corresponding KSLM $V\{\mu\}$ is in $C(B^0 \cup \Omega^\sigma)$.

P R O O F. We have pointed out, in [IV.3.c], that the inclusion $\mu \in C(\partial B)$ implies that $V\{\mu\}$ is defined on $B^0 \cup \Omega^\sigma$. Letting (R, x) be a reference pair for M as in Definition [I.3.25], we have the representation (IV.3.5):

$$V\{\mu\}(X, t) = \frac{1}{4\pi} \int_{\partial R} (\kappa[\hat{\mu} \cdot \hat{J}_X])(\cdot; X, t) d\lambda_{\partial R}, \quad (1)$$

for $(X, t) \in B^0 \cup \Omega^\sigma$.

Now, $\kappa|_{\partial R \times (B^0 \cup \Omega^\sigma)}$ is continuous (cf., [IV.3.d]); $\hat{\mu} := \mu \circ x^*$ and $\hat{J}_X$ are continuous on $\partial R \times \mathbb{R}$ (for the latter, recall [I.3.26.d]), while $(P, Y, s) \mapsto (P, s - \tau(P; Y, s))$ is clearly continuous on $\partial R \times (B^0 \cup \Omega^\sigma)$ into

$\partial R \times \mathbb{R}$, by [I.3.16.ii], so $[\overset{0}{\mu} \cdot \hat{J}\chi]$ is continuous on $\partial R \times \{B^0 \cup \Omega^0\}$.

Noting that ∂R is compact, the desired conclusion follows directly from [IV.6]. $\square$.

Consider next how we can use Corollary [IV.8] and (IV.9.1) to show that, under appropriate additional conditions on $M \in \mathcal{M}(1)$ and $\mu \in C(\partial B)$, $V\{\mu\} \in C^k(B^0 \cup \Omega^0)$ for some $k \in \mathbb{N}$. With (R, X) as in the proof of [IV.9], we recall that the differentiability of κ and τ in their second set of arguments depends on that of χ in its fourth argument. Also, since, for example,

$$[\hat{J}\chi](P; X, t) := \hat{J}\chi(P, t - \tau(P; X, t)) := J\chi_{t - \tau(P; X, t)}(P)$$

for $P \in \partial R$ and $(X, t) \in \mathbb{R}^4$, we should impose differentiability conditions on $\overset{0}{\mu}$ and $\hat{J}\chi$ in their fourth arguments. Now, if we require $M \in \mathcal{M}(q)$ for some $q \geq 2$, we can suppose that $D_4^j \chi \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$ for $j = 1, \dots, q$, and $D_4^j \hat{J}\chi \in C(\partial R \times \mathbb{R})$ for $j = 1, \dots, q-1$ (or for each $j \in \mathbb{N}$, if $q = \infty$), but this imposes unnecessary smoothness conditions on the various manifolds associated with M . Thus, we are motivated to consider classes of motions for which the two types of smoothness are, to some extent, "uncoupled." Specifically, we make the following definition:

[IV.10] D E F I N I T I O N. Let $q \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ [$m = \infty$]. If M is a motion, then $M \in \mathcal{M}(q; m)$ iff M possesses a reference pair (R, X) such that

- (i) conditions [I.3.25.i, ii, and iii] hold;

$$(ii) \quad D_4^k \chi \in C(\partial R \times \mathbb{R}; \mathbb{R}^3) \quad \text{for } k = q+1, \dots, q+m+1 \quad [k = q+1, q+2, \dots];$$

and

(iii) there exists a covering collection of coordinate systems for ∂R , $\{(U_i, h_i)\}_{i \in I}$, such that

$$D_4^{k_0} N_i \in C(U_i \times \mathbb{R}; \mathbb{R}^3) \quad \text{for } k = q, \dots, q+m \quad [k = q, q+1, \dots]$$

and each $i \in I$,

wherein the functions $N_i^0: U_i \times \mathbb{R} \rightarrow \mathbb{R}^3$ are given by

$$\begin{aligned} N_i^0(P, \zeta) &:= \epsilon_{ijk} \{ (X_{\zeta}^{j_0 h_i^{-1}})_{,1}(h_i(P)) \} \cdot \{ (X_{\zeta}^{k_0 h_i^{-1}})_{,2}(h_i(P)) \} \\ &= \epsilon_{ijk} \{ (X^{j_0 h_i^{*-1}})_{,1}(h_i(P), \zeta) \} \cdot \{ (X^{k_0 h_i^{*-1}})_{,2}(h_i(P), \zeta) \} \end{aligned} \quad (1)$$

for each $P \in U_i$, $\zeta \in \mathbb{R}$, and $i \in I$. ■

[IV.11] R E M A R K S. Suppose that $M \in \mathcal{M}(q; m)$ for some $q \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ [$m = \infty$]; let (R, χ) be a reference pair for M as in [IV.10], and $\{(U_i, h_i)\}_{i \in I}$ as in [IV.10.iii].

(a) Obviously, $\mathcal{M}(q; m) \subset \mathcal{M}(q)$, so $M \in \mathcal{M}(q)$, and we already know that $D_4^k \chi \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$ for $k = 0, \dots, q$, while $D_4^{k_0} N_i \in C(U_i \times \mathbb{R}; \mathbb{R}^3)$ for $k = 0, \dots, q-1$ and $i \in I$. These inclusions follow from the observations made in [I.3.26.b and c].

(b) It is important to point out that we have

$$\left. \begin{array}{l} D_4^{k0} \in C(\partial R \times R; R^3) \\ \text{and} \\ D_4^{kj} \in C(\partial R \times R) \end{array} \right\} \text{ for } k = 0, \dots, q+m \quad [k \in \mathbb{N} \setminus \{0\}]. \quad (1)$$

Indeed, for $k = 0, \dots, q-1$, these inclusions follow because $M \in M(q)$. By reasoning as in, for example, the proof of [I.3.27], it is easy to see that, for each $i \in I$,

$$v(P, \zeta) := v(\chi(P, \zeta), \zeta) = \eta_1 \cdot \frac{N_1^0(P, \zeta)}{|N_1^0(P, \zeta)|_3} \quad \text{for } P \in U_1, \quad \zeta \in \mathbb{R}, \quad (3)$$

with $|\eta_1|$ equal to 1, and

$$\hat{j}_\chi(P, \zeta) = \frac{|N_1^0(P, \zeta)|_3}{|(h_1^{-1})_{,1}(h_1(P)) \times (h_1^{-1})_{,2}(h_1(P))|_3}, \quad (4)$$

for $P \in U_1, \quad \zeta \in \mathbb{R},$

so that (1) and (2) hold for $k = q, \dots, q+m$ [$k = q, q+1, \dots$] by [IV.10.iii]. Since

$$\begin{aligned} v(P, \zeta) &:= v(\chi(P, \zeta), \zeta) = v^j(\chi(P, \zeta), \zeta) \cdot \chi_{,4}^j(P, \zeta) \\ &= v^j(P, \zeta) \cdot \chi_{,4}^j(P, \zeta) \quad \text{for } P \in \partial R, \quad \zeta \in \mathbb{R}, \end{aligned} \quad (5)$$

it must then also be true that

$$D_4^{k0} \in C(\partial R \times R) \quad \text{for } k = 0, \dots, q+m \quad [k \in \mathbb{N} \setminus \{0\}], \quad (6)$$

in view of [IV.10.1, ii] and (1).

[IV.12] P R O P O S I T I O N. Let M be a motion, $k \in \mathbb{N}$ [$k = \infty$],

and $\mu \in C(\partial B)$. Suppose that either

$$(i) \quad M \in \mathcal{M}(q) \text{ for some } q \geq k+1 \quad [M \in \mathcal{M}(\infty)]$$

or

$$(i)' \quad M \in \mathcal{M}(q;m), \text{ where } q+m \geq k \quad [M \in \mathcal{M}(q;\infty) \text{ for some } q \in \mathbb{N}],$$

and

$$(ii) \quad \text{for some reference pair } (R, \chi) \text{ for } M \text{ as in [I.3.25]} \\ \text{in case (i) holds, or as in [IV.10] if (i)' is true,}$$

$$D_4^{j0} \mu \in C(\partial R \times \mathbb{R}) \quad \text{for } j = 1, \dots, k \quad [\text{for each } j \in \mathbb{N}]. \quad (1)$$

Then $V\{\mu\} \in C^k(\mathbb{B}^0 \cup \Omega^\sigma)$, and the partial derivatives of $V\{\mu\}$ in $\mathbb{B}^0 \cup \Omega^\sigma$ can be computed from the representation (IV.9.1) generated by (R, χ) , by "differentiation under the integral." Moreover, if $k \geq 2$, then

$$\square_c V\{\mu\} = 0 \quad \text{in } \mathbb{B}^0 \cup \Omega^\sigma. \quad (2)$$

P R O O F. With $\mu^0 := \mu \circ \chi^*$, the reference pair (R, χ) for M is such that the inclusions in (1) obtain, while the conditions set forth in [I.3.25] hold if (i) prevails, whereas the requirements of [IV.10] are fulfilled if (i)' is true. From the remarks in [IV.3.c and d], noting that we have at least $M \in \mathcal{M}(1)$ and $\mu \in C(\partial B)$ in any case, it is clear that $V\{\mu\}$ is defined on $\mathbb{B}^0 \cup \Omega^\sigma$ and (IV.9.1) is valid. Let us convince ourselves that, under each set of hypotheses listed,

$$(\kappa[\hat{u} \cdot \hat{j}\chi]);_{\alpha} | \partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\} \in C(\partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\}), \quad (2)$$

whenever the degree of the multi-index

$\alpha = (\alpha_1, \dots, \alpha_4)$ is $\leq k$, if $k \in \mathbb{N}$, or

for each such multi-index α , if $k = \infty$.

Indeed, suppose $k \in \mathbb{N}$: under either (i) or (i)', we see that $D_4^j \chi \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$, for $j = 0, 1, \dots, k+1$ (cf., [I.3.26.c] and [IV.10.i]), so $\tau;_{\alpha} | \partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\} \in C(\partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\})$ whenever the degree of the multi-index α is $\leq k+1$ ([I.3.22.ii]), and $\kappa;_{\alpha} | \partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\} \in C(\partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\})$ if the degree of α is $\leq k$ ([IV.3.e]). Further, $D_4^{j\sigma}$ and $D_4^j \hat{j}\chi \in C(\partial R \times \mathbb{R})$ for $j = 0, 1, \dots, k$, the first by (ii), the second under either (i) or (i)', by [I.3.27.iv] or [IV.10.ii], respectively. Recalling that $[\hat{u}](P; Y, s) := \hat{u}(P, s - \tau(P; Y, s))$ and $[\hat{j}\chi](P; Y, s) := \hat{j}\chi(P, s - \tau(P; Y, s))$ for $P \in \partial R$, $(Y, s) \in \mathbb{R}^4$, the chain rule now obviously implies that $[\hat{u} \cdot \hat{j}\chi];_{\alpha} | \partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\} \in C(\partial R \times \{\mathbb{B}^0 \cup \Omega^{\sigma}\})$ if the degree of α is $\leq k$. Thus, (2) is true for $k \in \mathbb{N}$; the verification in case $k = \infty$ is quite similar, and we shall omit the details. In view of the compactness of ∂R , Corollary [IV.8] now allows us to assert that $V\{\mu\} \in C^1(\mathbb{B}^0 \cup \Omega^{\sigma})$ with

$$V\{\mu\}_{,1}(X, t) = \frac{1}{4\pi} \int_{\partial R} (\kappa[\hat{u} \cdot \hat{j}\chi]);_1(\cdot; X, t) d\lambda_{\partial R}, \quad (3)$$

for $(X, t) \in \mathbb{B}^0 \cup \Omega^{\sigma}$, $i = 1, 2, 3, 4$,

whichever set of hypotheses be in force, and whatever the value of k .

Further, if $k \geq 2$, reapplication of (2) and [IV.8] to each $V\{\mu\}_{,1}$, $i \in \{1, 2, 3, 4\}$, in the form given by (3), leads to the inclusion

$V\{\mu\} \in C^2(\mathbb{B}^0 \cup \Omega^\sigma)$ and the expressions for $V\{\mu\}_{,ij}$, $i, j \in \{1, \dots, 4\}$, obtained from (3) by differentiation of the integrand. It is now clear how we can complete the proof of the first assertion of the proposition by induction; once again, we shall not supply the particulars.

Finally, if $k \geq 2$, the first statement of the proposition, along with (IV.3.19) (taking $f = \overset{\circ}{\mu} \cdot \hat{J}X$ in the latter), produces

$$\square_c V\{\mu\}(X, t) = \frac{1}{4\pi} \int_{\partial R} \square_c (\kappa \cdot [\overset{\circ}{\mu} \cdot \hat{J}X])(\cdot; X, t) d\lambda_{\partial R} = 0,$$

if $(X, t) \in \mathbb{B}^0 \cup \Omega^\sigma$. □.

[IV.13] R E M A R K S. Let us agree that throughout this and the coming section [IV.14], M is in $\mathcal{M}(1;0)$ (e.g., $M \in \mathcal{M}(2)$ will do), (R, χ) is a reference pair for M as in [IV.10], and μ is a function in $C(\partial B)$ with $D_4 \overset{\circ}{\mu} \in C(\partial R \times \mathbb{R})$; τ and κ are the usual functions associated with (R, χ) . Then, using the notations established in [IV.2], we know by [IV.12] that $V^I\{\mu\} \in C^1(\mathbb{B}^0)$ and $V^O\{\mu\} \in C^1(\Omega^\sigma)$. Our next objective is the identification of additional conditions on χ and μ sufficient to ensure that each first-order partial derivative of $V^I\{\mu\}$ possesses a continuous extension to $\mathbb{B} = \mathbb{B}^{0-}$, and each first-order partial derivative of $V^O\{\mu\}$ possesses a continuous extension to $\Omega^{\sigma-}$; whenever this is the case, we also wish to determine the values on ∂B of these continuous extensions, i.e., the limiting values on ∂B of each of the first-order partial derivatives of $V^I\{\mu\}$ and $V^O\{\mu\}$. In the course of this reasoning, we shall discover conditions

under which $V\{\mu\}$ itself can be extended continuously from $\mathbb{B}^0 \cup \Omega^\sigma$ to all of $\mathbb{R}^4$. We begin this rather lengthy investigation by explicitly computing the partial derivatives of $V\{\mu\}$ in $\mathbb{B}^0 \cup \Omega^\sigma$ and recasting the resultant expressions into forms more suitable for the scheme which we intend to employ in the analysis.

[IV.14] EXPLICIT COMPUTATION: PARTIAL DERIVATIVES OF $V\{\mu\}$ IN $\mathbb{B}^0 \cup \Omega^\sigma$. Under the conditions which we have just posed in Section [IV.13], [IV.12] states that we can compute the partial derivatives of $V\{\mu\}$ in $\mathbb{B}^0 \cup \Omega^\sigma$ by finding the appropriate partial derivatives of the integrand in (IV.9.1),

$$V\{\mu\}(X,t) = \frac{1}{4\pi} \int_{\partial R} (\kappa[\mu \cdot \hat{J}X])(\cdot; X,t) d\lambda_{\partial R}, \quad (X,t) \in \mathbb{B}^0 \cup \Omega^\sigma.$$

By using first the chain rule, then appealing to (I.3.22.4), we find, for $i = 1, 2, 3$, and $(X,t) \in \mathbb{B}^0 \cup \Omega^\sigma$,

$$\begin{aligned} V\{\mu\}_{,i}(X,t) &= \frac{1}{4\pi} \int_{\partial R} \{ \kappa_{,i}(\cdot; X,t) \cdot [\mu \cdot \hat{J}X]_{(X,t)} \\ &\quad - \kappa(\cdot; X,t) \cdot \tau_{,i}(\cdot; X,t) \cdot [(\mu \cdot \hat{J}X)_{,4}]_{(X,t)} \} d\lambda_{\partial R} \\ &= \frac{1}{4\pi} \int_{\partial R} \{ \kappa_{,i}(\cdot; X,t) \cdot [\mu \cdot \hat{J}X]_{(X,t)} \\ &\quad + \frac{1}{c} \kappa(\cdot; X,t) \cdot \{ 1 - \tau_{,4}(\cdot; X,t) \} \cdot r_{X,i}^0[x]_{(X,t)} \\ &\quad \cdot [(\mu \cdot \hat{J}X)_{,4}]_{(X,t)} \} d\lambda_{\partial R}. \end{aligned} \tag{1}$$

More directly, we also have, for $(X,t) \in \mathbb{B}^0 \cup \Omega^\sigma$,

$$V\{u\}_{,4}(X,t) = \frac{1}{4\pi} \int_{\partial R} \{ \kappa;_4(\cdot;X,t) \cdot [\hat{L} \cdot \hat{J}X]_{(X,t)} + \kappa(\cdot;X,t) \cdot (1-\tau;_4(\cdot;X,t)) \cdot [(\hat{L} \cdot \hat{J}X)_{,4}]_{(X,t)} \} d\lambda_{\partial R}. \quad (2)$$

Now, whenever $t \in \mathbb{R}$, it is known, by [I.3.27.1.1], that $\chi_t: \partial R \rightarrow \mathbb{R}^3$ is (at least) a 1-imbedding, carrying ∂R onto ∂B_t . Then, according to [I.2.17.iii], $\chi_t^{-1}: \partial B_t \rightarrow \mathbb{R}^3$ is a 1-imbedding, taking ∂B_t onto ∂R , and Theorem [I.2.25] allows us to write

$$\int_{\partial R} f d\lambda_{\partial R} = \int_{\partial B_t} f \circ \chi_t^{-1} \cdot J\chi_t^{-1} d\lambda_{\partial B_t}, \quad (3)$$

for any $f \in L_1(\partial R)$. Let us use (IV.3.14) and (IV.3.15) in (1) and (2), respectively, along with (I.3.22.2) and the defining relation for τ ; upon applying (3) to transform each resulting integral over ∂R , a routine calculation leads to the equalities

$$V\{u\}_{,i}(X,t) = \sum_{j=1}^3 V\{u\}_i^j(X,t), \quad \text{for } (X,t) \in B^0 \cup \Omega^0, \quad (4)$$

$$i \in \{1,2,3,4\},$$

wherein,[†] for $i \in \{1,2,3\}$,

[†]We use the notation established in [I.3.23.c], viz., $\chi^c := (1/c^2)\chi$.

$$\begin{aligned}
 V\{\mu\}_1^1(X,t) &:= \frac{1}{4\pi} \int_{\partial B_t} \{r_X^0[x](X,t) \cdot \{1+r_{X,\ell}^0[x](X,t) \cdot [x_{,4}^c](X,t)\} \}^{-2} \circ x_t^{-1} \\
 &\cdot \left\{ \frac{1 - |[x_{,4}^c](X,t)|_3^2}{1+r_{X,k}^0[x](X,t) \cdot [x_{,4}^k](X,t)} r_{X,i}^0[x](X,t) \right. \\
 &\left. + [x_{,4}^i](X,t) \right\} \circ x_t^{-1} \cdot [\hat{\mu} \cdot \hat{J}X](X,t) \circ x_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 V\{\mu\}_1^2(X,t) &:= -\frac{1}{4\pi} \int_{\partial B_t} \{r_X^{-1}[x](X,t) \cdot \{1+r_{X,\ell}^0[x](X,t) \cdot [x_{,4}^{\ell c}](X,t)\}^{-3}\} \\
 &\circ x_t^{-1} \cdot \{r_{X,k}^0[x](X,t) \cdot [x_{,44}^{kc^2}](X,t) \cdot r_{X,i}^0[x](X,t)\} \circ x_t^{-1} \\
 &\cdot [\hat{\mu} \cdot \hat{J}X](X,t) \circ x_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{6}$$

$$\begin{aligned}
 V\{\mu\}_1^3(X,t) &:= \frac{1}{4\pi} \int_{\partial B_t} \{r_X^{-1}[x](X,t) \cdot \{1+r_{X,\ell}^0[x](X,t) \cdot [x_{,4}^{\ell c}](X,t)\}^{-2}\} \\
 &\circ x_t^{-1} \cdot r_{X,i}^0[x](X,t) \circ x_t^{-1} \cdot [(\hat{\mu} \cdot \hat{J}X)_{,4}^c](X,t) \circ x_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{7}$$

and

$$\begin{aligned}
 V\{\mu\}_4^1(X,t) &:= -\frac{c}{4\pi} \int_{\partial B_t} \{r_X^{-2}[x](X,t) \cdot \{1+r_{X,\ell}^0[x](X,t) \cdot [x_{,4}^{\ell c}](X,t)\}^{-3}\} \\
 &\circ x_t^{-1} \cdot \{r_{X,k}^0[x](X,t) \cdot [x_{,4}^{kc}](X,t) + |[x_{,4}^c](X,t)|_3^2\} \circ x_t^{-1} \\
 &\cdot [\hat{\mu} \cdot \hat{J}X](X,t) \circ x_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{8}$$

$$\begin{aligned}
 V(\mu)_4^2(X,t) &:= -\frac{c}{4\pi} \int_{\partial B_t} \{r_X^{-1} \circ [X](X,t) \cdot (1+r_{X,\ell} \circ [X](X,t) \cdot [\lambda,4]^c(X,t))^{-3}\} \\
 &\quad \circ \chi_t^{-1} \cdot (r_{X,k} \circ [X](X,t) \cdot [\lambda,44]^c(X,t)) \circ \chi_t^{-1} \\
 &\quad \cdot [\hat{\mu} \cdot \hat{J}_X](X,t) \circ \chi_t^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 V(\mu)_4^3(X,t) &:= \frac{c}{4\pi} \int_{\partial B_t} \{r_X^{-1} \circ [X](X,t) \cdot (1+r_{X,\ell} \circ [X](X,t) \cdot [\lambda,4]^c(X,t))^{-2}\} \\
 &\quad \circ \chi_t^{-1} \cdot [(\hat{\mu} \cdot \hat{J}_X)^c,4](X,t) \circ \chi_t^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t}.
 \end{aligned} \tag{10}$$

We require the introduction of an auxiliary function prior to the further manipulation of the functions given in $B^0 \cup \Omega^0$ by (5)-(10), to produce the final forms to be examined. In this definition, we need only $x_{,4} \in C(\partial R \times R; \mathbb{R}^3)$, although we have already required (at least) that $x_{,44} \in C(\partial R \times R; \mathbb{R}^3)$. Suppose, then, that $(Y,t) \in \partial B$ (so $t \in \mathbb{R}$ and $Y \in \partial B_t$) and $X \in \mathbb{R}^3$: if $X \neq Y$, observe that $\tau(\chi_t^{-1}(Y); X, t) > 0$, while $\tau(\chi_t^{-1}(Y); X, t) = 0$ if $X = Y$. We define $V: \partial B \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$V(Y,t;X) := \begin{cases} \frac{1}{\tau(\chi_t^{-1}(Y); X, t)} \cdot \{Y - [X](X,t) \circ \chi_t^{-1}(Y)\}, \\ \text{if } (Y,t) \in \partial B, \quad X \in \mathbb{R}^3 \cap \{Y\}', \\ x_{,4}(\chi_t^{-1}(Y), t), \quad \text{if } (Y,t) \in \partial B, \quad X = Y. \end{cases} \tag{11}$$

Let us show that V is continuous. The continuity of V at each point of the set $\{(Y,t,X) \mid (Y,t) \in \partial B, X \in \mathbb{R}^3 \cap \{Y\}'\}$ follows easily from

the positivity of $(Y, t, X) \mapsto \tau(\chi_t^{-1}(Y); X, t)$ on this set, the continuity of τ on $\partial R \times \mathbb{R}^4$ (cf., [I.3.16]), the continuity of the map $(Y, t) \mapsto \chi_t^{-1}(Y)$ on ∂B onto ∂R (cf., [I.3.7]), and the inclusion $X \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$. It remains then to consider a point $(Y, t, Y) \in \partial B \times \mathbb{R}^3$: let $((Y_n, t_n, X_n))_{n=1}^\infty$ be a sequence in $\partial B \times \mathbb{R}^3$ converging to (Y, t, Y) , so $Y_n \rightarrow Y$ and $X_n \rightarrow Y$ in $\mathbb{R}^3$, while $t_n \rightarrow t$ in $\mathbb{R}$. By the continuity properties just cited, we have then

$$\lim_{n \rightarrow \infty} \tau(\chi_{t_n}^{-1}(Y_n); X_n, t_n) = \tau(\chi_t^{-1}(Y); Y, t) = 0. \quad (12)$$

Now, whenever $n \in \mathbb{N}$ and $X_n \neq Y_n$, we can write, for $i \in \{1, 2, 3\}$,

$$\begin{aligned} v^i(Y_n, t_n; X_n) &= \frac{1}{\tau(\chi_{t_n}^{-1}(Y_n); X_n, t_n)} \\ &\quad \cdot \{ \chi^i(\chi_{t_n}^{-1}(Y_n), t_n) - \chi^i(\chi_{t_n}^{-1}(Y_n), t_n - \tau(\chi_{t_n}^{-1}(Y_n); X_n, t_n)) \} \\ &= \chi_{\frac{1}{4}}^i(\chi_{t_n}^{-1}(Y_n), \hat{t}_n^i), \end{aligned} \quad (13)$$

via the mean-value theorem, where

$$\hat{t}_n^i \in (t_n - \tau(\chi_{t_n}^{-1}(Y_n); X_n, t_n), t_n), \quad (14)$$

and the underscoring of indices in (13) signifies the suspension of the summation convention; on the other hand, if $X_n = Y_n$, then

$$v^i(Y_n, t_n; X_n) = \chi_{\frac{1}{4}}^i(\chi_{t_n}^{-1}(Y_n), t_n), \quad i \in \{1, 2, 3\}. \quad (15)$$

Defining the sequence $(t_n^i)_{n=1}^\infty$ in $\mathbb{R}$ by $t_n^i := \hat{t}_n^i$ if $X_n \neq Y_n$, $t_n^i := t_n$ if $X_n = Y_n$, for $i \in \{1, 2, 3\}$, it is clear that

$\lim_{n \rightarrow \infty} t_n^i = t$, by (12), (14), and the fact that $t_n \rightarrow t$. Then the continuity of $\chi_{,4}$ on $\partial B \times \mathbb{R}$ and of $(Z, \zeta) \mapsto \chi_\zeta^{-1}(Z)$ on ∂B give, with (13) and (15),

$$\begin{aligned} \lim_{n \rightarrow \infty} V^i(Y_n, t_n; X_n) &= \lim_{n \rightarrow \infty} \chi_{,4}^i(\chi_{t_n}^{-1}(Y_n), t_n^i) = \chi_{,4}^i(\chi_t^{-1}(Y), t) \\ &= V^i(Y, t; Y), \quad \text{for } i \in \{1, 2, 3\}. \end{aligned}$$

This implies the continuity of V at $(Y, t; Y)$.

It is important to note the bound

$$|V|_3 \leq c^* \quad \text{on} \quad \partial B \times \mathbb{R}^3, \quad (16)$$

following readily from (11), (I.3.1.1), and (I.3.26.3) (and the fact that $Y - [\chi]_{(X, t)} \circ \chi_t^{-1}(Y) = \chi(\chi_t^{-1}(Y), t) - \chi(\chi_t^{-1}(Y), t - \tau(\chi_t^{-1}(Y); X, t))$ if $(Y, t) \in \partial B$ and $X \in \mathbb{R}^3$).

We shall frequently employ the alternate notation

$$V_{(X, t)}(Y) := V(Y, t; X), \quad \text{for } (Y, t) \in \partial B \quad \text{and} \quad X \in \mathbb{R}^3. \quad (17)$$

Consider next the manner in which various combinations of functions appearing in the integrands in (5)-(10) can be rewritten in a form involving V . First, it is easy to see, directly from (11), that

$$\begin{aligned} [\chi]_{(X, t)} \circ \chi_t^{-1}(Y) &= Y - \{r_X \circ [\chi]_{(X, t)} \circ \chi_t^{-1}(Y)\} \cdot V_{(X, t)}^c(Y), \\ \text{for } (Y, t) \in \partial B, \quad X \in \mathbb{R}^3, \end{aligned} \quad (18)$$

since $\tau(\chi_t^{-1}(Y); Y, t) = 0$ gives $[\chi]_{(Y, t)} \circ \chi_t^{-1}(Y) = Y$ if $(Y, t) \in \partial B$.

Thus,

$$\begin{aligned} r_X^{2o}[X]_{(X,t)} \circ x_t^{-1}(Y) &= |(Y-X) - \{r_X^{o}[X]_{(X,t)} \circ x_t^{-1}(Y)\} \cdot v_{(X,t)}^c(Y)|_3^2 \\ &= r_X^2(Y) - 2\{r_X^{o}[X]_{(X,t)} \circ x_t^{-1}(Y)\} \cdot (Y^i - X^i) \cdot v_{(X,t)}^{ic}(Y) \quad (19) \\ &\quad + \{r_X^{2o}[X]_{(X,t)} \circ x_t^{-1}(Y)\} \cdot |v_{(X,t)}^c(Y)|_3^2, \end{aligned}$$

from which

$$\begin{aligned} r_X^{o}[X]_{(X,t)} \circ x_t^{-1}(Y) &= c\tau(x_t^{-1}(Y); X, t) \\ &= \frac{1}{1 - |v_{(X,t)}^c(Y)|_3^2} \cdot \{-(Y^i - X^i) \cdot v_{(X,t)}^{ic}(Y) \\ &\quad + \{ \{ (Y^i - X^i) \cdot v_{(X,t)}^{ic}(Y) \}^2 + \{ 1 - |v_{(X,t)}^c(Y)|_3^2 \} \\ &\quad \cdot r_X^2(Y) \}^{1/2} \} \quad (20) \\ &= \frac{r_X(Y)}{1 - |v_{(X,t)}^c(Y)|_3^2} \cdot \{ -r_{X,l}(Y) \cdot v_{(X,t)}^{lc}(Y) \\ &\quad + \{ \{ r_{X,k}(Y) \cdot v_{(X,t)}^{kc}(Y) \}^2 + \{ 1 - |v_{(X,t)}^c(Y)|_3^2 \} \}^{1/2} \}, \end{aligned}$$

$$\text{for } (Y, t) \in \partial B, \quad X \in \mathbb{R}^3 \cap \{Y\}',$$

having taken note of the inequality $(1 - |v^c|_3^2) \geq 1 - (c^*/c)^2 > 0$, from (16), and having chosen the non-negative root of (19). In turn, (20) produces, with (18),

$$\begin{aligned}
 r_{X,i} \circ [x]_{(X,t)} \circ x_t^{-1}(Y) &= \frac{[x^i]_{(X,t)} \circ x_t^{-1}(Y) - x^i}{r_{X^0[x]}_{(X,t)} \circ x_t^{-1}(Y)} \\
 &= \frac{Y^i - x^i}{r_{X^0[x]}_{(X,t)} \circ x_t^{-1}(Y)} - v_{(X,t)}^{i^c}(Y) \\
 &= \{1 - |v_{(X,t)}^c(Y)|_3^2\} \cdot r_{X,i}(Y) \cdot \{-r_{X,\ell}(Y) \cdot v_{(X,t)}^{\ell^c}(Y) \\
 &\quad + \{(r_{X,k}(Y) \cdot v_{(X,t)}^{k^c}(Y))^2 + \{1 - |v_{(X,t)}^c(Y)|_3^2\}^{1/2}\}^{-1} \\
 &\quad - v_{(X,t)}^{i^c}(Y)\},
 \end{aligned} \tag{21}$$

whenever $(Y,t) \in \partial B$, $x \in \mathbb{R}^3 \cap \{Y\}'$.

For ready reference, we shall provide various consequences of (20) and (21), obtained by routine computations: if $(X,t) \in \mathbb{R}^4$, then, on $\partial \delta_t \cap \{X\}'$,

$$\begin{aligned}
 &1 + (r_{X,k} \circ [x]_{(X,t)} \circ x_t^{-1}) \cdot ([x^k_4]_{(X,t)} \circ x_t^{-1}) \\
 &= \{-r_{X,\ell} \cdot v_{(X,t)}^{\ell^c} + \{(r_{X,k} \cdot v_{(X,t)}^{k^c})^2 + \{1 - |v_{(X,t)}^c|_3^2\}^{1/2}\}^{-1} \\
 &\quad \cdot \{1 - |v_{(X,t)}^c|_3^2\} \cdot r_{X,j} \cdot ([x^j_4]_{(X,t)} \circ x_t^{-1}) \\
 &\quad - \{1 - v_{(X,t)}^{k^c} \cdot [x^k_4]_{(X,t)} \circ x_t^{-1}\} \cdot r_{X,j} \cdot v_{(X,t)}^{j^c} \\
 &\quad + \{1 - v_{(X,t)}^{k^c} \cdot [x^k_4]_{(X,t)} \circ x_t^{-1}\} \cdot \{(r_{X,\ell} \cdot v_{(X,t)}^{\ell^c})^2 + \{1 - |v_{(X,t)}^c|_3^2\}^{1/2}\},
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 & \{r_{X^0}[x](X, t) \circ x_t^{-1}\} \cdot \{1 + (r_{X, k^0}[x](X, t) \circ x_t^{-1}) \cdot ([x, 4^k](X, t) \circ x_t^{-1})\} \\
 & = \frac{r_X}{1 - |v_{(X, t)}^c|^2} \cdot \{ \{1 - |v_{(X, t)}^c|^2\} \cdot r_{X, j} \cdot \{[x, 4^j](X, t) \circ x_t^{-1}\} \\
 & \quad - \{1 - v_{(X, t)}^{k^c} \cdot [x, 4^k](X, t) \circ x_t^{-1}\} \cdot r_{X, j} \cdot v_{(X, t)}^{j^c} \\
 & \quad + \{1 - v_{(X, t)}^{k^c} \cdot [x, 4^k](X, t) \circ x_t^{-1}\} \cdot \{ \{r_{X, \ell} \cdot v_{(X, t)}^{\ell^c}\}^2 + \{1 - |v_{(X, t)}^c|^2\} \}^{1/2} \},
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 & \{1 + (r_{X, k^0}[x](X, t) \circ x_t^{-1}) \cdot ([x, 4^k](X, t) \circ x_t^{-1})\}^{-1} \cdot r_{X, i^0}[x](X, t) \circ x_t^{-1} \\
 & = \{ \{1 - |v_{(X, t)}^c|^2\} \cdot r_{X, j} \cdot \{[x, 4^j](X, t) \circ x_t^{-1}\} \\
 & \quad - \{1 - v_{(X, t)}^{k^c} \cdot [x, 4^k](X, t) \circ x_t^{-1}\} \cdot r_{X, j} \cdot v_{(X, t)}^{j^c} \\
 & \quad + \{1 - v_{(X, t)}^{k^c} \cdot [x, 4^k](X, t) \circ x_t^{-1}\} \cdot \{ \{r_{X, \ell} \cdot v_{(X, t)}^{\ell^c}\}^2 + \{1 - |v_{(X, t)}^c|^2\} \}^{1/2} \}^{-1} \tag{24} \\
 & \quad \cdot \{ \{1 - |v_{(X, t)}^c|^2\} \cdot r_{X, i} + \{r_{X, j} \cdot v_{(X, t)}^{j^c}\} v_{(X, t)}^{i^c} \\
 & \quad - \{ \{r_{X, \ell} \cdot v_{(X, t)}^{\ell^c}\}^2 + \{1 - |v_{(X, t)}^c|^2\} \}^{1/2} \cdot v_{(X, t)}^{i^c} \},
 \end{aligned}$$

$$\begin{aligned}
 & \{1 - |v_{(X, t)}^c|^2\} \cdot \{1 + (r_{X, k^0}[x](X, t) \circ x_t^{-1}) \cdot ([x, 4^k](X, t) \circ x_t^{-1})\}^{-1} \\
 & \quad \cdot r_{X, i^0}[x](X, t) \circ x_t^{-1} + [x, 4^i](X, t) \circ x_t^{-1} \\
 & = \{ \{1 - |v_{(X, t)}^c|^2\} \cdot r_{X, j} \cdot \{[x, 4^j](X, t) \circ x_t^{-1}\} \\
 & \quad - \{1 - v_{(X, t)}^{k^c} \cdot [x, 4^k](X, t) \circ x_t^{-1}\} \cdot r_{X, j} \cdot v_{(X, t)}^{j^c} \\
 & \quad + \{1 - v_{(X, t)}^{k^c} \cdot [x, 4^k](X, t) \circ x_t^{-1}\} \cdot \{ \{r_{X, \ell} \cdot v_{(X, t)}^{\ell^c}\}^2 + \{1 - |v_{(X, t)}^c|^2\} \}^{1/2} \}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \{ (1 - |[\chi^c_{,4}](X, t)|^2)^{-1} \cdot \{ (1 - |v^c_{(X, t)}|^2)^{-1} \} r_{X,1} \\
 & + (1 - |[\chi^c_{,4}](X, t)|^2)^{-1} \cdot \{ r_{X,j} \cdot v^j_{(X, t)} \} v^i_{(X, t)} \\
 & - (1 - |[\chi^c_{,4}](X, t)|^2)^{-1} \cdot \{ \{ r_{X,\ell} \cdot v^\ell_{(X, t)} \}^2 + (1 - |v^c_{(X, t)}|^2)^{-1} \}^{1/2} \cdot v^i_{(X, t)} \\
 & + \{ (1 - |v^c_{(X, t)}|^2)^{-1} \} \cdot r_{X,\ell} \cdot \{ [\chi^c_{,4}](X, t) \}^{\circ \chi^{-1}_t} \\
 & - \{ (1 - v^k_{(X, t)} \cdot [\chi^k_{,4}](X, t) \}^{\circ \chi^{-1}_t} \cdot \{ r_{X,\ell} \cdot v^\ell_{(X, t)} \} \cdot [\chi^i_{,4}](X, t) \}^{\circ \chi^{-1}_t} \\
 & + \{ (1 - v^k_{(X, t)} \cdot [\chi^k_{,4}](X, t) \}^{\circ \chi^{-1}_t} \cdot \{ \{ r_{X,\ell} \cdot v^\ell_{(X, t)} \}^2 \\
 & + (1 - |v^c_{(X, t)}|^2)^{-1} \}^{1/2} \cdot [\chi^i_{,4}](X, t) \}^{\circ \chi^{-1}_t},
 \end{aligned} \tag{25}$$

and

$$\begin{aligned}
 & \{ r_{X,k} \cdot [\chi^k_{,4}](X, t) \}^{\circ \chi^{-1}_t} \cdot \{ [\chi^k_{,4}](X, t) \}^{\circ \chi^{-1}_t} + |[\chi^c_{,4}](X, t)|^2 \\
 & = \{ -r_{X,k} \cdot v^k_{(X, t)} + \{ \{ r_{X,\ell} \cdot v^\ell_{(X, t)} \}^2 + (1 - |v^c_{(X, t)}|^2)^{-1} \}^{1/2} \}^{-1} \\
 & \cdot \{ (1 - |v^c_{(X, t)}|^2)^{-1} \} \cdot r_{X,j} \cdot \{ [\chi^j_{,4}](X, t) \}^{\circ \chi^{-1}_t} \\
 & + \{ r_{X,j} \cdot v^j_{(X, t)} \} \cdot \{ v^k_{(X, t)} \cdot [\chi^k_{,4}](X, t) \}^{\circ \chi^{-1}_t} \\
 & - \{ \{ r_{X,\ell} \cdot v^\ell_{(X, t)} \}^2 + (1 - |v^c_{(X, t)}|^2)^{-1} \}^{1/2} \cdot \{ v^k_{(X, t)} \cdot [\chi^k_{,4}](X, t) \}^{\circ \chi^{-1}_t} \\
 & - \{ r_{X,j} \cdot v^j_{(X, t)} \} \cdot |[\chi^c_{,4}](X, t)|^2 \\
 & + \{ \{ r_{X,\ell} \cdot v^\ell_{(X, t)} \}^2 + (1 - |v^c_{(X, t)}|^2)^{-1} \}^{1/2} \cdot |[\chi^c_{,4}](X, t)|^2.
 \end{aligned} \tag{26}$$

Now, let us use the results (20)-(26) to rewrite the integrals appearing on the right-hand sides of (5)-(10); the latter are taken over

∂B_t for an $(X,t) \in B^0 \cup \Omega^0$, so that, in particular, $X \notin \partial B_t$, whence it is clear that (20)-(26) may indeed be applied for this purpose. Thus, for example, we shall use (23) and (25) in (5), (20), (21), and (22) in (6), (20), (22), and (26) in (8), etc. Further, in the manipulations involving (5) and (8), we shall use the decomposition of $\text{grad } r_X(Y)$ into components in $N_{\partial B_t}(Y)$ and $T_{\partial B_t}(Y)$, where $(X,t) \in B^0 \cup \Omega^0$ and $Y \in \partial B_t$, which is given by

$$r_{X,i}(Y) = \{r_{X,k}(Y) \cdot v_{\partial B_t}^k(Y)\} \cdot v_{\partial B_t}^i(Y) - \epsilon_{ijk} \epsilon_{kpq} v_{\partial B_t}^j(Y) \cdot v_{\partial B_t}^p(Y) \cdot r_{X,q}(Y), \quad (27)$$

for $i \in \{1,2,3\}$.

We shall give only the results of these somewhat lengthy rearrangements: we define $W: \partial B \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$W(Y,t;X) := \frac{1}{1 - v_{(X,t)}^k(Y) \cdot [x_{,4}^k]_{(X,t)} \circ x_t^{-1}(Y)} \cdot \{(1 - |v_{(X,t)}^c(Y)|^2) \cdot [x_{,4}^c]_{(X,t)} \circ x_t^{-1}(Y) - (1 - v_{(X,t)}^j(Y) \cdot [x_{,4}^j]_{(X,t)} \circ x_t^{-1}(Y)) \cdot v_{(X,t)}^c(Y)\}, \quad (28)$$

for $(Y,t) \in \partial B$ and $X \in \mathbb{R}^3$,

and, for $m,n \in \mathbb{N} \cup \{0\}$, $\Gamma^{mn}: \{(Y,t,X) \mid (Y,t) \in \partial B, X \in \mathbb{R}^3 \cap \{Y\}'\} \rightarrow \mathbb{R}$ by

$$\Gamma^{mn}(Y, t; X)$$

$$:= \frac{\{-v_{(X,t)}^{kc}(Y) \cdot r_{X,k}(Y) + \{(r_{X,l}(Y) \cdot v_{(X,t)}^{lc}(Y))^2 + (1 - |v_{(X,t)}^c(Y)|^2)^{1/2}\}^m}{\{w_{(X,t)}^k(Y) \cdot r_{X,k}(Y) + \{(r_{X,l}(Y) \cdot v_{(X,t)}^{lc}(Y))^2 + (1 - |v_{(X,t)}^c(Y)|^2)^{1/2}\}^n}$$

(29)

$$\text{for } (Y, t) \in \partial B \quad \text{and} \quad X \in \mathbb{R}^3 \cap \{Y\}';$$

in (29), we have introduced the notation

$$w_{(X,t)}(Y) := w(Y, t; X), \quad (30)$$

which we shall continue to use along with the alternate symbolism

$$\Gamma_{(X,t)}^{mn}(Y) := \Gamma^{mn}(Y, t; X). \quad (31)$$

We should point out that the denominator in (28) is bounded below by the positive number $1 - (c^*/c)^2$ (cf., (16) and (I.3.26.3)); we defer until later a proof of the less obvious fact that the denominator in (29) is also bounded below by a positive number. With these additional auxiliary functions, it can be shown that (5)-(7) can be rewritten as, for each $(X, t) \in B^0 \cup \Omega^\sigma$ and $i \in \{1, 2, 3\}$,

$$\begin{aligned} V\{\mu\}_i^1(X, t) &= \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \Gamma_{(X,t)}^{03} \cdot r_{X,j} v_{\partial B_t}^j \cdot \Lambda_{1i}^1\{\mu\}(X, t) \circ x_t^{-1} \cdot J\chi_t^{-1} d\lambda_{\partial B_t} \\ &\quad - \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \Gamma_{(X,t)}^{03} \cdot \epsilon_{ijk} \epsilon_{kpq} v_{\partial B_t}^j v_{\partial B_t}^p r_{X,q} \\ &\quad \cdot \Lambda_{2i}^1\{\mu\}(X, t) \circ x_t^{-1} \cdot J\chi_t^{-1} d\lambda_{\partial B_t} \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{03} \cdot \epsilon_{ljk} \epsilon_{kpq} v_{\partial B_t}^j v_{\partial B_t}^p r_{X,q} \\
 & \cdot \Lambda_{31l}^1 \{ \mu \} (X,t) \circ x_t^{-1} \cdot J x_t^{-1} d\lambda_{\partial B_t} \\
 & - \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{13} \cdot \Lambda_{41}^1 \{ \mu \} (X,t) \circ x_t^{-1} \cdot J x_t^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{32}$$

having defined auxiliary functions on ∂R by

$$\begin{aligned}
 \Lambda_{11}^1 \{ \mu \} (X,t) & := \{ (1 - |x_{,4}^c|(X,t)|_3^2) \cdot v_{\partial B_t}^1 \circ x_t + (v_{\partial B_t}^j \circ x_t \cdot [x_{,4}^j]^c(X,t)) \cdot [x_{,4}^1]^c(X,t) \} \\
 & \cdot \left\{ \frac{1 - |v_{(X,t)}^c \circ x_t|_3^2}{1 - v_{(X,t)}^{kc} \circ x_t \cdot [x_{,4}^k]^c(X,t)} \right\}^3 \cdot [\hat{\mu} \cdot Jx](X,t),
 \end{aligned} \tag{33}$$

$$\Lambda_2^1 \{ \mu \} (X,t) := \frac{\{ 1 - |x_{,4}^c|(X,t)|_3^2 \} \cdot \{ 1 - |v_{(X,t)}^c \circ x_t|_3^2 \}^3}{\{ 1 - v_{(X,t)}^{kc} \circ x_t \cdot [x_{,4}^k]^c(X,t) \}^3} \cdot [\hat{\mu} \cdot Jx](X,t), \tag{34}$$

$$\begin{aligned}
 \Lambda_{31l}^1 \{ \mu \} (X,t) & := \left\{ \frac{1 - |v_{(X,t)}^c \circ x_t|_3^2}{1 - v_{(X,t)}^{kc} \circ x_t \cdot [x_{,4}^k]^c(X,t)} \right\}^3 \cdot [x_{,4}^1]^c(X,t) \cdot [x_{,4}^l]^c(X,t) \\
 & \cdot [\hat{\mu} \cdot Jx](X,t),
 \end{aligned} \tag{35}$$

and

$$\Lambda_{4i}^1\{\mu\}(X,t) := \left\{ \left(1 - |[\chi_{,4}^c](X,t)|^2 \right) \cdot v_{(X,t)}^{ic} \cdot \{1 - v_{(X,t)}^{kc} \cdot [\chi_{,4}^k](X,t) \} \right. \\ \left. \cdot [\chi_{,4}^i](X,t) \right\} \cdot \frac{\{1 - |v_{(X,t)}^c \cdot \chi_t|^2\}^2}{\{1 - v_{(X,t)}^{kc} \cdot [\chi_{,4}^k](X,t)\}^3} \cdot [\hat{u} \cdot \hat{j}_X](X,t), \quad (36)$$

$$V\{\mu\}_i^2(X,t) = -\frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,i}^{03}(X,t) \cdot r_{X,\ell} \cdot \Lambda_{1\ell}^2\{\mu\}(X,t) \cdot \chi_t^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t} \\ + \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,\ell}^{13}(X,t) \cdot r_{X,i} \cdot \Lambda_{2i\ell}^2\{\mu\}(X,t) \cdot \chi_t^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t} \\ + \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,i}^{13}(X,t) \cdot r_{X,\ell} \cdot \Lambda_{3\ell}^2\{\mu\}(X,t) \cdot \chi_t^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t} \\ - \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,\ell}^{23}(X,t) \cdot \Lambda_{4i}^2\{\mu\}(X,t) \cdot \chi_t^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t}, \quad (37)$$

in which

$$\Lambda_{1\ell}^2\{\mu\}(X,t) := \left\{ \frac{1 - |v_{(X,t)}^c \cdot \chi_t|^2}{1 - v_{(X,t)}^{kc} \cdot [\chi_{,4}^k](X,t)} \right\}^3 \cdot [\chi_{,44}^{\ell c^2}](X,t) \cdot [\hat{u} \cdot \hat{j}_X](X,t), \quad (38)$$

$$\Lambda_{2i\ell}^2\{\mu\}(X,t) := \frac{\{1 - |v_{(X,t)}^c \cdot \chi_t|^2\}^2}{\{1 - v_{(X,t)}^{kc} \cdot [\chi_{,4}^k](X,t)\}^3} \cdot v_{(X,t)}^{ic} \cdot \chi_t \cdot [\chi_{,44}^{\ell c^2}](X,t) \\ \cdot [\hat{u} \cdot \hat{j}_X](X,t), \quad (39)$$

$$\Lambda_{3\{\mu\}}^2(X, t) := \frac{\{1 - |v_{(X, t)}^c \circ x_t|^2\}_3^2}{\{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c]_{(X, t)}\}_3^3} \cdot \{v_{(X, t)}^{lc} \circ x_t \cdot [x_{,44}^{lc^2}]_{(X, t)}\} \cdot [\hat{u} \cdot \hat{j}x]_{(X, t)}, \quad (40)$$

and

$$\Lambda_{4i\{\mu\}}^2(X, t) := \frac{\{1 - |v_{(X, t)}^c \circ x_t|^2\}_3^2}{\{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c]_{(X, t)}\}_3^3} \cdot \{v_{(X, t)}^{lc} \circ x_t \cdot [x_{,44}^{lc^2}]_{(X, t)}\} \cdot v_{(X, t)}^{ic} \circ x_t \cdot [\hat{u} \cdot \hat{j}x]_{(X, t)}, \quad (41)$$

and

$$\begin{aligned} v_{\{\mu\}}^3(X, t) &= \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,1}^{02}(X, t) \cdot r_{X,1} \cdot \Lambda_{1\{\mu\}}^3(X, t) \circ x_t^{-1} \cdot jx_t^{-1} d\lambda_{\partial B_t} \\ &\quad - \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,1}^{12}(X, t) \cdot \Lambda_{2i\{\mu\}}^3(X, t) \circ x_t^{-1} \cdot jx_t^{-1} d\lambda_{\partial B_t}, \end{aligned} \quad (42)$$

with

$$\Lambda_{1\{\mu\}}^3(X, t) := \left\{ \frac{1 - |v_{(X, t)}^c \circ x_t|^2\}_3^2}{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c]_{(X, t)}} \right\}^2 \cdot [(\hat{u} \cdot \hat{j}x)_{,4}^c]_{(X, t)}, \quad (43)$$

and

$$\Lambda_{2i\{\mu\}}^3(X, t) := \frac{\{1 - |v_{(X, t)}^c \circ x_t|^2\}_3^2}{\{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c]_{(X, t)}\}_2^2} \cdot v_{(X, t)}^{ic} \circ x_t \cdot [(\hat{u} \cdot \hat{j}x)_{,4}^c]_{(X, t)}. \quad (44)$$

Further, (8)-(10) become, again for each $(X, t) \in B^{\text{out}}$,

$$\begin{aligned}
 v_{\{u\}}^1(X, t) = & -\frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X^2} \cdot r_{(X, t)}^{03} \cdot r_{X, j} \cdot v_{\partial B_t}^j \cdot \tilde{\lambda}_{1\{u\}}^1(X, t) \cdot \circ x_t^{-1} \cdot J_{X_t}^{-1} d\lambda_{\partial B_t} \\
 & + \frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X^2} \cdot r_{(X, t)}^{03} \cdot \varepsilon_{ljk} \varepsilon_{kpq} v_{\partial B_t}^j v_{\partial B_t}^p r_{X, q} \\
 & \cdot \tilde{\lambda}_{2l\{u\}}^1(X, t) \cdot \circ x_t^{-1} \cdot J_{X_t}^{-1} d\lambda_{\partial B_t} \\
 & - \frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X^2} \cdot r_{(X, t)}^{13} \cdot \tilde{\lambda}_{3\{u\}}^1(X, t) \cdot \circ x_t^{-1} \cdot J_{X_t}^{-1} d\lambda_{\partial B_t},
 \end{aligned} \tag{45}$$

having set

$$\begin{aligned}
 \tilde{\lambda}_{1\{u\}}^1(X, t) := & \left\{ \frac{1 - |v_{(X, t)}^c \circ x_t|^2}{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c](X, t)} \right\}^3 \cdot \{v_{\partial B_t}^j \circ x_t \cdot [x_{,4}^c](X, t)\} \\
 & \cdot [\mu \cdot \hat{J}X](X, t),
 \end{aligned} \tag{46}$$

$$\tilde{\lambda}_{2l\{u\}}^1(X, t) := \left\{ \frac{1 - |v_{(X, t)}^c \circ x_t|^2}{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c](X, t)} \right\}^3 \cdot [x_{,4}^c](X, t) \cdot [\mu \cdot \hat{J}X](X, t), \tag{47}$$

and

$$\begin{aligned}
 \tilde{\lambda}_{3\{u\}}^1(X, t) := & \frac{\{1 - |v_{(X, t)}^c \circ x_t|^2\}^2}{\{1 - v_{(X, t)}^{kc} \circ x_t \cdot [x_{,4}^c](X, t)\}^3} \{ |[x_{,4}^c](X, t)|^2 \\
 & - v_{(X, t)}^{jc} \circ x_t \cdot [x_{,4}^c](X, t) \} \cdot [\mu \cdot \hat{J}X](X, t),
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 V\{\mu\}_4^2(X,t) = & -\frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{13} \cdot r_{X,2} \cdot \tilde{\Lambda}_{12}^2\{\mu\}(X,t) \circ x_t^{-1} \cdot Jx_t^{-1} d\lambda_{\partial B_t} \\
 & + \frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{23} \cdot \tilde{\Lambda}_2^2\{\mu\}(X,t) \circ x_t^{-1} \cdot Jx_t^{-1} d\lambda_{\partial B_t},
 \end{aligned} \quad (49)$$

wherein

$$\tilde{\Lambda}_{12}^2\{\mu\}(X,t) := \frac{\{1 - |v_{(X,t)}^c \circ x_t|^2\}_3^2}{\{1 - v_{(X,t)}^k \circ x_t \cdot [x_{,4}^k](X,t)\}^3} \cdot [x_{,44}^{\ell c^2}](X,t) \cdot [\hat{u} \cdot \hat{j}_X](X,t), \quad (50)$$

and

$$\begin{aligned}
 \tilde{\Lambda}_2^2\{\mu\}(X,t) := & \frac{\{1 - |v_{(X,t)}^c \circ x_t|^2\}_3^2}{\{1 - v_{(X,t)}^k \circ x_t \cdot [x_{,4}^k](X,t)\}^3} \cdot \{v_{(X,t)}^{\ell c^2} \circ x_t \cdot [x_{,44}^{\ell c^2}](X,t)\} \\
 & \cdot [\hat{u} \cdot \hat{j}_X](X,t),
 \end{aligned} \quad (51)$$

and

$$V\{\mu\}_4^3(X,t) = \frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{12} \cdot \tilde{\Lambda}_1^3\{\mu\}(X,t) \circ x_t^{-1} \cdot Jx_t^{-1} d\lambda_{\partial B_t}, \quad (52)$$

with the definition

$$\tilde{\Lambda}_1^3\{\mu\}(X,t) := \frac{\{1 - |v_{(X,t)}^c \circ x_t|^2\}_3^2}{\{1 - v_{(X,t)}^k \circ x_t \cdot [x_{,4}^k](X,t)\}^3} \cdot [(\hat{u} \cdot \hat{j}_X)^c_{,4}](X,t). \quad (53)$$

Now, to discover additional properties of the partial derivatives of $V\{\mu\}$ near ∂B , we intend to examine the functions in $B^{\circ \cup \partial B}$.

given by the individual integral-terms on the right-hand sides of (37), (42), (45), (49), and (52). Observe that the integrands in these terms involve, for $(X,t) \in B^0 \cup \Omega^\sigma$, the distance function r_X and its gradient, on ∂B_t ; indeed, the desire to achieve such forms was the motivation for introducing the function V , so that the functions appearing more clearly constitute variants and generalizations of those classical ones already extensively studied.

It should also be pointed out that the various functions $(P,X,t) \mapsto \Lambda_{11}^1\{\mu\}_{(X,t)}(P)$, $(P,X,t) \mapsto \Lambda_{22}^1\{\mu\}_{(X,t)}(P)$, etc., on $\partial R \times \mathbb{R}^4$, given by (33)-(36), (38)-(41), (43), (44), (46)-(48), (50), (51), and (53), are continuous, as it is easy to verify: we have seen that $V \in C(\partial B \times \mathbb{R}^3; \mathbb{R}^3)$, $v \in C(\partial B; \mathbb{R}^3)$, $\hat{J}_X \in C(\partial R \times \mathbb{R})$, and we know that $X, X_{,4}, X_{,44} \in C(\partial R \times \mathbb{R}; \mathbb{R}^3)$ and $\bar{u} \in C(\partial R \times \mathbb{R})$. Since $\tau \in C(\partial R \times \mathbb{R}^4)$, the various retardations appearing, $[X_{,4}]$, $[X_{,44}]$, and $[\bar{u} \cdot \hat{J}_X]$, are continuous on $\partial R \times \mathbb{R}^4$. The inequality

$$1 - v_{(X,t)}^{k^c} \circ x_t(P) \cdot [X_{,4}^{k^c}]_{(X,t)}(P) \geq 1 - (c^*/c)^2 > 0, \quad (54)$$

for $P \in \partial R, \quad (X,t) \in \mathbb{R}^4,$

is plain enough, by (16) and (I.3.26.3). Upon combining these facts, it is clear that our original assertion can be verified. Moreover, the map $(Y,t) \mapsto X_t^{-1}(Y)$ is continuous on ∂B ([I.3.7]), and we can use [I.2.17.v] to deduce that $(Y,t) \mapsto JX_t^{-1}(Y)$ is in $C(\partial B)$, since $\hat{J}_X$ is positive and continuous on $\partial R \times \mathbb{R}$, while

$$JX_t^{-1}(Y) = \{JX_t(X_t^{-1}(Y))\}^{-1} = \{\hat{J}_X(X_t^{-1}(Y), t)\}^{-1} \quad \text{for } (Y,t) \in \partial B. \quad (55)$$

Thus, the maps $(Y, t, X) \mapsto \Lambda_{11}^1 \{u\}_{(X, t)} \circ x_t^{-1}(Y) \cdot Jx_t^{-1}(Y)$, etc., are in $C(\partial B \times \mathbb{R}^3)$.

Obviously, using the facts adduced in the preceding paragraph, the inclusion $W \in C(\partial B \times \mathbb{R}^3; \mathbb{R}^3)$ must hold.

Let us return to the functions Γ^{mn} , defined by (29) on the set $\{(Y, t, X) \mid (Y, t) \in \partial B, X \in \mathbb{R}^3 \cap \{Y\}'\}$; as promised, we shall first show that the denominator in (29) is non-zero on this set. For this, choose $(Y, t) \in \partial B$, then $X \in \mathbb{R}^3 \cap \{Y\}'$. It is easy to see from (22) that

$$\begin{aligned} & W_{(X, t)}^k(Y) \cdot r_{X, k}(Y) + \{ \{ r_{X, l}(Y) \cdot v_{(X, t)}^{lc}(Y) \}^2 + \{ 1 - |v_{(X, t)}^c(Y)|_3^2 \} \}^{1/2} \\ &= \{ 1 - v_{(X, t)}^{jc}(Y) \cdot \{ x_{, 4}^j \}_{(X, t)} (x_t^{-1}(Y)) \}^{-1} \{ 1 + r_{X, k}([x]_{(X, t)}(x_t^{-1}(Y))) \\ &\quad \cdot [x_{, 4}^k]_{(X, t)}(x_t^{-1}(Y)) \} \{ -r_{X, p}(Y) \cdot v_{(X, t)}^{pc}(Y) + \{ \{ r_{X, l}(Y) \cdot v_{(X, t)}^{lc}(Y) \}^2 \\ &\quad + \{ 1 - |v_{(X, t)}^c(Y)|_3^2 \} \}^{1/2} \} \\ &\geq \{ 1 + (c^*/c)^2 \}^{-1} \cdot \{ 1 - (c^*/c) \} \\ &\quad \cdot \{ -r_{X, p}(Y) \cdot v_{(X, t)}^{pc}(Y) + \{ \{ r_{X, l}(Y) \cdot v_{(X, t)}^{lc}(Y) \}^2 + \{ 1 - (c^*/c)^2 \} \}^{1/2} \}. \end{aligned} \tag{56}$$

Now, consider the function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\psi(s) := -s + \{ s^2 + 1 - (c^*/c)^2 \}^{1/2} \quad \text{for } s \in \mathbb{R}.$$

It can be checked that $\psi(c^*/c) = 1 - (c^*/c)$ and $\psi' < 0$ on $\mathbb{R}$, so

$$\psi(s) \geq 1 - (c^*/c) \quad \text{if } s \leq c^*/c. \quad \text{Since } r_{X, p}(Y) \cdot v_{(X, t)}^{pc}(Y) \leq c^*/c, \tag{56}$$

gives

$$\begin{aligned} & W_{(X,t)}^k(Y) \cdot r_{X,k}(Y) + \{ \{ r_{X,l}(Y) \cdot v_{(X,t)}^l(Y) \}^2 + \{ 1 - |v_{(X,t)}^c(Y)|_3^2 \} \}^{1/2} \\ & \geq \{ 1 + (c^*/c)^2 \}^{-1} \{ 1 - (c^*/c)^2 \}^2, \end{aligned} \quad (57)$$

which clearly substantiates our claim concerning the denominator in (29). For any m and n chosen from $\mathbb{N} \cup \{0\}$, the continuity of Γ^{mn} on $\{(Y,t,X) \mid (Y,t) \in \partial B, X \in \mathbb{R}^3 \cap \{Y\}'\}$ is now a direct consequence of the continuity of V and W on $\partial B \times \mathbb{R}^3$ and of the map $(Y,X) \mapsto \text{grad } r_X(Y)$ on $\{(Y,X) \mid Y, X \in \mathbb{R}^3, Y \neq X\}$. Observe also that Γ^{mn} is positive and bounded: for any (Y,t,X) in the domain of Γ^{mn} , we obviously have

$$\begin{aligned} & -v_{(X,t)}^k(Y) \cdot r_{X,k}(Y) + \{ \{ r_{X,l}(Y) \cdot v_{(X,t)}^l(Y) \}^2 + \{ 1 - |v_{(X,t)}^c(Y)|_3^2 \} \}^{1/2} \\ & \leq (c^*/c) + \{ 1 + (c^*/c)^2 \}^{1/2}, \end{aligned} \quad (58)$$

which, with (57), shows that

$$\Gamma^{mn} \leq \{ 1 + (c^*/c)^2 \}^n \cdot \{ 1 - (c^*/c)^2 \}^{-2n} \cdot \{ (c^*/c) + \{ 1 + (c^*/c)^2 \}^{1/2} \}^m. \quad (59)$$

The positivity of Γ^{mn} follows simply from (57) and the form of the numerator in (29). In fact, using reasoning similar to that just completed, it can be shown that

$$\Gamma^{mn} \geq \{ 1 - (c^*/c)^2 \}^{m+n} \cdot \{ (c^*/c) + \{ 1 + (c^*/c)^2 \}^{1/2} \}^{-n}. \quad (60)$$

Anticipating the analysis of the fourth terms on the right in (32) and (45), let us point out here that

$$\Lambda_{4i}^1\{\mu\}(Y,t) \circ x_t^{-1}(Y) = \tilde{\Lambda}_3^1\{\mu\}(Y,t) \circ x_t^{-1}(Y) = 0 \quad (61)$$

for each $(Y,t) \in \partial B$,

for, whenever $(Y,t) \in \partial B$, then $v_{(Y,t)}^c(Y) = x_{,4}^c(x_t^{-1}(Y),t)$, while $\tau(x_t^{-1}(Y);Y,t) = 0$, so $[x_{,4}^c]_{(Y,t)} \circ x_t^{-1}(Y) = x_{,4}^c(x_t^{-1}(Y),t)$, as well.

Finally, in passing, we note that each function on $B^0 \cup \Omega^\sigma$ given by an integral appearing on the right-hand side of (32), (37), (42), (45), (49), or (52) is in $C(B^0 \cup \Omega^\sigma)$. This is most easily seen by transforming back to integration over ∂R and appealing to [IV.6], utilizing the continuity facts already presented. As we shall see, however, the behavior of these functions near ∂B can be quite disparate.

Upon inspecting the integrals in (IV.14.32, 37, 42, 45, 49, and 52), we discover certain prototypic forms which we shall study; the functions on $B^0 \cup \Omega^\sigma$ which we now define are generated by these recurring forms.

[IV.15] DEFINITIONS. Let $M \in \mathcal{M}(1)$; let (R,x) be a reference pair for M possessing the properties of [I.3.25]. Suppose that $\phi \in C(\partial R \times \mathbb{R}^4)$; write

$$\phi_{(X,t)}(P) := \phi(P,X,t) \quad \text{for } P \in \partial R \text{ and } (X,t) \in \mathbb{R}^4.$$

(i) Define $w_1\{\phi\}: B^0 \cup \Omega^\sigma \rightarrow \mathbb{K}$ by

$$\omega_1\{\phi\}(X,t) := \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,j} v_{\partial B_t}^j \cdot \Gamma^{03}(X,t) \cdot \phi(X,t) \circ X_t^{-1} \cdot J_t^{-1} d\lambda_{\partial B_t}, \quad (1)$$

for each $(X,t) \in B^0 \cup \Omega^\sigma$,

and set

$$\omega_1^I\{\phi\} := \omega_1\{\phi\}|_{B^0}, \quad \omega_1^O\{\phi\} := \omega_1\{\phi\}|_{\Omega^\sigma}.$$

(ii) Define, for each $i \in \{1,2,3\}$, $\omega_{2i}\{\phi\}: B^0 \cup \Omega^\sigma \rightarrow \mathbb{R}$ by

$$\omega_{2i}\{\phi\}(X,t) := \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot r_{X,q} T_t^{iq} \cdot \Gamma^{03}(X,t) \cdot \phi(X,t) \circ X_t^{-1} \cdot J_t^{-1} d\lambda_{\partial B_t}, \quad (2)$$

for each $(X,t) \in B^0 \cup \Omega^\sigma$,

wherein $T_t^{iq}: \partial B \rightarrow \mathbb{R}$ is given by

$$T_t^{iq} := \epsilon_{ijk} \epsilon_{kpq} v_{\partial B}^j v^p = v_{\partial B}^i v^q - \delta_{iq}, \quad \text{for each } q \in \{1,2,3\}, \quad (3)$$

and we have written

$$T_t^{iq}(Y) := T_t^{iq}(Y,t) \quad \text{for } (Y,t) \in \partial B, \quad \text{i.e., whenever } t \in \mathbb{R}$$

and $Y \in \partial B_t$.

Further, define

$$\omega_{2i}^I\{\phi\} := \omega_{2i}\{\phi\}|_{B^0}, \quad \omega_{2i}^O\{\phi\} := \omega_{2i}\{\phi\}|_{\Omega^\sigma}, \quad \text{for } i \in \{1,2,3\}.$$

(iii) Let $\Gamma: \{(Y,t,X) \mid (Y,t) \in \partial B, X \in \mathbb{R}^3 \cap \{Y\}'\} \rightarrow \mathbb{R}$ be bounded and continuous; write

$$\Gamma_{(X,t)}(Y) := \Gamma(Y,t,X).$$

Let $\beta \in \mathbb{R}$. Define $w_{3\beta}\{\phi\}: \mathbb{B}^0 \cup \Omega^\sigma \rightarrow \mathbb{K}$ according to

$$w_{3\beta}\{\phi\}(X,t) := \int_{\partial \mathcal{B}_t} \frac{1}{r_X^\beta} \cdot \Gamma_{(X,t)} \cdot \phi(X,t) \circ \chi_t^{-1} \cdot J\chi_t^{-1} d\lambda_{\partial \mathcal{B}_t}, \quad (4)$$

for each $(X,t) \in \mathbb{B}^0 \cup \Omega^\sigma$. ■.

[IV.16] R E M A R K. Notation is as in [IV.15]. If $(X,t) \in \mathbb{B}^0 \cup \Omega^\sigma$, it is easy to check that each of the integrands appearing in (IV.15.1, 2, and 4) is a continuous function on $\partial \mathcal{B}_t$. Thus, we have indeed defined functions $w_1\{\phi\}$, $w_{21}\{\phi\}$, and $w_{3\beta}\{\phi\}$ on $\mathbb{B}^0 \cup \Omega^\sigma$ in the preceding section. In fact, each is an element of $C(\mathbb{B}^0 \cup \Omega^\sigma)$: to see this, consider transforming the integrals in (IV.15.1, 2, and 4) to integrals over $\partial \mathcal{R}$, by using the 1-imbedding χ_t carrying $\partial \mathcal{R}$ onto $\partial \mathcal{B}_t$, for each $(X,t) \in \mathbb{B}^0 \cup \Omega^\sigma$. The resultant integrands will be functions continuous on $\partial \mathcal{R} \times (\mathbb{B}^0 \cup \Omega^\sigma)$, so that application of Corollary [IV.6] assures us that $w_1\{\phi\}$, $w_{21}\{\phi\}$, and $w_{3\beta}\{\phi\}$ are continuous on $\mathbb{B}^0 \cup \Omega^\sigma$.

The study of the behavior of these types of functions near $\partial \mathcal{B}$ will occupy us in the upcoming sections of this chapter. Before launching into this lengthy analysis, let us summarize, in the following proposition, the results of the computation carried out in [IV.14]:

[IV.17] P R O P O S I T I O N. Let M be a motion in $\mathcal{M}(1;0)^+, (R,x)$

⁺ $M \in \mathcal{M}(2)$ will do, since $\mathcal{M}(2) \subset \mathcal{M}(1;0)$.

a reference pair for M as in [IV.10], and $\mu \in C(\partial B)$ with $D_4^0 \mu \in C(\partial B \times \mathbb{R})$, where, as usual, $\mu^0 := \mu \circ X^*$. For $m, n \in \mathbb{N} \setminus \{0\}$, let $\Gamma^{mn}: \{(Y, t, X) \mid (Y, t) \in \partial B, X \in \mathbb{R}^3 \cap \{Y\}^c\} \rightarrow \mathbb{R}$ be defined by (IV.14.29), with $V: \partial B \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $W: \partial B \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by (IV.14.11) and (IV.14.29), respectively,† writing

$$V_{(X,t)}(Y) := V(Y, t, X),$$

$$W_{(X,t)}(Y) := W(Y, t, X),$$

and

$$\Gamma_{(X,t)}^{mn}(Y) := \Gamma^{mn}(Y, t, X).$$

Then $V\{\mu\} \in C^1(\partial B^0 \cup \partial B^c)$, and the partial derivatives of this function are given by

$$\begin{aligned} V\{\mu\}_{,i}(X,t) &= W_1\{\Lambda_{1i}^1\{\mu\}\}(X,t) - W_{2i}\{\Lambda_2^1\{\mu\}\}(X,t) - W_{2j}\{\Lambda_{3ij}^1\{\mu\}\}(X,t) \\ &\quad - \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{13} \cdot \Lambda_{4i}^1\{\mu\}(X,t) \circ X_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t} \quad (1) \\ &\quad + V\{\mu\}_i^2(X,t) + V\{\mu\}_i^3(X,t), \quad \text{for } i = 1, 2, 3, \end{aligned}$$

and

†Note that V , W , and Γ^{mn} can be defined even if only $M \in \mathcal{M}(1)$.

$$\begin{aligned}
 V\{\mu\}_{,4}(X,t) = & -c\omega_1\{\tilde{\lambda}_1^1\{\mu\}\}(X,t) + c\omega_{2j}\{\tilde{\lambda}_{2j}^1\{\mu\}\}(X,t) \\
 & - \frac{c}{4\pi} \int_{\partial B_t} \frac{1}{r_X^2} \cdot \Gamma_{(X,t)}^{13} \cdot \tilde{\lambda}_3^1\{\mu\}(X,t) \circ X_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t} \\
 & + V\{\mu\}_4^2(X,t) + V\{\mu\}_4^3(X,t),
 \end{aligned} \tag{2}$$

for each $(X,t) \in B^0 \cup \Omega^\sigma$.

The functions $V\{\mu\}_1^2$, $V\{\mu\}_1^3$, $V\{\mu\}_4^2$, and $V\{\mu\}_4^3$ can be obtained from (IV.14.37, 42, 49, and 52), respectively, where they are displayed as sums of functions of the form $W_{31}\{\phi\}$.

[IV.18] R E M A R K S. (a) In spite of the fact that we employ a particular reference pair for the motion in the explicit computation of the partial derivatives of $V\{\mu\}$ as in [IV.17], the resultant expressions in (IV.17.1 and 2) must be independent of the reference pair chosen, since the definition of $V\{\mu\}$ involves sets and functions which are, with the exception of the density μ , intrinsic to the motion. Of course, the auxiliary functions introduced in [IV.15] do, in general, depend on the reference pair selected for the motion.

(b) We intend to study the properties of functions of the forms $W_1\{\phi\}$, $W_{21}\{\phi\}$, and $W_{38}\{\phi\}$, subsequently using the results in conjunction with [IV.17] in order to deduce facts concerning the partial derivatives $V^I\{\mu\}_{,1}$ and $V^O\{\mu\}_{,1}$, $i \in \{1,2,3,4\}$. In order to carry out this investigation, we shall find it convenient to impose on the underlying motion $M \in \mathcal{M}(1)$ at least the requirement

that $\{\mathcal{B}_\zeta^0\}_{\zeta \in \mathbb{R}}$ be "locally uniformly Lyapunov," in the sense that whenever K is a compact subset of $\mathbb{R}$, $\{\mathcal{B}_\zeta^0\}_{\zeta \in K}$ is a uniformly Lyapunov family of domains; it is not too difficult to formulate a condition on a reference pair (R, X) for M which is sufficient to imply that this requirement be fulfilled, but we choose to make the more direct postulate our primary one, in the interest of simplicity. In particular, it will be clear that the inclusion $M \in \mathcal{M}(2)$ certainly provides for the validity of the analyses in this chapter.

(c) We shall make full use of the definitions and results outlined in Chapter [I.2] concerning the geometry of the boundary of a Lyapunov domain. Let us recall the most important of these facts, in the context of our present interest: let $M \in \mathcal{M}(1)$, and suppose that $s \in \mathbb{R}$ is such that $\mathcal{B}_s^0$ is a Lyapunov domain, so

$$|v(Z_2, s) - v(Z_1, s)|_3 \leq a |Z_2 - Z_1|_3^\alpha \quad \text{for } Z_1, Z_2 \in \partial \mathcal{B}_s,$$

where $a > 0$ and $\alpha \in (0, 1]$. Let d be any positive number such that $ad^\alpha < 1/2$. Choose any $Y \in \partial \mathcal{B}_s$. Then the restriction to $\partial \mathcal{B}_s \cap \mathcal{B}_d^3(Y)$ of the orthogonal projection map of $\mathbb{R}^3$ onto $Y + T_{\partial \mathcal{B}_s}(Y)$, denoted by $\Pi_Y: \partial \mathcal{B}_s \cap \mathcal{B}_d^3(Y) \rightarrow Y + T_{\partial \mathcal{B}_s}(Y)$, is an injection. Corresponding to a preassigned orthonormal basis $\{\hat{e}_1, \hat{e}_2\}$ for $T_{\partial \mathcal{B}_s}(Y)$, $A_Y: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the (linear) isometry such that

$$A_Y \hat{e}_i = e_i^{(3)}, \quad i = 1, 2, \quad \text{and} \quad A_Y v(Y, s) = e_3^{(3)}.$$

We then define $\mathcal{K}_Y: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ according to

$$\mathcal{K}_Y(Z) := A_Y(Z-Y) \quad \text{for each } Z \in \mathbb{R}^3,$$

and $\hat{\mathcal{K}}_Y: Y+T_{\partial B_s}(Y) \rightarrow \mathbb{R}^2$ by

$$\hat{\mathcal{K}}_Y(Z) := (\mathcal{K}_Y^1(Z), \mathcal{K}_Y^2(Z)) \quad \text{for each } Z \in Y+T_{\partial B_s}(Y).$$

It follows easily that $r_Y(Z) = |\hat{\mathcal{K}}_Y(Z)|_2$ whenever $Z \in Y+T_{\partial B_s}(Y)$, and so also

$$r_Y \circ \hat{\mathcal{K}}_Y^{-1}(\xi) = |\xi|_2 \quad \text{for each } \xi \in \mathbb{R}^2.$$

Setting

$$h_Y := \hat{\mathcal{K}}_Y \circ \pi_Y: \partial B_s \cap B_d^3(Y) \rightarrow \mathbb{R}^2,$$

we generate a coordinate system $(\partial B_s \cap B_d^3(Y), h_Y)$ in ∂B_s ; $h_Y(\partial B_s \cap B_d^3(Y))$ is an open neighborhood of 0 in $\mathbb{R}^2$ which is starlike with respect to 0, and

$$B_{(7/9)d}^2(0) \subset h_Y(\partial B_s \cap B_d^3(Y)) \subset B_d^2(0).$$

Obviously,

$$h_Y^{-1} = \pi_Y^{-1} \circ (\hat{\mathcal{K}}_Y^{-1}|_{h_Y(\partial B_s \cap B_d^3(Y))}).$$

We also find that

$$Jh_Y^{-1}(\xi) = \{v^j(h_Y^{-1}(\xi), s)v^j(Y, s)\}^{-1} \quad \text{for each } \xi \in h_Y(\partial B_s \cap B_d^3(Y)),$$

with

$$Jh_Y^{-1} < \sqrt{2} \quad \text{on } h_Y(\partial B_s \cap B_d^3(Y)).$$

Further, the following inequalities hold: if $Z \in \partial B_s \cap B_d^3(Y)$, then

$$|Z - \pi_Y(Z)|_3 < \tilde{a} r_Y^{1+\alpha}(\pi_Y(Z)) \leq \tilde{a} r_Y^{1+\alpha}(Z),$$

$$|r_{Y,k}(Z) \cdot v^k(Z,s)| < \hat{a} r_Y^\alpha(Z), \quad \text{if } Z \neq Y,$$

and

$$r_Y(Z) \geq r_Y(\pi_Y(Z)) > \frac{7}{9} r_Y(Z),$$

with

$$\tilde{a} := \frac{8}{7} a \cdot \left(\frac{65}{49}\right)^{\alpha/2} \cdot (1+\alpha)^{-1},$$

and

$$\hat{a} := \frac{16}{7} a + \tilde{a}.$$

It is also important to recall that there exists a $d_0 > 0$, depending only on a and α , such that if we also require $d < d_0$, then there exists a $\gamma_d \in (0,1)$, depending only on a , α , and d , for which

$$\gamma_d < \frac{r_X(Z)}{r_X(\pi_Y(Z))} < \frac{1}{\gamma_d} \quad \text{whenever } X \in \{Y + \zeta v(Y,s) \mid \zeta \in \mathbb{R}\},$$

$$\text{and } Z \in \partial B_s \cap B_d^3(Y) \cap \{X\}'.$$

For more details concerning these statements, including their proofs, Sections [VI.62-67] should be consulted.

It will prove convenient to have available certain additional notations in the present setting. First, we denote by $L_Y(Y,s)$ the

line normal to ∂B_s at Y :

$$L_v^-(Y, s) := Y + N_{\partial B_s}(Y) = \{Y + \zeta v(Y, s) \mid \zeta \in \mathbb{R}\}, \quad (1)$$

writing also

$$L_v^+(Y, s) := \{Y + \zeta v(Y, s) \mid \zeta > 0\} \quad (2)$$

and

$$L_v^-(Y, s) := \{Y + \zeta v(Y, s) \mid \zeta < 0\}. \quad (3)$$

Since B_s^0 is, in particular, a 1-regular domain, we know that $Y + \zeta v(Y, s) \in B_s^0$ for all sufficiently small $\zeta > 0$ [for all $\zeta < 0$ with $|\zeta|$ sufficiently small]. Suppose further that $\rho > 0$: define

$$C_\rho^\Pi(Y, s) := \{Z \in Y + T_{\partial B_s}(Y) \mid |Z - Y|_3 < \rho\}, \quad (4)$$

which is just a disk of radius ρ centered at Y and lying in the tangent plane to ∂B_s at Y . Now, it is easy to show that

$$C_\rho^\Pi(Y, s) \subset \Pi_Y(\partial B_s \cap B_d^3(Y)) \quad \text{whenever} \quad 0 < \rho < \frac{7}{9} d. \quad (5)$$

For, suppose that $Z \in Y + T_{\partial B_s}(Y)$, with $r_Y(Z) < \frac{7}{9} d$. Then $|\hat{\mathcal{K}}_Y(Z)|_2 = r_Y(Z) < \frac{7}{9} d$, so $\hat{\mathcal{K}}_Y(Z) \in B_{(7/9)d}^2(0) \subset h_Y(\partial B_s \cap B_d^3(Y))$, i.e., we have $h_Y^{-1}(\hat{\mathcal{K}}_Y(Z)) \in \partial B_s \cap B_d^3(Y)$. Consequently,

$$\Pi_Y(h_Y^{-1}(\hat{\mathcal{K}}_Y(Z))) = \Pi_Y(\Pi_Y^{-1} \circ \hat{\mathcal{K}}_Y^{-1}(\hat{\mathcal{K}}_Y(Z))) = Z,$$

whence (5) follows. Using (5), it is legitimate to define the subset

$C_\rho^\partial(Y, s)$ of $\partial B_s \cap B_d^3(Y)$ via

$$C_\rho^\partial(Y, s) := \pi_Y^{-1}(C_\rho^\Pi(Y, s)) \quad \text{whenever} \quad 0 < \rho < \frac{7}{9} d. \quad (6)$$

Observe that

$$\begin{aligned} h_Y(C_\rho^\partial(Y, s)) &= \hat{\mathcal{H}}_Y \circ \pi_Y(\pi_Y^{-1}(C_\rho^\Pi(Y, s))) = \hat{\mathcal{H}}_Y(C_\rho^\Pi(Y, s)) \\ &= \{\hat{\mathcal{H}}_Y(Z) \mid Z \in Y + T_{\partial B_s}(Y), \quad |\hat{\mathcal{H}}_Y(Z)|_2 = r_Y(Z) < \rho\} \quad (7) \\ &= B_\rho^2(0), \quad \text{for} \quad 0 < \rho < \frac{7}{9} d; \end{aligned}$$

similarly,

$$h_Y(C_{\rho_2}^\partial(Y, s) \cap C_{\rho_1}^\partial(Y, s)') = B_{\rho_2}^2(0) \cap B_{\rho_1}^2(0)', \quad \text{if} \quad 0 < \rho_1 < \rho_2 < \frac{7}{9} d. \quad (8)$$

If $0 < \rho < \frac{7}{9} d$ and $Z \in C_\rho^\partial(Y, s)$, then

$$r_Y(Z) < \frac{9}{7} r_Y(\pi_Y(Z)) < \frac{9}{7} \rho;$$

on the other hand, if $Z \in \partial B_s \cap C_\rho^\partial(Y, s)'$, then

$$r_Y(Z) \geq \rho,$$

since $r_Y(Z) \geq d > \rho$ if $Z \in B_d^3(Y)'$, while $r_Y(Z) \geq r_Y(\pi_Y(Z)) \geq \rho$ if $Z \in B_d^3(Y)$.

(d) Suppose that M is a motion in $M(1)$. Whenever s and $\hat{s} \in \mathbb{R}$, we define $\chi_{ss} : \partial B_s \rightarrow \mathbb{R}^3$ to be just

$$\chi_{ss} := \chi_s \circ \chi_s^{-1}. \quad (9)$$

Since χ_s^{-1} is a 1-embedding taking ∂B_s onto ∂R , while χ_s is a 1-embedding carrying ∂R onto ∂B_s (cf., [I.3.27]), it follows that χ_{ss} is also a 1-embedding, with $\chi_{ss}(\partial B_s) = \partial B_s$, in view of [I.2.19]. Clearly,

$$\chi_{ss}^{-1} = \chi_{ss}, \quad (10)$$

and χ_{ss} is the identity map on ∂B_s . Further, [I.2.17.v] and [I.2.19.iii] show that

$$J\chi_{ss} = \{(J\chi_s) \circ \chi_s^{-1}\} \cdot J\chi_s^{-1} = \{(J\chi_s) \circ \chi_s^{-1}\} \cdot \{(J\chi_s) \circ \chi_s^{-1}\}^{-1}. \quad (11)$$

[IV.19] E X A M P L E. We shall provide here a typical example demonstrating how the results accumulated for the geometry of the boundary of a Lyapunov domain can be applied in the investigation of integrals on such a manifold. Let M be a motion in $M(1)$, and suppose that $s \in \mathbb{R}$ and B_s^0 is a Lyapunov domain, with constants (a, α, d) . Choose $\beta \in (0, 2)$ and $Y \in \partial B_s$: we shall first show that $r_Y^{-\beta} \in L_1(\partial B_s)$. For this, it clearly suffices to select $\delta \in (0, (7/9)d)$ and verify that

$$\int_{C_\delta^\partial(Y, s)} \frac{1}{r_Y^\beta} d\mu_{\partial B_s} < \infty, \quad (1)$$

since the function $Z \mapsto r_Y^{-\beta}(Z)$ is continuous on the compact set $\partial B_s \cap C_\delta^\partial(Y, s)'$. With recourse to the properties of the coordinate system $(\partial B_s \cap B_d^3(Y), h_Y)$, reviewed in [IV.18.b and c], and noting that $r_Y \geq r_Y \circ \pi_Y$ on $\partial B_s \cap B_d^3(Y)$, we have

$$\begin{aligned}
 \int_{C_\delta^\partial(Y,s)} \frac{1}{r_Y^\beta} d\lambda_{\partial B_s} &\leq \int_{C_\delta^\partial(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^\beta} d\lambda_{\partial B_s} \\
 &= \int_{h_Y(C_\delta^\partial(Y,s))} \frac{1}{(r_Y \circ \Pi_Y \circ h_Y^{-1})^\beta} \cdot Jh_Y^{-1} d\lambda_2 \\
 &= \int_{B_\delta^2(0)} \frac{1}{(r_Y \circ \tilde{h}_Y^{-1})^\beta} \cdot Jh_Y^{-1} d\lambda_2 \\
 &= \int_{B_\delta^2(0)} \frac{1}{|\xi|_2^\beta} \cdot Jh_Y^{-1}(\xi) d\lambda_2(\xi) \\
 &< 2^{1/2} \int_{B_\delta^2(0)} \frac{1}{|\xi|_2^\beta} d\lambda_2(\xi) \\
 &= 2^{3/2} \pi \int_0^\delta \frac{1}{\zeta^{\beta-1}} d\lambda_1(\zeta) < \infty,
 \end{aligned}$$

completing the proof.

Still supposing that B_s^0 is a Lyapunov domain, let ϕ be in $C(\partial R \times \partial B)$, and employ the notation of [IV.15]. Again, let $Y \in \partial B_s$, and $\delta \in (0, (7/9)d)$. $\phi_{(Y,s)} \circ \chi_s^{-1}$ and $J\chi_s^{-1}$ are certainly bounded on ∂B_s , while $\Gamma_{(Y,s)}^{03}$ is bounded on $\partial B_s \cap \{Y\}'$. Upon applying the estimate displayed in [I.2.37.iii.4], it follows that, for some positive

$$\int_{C_0^\partial(Y,s)} \left| \frac{1}{r_Y} r_{Y,k} v_{\partial B_s}^k \cdot r_{(Y,s)}^{03} \cdot \phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} \right| d\lambda_{\partial B_s}$$

$$\leq M \int_{C_0^\partial(Y,s)} \frac{1}{r_Y} |r_{Y,k} v_{\partial B_s}^k| d\lambda_{\partial B_s}$$

$$< M\hat{a} \int_{C_0^\partial(Y,s)} \frac{1}{r_Y^{2-\alpha}} d\lambda_{\partial B_s} < \infty,$$

by the result just obtained. Utilizing the continuity of the function $r_Y^{-2} \cdot r_{Y,k} v_{\partial B_s}^k \cdot r_{(Y,s)}^{03} \cdot \phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1}$ on $\partial B_s \cap \{Y\}'$, we can conclude that this function is in fact in $L_1(\partial B_s)$. In consequence, the following definition is legitimate.

[IV.20] D E F I N I T I O N. Let $M \in \mathcal{M}(1)$, and suppose that B_ζ^0 is a Lyapunov domain for each $\zeta \in \mathbb{R}$.[†] Let (R, χ) be a reference pair for M as in [I.3.25], and $(P, Y, s) \mapsto \phi_{(Y,s)}(P)$ an element of $C(\partial R \times \partial B)$. We define $\omega_1^*(\phi): \partial B \rightarrow \mathbb{K}$ by

$$\omega_1^*(\phi)(Y, s) := \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Y} r_{Y,k} v_{\partial B_s}^k \cdot r_{(Y,s)}^{03} \cdot \phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}$$

for each $(Y, s) \in \partial B$. ■.

The following basic statement is to be used in the identification of conditions on a motion M and a function ϕ , as in [IV.15], which

[†]Certainly, this holds if $M \in \mathcal{M}(2)$, when B_ζ^0 is 2-regular for each $\zeta \in \mathbb{R}$; cf., [I.2.34.a] and [I.3.27.i.2].

suffice to ensure that various of the inclusions $w_1^I\{\cdot\}, w_{21}^I\{\cdot\} \in C(\mathbb{B})$, $w_1^O\{\phi\}, w_{21}^O\{\phi\} \in C(\Omega^{\sigma-})$, $w_{31}\{\phi\} \in C(\mathbb{R}^4)$, or $w_{32}\{\cdot\} \in C(\mathbb{R}^4)$ obtain.

[IV.21] L E M M A. Let M be a motion in $\mathbb{M}(1)$, and $f \in C(\Omega^{\sigma})$ [$f \in C(\mathbb{B}^{\circ})$]. Suppose that there exists a function $f_{\partial} \in C(\partial\mathbb{B})$ such that for each $(Z, \zeta) \in \partial\mathbb{B}$

$$\begin{aligned} \lim_{\substack{X \rightarrow Z \\ X \in L_V^+(Z, \zeta) \\ [X \in L_V^-(Z, \zeta)]}} f(X, \zeta) &= f_{\partial}(Z, \zeta) \end{aligned}$$

locally uniformly in (Z, ζ) , i.e., whenever $(Z, \zeta) \in \partial\mathbb{B}$, if $\epsilon > 0$ there exists a $\delta(\epsilon, Z, \zeta) > 0$ such that for each $(\hat{Z}, \hat{\zeta}) \in \partial\mathbb{B}$ with $|(\hat{Z}, \hat{\zeta}) - (Z, \zeta)|_4 < \delta(\epsilon, Z, \zeta)$, the inequality $|f(X, \hat{\zeta}) - f_{\partial}(\hat{Z}, \hat{\zeta})| < \epsilon$ holds whenever $X \in L_V^+(\hat{Z}, \hat{\zeta}) \cap \mathbb{B}_{\hat{\zeta}}^+$ [$X \in L_V^-(\hat{Z}, \hat{\zeta}) \cap \mathbb{B}_{\hat{\zeta}}^{\circ}$] and $|X - \hat{Z}|_3 < \delta(\epsilon, Z, \zeta)$. Then

$$(i) \quad \lim_{\substack{(X, t) \rightarrow (Z, \zeta) \\ (X, t) \in \Omega^{\sigma} \\ [(X, t) \in \mathbb{B}^{\circ}]}} f(X, t) = f_{\partial}(Z, \zeta) \quad \text{for each } (Z, \zeta) \in \partial\mathbb{B},$$

and

(ii) the function $\tilde{f}$ given on $\Omega^{\sigma-}[\mathbb{B}]$ by

$$\tilde{f}(X, t) := \begin{cases} f(X, t) & \text{if } (X, t) \in \Omega^{\sigma}[\mathbb{B}^{\circ}] \\ f_{\partial}(X, t) & \text{if } (X, t) \in \partial\mathbb{B} \end{cases}$$

is in $C(\Omega^{\sigma-}) [C(\mathbb{B})]$.

P R O O F. We shall give the proof in case $f \in C(\Omega^\sigma)$, the verification for $f \in C(B^0)$ being quite similar. Then, choose $(Z, \zeta) \in \partial B$. Let $\epsilon > 0$. Since $f_\partial \in C(\partial B)$, there exists $\delta'(\epsilon, Z, \zeta) > 0$ for which

$$|f_\partial(\hat{Z}, \hat{\zeta}) - f_\partial(Z, \zeta)| < \epsilon \quad \text{whenever} \quad (\hat{Z}, \hat{\zeta}) \in \partial B \cap B_{\delta'(\epsilon, Z, \zeta)}^4(Z, \zeta).$$

Setting

$$\eta := \begin{cases} 1 & \text{if } c^* = 0, \\ \min \{1, 1/c^*\} & \text{if } c^* > 0, \end{cases}$$

suppose that $(X, t) \in \Omega^\sigma$, with

$$|(X, t) - (Z, \zeta)|_4 < \frac{1}{3} \eta \cdot \min \{\delta(\epsilon/2, Z, \zeta), \delta'(\epsilon/2, Z, \zeta)\}.$$

Letting (R, X) denote a reference pair for $M \in \mathcal{M}(1)$ with the properties of [I.3.25], we have

$$\begin{aligned} |X - \chi(\chi_\zeta^{-1}(Z), t)|_3 &\leq |X - Z|_3 + |\chi(\chi_\zeta^{-1}(Z), \zeta) - \chi(\chi_\zeta^{-1}(Z), t)|_3 \\ &\leq |X - Z|_3 + c^* |\zeta - t| \\ &< \frac{2}{3} \min \{\delta(\epsilon/2, Z, \zeta), \delta'(\epsilon/2, Z, \zeta)\}, \end{aligned}$$

from which we deduce that

$$\begin{aligned} \text{dist}(X, \partial B_t) &:= \inf \{|Y - X|_3 \mid Y \in \partial B_t\} \\ &< \frac{2}{3} \min \{\delta(\epsilon/2, Z, \zeta), \delta'(\epsilon/2, Z, \zeta)\}. \end{aligned}$$

According to [I.2.20], we can select $Z_X \in \partial B_t$ such that

$|X - Z_X|_3 = \text{dist}(X, \partial B_t)$, and we also have $X \in L_V^+(Z_X, t) \cap B_t'$ (of course, X must be in B_t' since $(X, t) \in \Omega^\sigma$). Then

$$\begin{aligned} |(Z_X, t) - (Z, \zeta)|_4 &\leq |(Z_X, t) - (X, t)|_4 + |(X, t) - (Z, \zeta)|_4 \\ &< \left(\frac{2}{3} + \frac{1}{3} \eta \right) \cdot \min \{ \delta(\epsilon/2, Z, \zeta), \delta'(\epsilon/2, Z, \zeta) \} \\ &\leq \min \{ \delta(\epsilon/2, Z, \zeta), \delta'(\epsilon/2, Z, \zeta) \}. \end{aligned}$$

To summarize, we have $(Z_X, t) \in \partial B$ with $|(Z_X, t) - (Z, \zeta)|_4 < \delta(\epsilon/2, Z, \zeta)$ and $< \delta'(\epsilon/2, Z, \zeta)$, while $X \in L_V^+(Z_X, t) \cap B_t'$ with $|X - Z_X|_3 < \delta(\epsilon/2, Z, \zeta)$, so

$$\begin{aligned} |f(X, t) - f_\partial(Z, \zeta)| &\leq |f(X, t) - f_\partial(Z_X, t)| + |f_\partial(Z_X, t) - f_\partial(Z, \zeta)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This reasoning clearly implies that (i) is correct.

Now (ii) is a simple consequence of (i) and the continuity of f_∂ on ∂B . $\square$.

Having defined, under certain conditions, the "direct value" function $W_1^*\{\phi\}$ on ∂B , we take up next the identification of the limiting values of $W_1^0\{\phi\}(\cdot, s)$ and $W_1^I\{\phi\}(\cdot, s)$, for fixed $s \in \mathbb{R}$, as their arguments approach a point $Y \in \partial B_s$ from along $L_V^+(Y, s)$ and $L_V^-(Y, s)$, respectively.

[IV.22] T H E O R E M. Let M be a motion in $M(1)$. Suppose further that

- (i) $\{B_\zeta^0\}_{\zeta \in \mathbb{R}}$ is "locally uniformly Lyapunov": whenever $\bar{K} \subset \mathbb{R}$ is compact, then $\{B_\zeta^0\}_{\zeta \in \bar{K}}$ is uniformly Lyapunov;
- (ii) there exists a reference pair (R, x) for M which possesses the properties of [I.3.25] and is also such that x_4 and $\hat{J}x$ are locally Hölder continuous on $\partial R \times \mathbb{R}$: whenever $\bar{K} \subset \mathbb{R}$ is compact, then $x_4|_{\partial R \times \bar{K}}$ and $\hat{J}x|_{\partial R \times \bar{K}}$ are Hölder continuous;
- (iii) $(P, X, t) \mapsto \phi_{(X, t)}(P)$ is a function in $C(\partial R \times \mathbb{R}^4)$ which satisfies the following local Hölder-type estimates: whenever $\bar{K} \subset \mathbb{R}$ is compact, there exist positive numbers $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3$, and $\eta_{\bar{K}}$, and numbers $\bar{\beta}_1, \bar{\beta}_2$, and $\bar{\beta}_3$ in $(0, 1]$, depending on ϕ, M , and (perhaps) $\bar{K}$, for which

$$|\phi_{(\bar{X}, \bar{s})} \circ x_{\bar{s}}^{-1}(Z) - \phi_{(\bar{Y}, \bar{s})} \circ x_{\bar{s}}^{-1}(Z)| \leq \bar{\kappa}_1 |\bar{X} - \bar{Y}|_3^{\bar{\beta}_1} \quad (1)$$

and

$$|\phi_{(\bar{X}, \bar{s})} \circ x_{\bar{s}}^{-1}(Z) - \phi_{(\bar{Y}, \bar{s})} \circ x_{\bar{s}}^{-1}(\bar{Y})| \leq \bar{\kappa}_2 |Z - \bar{Y}|_3^{\bar{\beta}_2} + \bar{\kappa}_3 |Z - \bar{X}|_3^{\bar{\beta}_3} \quad (2)$$

whenever $\bar{s} \in \bar{K}$, $\bar{Y} \in \partial B_{\bar{s}}$, $Z \in \partial B_{\bar{s}}$, and

$$\bar{X} \in L_{\vee}(\bar{Y}, \bar{s}) \cap B_{\eta_{\bar{K}}}^3(\bar{Y}).$$

Then, whenever $K \subset \mathbb{R}$ is compact, there exist $\Delta > 0$, $A > 0$, and $\lambda \in (0, 1]$, depending on ϕ, M , and K , such that, for each $s \in K$, $Y \in \partial B_s$, and

$$X \in L_V^+(Y, s) \quad [X \in L_V^-(Y, s)] \quad \text{with} \quad |X-Y|_3 < \Delta,$$

$$\left| w_1^{[I]} \{ \phi \} (X, s) - \left\{ \begin{array}{l} \bar{+} \\ + \end{array} \right\} \frac{1}{2(1-|X_s^c(X_s^{-1}(Y), s)|_3^2)} \phi(Y, s)^{\circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y)} \right. \\ \left. + w_1^* \{ \phi \} (Y, s) \right\} \Big| \leq A |X-Y|_3^\lambda. \quad (3)$$

In particular, it follows that

$$\lim_{\substack{X \rightarrow Y \\ X \in L_V^+(Y, s) \\ [X \in L_V^-(Y, s)]}} w_1^{[I]} \{ \phi \} (X, s) \quad (4)$$

$$= \left\{ \begin{array}{l} \bar{+} \\ + \end{array} \right\} \frac{1}{2(1-|X_s^c(X_s^{-1}(Y), s)|_3^2)} \phi(Y, s)^{\circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y)} + w_1^* \{ \phi \} (Y, s),$$

uniformly for $(Y, s) \in \bigcup_{\zeta \in K} \{ \partial B_\zeta \times \{ \zeta \} \}$.

Assume, moreover, that

(iv) $\mu: \partial B \rightarrow K$ is locally Hölder continuous: whenever $\tilde{K} \subset \mathbb{R}$ is compact, then $\mu|_{\bigcup_{\zeta \in \tilde{K}} \{ \partial B_\zeta \times \{ \zeta \} \}}$ is Hölder continuous.

Then requirement (iii) is fulfilled by taking either $\phi = \lambda_{11}^1 \{ \mu \}$ or $\phi = \bar{\lambda}_{11}^1 \{ \mu \}$ (cf., (IV.14.33) and (IV.14.46)), whence the assertions made above hold for either of these choices for ϕ . In this regard, we record the expressions

$$\begin{aligned} \Lambda_{11}^1\{u\}_{(Y,s)} \circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y) &= \{(1 - |X_{,4}^c(X_s^{-1}(Y), s)|_3^2)^{1/2} v^1(Y, s) \\ &+ u^c(Y, s) \cdot X_{,4}^{1c}(X_s^{-1}(Y), s)\} u(Y, s), \end{aligned} \quad (5)$$

and

$$\tilde{\Lambda}_1^1\{u\}_{(Y,s)} \circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y) = u^c(Y, s) \cdot u(Y, s), \quad (6)$$

holding for each $(Y, s) \in \partial B$.

Finally, if $M \in \mathbb{M}(2)$, then M satisfies requirements (i) and (ii).

P R O O F. It is clear, from [IV.20], that $\omega_1^*\{\phi\}$ is defined on ∂B .

It obviously suffices to prove the theorem in the case in which K is a closed interval $[t_1, t_2] \subset \mathbb{R}$, which we shall do. From (i),

$\{B_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov; let (a_K, α_K, d_K) be a set of Lyapunov constants for B_ζ^0 for each $\zeta \in K$. That is, $a_K > 0$, $d_K > 0$,

$\alpha_K \in (0, 1]$, $a_K d_K^{\alpha_K} < 1/2$, and

$$|v(Z_2, \zeta) - v(Z_1, \zeta)|_3 \leq a_K |Z_2 - Z_1|_3^{\alpha_K} \quad \text{for } \zeta \in K, \quad Z_1, Z_2 \in \partial B_\zeta. \quad (7)$$

We may, and shall, also suppose that there exists a number $\gamma_K \in (0, 1)$, depending upon only a_K , α_K , and d_K , such that

$$\gamma_K < \frac{r_X(Z)}{r_X(\Pi_Y(Z))} < \frac{1}{\gamma_K} \quad \text{whenever } s \in K, \quad Y \in \partial B_s, \quad (8)$$

$$X \in L_v(Y, s), \quad \text{and} \quad Z \in \partial B_s \cap B_{d_K}^3(Y) \cap \{X\}',$$

(by Lemma [I.2.38] and the fact that $\{B_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov),

and that

$$d_K \leq \eta_K. \quad (9)$$

Fix any $d \in (0, (7/9)d_K)$. Choose any $s \in K$, then select $Y \in \partial B_s$; let $X \in L_V^+(Y, s)$ [$X \in L_V^-(Y, s)$], and write $\delta := |X-Y|_3$. Then $\delta > 0$, and we suppose also that

$$\delta < \min \left\{ \frac{d}{2}, 1 \right\}. \quad (10)$$

Recalling the function $\Gamma_Z\{\xi\}: \mathbb{R}^3 \cap \{Z\}' \rightarrow \mathbb{R}$, defined in Lemma [I.2.44] whenever $Z \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ with $|\xi|_3 < 1$, and noting that $|x_{,4}^c(x_s^{-1}(Y), s)|_3 \leq c^*/c < 1$, we write first

$$\begin{aligned} & \omega_1^{[I]} \{\phi\}(X, s) - \omega_1^* \{\phi\}(Y, s) \\ &= \phi(Y, s) \circ x_s^{-1}(Y) \cdot Jx_s^{-1}(Y) \cdot \left\{ \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_X} r_{X,i} \nu_{\partial B_s}^i \cdot \Gamma_X\{x_{,4}^c(x_s^{-1}(Y), s)\} d\lambda_{\partial B_s} \right. \\ & \quad \left. - \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Y} r_{Y,i} \nu_{\partial B_s}^i \cdot \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y), s)\} d\lambda_{\partial B_s} \right\} \\ & \quad + \phi(Y, s) \circ x_s^{-1}(Y) \cdot Jx_s^{-1}(Y) \cdot \left\{ \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_X} r_{X,i} \nu_{\partial B_s}^i \right. \\ & \quad \left. \cdot \{\Gamma_{(X,s)}^{03} - \Gamma_X\{x_{,4}^c(x_s^{-1}(Y), s)\}\} d\lambda_{\partial B_s} \right. \\ & \quad \left. - \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Y} r_{Y,i} \nu_{\partial B_s}^i \{\Gamma_{(Y,s)}^{03} - \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y), s)\}\} d\lambda_{\partial B_s} \right\} \end{aligned} \quad (11)$$

$$\begin{aligned}
 & + \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_X^2} r_{X,i} \nu_{\partial B_s}^i \cdot \Gamma_{(X,s)}^{03} \cdot \{\phi(X,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} \\
 & - \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)\} d\lambda_{\partial B_s} \\
 & - \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Y^2} r_{Y,i} \nu_{\partial B_s}^i \cdot \Gamma_{(Y,s)}^{03} \cdot \{\phi(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} \\
 & - \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)\} d\lambda_{\partial B_s}.
 \end{aligned}$$

Referring to Lemma [I.2.44], we find that, if $Z \in \mathbb{R}^3$,

$$\frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Z^2} r_{Z,i} \nu_{\partial B_s}^i \cdot \Gamma_Z \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} d\lambda_{\partial B_s} = \nu_Z \{ 1 - | \chi_{,4}^c(\chi_s^{-1}(Y), s) |_3^2 \}^{-1},$$

where $\nu_Z = 0$ if $Z \in B'_s$, $\nu_Z = 1/2$ if $Z \in \partial B_s$, and $\nu_Z = 1$ if $Z \in B_s^0$. Consequently, the first term on the right in (11) is computed to be $[+] \frac{1}{2} \{ 1 - | \chi_{,4}^c(\chi_s^{-1}(Y), s) |_3^2 \}^{-1} \cdot \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)$. Using this in (11) and splitting the integrals remaining on the right, we arrive at the inequality

$$\begin{aligned}
 & \left| \omega_1^{[I]} \{ \phi \} (X,s) - \left\{ [+] \frac{1}{2(1 - | \chi_{,4}^c(\chi_s^{-1}(Y), s) |_3^2)} \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \right. \right. \\
 & \left. \left. + \omega_1^{[I]} \{ \phi \} (Y,s) \right\} \right| \leq \frac{1}{4\pi} \left\{ M_\phi^K \sum_{j=1}^4 I_j + \sum_{j=5}^8 I_j \right\},
 \end{aligned} \tag{12}$$

in which

$$\begin{aligned}
 & M_\phi^K := \max \{ | \phi(Z, \zeta) \circ \chi_\zeta^{-1}(Z) \cdot J\chi_\zeta^{-1}(Z) | \mid (Z, \zeta) \in (\partial B)_{[t_1, t_2]} \} \\
 & (\phi \in C(\partial \mathbb{R} \times \mathbb{R}^4), (Z, \zeta) \mapsto \chi_\zeta^{-1}(Z) \text{ and } (Z, \zeta) \mapsto J\chi_\zeta^{-1}(Z) \text{ are continuous} \\
 & \text{on } \partial B, \text{ while } (\partial B)_{[t_1, t_2]} \text{ is compact}),
 \end{aligned}$$

$$I_1 := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \left\{ \frac{1}{2} r_{X,i} \nu_{\partial B_s}^1 \{ \Gamma_{(X,s)}^{03} - \Gamma_X \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \} \right. \right. \\ \left. \left. - \frac{1}{2} r_{Y,i} \nu_{\partial B_s}^1 \{ \Gamma_{(Y,s)}^{03} - \Gamma_Y \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \} \right\} d\lambda_{\partial B_s} \right|, \quad (13)$$

$$I_2 := \left| \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \left\{ \frac{1}{2} r_{X,i} \nu_{\partial B_s}^1 \{ \Gamma_{(X,s)}^{03} - \Gamma_X \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \} \right. \right. \\ \left. \left. - \frac{1}{2} r_{Y,i} \nu_{\partial B_s}^1 \{ \Gamma_{(Y,s)}^{03} - \Gamma_Y \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \} \right\} d\lambda_{\partial B_s} \right|, \quad (14)$$

$$I_3 := \left| \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{2} r_{X,i} \nu_{\partial B_s}^1 \{ \Gamma_{(X,s)}^{03} - \Gamma_X \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \} d\lambda_{\partial B_s} \right|, \quad (15)$$

$$I_4 := \left| \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{2} r_{Y,i} \nu_{\partial B_s}^1 \{ \Gamma_{(Y,s)}^{03} - \Gamma_Y \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \} d\lambda_{\partial B_s} \right|, \quad (16)$$

$$I_5 := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \left\{ \frac{1}{2} r_{X,i} \nu_{\partial B_s}^1 \cdot \Gamma_{(X,s)}^{03} \{ \phi(X,s)^{\circ \chi_s^{-1} \cdot J \chi_s^{-1}} \right. \right. \\ \left. \left. - \phi(Y,s)^{\circ \chi_s^{-1}(Y) \cdot J \chi_s^{-1}(Y)} \right\} \right. \\ \left. - \frac{1}{2} r_{Y,i} \nu_{\partial B_s}^1 \cdot \Gamma_{(Y,s)}^{03} \{ \phi(Y,s)^{\circ \chi_s^{-1} \cdot J \chi_s^{-1}} \right. \right. \\ \left. \left. - \phi(Y,s)^{\circ \chi_s^{-1}(Y) \cdot J \chi_s^{-1}(Y)} \right\} d\lambda_{\partial B_s} \right|, \quad (17)$$

$$I_6 := \left| \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \left\{ \frac{1}{2} r_{X,i} v_{\partial B_s}^i \cdot r_{(X,s)}^{03} \{ \phi(X,s)^{ox_s^{-1}} \cdot JX_s^{-1} \right. \right. \\ \left. \left. - \phi(Y,s)^{ox_s^{-1}}(Y) \cdot JX_s^{-1}(Y) \right\} \right. \\ \left. - \frac{1}{2} r_{Y,i} v_{\partial B_s}^i \cdot r_{(Y,s)}^{03} \{ \phi(Y,s)^{ox_s^{-1}} \cdot JX_s^{-1} \right. \\ \left. \left. - \phi(Y,s)^{ox_s^{-1}}(Y) \cdot JX_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \right|, \quad (18)$$

$$I_7 := \left| \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{2} r_{X,i} v_{\partial B_s}^i \cdot r_{(X,s)}^{03} \{ \phi(X,s)^{ox_s^{-1}} \cdot JX_s^{-1} \right. \\ \left. \left. - \phi(Y,s)^{ox_s^{-1}}(Y) \cdot JX_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \right|, \quad (19)$$

and

$$I_8 := \left| \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{2} r_{Y,i} v_{\partial B_s}^i \cdot r_{(Y,s)}^{03} \{ \phi(Y,s)^{ox_s^{-1}} \cdot JX_s^{-1} \right. \\ \left. \left. - \phi(Y,s)^{ox_s^{-1}}(Y) \cdot JX_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \right|; \quad (20)$$

for the integrals in (14) and (18), note that we have ensured that $2\delta < d$. Clearly, the first assertion of the theorem can be proven by producing positive numbers $A(j)$ and numbers $\lambda(j) \in (0,1]$, $j = 1, \dots, 8$, depending only on ϕ , M , and K , such that $I_j \leq A(j)\delta^{\lambda(j)}$ (no sum) for each $j \in \{1, \dots, 8\}$, which we shall do, following the preparation of various simple estimates.

Suppose first that $Z \in \partial E_s$ and $Z \neq Y$. Certainly, $X \notin \partial E_s$, so $Z \neq X$, and we can write

$$\begin{aligned} |\text{grad } r_X(Z) - \text{grad } r_Y(Z)|_3 &= \left| \frac{1}{r_X(Z)} (Z-X) - \frac{1}{r_Y(Z)} (Z-Y) \right|_3 \\ &= \left| \frac{1}{r_Y(Z)} (Y-X) + (Z-X) \left(\frac{1}{r_X(Z)} - \frac{1}{r_Y(Z)} \right) \right|_3 \\ &\leq \frac{\delta}{r_Y(Z)} + r_X(Z) \left| \frac{1}{r_X(Z)} - \frac{1}{r_Y(Z)} \right| \quad (21) \\ &= \{\delta + |r_Y(Z) - r_X(Z)|\} \cdot \frac{1}{r_Y(Z)} \\ &\leq \frac{2\delta}{r_Y(Z)}. \end{aligned}$$

Now, let $\xi \in \mathbb{R}^3$ with $|\xi|_3 \leq c^*/c$, and $\hat{X} \in \mathbb{R}^3$. For $\hat{X} \neq Z$, we have

$$\Gamma_{\hat{X}}\{\xi\}(Z) := \{(\xi^L r_{\hat{X},l}(Z))^2 + (1 - |\xi|_3^2)\}^{-3/2}$$

(cf., (I.2.44.1)), and it is easy to see (as in [VI.69]) that

$$1 - (c^*/c)^2 \leq (\xi^L r_{\hat{X},l}(Z))^2 + (1 - |\xi|_3^2) \leq 1. \quad (22)$$

A bit of algebra shows that, for positive numbers a and b ,

$$a^{-3/2} - b^{-3/2} = \frac{b+a^{1/2}b^{1/2}+a}{a^{3/2}b^{3/2}(a^{1/2}+b^{1/2})} (b-a), \quad (23)$$

whence (22) gives

$$\begin{aligned} |\Gamma_X\{\xi\}(Z) - \Gamma_Y\{\xi\}(Z)| &\leq \frac{3}{2} \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-7/2} |(\xi^L r_{X,l}(Z))^2 - (\xi^L r_{Y,l}(Z))^2| \\ &\leq 3 \left(\frac{c^*}{c} \right) \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-7/2} |\xi^L r_{X,l}(Z) - \xi^L r_{Y,l}(Z)| \end{aligned}$$

$$\leq 3 \left(\frac{c^*}{c} \right)^2 \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-7/2} |\text{grad } r_X(Z) - \text{grad } r_Y(Z)|_3. \quad (24)$$

Taking $\xi = \chi_{s,4}^c(\chi_s^{-1}(Y), s)$ and combining (21) and (24),

$$\begin{aligned} & |\Gamma_X\{\chi_{s,4}^c(\chi_s^{-1}(Y), s)\}(Z) - \Gamma_Y\{\chi_{s,4}^c(\chi_s^{-1}(Y), s)\}(Z)| \\ & \leq 6 \left(\frac{c^*}{c} \right)^2 \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-7/2} \cdot \frac{\delta}{r_Y(Z)}, \end{aligned} \quad (25)$$

for each $Z \in \partial B_s \cap \{Y\}'$.

Next, let $Z \in \partial B_s \cap C_{2\delta}^{\partial}(Y, s)'$: if $Z \in B_d^3(Y)$, then $r_Y(Z) \geq r_Y(\pi_Y(Z)) \geq 2\delta$, while if $Z \in B_d^3(Y)'$, then $r_Y(Z) \geq d > 2\delta$, showing that $\delta/r_Y(Z) \leq 1/2$. Consequently, from the inequality

$|r_X(Z) - r_Y(Z)| \leq |X - Y|_3 = \delta$, we find

$$1 - \frac{\delta}{r_Y(Z)} \leq \frac{r_X(Z)}{r_Y(Z)} \leq 1 + \frac{\delta}{r_Y(Z)},$$

so

$$\frac{1}{2} \leq \frac{r_X(Z)}{r_Y(Z)} \leq \frac{3}{2}, \quad \text{for each } Z \in \partial B_s \cap C_{2\delta}^{\partial}(Y, s)'. \quad (26)$$

In turn, (26) allows us to estimate

$$\begin{aligned} & \left| \left\{ \frac{1}{r_X^2(Z)} r_{X,i}(Z) - \frac{1}{r_Y^2(Z)} r_{Y,i}(Z) \right\} v^1(Z, s) \right| \\ & \leq \left| \frac{1}{r_X^2(Z)} (Z - X) - \frac{1}{r_Y^2(Z)} (Z - Y) \right|_3 \\ & = \left| \frac{1}{r_Y^2(Z)} (Y - X) + (Z - X) \left(\frac{1}{r_X^2(Z)} - \frac{1}{r_Y^2(Z)} \right) \right|_3 \\ & \leq \frac{\delta}{r_Y^2(Z)} + \frac{1}{r_Y^2(Z)} \cdot |r_Y(Z) - r_X(Z)| \cdot \left\{ 1 + \frac{r_Y(Z)}{r_X(Z)} + \frac{r_Y^2(Z)}{r_X^2(Z)} \right\} \end{aligned}$$

$$\leq \frac{8\delta}{r_Y(Z)}, \quad \text{for each } Z \in \partial B_s \cap C_{2\delta}^\partial(Y, s). \quad (27)$$

Turning to an examination of the function $|\chi_{,4}^c|_{(X,s)} \circ \chi_s^{-1} - |\chi_{,4}^c|_{(Y,s)} \circ \chi_s^{-1}|_3$ on ∂B_s , we have, more explicitly,

$$\begin{aligned} & |[\chi_{,4}^c]_{(X,s)} \circ \chi_s^{-1}(Z) - [\chi_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z)|_3 \\ &= |\chi_{,4}^c(\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); X, s)) - \chi_{,4}^c(\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); Y, s))|_3 \end{aligned} \quad (28)$$

for each $Z \in \partial B_s$.

Choose $Z \in \partial B_s$: since $\tau(\chi_s^{-1}(Z); Z, s) = 0$, (I.3.16.1) shows that

$$\begin{aligned} \tau(\chi_s^{-1}(Z); X, s) &= \tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Z, s) \\ &\leq \frac{1}{c-c^*} |Z-X|_3 \leq \frac{1}{c-c^*} \{|X-Y|_3 + |Z-Y|_3\} \\ &\leq \frac{1}{c-c^*} \left\{ \frac{d}{2} + \text{diam } B_s \right\}, \end{aligned} \quad (29)$$

and, similarly,

$$\tau(\chi_s^{-1}(Z); Y, s) \leq \frac{1}{c-c^*} |Z-Y|_3 \leq \frac{1}{c-c^*} \text{diam } B_s. \quad (30)$$

Set

$$t_0 := \frac{1}{c-c^*} \left\{ \frac{d}{2} + \max_{t_1 \leq \tau \leq t_2} \text{diam } B_t \right\},$$

and

$$\tilde{K} := [t_1 - t_0, t_2].$$

Then, (29) and (30) imply that $s - \tau(\chi_s^{-1}(Z); X, s) \in \tilde{K}$ and

$s - \tau(\chi_s^{-1}(Z); Y, s) \in \bar{K}$ for each $Z \in \partial B_s$. Appealing to hypothesis (ii), $\chi_{,4}^c|_{\partial R \times \bar{K}}$ is Hölder continuous: there exist $\tilde{A} > 0$ and $\tilde{\alpha} \in (0, 1]$ such that

$$|\chi_{,4}^c(P_2, \zeta_2) - \chi_{,4}^c(P_1, \zeta_1)|_3 \leq \tilde{A} |(P_2, \zeta_2) - (P_1, \zeta_1)|_4^{\tilde{\alpha}} \quad (31)$$

whenever $P_1, P_2 \in \partial R$ and $\zeta_1, \zeta_2 \in \bar{K}$.

In view of (28), (31) leads, with (I.3.16.1), to the estimate

$$\begin{aligned} & |[\chi_{,4}^c]_{(X,s)} \circ \chi_s^{-1}(Z) - [\chi_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z)|_3 \\ & \leq \tilde{A} |\tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Y, s)|^{\tilde{\alpha}} \\ & \leq \frac{\tilde{A}}{(c-c^*)^{\tilde{\alpha}}} \delta^{\tilde{\alpha}}, \quad \text{for each } Z \in \partial B_s; \end{aligned} \quad (32)$$

it is evident from the derivation just given that (32) remains valid whatever the originally chosen $s \in K$, $Y \in \partial B_s$, and $X \in L_V^+(Y, s)$ [$X \in L_V^-(Y, s)$], with δ satisfying (10).

Consider next $|V_{(X,s)} - V_{(Y,s)}|_3$ on ∂B_s . Supposing first that $Z \in \partial B_s \cap \{Y\}'$, we have, by (IV.14.11),

$$\begin{aligned} & |V_{(X,s)}^c(Z) - V_{(Y,s)}^c(Z)|_3 \\ & = \frac{1}{c} \left| \frac{1}{\tau(\chi_s^{-1}(Z); X, s)} \{X(\chi_s^{-1}(Z), s) - X(\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); X, s))\} \right. \\ & \quad \left. - \frac{1}{\tau(\chi_s^{-1}(Z); Y, s)} \{X(\chi_s^{-1}(Z), s) - X(\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); Y, s))\} \right|; \end{aligned}$$

temporarily writing, for brevity, $Z_s := \chi_s^{-1}(Z)$, $\tau_X := \tau(\chi_s^{-1}(Z); X, s)$, and $\tau_Y := \tau(\chi_s^{-1}(Z); Y, s)$, the latter expression is

$$\begin{aligned}
 &= \left| \frac{1}{\tau_X} \int_{s-\tau_X}^s \chi_{,4}^c(Z_s, \sigma) d\sigma - \frac{1}{\tau_Y} \int_{s-\tau_Y}^s \chi_{,4}^c(Z_s, \sigma) d\sigma \right| \\
 &= \left| \int_0^1 \chi_{,4}^c(Z_s, s+(\sigma-1)\tau_X) d\sigma - \int_0^1 \chi_{,4}^c(Z_s, s+(\sigma-1)\tau_Y) d\sigma \right| \\
 &\leq \int_0^1 |\chi_{,4}^c(\chi_s^{-1}(Z), s+(\sigma-1)\tau(\chi_s^{-1}(Z); X, s)) \\
 &\quad - \chi_{,4}^c(\chi_s^{-1}(Z), s+(\sigma-1)\tau(\chi_s^{-1}(Z); Y, s))| d\sigma.
 \end{aligned} \tag{33}$$

It is easy to see that $s+(\sigma-1)\tau(\chi_s^{-1}(Z); X, s)$ and $s+(\sigma-1)\tau(\chi_s^{-1}(Z); Y, s)$ are in $\bar{K} = [t_1 - t_0, t_2]$ for each $\sigma \in [0, 1]$, so we can apply (31) and (I.3.16.1) once more to obtain, from (33),

$$\begin{aligned}
 |v_{(X,s)}^c(Z) - v_{(Y,s)}^c(Z)|_3 &\leq \tilde{A} \int_0^1 |(\sigma-1)\{\tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Y, s)\}|^{\tilde{\alpha}} d\sigma \\
 &= \tilde{A} \int_0^1 (1-\sigma)^{\tilde{\alpha}} d\sigma \cdot |\tau(\chi_s^{-1}(Z); X, s) - \\
 &\quad - \tau(\chi_s^{-1}(Z); Y, s)|^{\tilde{\alpha}} \\
 &\leq \frac{\tilde{A}}{(1+\tilde{\alpha})(c-c^*)^{\tilde{\alpha}}} \delta^{\tilde{\alpha}}, \quad \text{for each } Z \in \partial\mathcal{B}_s
 \end{aligned} \tag{34}$$

((34) was originally derived for $Z \in \partial\mathcal{B}_s \cap \{Y\}'$, but the continuity of V shows that it is also valid for $Z = Y$). We remark that (34) can be improved for $Z \in \partial\mathcal{B}_s \cap C_d^{\partial}(Y, s)'$, and without a local Hölder-continuity condition on $\chi_{,4}$; such an improvement will not help us in the subsequent computations, so we shall not develop this inequality.

Let us turn our attention to $|\Gamma_{(X,s)}^{03}(Z) - \Gamma_{(Y,s)}^{03}(Z)|$, for $Z \in \partial B_s \cap \{Y\}'$: with $\hat{X}$ denoting either X or Y , we have, from (IV.14.29),

$$\begin{aligned} \Gamma_{(\hat{X},s)}^{03}(Z) &= \{W_{(\hat{X},s)}^k(Z) \cdot r_{\hat{X},k}(Z) + (r_{\hat{X},\ell}(Z) \cdot v_{(\hat{X},s)}^{\ell c}(Z))^2 \\ &\quad + (1 - |v_{(\hat{X},s)}^c(Z)|_3^2)^{1/2}\}^{-3}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} W_{(\hat{X},s)}(Z) &= \frac{1}{1 - v_{(\hat{X},s)}^k(Z) \cdot [\chi_{,4}^k]_{(\hat{X},s)} \circ \chi_s^{-1}(Z)} \{ \{1 - |v_{(\hat{X},s)}^c(Z)|_3^2\} \\ &\quad \cdot [\chi_{,4}^c]_{(\hat{X},s)} \circ \chi_s^{-1}(Z) - \{1 - v_{(\hat{X},s)}^j(Z) \cdot [\chi_{,4}^j]_{(\hat{X},s)} \circ \chi_s^{-1}(Z)\} \\ &\quad \cdot v_{(\hat{X},s)}^c(Z) \}, \end{aligned} \quad (36)$$

by (IV.14.28). Now,

$$\begin{aligned} c_1^* &\leq W_{(\hat{X},s)}^k(Z) \cdot r_{\hat{X},k}(Z) + (r_{\hat{X},\ell}(Z) \cdot v_{(\hat{X},s)}^{\ell c}(Z))^2 + (1 - |v_{(\hat{X},s)}^c(Z)|_3^2)^{1/2} \\ &\leq c_2^*, \end{aligned} \quad (37)$$

where

$$c_1^* := \{1 + (c^*/c)^2\}^{-1} \{1 - (c^*/c)\}^2$$

and

$$c_2^* := 1 + \{1 - (c^*/c)^2\}^{-1} \cdot (c^*/c) \cdot \{2 + (c^*/c)^2\},$$

the first inequality in (37) following from (IV.14.57), the second being an easy consequence of (36) and the bounds $|v|_3 \leq c^*$, $|\chi_{,4}|_3 \leq c^*$.

Noting that

$$\{1-(c^*/c)^2\}^{1/2} \leq \{(r_{\hat{X},\ell}(Z) \cdot v_{(\hat{X},s)}^{\ell^c}(Z))^2 + (1-|v_{(\hat{X},s)}^c(Z)|_3^2)\}^{1/2},$$

we can write

$$\begin{aligned} & |\Gamma_{(X,s)}^{03}(Z) - \Gamma_{(Y,s)}^{03}(Z)| \\ & \leq \frac{3c_2^{*2}}{c_1^{*6}} \cdot \{w_{(Y,s)}^k(Z) \cdot r_{Y,k}(Z) + \{(r_{Y,\ell}(Z) \cdot v_{(Y,s)}^{\ell^c}(Z))^2 + (1-|v_{(Y,s)}^c(Z)|_3^2)\}^{1/2}; \\ & \quad - \{w_{(X,s)}^k(Z) \cdot r_{X,k}(Z) + \{(r_{X,\ell}(Z) \cdot v_{(X,s)}^{\ell^c}(Z))^2 + (1-|v_{(X,s)}^c(Z)|_3^2)\}^{1/2}\} \\ & \leq \frac{3c_2^{*2}}{c_1^{*6}} \{ |w_{(Y,s)}(Z) - w_{(X,s)}(Z)|_3 \\ & \quad + \frac{1}{2} \{1-(c^*/c)^2\}^{-1/2} \{ |(r_{Y,\ell}(Z) \cdot v_{(Y,s)}^{\ell^c}(Z))^2 + (1-|v_{(Y,s)}^c(Z)|_3^2) \} \\ & \quad - \{ (r_{X,\ell}(Z) \cdot v_{(X,s)}^{\ell^c}(Z))^2 + (1-|v_{(X,s)}^c(Z)|_3^2) \} \} \\ & \leq \frac{3c_2^{*2}}{c_1^{*6}} \{ |w_{(Y,s)}(Z) - w_{(X,s)}(Z)|_3 \\ & \quad + \frac{1}{2} \{1-(c^*/c)^2\}^{-1/2} \{ |(r_{Y,\ell}(Z) \cdot v_{(Y,s)}^{\ell^c}(Z))^2 - (r_{X,\ell}(Z) \cdot v_{(X,s)}^{\ell^c}(Z))^2| \\ & \quad + ||v_{(Y,s)}^c(Z)|_3^2 - |v_{(X,s)}^c(Z)|_3^2| \} \} \\ & \leq \frac{3c_2^{*2}}{c_1^{*6}} |w_{(Y,s)}(Z) - w_{(X,s)}(Z)|_3 \\ & \quad + \frac{3c_2^{*2}}{c_1^{*6}} \cdot \{1-(c^*/c)^2\}^{-1/2} \cdot (c^*/c) \{ 2|v_{(Y,s)}^c(Z) - v_{(X,s)}^c(Z)|_3 \\ & \quad + (c^*/c) |\text{grad } r_Y(Z) - \text{grad } r_X(Z)|_3 \}. \end{aligned} \tag{38}$$

Now, from (36), a short computation yields

$$\begin{aligned}
 & |w_{(Y,s)}(Z) - w_{(X,s)}(Z)|_3 \\
 & \leq \{1 - (c^*/c)^2\}^{-1} \{1 + 6(c^*/c)^2 + (c^*/c)^4\} \cdot |v_{(Y,s)}^c(Z) - v_{(X,s)}^c(Z)|_3 \\
 & \quad + \{1 - (c^*/c)^2\}^{-1} \{1 + 3(c^*/c)^2 + (c^*/c)^4\} \cdot |[x_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z) \\
 & \quad - [x_{,4}^c]_{(X,s)} \circ \chi_s^{-1}(Z)|_3.
 \end{aligned} \tag{39}$$

Upon combining (38) and (39), we can assert that there exist positive numbers c_3^* , c_4^* , and c_5^* , depending only on the ratio c^*/c , such that

$$\begin{aligned}
 |\Gamma_{(X,s)}^{03}(Z) - \Gamma_{(Y,s)}^{03}(Z)| & \leq c_3^* |[x_{,4}^c]_{(X,s)} \circ \chi_s^{-1}(Z) - [x_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z)|_3 \\
 & \quad + c_4^* |v_{(X,s)}^c(Z) - v_{(Y,s)}^c(Z)|_3 \\
 & \quad + c_5^* |\text{grad } r_X(Z) - \text{grad } r_Y(Z)|_3,
 \end{aligned} \tag{40}$$

whence, in view of (21), (32), and (34),

$$|\Gamma_{(X,s)}^{03}(Z) - \Gamma_{(Y,s)}^{03}(Z)| \leq k_1 \delta^{\tilde{\alpha}} + k_2 \frac{\delta}{r_Y(Z)} \quad \text{for each } Z \in \partial B_s \cap \{Y\}', \tag{41}$$

in which $k_1 > 0$ and depends only on M and K , while $k_2 > 0$ and depends only on M .

In our final preliminary estimation, we shall examine

$|\Gamma_{(\hat{X},s)}^{03}(Z) - \Gamma_{\hat{X}}\{x_{,4}^c(\chi_s^{-1}(Y),s)\}(Z)|$, where $\hat{X}$ again denotes either X or Y , while $Z \in \partial B_s \cap \{\hat{X}\}'$ (i.e., $Z \in \partial B_s$ if $\hat{X} = X$, $Z \in \partial B_s \cap \{Y\}'$ if $\hat{X} = Y$): we begin by observing that

$$\begin{aligned}
 & \left| \Gamma_{(\hat{X},s)}^{03}(Z) - \Gamma_{\hat{X}}\{X_{,4}^c(X_s^{-1}(Y),s)\}(Z) \right| \\
 & \leq \frac{1+c_2^*+c_2^{*2}}{c_1^{*3}\left\{1-\left(\frac{c^*}{c}\right)^2\right\}^{3/2}} \cdot |W_{(\hat{X},s)}^k(Z) r_{\hat{X},k}(Z) + \{(r_{\hat{X},l}(Z) \cdot v_{(\hat{X},s)}^{lc}(Z))^2 \\
 & \quad + (1-|v_{(\hat{X},s)}^c(Z)|_3^2)\}^{1/2} - \{(r_{\hat{X},l}(Z) \cdot X_{,4}^{lc}(X_s^{-1}(Y),s))\}^2 \\
 & \quad + (1-|X_{,4}^c(X_s^{-1}(Y),s)|_3^2)\}^{1/2}| \\
 & \leq \frac{1+c_2^*+c_2^{*2}}{c_1^{*3}\left\{1-\left(\frac{c^*}{c}\right)^2\right\}^{3/2}} \cdot |W_{(\hat{X},s)}(Z)|_3 \quad (42) \\
 & \quad + \frac{1+c_2^*+c_2^{*2}}{2c_1^{*3}\left\{1-\left(\frac{c^*}{c}\right)^2\right\}^2} \cdot \{ |(r_{\hat{X},l}(Z) \cdot v_{(\hat{X},s)}^{lc}(Z))^2 - (r_{\hat{X},l}(Z) \cdot X_{,4}^{lc}(X_s^{-1}(Y),s))^2 | \\
 & \quad + | |v_{(\hat{X},s)}^c(Z)|_3^2 - |X_{,4}^c(X_s^{-1}(Y),s)|_3^2 | \} \\
 & \leq \frac{1+c_2^*+c_2^{*2}}{c_1^{*3}\left\{1-\left(\frac{c^*}{c}\right)^2\right\}^{3/2}} \cdot |W_{(\hat{X},s)}(Z)|_3 \\
 & \quad + \frac{2(1+c_2^*+c_2^{*2})}{c_1^{*3}\left\{1-\left(\frac{c^*}{c}\right)^2\right\}^2} \cdot \left(\frac{c^*}{c}\right) \cdot |v_{(\hat{X},s)}^c(Z) - X_{,4}^c(X_s^{-1}(Y),s)|_3;
 \end{aligned}$$

from (36), it is easy to see that

$$|W_{(\hat{X},s)}(Z)|_3 \leq \frac{1+(c^*/c)^2}{1-(c^*/c)^2} \cdot |v_{(\hat{X},s)}^c(Z) - [X_{,4}^c]_{(\hat{X},s)} \circ X_s^{-1}(Z)|_3,$$

so that there are positive numbers c_6^* and c_7^* , depending only on the ratio c^*/c , such that

$$\begin{aligned}
 & |\Gamma_{(\hat{X},s)}^{03}(Z) - \Gamma_{\hat{X}}^c(\chi_{s,4}^{-1}(Y),s)(Z)| \\
 & \leq c_6^* |V_{(\hat{X},s)}^c(Z) - [\chi_{s,4}^c]_{(\hat{X},s)} \circ \chi_s^{-1}(Z)|_3 + c_7^* |V_{(\hat{X},s)}^c(Z) - \chi_{s,4}^c(\chi_s^{-1}(Y),s)|_3 \\
 & \leq (c_6^* + c_7^*) \cdot |V_{(\hat{X},s)}^c(Z) - \chi_{s,4}^c(\chi_s^{-1}(Z),s)|_3 \\
 & \quad + c_7^* |\chi_{s,4}^c(\chi_s^{-1}(Z),s) - \chi_{s,4}^c(\chi_s^{-1}(Y),s)|_3 \\
 & \quad + c_6^* |\chi_{s,4}^c(\chi_s^{-1}(Z),s) - [\chi_{s,4}^c]_{(\hat{X},s)} \circ \chi_s^{-1}(Z)|_3.
 \end{aligned} \tag{43}$$

Since $s - \tau(\chi_s^{-1}(Z); \hat{X}, s)$ is, with s , in $\bar{K}$, we can write

$$\begin{aligned}
 & |\chi_{s,4}^c(\chi_s^{-1}(Z),s) - [\chi_{s,4}^c]_{(\hat{X},s)} \circ \chi_s^{-1}(Z)|_3 \\
 & = |\chi_{s,4}^c(\chi_s^{-1}(Z),s) - \chi_{s,4}^c(\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); \hat{X}, s))|_3 \\
 & \leq \tilde{A} \tau(\chi_s^{-1}(Z); \hat{X}, s)^{\tilde{\alpha}} = \tilde{A} \{\tau(\chi_s^{-1}(Z); \hat{X}, s) - \tau(\chi_s^{-1}(Z); Z, s)\}^{\tilde{\alpha}} \\
 & \leq \frac{\tilde{A}}{(c - c^*)^{\tilde{\alpha}}} |Z - \hat{X}|_3^{\tilde{\alpha}}.
 \end{aligned} \tag{44}$$

Now, the function $(\tilde{Z}, \tilde{s}) \mapsto \chi_{\tilde{s}}^{-1}(\tilde{Z})$ is in $C^1(\partial B; \mathbb{R}^3)$ (because the function $(\tilde{Z}, \tilde{s}) \mapsto \chi^{*-1}(\tilde{Z}, \tilde{s}) = (\chi_{\tilde{s}}^{-1}(\tilde{Z}), \tilde{s})$ is in $C^1(\partial B; \mathbb{R}^4)$), and thus is Lipschitz continuous on the compact subset $\chi^*(\partial R \times K) = \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$ of ∂B , by [I.2.13]: there exists a positive A_K for which

$$|\chi_{s_2}^{-1}(Z_2) - \chi_{s_1}^{-1}(Z_1)|_3 \leq A_K |(Z_2, s_2) - (Z_1, s_1)|_4, \tag{45}$$

whenever $s_1, s_2 \in K$ and $Z_1 \in \partial B_{s_1}$, $Z_2 \in \partial B_{s_2}$.

Thus, from (31) and (45),

$$\begin{aligned} |x_{,4}^c(x_s^{-1}(Z), s) - x_{,4}^c(x_s^{-1}(Y), s)|_3 &\leq \tilde{A} |x_s^{-1}(Z) - x_s^{-1}(Y)|_3^{\tilde{\alpha}} \\ &\leq \tilde{A} \tilde{A}_K^{\tilde{\alpha}} |Z - Y|_3^{\tilde{\alpha}}. \end{aligned} \quad (46)$$

Since $v_{(\hat{X}, s)}^i(Z) = \frac{1}{\tau(x_s^{-1}(Z); \hat{X}, s)} \{x(x_s^{-1}(Z), s) - x(x_s^{-1}(Z), s - \tau(x_s^{-1}(Z); \hat{X}, s))\}$,

the mean-value theorem shows that, for each $i \in \{1, 2, 3\}$, there exists $\hat{t}^i \in (s - \tau(x_s^{-1}(Z); \hat{X}, s), s)$, depending on Z , s , and $\hat{X}$, such that

$$v_{(\hat{X}, s)}^i(Z) = x_{,4}^i(x_s^{-1}(Z), \hat{t}^i).$$

We obviously have $\hat{t}^i \in \tilde{K}$ and $|s - \hat{t}^i| < \tau(x_s^{-1}(Z); \hat{X}, s)$, so

$$\begin{aligned} |v_{(\hat{X}, s)}^i(Z) - x_{,4}^i(x_s^{-1}(Z), s)| &\leq \tilde{A} |s - \hat{t}^i|^{\tilde{\alpha}} < \tilde{A} \cdot \tau(x_s^{-1}(Z); \hat{X}, s)^{\tilde{\alpha}} \\ &\leq \frac{\tilde{A}}{(c - c^*)^{\tilde{\alpha}}} |Z - \hat{X}|_3^{\tilde{\alpha}}. \end{aligned} \quad (47)$$

Using (44), (46), and (47) in (43) produces the desired inequality

$$\begin{aligned} &|\Gamma_{(\hat{X}, s)}^{03}(Z) - \Gamma_{\hat{X}}\{x_{,4}^c(x_s^{-1}(Y), s)\}(Z)| \\ &\leq (c_6^* + c_7^*) \cdot \sqrt{3} \cdot \frac{\tilde{A}}{(c - c^*)^{\tilde{\alpha}}} |Z - \hat{X}|_3^{\tilde{\alpha} + c_7^*} \cdot \tilde{A} \tilde{A}_K^{\tilde{\alpha}} |Z - Y|_3^{\tilde{\alpha} + c_6^*} \frac{\tilde{A}}{(c - c^*)^{\tilde{\alpha}}} |Z - \hat{X}|_3^{\tilde{\alpha}} \\ &= k_3 |Z - \hat{X}|_3^{\tilde{\alpha} + k_4} |Z - Y|_3^{\tilde{\alpha}}, \quad \text{for each } Z \in \partial B_s \cap \{\hat{X}\}', \end{aligned} \quad (48)$$

wherein k_3 and k_4 are positive and depend upon only M and K .

Returning to the main line of argument, recall that our task is the demonstration of the existence of $A(j) > 0$ and $\lambda(j) \in (0, 1]$, depending on only ϕ , M , and K , such that $I_j \leq A(j) \delta^{\lambda(j)}$ (no

sum) for $j = 1, \dots, 8$, where $s \in K$, $Y \in \partial B_s$, and $X \in L_V^+(Y, s)$ [$X \in L_V^-(Y, s)$], with $\delta := |X-Y|_3$ satisfying (10), have been selected. All of the just-derived estimates are uniformly valid for such s , Y , and X .

I₁: If $Z \in \partial B_s \cap C_d^\partial(Y, s)'$, it is clear that $r_Y(Z) \geq d$, $|r_{Y,1}(Z) \cdot \nu_{\partial B_s}^1(Z)| \leq 1$, and $Z \in \partial B_s \cap C_{2\delta}^\partial(Y, s)'$ (so that (27) is valid). Let M denote an upper bound for both Γ^{03} and $\Gamma_{\tilde{X}}\{\xi\}$, for $\tilde{X} \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ with $|\xi|_3 \leq c^*/c$ (cf., (IV.14.59) and (VI.69.4)).

Then, using (25), (27), and (41), we can write

$$\begin{aligned} I_1 &\leq \left| \int_{\partial B_s \cap C_d^\partial(Y, s)'} \left\{ \frac{1}{r_X} r_{X,1} - \frac{1}{r_Y} r_{Y,1} \right\} \nu_{\partial B_s}^1 \cdot \{\Gamma^{03}(X, s) \right. \\ &\quad \left. - \Gamma_X\{x_{,4}^c(x_s^{-1}(Y), s)\}\} d\lambda_{\partial B_s} \right. \\ &\quad + \int_{\partial B_s \cap C_d^\partial(Y, s)'} \frac{1}{r_Y} r_{Y,1} \nu_{\partial B_s}^1 \{ \{\Gamma^{03}(X, s) - \Gamma^{03}(Y, s)\} \\ &\quad \left. - \{\Gamma_X\{x_{,4}^c(x_s^{-1}(Y), s)\} - \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y), s)\}\} \} d\lambda_{\partial B_s} \right| \\ &\leq 2M \int_{\partial B_s \cap C_d^\partial(Y, s)'} \left| \left\{ \frac{1}{r_X} r_{X,1} - \frac{1}{r_Y} r_{Y,1} \right\} \nu_{\partial B_s}^1 \right| d\lambda_{\partial B_s} \\ &\quad + \int_{\partial B_s \cap C_d^\partial(Y, s)'} \frac{1}{r_Y} \cdot |\Gamma^{03}(X, s) - \Gamma^{03}(Y, s)| d\lambda_{\partial B_s} \\ &\quad + \int_{\partial B_s \cap C_d^\partial(Y, s)'} \frac{1}{r_Y} \cdot |\Gamma_X\{x_{,4}^c(x_s^{-1}(Y), s)\} - \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y), s)\}| d\lambda_{\partial B_s} \end{aligned}$$

$$\leq \left\{ \frac{16M}{d^3} \delta + \frac{1}{d^2} \{k_1 \delta^{\tilde{\alpha}} + \frac{k_2}{d} \delta\} + \frac{1}{d^3} k_5 \cdot \delta \right\} \cdot \sup_{\zeta \in K} \lambda_{\partial \delta_{\zeta}} (\partial B_{\zeta}), \quad (49)$$

where k_5 is the coefficient appearing in (25). Since $\delta \leq \delta^{\tilde{\alpha}}$, it is clear from (49) that I_1 satisfies an inequality of the desired form, with $\lambda(1) = \tilde{\alpha}$.

I_2 : Proceeding initially as for I_1 , we find

$$\begin{aligned} I_2 \leq & \int_{C_d^{\partial}(Y,s) \cap C_{2\delta}^{\partial}(Y,s)} \left| \left\{ \frac{1}{r_X} r_{X,i} - \frac{1}{r_Y} r_{Y,i} \right\} v_{\partial B_s}^1 \right| \\ & \cdot |\Gamma_{(X,s)}^{03} - \Gamma_X^c(\chi_{,4}^{-1}(Y), s)| \, d\lambda_{\partial B_s} \\ & + \int_{C_d^{\partial}(Y,s) \cap C_{2\delta}^{\partial}(Y,s)} \frac{1}{r_Y} |r_{Y,i} v_{\partial B_s}^1| \cdot |\Gamma_{(X,s)}^{03} - \Gamma_{(Y,s)}^{03}| \\ & + |\Gamma_X^c(\chi_{,4}^{-1}(Y), s) - \Gamma_Y^c(\chi_{,4}^{-1}(Y), s)| \, d\lambda_{\partial B_s}. \end{aligned}$$

Use of (27) and (48) in the first integral, and (25), (41), and (I.2.37.7) in the second produces

$$\begin{aligned} I_2 \leq & \int_{C_d^{\partial}(Y,s) \cap C_{2\delta}^{\partial}(Y,s)} \frac{8\delta}{r_Y} \cdot \{k_3 r_X^{\tilde{\alpha}} + k_4 r_Y^{\tilde{\alpha}}\} \, d\lambda_{\partial B_s} \\ & + \int_{C_d^{\partial}(Y,s) \cap C_{2\delta}^{\partial}(Y,s)} \frac{1}{r_Y} \cdot \hat{a}_K r_Y^{\alpha_K} \left\{ \left\{ k_1 \delta^{\tilde{\alpha}} + k_2 \frac{\delta}{r_Y} \right\} + k_5 \frac{\delta}{r_Y} \right\} \, d\lambda_{\partial B_s} \end{aligned}$$

$$\begin{aligned}
 &= 8k_3 \delta \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y} r_X^{\tilde{\alpha}} d\lambda_{\partial B_s} \\
 &+ 8k_4 \delta \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y^{3-\tilde{\alpha}}} d\lambda_{\partial B_s} \\
 &+ \hat{a}_K k_1 \delta^{\tilde{\alpha}} \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &+ \hat{a}_K (k_2 + k_5) \delta \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y^{3-\alpha_K}} d\lambda_{\partial B_s};
 \end{aligned} \tag{50}$$

note that, from [I.2.37.iii.3,4],

$$\hat{a}_K = \frac{8}{7} \left\{ 2 + \frac{1}{1+\alpha_K} \left(\frac{65}{49} \right)^{\alpha_K/2} \right\} a_K$$

depends only on M and K , i.e., is independent of the s , Y , and X chosen as specified, because $\{\mathcal{B}_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov.

Using (26), for each $Z \in \partial B_s \cap C_{2\delta}^\partial(Y,s)'$,

$$r_X(Z) \leq \frac{3}{2} r_Y(Z),$$

so

$$\int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y} r_X^{\tilde{\alpha}} d\lambda_{\partial B_s} \leq \left(\frac{3}{2} \right)^{\tilde{\alpha}} \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y^{3-\tilde{\alpha}}} d\lambda_{\partial B_s}, \tag{51}$$

showing that we can estimate the integral in the first term on the right in (50) by estimating the integral in the second term. Now suppose that $\alpha \in (0,1]$: then

$$\begin{aligned}
& \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y^{3-\alpha}} d\lambda_{\partial B_s} \\
& \leq \delta \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^{3-\alpha}} d\lambda_{\partial B_s} \\
& = \delta \int_{h_Y(C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s))} \frac{1}{(r_Y \circ \Pi_Y \circ h_Y^{-1})^{3-\alpha}} Jh_Y^{-1} d\lambda_2 \\
& = \delta \int_{\hat{h}_Y \circ \Pi_Y(C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s))} \frac{1}{(r_Y \circ \hat{h}_Y^{-1})^{3-\alpha}} Jh_Y^{-1} d\lambda_2 \quad (52) \\
& = \delta \int_{B_d^2(0) \cap B_{2\delta}^2(0)} \frac{1}{|\hat{\xi}|_2^{3-\alpha}} Jh_Y^{-1}(\hat{\xi}) d\lambda_2(\hat{\xi}) \\
& \leq 2^{1/2} \delta \int_{B_d^2(0) \cap B_{2\delta}^2(0)} \frac{1}{|\hat{\xi}|_2^{3-\alpha}} d\lambda_2(\hat{\xi}) \\
& = 2^{3/2} \pi \delta \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha}} d\zeta \\
& = \begin{cases} \frac{2^{3/2} \pi}{1-\alpha} \delta \{ (2\delta)^{\alpha-1} - d^{\alpha-1} \} < \frac{2^{3/2} \pi}{1-\alpha} \delta^\alpha, & \text{if } \alpha \in (0,1), \\ 2^{3/2} \pi \cdot \delta \ln \frac{d}{2\delta}, & \text{if } \alpha = 1. \end{cases}
\end{aligned}$$

In connection with the case $\alpha = 1$ in (52), since $\lim_{\zeta \rightarrow 0^+} \zeta^\lambda \ln \frac{d}{2\zeta} = 0$ for any positive λ , there exists, for each $\lambda > 0$, an $M'_\lambda > 0$ such that $\zeta^\lambda \cdot \ln \frac{d}{2\zeta} \leq M'_\lambda$ for $0 < \zeta \leq d/2$. Consequently, choosing any $\alpha \in (0,1)$, we have

$$\delta \ln \frac{d}{2\delta} = \delta^\alpha \cdot \delta^{1-\alpha} \ln \frac{d}{2\delta} \leq M'_{1-\alpha} \delta^\alpha. \quad (53)$$

Following the reasoning employed in deriving (52),

$$\begin{aligned} \delta^{\tilde{\alpha}} \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} &\leq 2^{3/2} \pi \delta^{\tilde{\alpha}} \int_{2\delta}^d \frac{1}{\zeta^{1-\alpha_K}} d\zeta \\ &\leq \frac{2^{3/2} \pi d^{\alpha_K}}{\alpha_K} \delta^{\tilde{\alpha}}. \end{aligned} \quad (54)$$

From (50), in view of the inequalities (51)-(54), it is evident that I_2 satisfies an estimate of the required form. Note that $0 < \lambda(2) < 1$, even if $\alpha_K = \tilde{\alpha} = 1$, because of (52) and (53).

I_3 : First, since $C_{2\delta}^\partial(Y,s) \subset C_d^\partial(Y,s) \subset B_{d_K}^3(Y)^+$ (because $d \in (0, (7/9)d_K)$; cf., [IV.18.c]), we may use (8) to conclude that

$$\frac{1}{r_X(Z)} = \frac{r_X(\pi_Y(Z))}{r_X(Z)} \cdot \frac{1}{r_X(\pi_Y(Z))} < \frac{1}{\gamma_K} \cdot \frac{1}{r_X(\pi_Y(Z))} < \frac{1}{\gamma_K} \cdot \frac{1}{r_Y(\pi_Y(Z))} \quad (55)$$

for each $Z \in C_{2\delta}^\partial(Y,s) \cap \{Y\}'$,

The second inequality following from the obvious fact that $r_X(\pi_Y(Z)) > r_Y(\pi_Y(Z))$ for $Z \in \partial B_s \cap B_{d_K}^3(Y)$. Another application of (8), with X replaced by Y , gives

$$r_Y(Z) = \frac{r_Y(Z)}{r_Y(\pi_Y(Z))} \cdot r_Y(\pi_Y(Z)) < \frac{1}{\gamma_K} r_Y(\pi_Y(Z)), \quad (56)$$

for each $Z \in C_{2\delta}^\partial(Y,s) \cap \{Y\}'$.

⁺ Note that the second of these inclusions has already been used tacitly in justifying the computations in (52) and (54).

Observing that $|r_{X,i} v_{\partial B_s}^1| \leq 1$ on ∂B_s , (55), (56), and (48) (with $\hat{X} = X$) imply

$$\begin{aligned}
 I_3 &\leq \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{r_X} \{k_3 r_X^{\tilde{\alpha}} + k_4 r_Y^{\tilde{\alpha}}\} d\lambda_{\partial B_s} \\
 &= \int_{C_{2\delta}^\partial(Y,s)} \left\{ k_3 \cdot \frac{1}{r_X^{2-\tilde{\alpha}}} + k_4 \cdot \frac{1}{r_X^2} r_Y^{\tilde{\alpha}} \right\} d\lambda_{\partial B_s} \\
 &\leq \int_{C_{2\delta}^\partial(Y,s)} \left\{ k_3 \cdot \frac{1}{\gamma_K^{2-\tilde{\alpha}}} \cdot \frac{1}{(r_Y \circ \Pi_Y)^{2-\tilde{\alpha}}} \right. \\
 &\quad \left. + k_4 \cdot \frac{1}{\gamma_K^2} \cdot \frac{1}{(r_Y \circ \Pi_Y)^2} \cdot \frac{1}{\gamma_K^{\tilde{\alpha}}} \cdot (r_Y \circ \Pi_Y)^{\tilde{\alpha}} \right\} d\lambda_{\partial B_s} \tag{57} \\
 &= \left\{ \frac{k_3}{\gamma_K^{2-\tilde{\alpha}}} + \frac{k_4}{\gamma_K^{2+\tilde{\alpha}}} \right\} \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^{2-\tilde{\alpha}}} d\lambda_{\partial B_s} \\
 &\leq \left\{ \frac{k_3}{\gamma_K^{2-\tilde{\alpha}}} + \frac{k_4}{\gamma_K^{2+\tilde{\alpha}}} \right\} \cdot 2^{3/2} \pi \int_0^{2\delta} \frac{1}{\zeta^{1-\tilde{\alpha}}} d\zeta \\
 &= \left\{ \frac{k_3}{\gamma_K^{2-\tilde{\alpha}}} + \frac{k_4}{\gamma_K^{2+\tilde{\alpha}}} \right\} \frac{2^{(3/2)+\tilde{\alpha}} \cdot \pi}{\tilde{\alpha}} \cdot \delta^{\tilde{\alpha}},
 \end{aligned}$$

an inequality of the form desired.

I_4 : Here, we can proceed more directly than in the reasoning for I_3 , obtaining, by [I.2.37.111.4] and (48) (with $\hat{X} = Y$),

$$I_4 \leq \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot \hat{a}_K^{\alpha_K} \cdot \{k_3 r_Y^{\tilde{\alpha}} + k_4 r_Y^{\tilde{\alpha}}\} d\lambda_{\partial B_s}$$

$$\begin{aligned}
 &= \hat{a}_K(k_3+k_4) \int_{C_{2\delta}^\partial(Y,s)} \frac{1}{2-(\tilde{\alpha}+\alpha_K)} d\lambda_{\partial B_s} \quad (5C) \\
 &\leq \hat{a}_K(k_3+k_4) \cdot 2^{(3/2)+\tilde{\alpha}+\alpha_K} \pi \cdot \frac{1}{\tilde{\alpha}+\alpha_K} \delta^{\tilde{\alpha}+\alpha_K}.
 \end{aligned}$$

I₅: We begin by splitting the integral in (17) into three terms:

$$\begin{aligned}
 I_5 \leq & \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \left\{ \frac{1}{r_X^2} r_{X,i} - \frac{1}{r_Y^2} r_{Y,i} \right\} v_{\partial B_s}^i \cdot \Gamma_{(X,s)}^{03} \right. \\
 & \cdot \{ \phi_{(X,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} - \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \\
 & + \int_{\partial B_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y^2} r_{Y,i} v_{\partial B_s}^i \{ \Gamma_{(X,s)}^{03} - \Gamma_{(Y,s)}^{03} \} \\
 & \cdot \{ \phi_{(X,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} - \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \\
 & + \left. \int_{\partial B_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y^2} r_{Y,i} v_{\partial B_s}^i \cdot \Gamma_{(Y,s)}^{03} \cdot \{ \phi_{(X,s)} \circ \chi_s^{-1} - \phi_{(Y,s)} \circ \chi_s^{-1} \} \right. \\
 & \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|.
 \end{aligned}$$

Using (27), and observing that $r_Y(Z) \geq d$, $|r_{Y,i}(Z) \cdot v_{\partial B_s}^i(Z)| \leq 1$ for each $Z \in \partial B_s \cap C_d^\partial(Y,s)$,

$$\begin{aligned}
 I_5 \leq & \frac{8M}{d^3} \cdot \delta \int_{\partial B_s \cap C_d^{\partial}(Y,s)} \{ |\phi_{(X,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1}| \\
 & + |\phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)| \} d\lambda_{\partial B_s} \\
 & + \frac{1}{d^2} \int_{\partial B_s \cap C_d^{\partial}(Y,s)} |\Gamma_{(X,s)}^{03} - \Gamma_{(Y,s)}^{03}| \cdot \{ |\phi_{(X,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1}| \\
 & + |\phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)| \} d\lambda_{\partial B_s} \\
 & + \frac{M}{d^2} \int_{\partial B_s \cap C_d^{\partial}(Y,s)} |\phi_{(X,s)} \circ \chi_s^{-1} - \phi_{(Y,s)} \circ \chi_s^{-1}| \cdot J\chi_s^{-1} d\lambda_{\partial B_s}.
 \end{aligned} \tag{59}$$

From [I.2.17.v], we have

$$\begin{aligned}
 J\chi_{\tilde{s}}^{-1}(\tilde{z}) &= \{ J\chi_{\tilde{s}}(\chi_{\tilde{s}}^{-1}(\tilde{z})) \}^{-1} = \{ \hat{J}\chi(\chi_{\tilde{s}}^{-1}(\tilde{z}), \tilde{s}) \}^{-1}, \\
 &\text{for each } (\tilde{z}, \tilde{s}) \in \partial B.
 \end{aligned} \tag{60}$$

Since $\hat{J}\chi$ is continuous and positive on $\partial R \times \mathbb{R}$, while $\partial R \times K$ is compact, there exist positive numbers M_J^K and m_J^K such that

$$m_J^K \leq \hat{J}\chi(P, \zeta) \leq M_J^K, \quad \text{for each } (P, \zeta) \in \partial R \times K. \tag{61}$$

Thus, from (60) and (61),

$$\begin{aligned}
 (M_J^K)^{-1} &\leq J\chi_{\tilde{s}}^{-1}(\tilde{z}) \leq (m_J^K)^{-1}, \\
 &\text{for each } (\tilde{z}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}.
 \end{aligned} \tag{62}$$

Now, set

$$\begin{aligned} \bar{M}_\phi^K &:= \max \{ |\phi_{(\tilde{X}, \tilde{s})} \circ \chi_{\tilde{s}}^{-1}(\tilde{Z}) \cdot J\chi_{\tilde{s}}^{-1}(\tilde{Z})| \mid \tilde{s} \in K, \quad \tilde{Z} \in \partial B_{\tilde{s}}, \\ &\quad \text{dist}(\tilde{X}, \partial B_{\tilde{s}}) \leq d/2 \}, \end{aligned} \quad (63)$$

and observe that, by (iii), since $\delta < d_K < \eta_K$, we have

$$|\phi_{(X,s)} \circ \chi_s^{-1}(Z) - \phi_{(Y,s)} \circ \chi_s^{-1}(Z)| \leq \kappa_1 \delta^{\beta_1} \quad \text{for each } Z \in \partial B_s, \quad (64)$$

where $\kappa_1 > 0$ and $\beta_1 \in (0, 1]$ depend upon only ϕ , K , and M .

With (41), (62), (63), and (64), inequality (59) leads to

$$\begin{aligned} I_5 &\leq \left\{ \frac{8M}{d^3} \cdot \delta \cdot \{\bar{M}_\phi^K + M_\phi^K\} + \frac{1}{d^2} \{\bar{M}_\phi^K + M_\phi^K\} \cdot \left\{ \kappa_1 \delta^{\beta_1} + \frac{k_2}{d} \delta \right\} \right. \\ &\quad \left. + \frac{M}{d^{\frac{2}{m_j} K}} \cdot \kappa_1 \delta^{\beta_1} \right\} \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta), \end{aligned}$$

with which it is clear that I_5 can be estimated in the required manner.

I_6 : Decomposing the integral in (18) just as we did that in (17),

$$\begin{aligned} I_6 &\leq \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \left| \left\{ \frac{1}{r_X} r_{X,i} - \frac{1}{r_Y} r_{Y,i} \right\} v_{\partial B_s}^1 \right| \cdot |\Gamma_{(X,s)}^{03}| \\ &\quad \cdot |\phi_{(X,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} - \phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1}| \, d\lambda_{\partial B_s} \\ &\quad + \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y} |r_{Y,i} v_{\partial B_s}^1| \cdot |\Gamma_{(X,s)}^{03} - \Gamma_{(Y,s)}^{03}| \\ &\quad \cdot \{ |\phi_{(X,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1}| + |\phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1}| \} \, d\lambda_{\partial B_s} \end{aligned}$$

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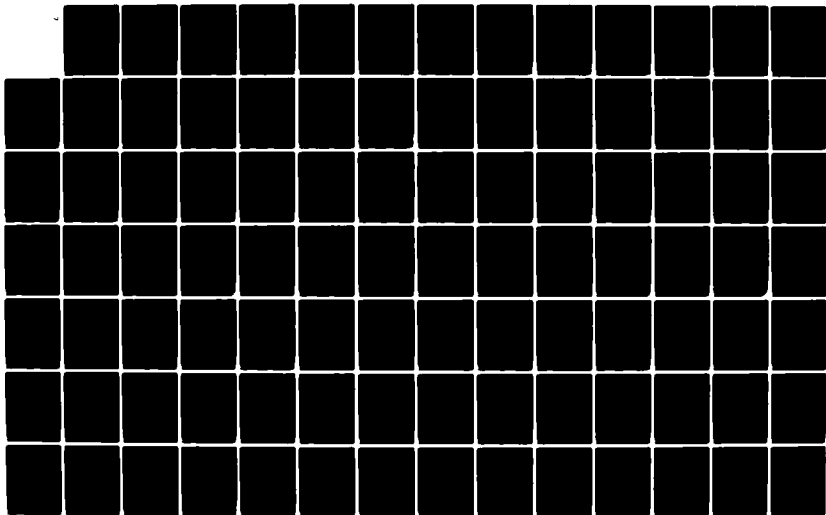
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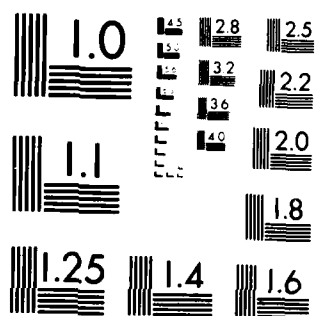
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$$\begin{aligned}
 & + \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y} |r_{Y,1}^1| \cdot |r_{(Y,s)}^{03}| \\
 & \cdot |\phi(X,s)^{\circ\chi_s^{-1}} - \phi(Y,s)^{\circ\chi_s^{-1}}| \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \\
 & \leq 8M\delta \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot |\phi(X,s)^{\circ\chi_s^{-1}} \cdot J\chi_s^{-1} \\
 & - \phi(Y,s)^{\circ\chi_s^{-1}}(Y) \cdot J\chi_s^{-1}(Y)| d\lambda_{\partial B_s} \\
 & + \hat{a}_K \{ \tilde{M}_\phi^K + M_\phi^K \} \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y} \left\{ k_1 \delta^{\tilde{\alpha}} + k_2 \frac{\delta}{r_Y} \right\} d\lambda_{\partial B_s} \\
 & + \hat{a}_K M \cdot (m_j^K)^{-1} \cdot \kappa_1 \delta^{\beta_1} \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \frac{1}{r_Y} d\lambda_{\partial B_s},
 \end{aligned} \tag{65}$$

the second inequality holding in view of (27), (I.2.37.7), (41), (62), (64), and the definitions of M , M_ϕ^K , and $\tilde{M}_\phi^K$. Now, according to hypothesis (ii), there exist $\hat{A} > 0$ and $\hat{\alpha} \in (0,1]$, depending only on K and M , such that

$$|\hat{J}\chi(P_2, \zeta) - \hat{J}\chi(P_1, \zeta)| \leq \hat{A} |P_2 - P_1|_3^{\hat{\alpha}}$$

(66)

whenever $P_1, P_2 \in \partial R$, and $\zeta \in K$.

Combining (60), (61), (66), and (45),

$$\begin{aligned}
 |J\chi_s^{-1}(Z) - J\chi_s^{-1}(Y)| & \leq (m_j^K)^{-2} |\hat{J}\chi(\chi_s^{-1}(Z), s) - \hat{J}\chi(\chi_s^{-1}(Y), s)| \\
 & \leq (m_j^K)^{-2} \hat{A} \cdot |\chi_s^{-1}(Z) - \chi_s^{-1}(Y)|_3^{\hat{\alpha}}
 \end{aligned}$$

$$\leq (m_j^K)^{-2} \hat{A} \hat{A}_K^{\hat{\alpha}} |Z-Y|_3^{\hat{\alpha}}, \quad \text{for each } Z \in \partial B_s. \quad (67)$$

Moreover, (iii) says that there exist $\kappa_2, \kappa_3 > 0$ and $\beta_2, \beta_3 \in (0, 1]$, depending only upon ϕ , K , and M , for which

$$|\phi(\hat{X}, s)^{\circ X_s^{-1}(Z)} - \phi(Y, s)^{\circ X_s^{-1}(Y)}| \leq \kappa_2 |Z-Y|_3^{\beta_2} + \kappa_3 |Z-\hat{X}|_3^{\beta_3}, \quad (68)$$

for each $Z \in \partial B_s$,

where, as usual, $\hat{X}$ denotes either X or Y . Thus, from (67) and (68)

$$\begin{aligned} & |\phi(\hat{X}, s)^{\circ X_s^{-1}(Z)} \cdot JX_s^{-1}(Z) - \phi(Y, s)^{\circ X_s^{-1}(Y)} \cdot JX_s^{-1}(Y)| \\ & \leq JX_s^{-1}(Z) \cdot |\phi(\hat{X}, s)^{\circ X_s^{-1}(Z)} - \phi(Y, s)^{\circ X_s^{-1}(Y)}| + |\phi(Y, s)^{\circ X_s^{-1}(Y)}| \\ & \quad \cdot |JX_s^{-1}(Z) - JX_s^{-1}(Y)| \\ & \leq (m_j^K)^{-1} \cdot \{\kappa_2 r_Y^{\beta_2}(Z) + \kappa_3 r_{\hat{X}}^{\beta_3}(Z)\} + M_{\phi}^K \cdot (m_j^K)^{-2} \cdot \hat{A} \hat{A}_K^{\hat{\alpha}} \cdot r_Y^{\hat{\alpha}}(Z), \end{aligned} \quad (69)$$

for each $Z \in \partial B_s$.

Since $r_X \leq \frac{3}{2} r_Y$ on $C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'$, by (26), (65) and (69) yield

$$\begin{aligned} I_6 \leq \delta \int_{C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'} & \left\{ \frac{k_6}{r_Y^{3-\beta_2}} + \frac{k_7}{r_Y^{3-\beta_3}} + \frac{k_8}{r_Y^{3-\hat{\alpha}}} + \frac{k_9}{r_Y^{3-\alpha_K}} \right\} d\lambda_{\partial B_s} \\ & + \{k_{10} \delta^{\hat{\alpha}} + k_{11} \delta^{\beta_1}\} \int_{C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s}, \end{aligned} \quad (70)$$

wherein $k_6, \dots, k_{11}$ are positive and depend only on ϕ , K , and M .

Upon referring to the computations (52), (53), and (54), we can conclude from (70) that I_6 satisfies an inequality of the desired form.

I_7 : The analysis here can be carried out using estimates already prepared: by (69), with $\hat{X} = X$,

$$I_7 \leq M \int_{C_{2\delta}^{\partial}(Y,s)} \frac{1}{r_X} \{ (m_j^K)^{-1} \{ \kappa_2 r_Y^{\beta_2} + \kappa_3 r_X^{\beta_3} \} + M_{\phi}^K \cdot (m_j^K)^{-2} \hat{A} \hat{A}_K^{\hat{\alpha}} r_Y^{\hat{\alpha}} \} d\lambda_{\partial B_s}$$

$$= \int_{C_{2\delta}^{\partial}(Y,s)} \left\{ k'_1 \frac{1}{r_X} r_Y^{\beta_2} + k'_2 \frac{1}{r_X^{2-\beta_3}} + k'_3 \frac{1}{r_X} r_Y^{\hat{\alpha}} \right\} d\lambda_{\partial B_s},$$

where the positive numbers k'_1 , k'_2 , and k'_3 depend upon only ϕ , K , and M . Recalling (55) and (56), we can write further

$$I_7 \leq \int_{C_{2\delta}^{\partial}(Y,s)} \left\{ \frac{k'_1}{\gamma_K^{2+\beta_2}} \cdot \frac{1}{(r_Y \circ \Pi_Y)^{2-\beta_2}} + \frac{k'_2}{\gamma_K^{2-\beta_3}} \cdot \frac{1}{(r_Y \circ \Pi_Y)^{2-\beta_3}} \right. \\ \left. + \frac{k'_3}{\gamma_K^{2+\hat{\alpha}}} \cdot \frac{1}{(r_Y \circ \Pi_Y)^{2-\hat{\alpha}}} \right\} d\lambda_{\partial B_s}. \quad (71)$$

Upon referring to the calculation performed in (57), for each $\alpha \in (0,1]$ we have

$$\int_{C_{2\delta}^{\partial}(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^{2-\alpha}} d\lambda_{\partial B_s} \leq \frac{2^{(3/2)+\alpha}}{\alpha} \pi \delta^{\alpha}. \quad (72)$$

In view of (72), it follows from (71) that I_7 possesses an estimate of the required form.

I₈: Since (69) is valid with $\hat{X} = Y$, it is easy to see that

$$I_8 \leq \hat{a}_K \int_{C_{2\delta}^\partial(Y,s)} \left\{ \frac{k'_1}{r_Y^{2-\beta_2-\alpha_K}} + \frac{k'_2}{r_Y^{2-\beta_3-\alpha_K}} + \frac{k'_3}{r_Y^{2-\hat{\alpha}-\alpha_K}} \right\} d\lambda_{\partial B_s},$$

from which the desired result for I_8 follows, with (72) and the inequality $r_Y^{-1}(Z) \leq r_Y^{-1}(\pi_Y(Z))$, holding for $Z \in C_{2\delta}^\partial(Y,s) \cap \{Y\}'$.

The proof of the first assertion of the theorem, i.e., that accompanying (3), is now complete. The second assertion, concerning (4), is an obvious consequence of the first.

Now, suppose that $\mu: \partial B \rightarrow K$ is locally Hölder continuous, as specified in hypothesis (iv): we wish to show that statement (iii) is true when either $\phi = \Lambda_{11}^1\{\mu\}$ or $\phi = \bar{\Lambda}_1^1\{\mu\}$. Recall that, from (IV.14.33) and (IV.14.46),

$$\begin{aligned} & \Lambda_{11}^1\{\mu\}_{(\tilde{X},s)} \circ \chi_s^{-1}(Z) \\ &= \{ \{1 - |[\chi_{,4}^c]_{(\tilde{X},s)} \circ \chi_s^{-1}(Z)|^2\} v^1(Z,s) + \{v^j(Z,s) \cdot [\chi_{,4}^j]_{(\tilde{X},s)} \circ \chi_s^{-1}(Z)\} \\ & \quad \cdot [\chi_{,4}^{1c}]_{(\tilde{X},s)} \circ \chi_s^{-1}(Z) \} \cdot \left\{ \frac{1 - |v_{(\tilde{X},s)}^c(Z)|^2}{1 - v_{(\tilde{X},s)}^k(Z) \cdot [\chi_{,4}^k]_{(\tilde{X},s)} \circ \chi_s^{-1}(Z)} \right\}^3 \\ & \quad \cdot [\mu \cdot \hat{J}\chi]_{(\tilde{X},s)} \circ \chi_s^{-1}(Z), \end{aligned} \tag{73}$$

and

$$\begin{aligned} & \tilde{\Lambda}_1^1\{\mu\}(\tilde{X}, s) \circ \chi_s^{-1}(Z) \\ &= \left\{ \frac{1 - |v_{(\tilde{X}, s)}^c(Z)|_3^2}{1 - v_{(\tilde{X}, s)}^{k^c}(Z) \cdot [\chi_{,4}^{k^c}](\tilde{X}, s) \circ \chi_s^{-1}(Z)} \right\}^3 \\ & \cdot \{v^j(Z, s) \cdot [\chi_{,4}^j](\tilde{X}, s) \circ \chi_s^{-1}(Z)\} \cdot [\hat{\mu} \cdot \hat{j}\chi](\tilde{X}, s) \circ \chi_s^{-1}(Z), \end{aligned} \quad (74)$$

for each $(Z, s) \in \partial B$, $\tilde{X} \in \mathbb{R}^3$,

where $\hat{\mu} = \mu \circ \chi^*$. We shall give the proof for $\Lambda_{1,1}^1\{\mu\}$; it will be clear that the proof for $\tilde{\Lambda}_1^1\{\mu\}$ requires no essential modification. Then, let $K \subset \mathbb{R}$ be compact; once again, it suffices to suppose that $K = [t_1, t_2]$, a compact interval. Choose $\eta > 0$, and set

$$\hat{t}_0 := \frac{1}{c-c^*} \left\{ \eta + \max_{\zeta \in K} \text{diam } B_\zeta \right\},$$

$$\hat{K} := [t_1 - \hat{t}_0, t_2].$$

By hypothesis (ii), $\chi_{,4}^c|_{\partial R \times \hat{K}}$ and $\hat{j}\chi|_{\partial R \times \hat{K}}$ are Hölder continuous. Moreover, $\chi^*|_{\partial R \times \hat{K}}$ is Lipschitz continuous (since $\chi^* \in C^1(\partial R \times \mathbb{R}; \mathbb{R}^4)$), while $\mu|_{\chi^*(\partial R \times \hat{K})}$ is Hölder continuous, so $\hat{\mu}|_{\partial R \times \hat{K}} = (\mu \circ \chi^*)|_{\partial R \times \hat{K}}$ is Hölder continuous. Thus, there exist positive κ' , κ'' , and κ''' , and β' , β'' , and $\beta''' \in (0, 1]$ such that

$$|\chi_{,4}^c(P_2, s_2) - \chi_{,4}^c(P_1, s_1)|_3 \leq \kappa' |(P_2, s_2) - (P_1, s_1)|_4^{\beta'}, \quad (75)$$

$$|\hat{j}\chi(P_2, s_2) - \hat{j}\chi(P_1, s_1)| \leq \kappa'' |(P_2, s_2) - (P_1, s_1)|_4^{\beta''}, \quad (76)$$

$$|\hat{\mu}(P_2, s_2) - \hat{\mu}(P_1, s_1)| \leq \kappa''' |(P_2, s_2) - (P_1, s_1)|_4^{\beta'''}, \quad (77)$$

whenever $P_1, P_2 \in \partial R$ and $s_1, s_2 \in \hat{K}$,

where κ' , κ'' , β' , and β'' depend only on η , K , and M , while κ''' and β''' depend only on μ , η , K , and M . Using reasoning like that of (29) and (30), it is clear that $\tau(\chi_s^{-1}(Z); \tilde{X}, s) \leq \hat{t}_0$ and so

$$s - \tau(\chi_s^{-1}(Z); \tilde{X}, s) \in \hat{K}, \quad \text{for } s \in K, \quad Z \in \partial B_s, \quad (78)$$

$$\text{and } \text{dist}(\tilde{X}, \partial B_s) < \eta.$$

Consequently, just as (34) was derived from (31), we can deduce here from (75) that

$$|v_{(X,s)}^c(Z) - v_{(Y,s)}^c(Z)|_3 \leq \kappa'_0 |X - Y|_3^{\beta'}, \quad (79)$$

$$\text{whenever } s \in K, \quad Y, Z \in \partial B_s, \quad \text{and } X \in B_\eta^3(Y).$$

Now, since $\hat{u}$ and $\hat{J}X$ are bounded on $\partial R \times \hat{K}$, $|\chi_{,4}^c|_3 \leq c^*/c$, and $|v^c|_3 \leq c^*/c$, it follows from (73), via a trivial computation, that there exist positive k'_1 , k'_2 , k'_3 , and k'_4 , dependent upon only μ , η , K , and M , such that

$$\begin{aligned} & |\Lambda_{11}^1\{u\}_{(X,s)} \circ \chi_s^{-1}(Z) - \Lambda_{11}^1\{u\}_{(Y,s)} \circ \chi_s^{-1}(Z)| \\ & \leq k'_1 \cdot |[X_{,4}^c]_{(X,s)} \circ \chi_s^{-1}(Z) - [X_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z)|_3 \\ & \quad + k'_2 |v_{(X,s)}^c(Z) - v_{(Y,s)}^c(Z)|_3 \\ & \quad + k'_3 |[\hat{J}X]_{(X,s)} \circ \chi_s^{-1}(Z) - [\hat{J}X]_{(Y,s)} \circ \chi_s^{-1}(Z)| \\ & \quad + k'_4 |[\hat{u}]_{(X,s)} \circ \chi_s^{-1}(Z) - [\hat{u}]_{(Y,s)} \circ \chi_s^{-1}(Z)|, \\ & \text{for } s \in K, \quad Y, Z \in \partial B_s, \quad \text{and } X \in B_\eta^3(Y), \end{aligned}$$

whereupon, in view of (78), we can use (75), (76), (77), and (79) to conclude that

$$\begin{aligned}
 & \left| \Lambda_{11}^1 \{u\} (X, s) \circ \chi_s^{-1}(Z) - \Lambda_{11}^1 \{u\} (Y, s) \circ \chi_s^{-1}(Z) \right| \\
 & \leq k_1' \kappa' \left| \tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Y, s) \right|^{\beta'} \\
 & \quad + k_2' \kappa_0' |X - Y|_3^{\beta'} \\
 & \quad + k_3' \kappa'' \left| \tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Y, s) \right|^{\beta''} \\
 & \quad + k_4' \kappa''' \left| \tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Y, s) \right|^{\beta'''} \\
 & \leq \left\{ \frac{k_1' \kappa'}{(c-c^*)^{\beta'}} + k_2' \kappa_0' \right\} |X - Y|_3^{\beta'} + \frac{k_3' \kappa''}{(c-c^*)^{\beta''}} |X - Y|_3^{\beta''} \\
 & \quad + \frac{k_4' \kappa'''}{(c-c^*)^{\beta'''}} |X - Y|_3^{\beta'''},
 \end{aligned} \tag{80}$$

whenever $s \in K$, $Y, Z \in \partial B_s$, and $X \in B_\eta^3(Y)$.

From the latter inequality, it is apparent that $\Lambda_{11}^1 \{u\}$ possesses the first property required in (iii).

Observe next that we can write, for any choices of $s \in K$, Y and $Z \in \partial B_s$, and $X \in B_\eta^3(Y)$,

$$\begin{aligned}
 & \left| \Lambda_{11}^1 \{u\} (X, s) \circ \chi_s^{-1}(Z) - \Lambda_{11}^1 \{u\} (Y, s) \circ \chi_s^{-1}(Y) \right| \\
 & \leq k_1'' \left| [\chi_{,4}^c] (X, s) \circ \chi_s^{-1}(Z) - [\chi_{,4}^c] (Y, s) \circ \chi_s^{-1}(Y) \right|_3 \\
 & \quad + k_2'' |v_{(X,s)}^c(Z) - v_{(Y,s)}^c(Y)|_3
 \end{aligned}$$

$$\begin{aligned}
 & +k_3'' |v(Z,s) - v(Y,s)|_3 \\
 & +k_4'' |[\bar{u}]_{(X,s)} \circ \chi_s^{-1}(Z) - [\bar{u}]_{(Y,s)} \circ \chi_s^{-1}(Y)| \\
 & +k_5'' |[\hat{J}\chi]_{(X,s)} \circ \chi_s^{-1}(Z) - [\hat{J}\chi]_{(Y,s)} \circ \chi_s^{-1}(Y)|,
 \end{aligned} \tag{81}$$

for certain positive numbers $k_1'', \dots, k_5''$, depending on only μ , η , K , and M . Since $\tau(\chi_s^{-1}(Y); Y, s) = 0$, it follows that $[\chi_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Y) = \chi_{,4}^c(\chi_s^{-1}(Y), s)$, $[\bar{u}]_{(Y,s)} \circ \chi_s^{-1}(Y) = \bar{u}(\chi_s^{-1}(Y), s)$, and $[\hat{J}\chi]_{(Y,s)} \circ \chi_s^{-1}(Y) = \hat{J}\chi(\chi_s^{-1}(Y), s)$. Supposing that $X \neq Z$, (IV.14.11) shows that

$$v_{(X,s)}^{i^c}(Z) = \chi_{,4}^{i^c}(\chi_s^{-1}(Z), t_0^1), \tag{82}$$

for some t_0^1 depending on s , X , and Z , and lying in the interval $(s - \tau(\chi_s^{-1}(Z); X, s), s)$, so that $|s - t_0^1| < \tau(\chi_s^{-1}(Z); X, s)$. If we set $t_0^1 := s$ for $X = Z$, it is clear that (82) remains valid. Of course, since $\{B_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov,

$$|v(Z,s) - v(Y,s)|_3 \leq a_K |Z - Y|_3^{\alpha_K}, \quad \text{for } s \in K, \quad Y, Z \in \partial B_s,$$

where $a_K > 0$ and $\alpha_K \in (0, 1]$. Also, the function $(Z, s) \mapsto \chi_s^{-1}(Z)$ is Lipschitz continuous on $X^*(\partial R \times K)$, so that inequality (45) is valid. Upon combining all of these facts with (75)-(78), (81) leads to the estimate

$$\begin{aligned}
 & \left| \Lambda_{11}^1 \{ \mu \} (X, s) \circ \chi_s^{-1}(Z) - \Lambda_{11}^1 \{ \mu \} (Y, s) \circ \chi_s^{-1}(Y) \right| \\
 & \leq k_1'' \left| (\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); X, s)) - (\chi_s^{-1}(Y), s) \right|_4^{\beta'} \\
 & \quad + k_2'' \left\{ \sum_{i=1}^3 \left| x_{,4}^i (\chi_s^{-1}(Z), t_0^i) - x_{,4}^i (\chi_s^{-1}(Y), s) \right|^2 \right\}^{1/2} \\
 & \quad + k_3'' a_K |Z - Y|_3^{\alpha_K} \\
 & \quad + k_4'' \left| (\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); X, s)) - (\chi_s^{-1}(Y), s) \right|_4^{\beta''} \\
 & \quad + k_5'' \left| (\chi_s^{-1}(Z), s - \tau(\chi_s^{-1}(Z); X, s)) - (\chi_s^{-1}(Y), s) \right|_4^{\beta'''} \\
 & \leq 2^{\beta'/2} k_1'' \left\{ \left| \chi_s^{-1}(Y) - \chi_s^{-1}(Z) \right|_3^{\beta'} + \tau(\chi_s^{-1}(Z); X, s)^{\beta'} \right\} \\
 & \quad + 3^{1/2} k_2'' \left\{ \left| \chi_s^{-1}(Y) - \chi_s^{-1}(Z) \right|_3^{\beta'} + \tau(\chi_s^{-1}(Z); X, s)^{\beta'} \right\} \\
 & \quad + k_3'' a_K |Z - Y|_3^{\alpha_K} \\
 & \quad + 2^{\beta''/2} k_4'' \left\{ \left| \chi_s^{-1}(Y) - \chi_s^{-1}(Z) \right|_3^{\beta''} + \tau(\chi_s^{-1}(Z); X, s)^{\beta''} \right\} \\
 & \quad + 2^{\beta'''/2} k_5'' \left\{ \left| \chi_s^{-1}(Y) - \chi_s^{-1}(Z) \right|_3^{\beta'''} + \tau(\chi_s^{-1}(Z); X, s)^{\beta'''} \right\},
 \end{aligned} \tag{83}$$

whenever $s \in K$, $Y, Z \in \partial B_s$, and $X \in B_n^3(Y)$.

Finally, using (45) and the inequality $\tau(\chi_s^{-1}(Z); X, s) \leq \frac{1}{c-c^*} |Z - X|_3$, it follows from (83) that $\Lambda_{11}^1 \{ \mu \}$ satisfies the second estimate required in (iii).

To verify (5) and (6), let $(Y, s) \in \partial B$, i.e., let $s \in \mathbb{R}$ and $Y \in \partial B_s$: then $\tau(\chi_s^{-1}(Y); Y, s) = 0$, so $[x_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Y) = x_{,4}^c(\chi_s^{-1}(Y), s)$, but also $v_{(Y,s)}^c(Y) = x_{,4}^c(\chi_s^{-1}(Y), s)$. Recalling the definition of the normal velocity v , (73) and (74) then give,

respectively,

$$\begin{aligned}
 & \Lambda_{11}^1\{\mu\}_{(Y,s)} \circ x_s^{-1}(Y) \cdot Jx_s^{-1}(Y) \\
 &= \{ \{1 - |x_{,4}^c(x_s^{-1}(Y), s)|^2\} v^1(Y, s) + \{v^j(Y, s) \cdot x_{,4}^{jc}(x_s^{-1}(Y), s)\} \\
 & \quad \cdot x_{,4}^{ic}(x_s^{-1}(Y), s)\} \cdot \mu(x_s^{-1}(Y), s) \cdot \hat{J}x(x_s^{-1}(Y), s) \cdot Jx_s^{-1}(Y) \\
 &= \{ \{1 - |x_{,4}^c(x_s^{-1}(Y), s)|^2\} v^1(Y, s) + v^c(Y, s) \cdot x_{,4}^{ic}(x_s^{-1}(Y), s)\} \\
 & \quad \cdot \mu(x^*(x_s^{-1}(Y), s)) \cdot Jx_s(x_s^{-1}(Y)) \cdot Jx_s^{-1}(Y),
 \end{aligned}$$

and

$$\begin{aligned}
 & \tilde{\Lambda}_1^1\{\mu\}_{(Y,s)} \circ x_s^{-1}(Y) \cdot Jx_s^{-1}(Y) \\
 &= \{v^j(Y, s) \cdot x_{,4}^{jc}(x_s^{-1}(Y), s)\} \cdot \mu(x_s^{-1}(Y), s) \cdot \hat{J}x(x_s^{-1}(Y), s) \cdot Jx_s^{-1}(Y) \\
 &= v^c(Y, s) \cdot \mu(x^*(x_s^{-1}(Y), s)) \cdot Jx_s(x_s^{-1}(Y)) \cdot Jx_s^{-1}(Y).
 \end{aligned}$$

To obtain (5) and (6) from these equalities, we have but to point out that $x^{*-1}(Y, s) = (x_s^{-1}(Y), s)$, so $x^*(x_s^{-1}(Y), s) = (Y, s)$, and that $Jx_s(x_s^{-1}(Y)) \cdot Jx_s^{-1}(Y) = 1$ (cf., (60)).

Finally, suppose that $M \in M(2)$: then $v \in C^1(\partial B; \mathbb{R}^3)$, hence is Lipschitz continuous on any compact subset of ∂B . In particular, whenever $\bar{K} \subset \mathbb{R}$ is compact, then $v|_{\cup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}}$ is Lipschitz continuous, so certainly the collection $\{\delta_\zeta^0\}_{\zeta \in \bar{K}}$ of 2-regular domains is uniformly Lyapunov (for which we need only know that $v(\cdot, \zeta)$ is Hölder continuous on ∂B_ζ , uniformly for $\zeta \in \bar{K}$). Thus, hypothesis (i) is satisfied. To see that (ii) holds, let (R, x) be

a reference pair for M with the properties listed in [I.3.25] for $q = 2$. Then $x \in C^2(\partial R \times \mathbb{R}; \mathbb{R}^3)$, so $\dot{x}_4 \in C^1(\partial R \times \mathbb{R}; \mathbb{R}^3)$ and is therefore Lipschitz continuous on $\partial R \times \bar{K}$ for any compact $\bar{K}$ in $\mathbb{R}$. Similarly, we shall be able to conclude that $\hat{J}x|_{\partial R \times \bar{K}}$ is Lipschitz continuous for each such $\bar{K}$ (and hence that (ii) holds) by showing that $\hat{J}x \in C^1(\partial R \times \mathbb{R})$; we have already noted the validity of this inclusion, in [I.3.26.d]. $\square$.

[IV.23] R E M A R K S. (a) Let us emphasize the implications of the inclusion $M \in \mathcal{M}(2)$ which were established in the closing paragraph of the proof of [IV.22]: supposing this condition to be fulfilled, we then have $v \in C^1(\partial B; \mathbb{R}^3)$, so that, for any compact $K \subset \mathbb{R}$, $v|_{\bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}}$ is Lipschitz continuous; in particular, it is obvious that $\{B_\zeta^0\}_{\zeta \in K}$ is a uniformly Lyapunov family. Moreover, letting (R, x) denote a reference pair for M as in [I.3.25] with $q = 2$, $x_4|_{\partial R \times K}$ and $\hat{J}x|_{\partial R \times K}$ are Lipschitz continuous whenever K is compact in $\mathbb{R}$. These facts will be used on a number of occasions in the sequel.

(b) An inspection of the proof of [IV.22] reveals that certain of the conclusions drawn there actually remain valid under hypotheses which are somewhat less stringent. For example, the first assertion of [IV.22] is true if it is known only that, for each compact $K \subset \mathbb{R}$, $\hat{J}x(\cdot, \zeta)$ is Hölder continuous on ∂R , uniformly for each $\zeta \in K$ (cf., (IV.22.66)); we have required the local Hölder continuity of $\hat{J}x$ on $\partial R \times \mathbb{R}$ in [IV.22.ii] as a convenient and simple hypothesis serving to provide for the proofs of all conclusions of [IV.22]. Remarks of

a similar nature apply to various upcoming statements.

(c) The locally uniform existence of the limit (IV.22.4) can be established if only $\phi \in C(\partial R \times \mathbb{R}^4)$, by a simple modification of the proof of [IV.22]. The estimate (IV.22.3) need not hold under this weakened hypothesis.

If a complete analysis of the scattering problem is to be carried out by means of the program being set up here, it is essential to have available a collection of results concerning the integral operators $\mu \mapsto \omega_1^* \{\Lambda_{11}^1 \{\mu\}\}$, $\mu \mapsto \omega_1^* \{\tilde{\Lambda}_1^1 \{\mu\}\}$, and corresponding ones constructed from $\omega_{2i} \{\cdot\}$ and $\omega_{3g} \{\cdot\}$, acting in spaces of functions on ∂B . As a first step in this direction, we offer the following result.

[IV.24] T H E O R E M. Let M be a motion in $M(1)$. Suppose further that

- (i) $\{B_\zeta^0\}_{\zeta \in \mathbb{R}}$ is "strongly locally uniformly Lyapunov": whenever $\tilde{K} \subset \mathbb{R}$ is compact, then $v|_{\cup_{\zeta \in \tilde{K}} \{\partial B_\zeta \times \{\zeta\}\}}$ is Hölder continuous, so that there exist $a_{\tilde{K}} > 0$ and $\alpha_{\tilde{K}} \in (0,1]$, depending on M and (perhaps) on $\tilde{K}$, for which

$$|v(Y_2, s_2) - v(Y_1, s_1)|_3 \leq a_{\tilde{K}} |(Y_2, s_2) - (Y_1, s_1)|_4^{\alpha_{\tilde{K}}} \quad (1)$$

whenever $s_1, s_2 \in \tilde{K}$, $Y_1 \in \partial B_{s_1}$, and $Y_2 \in \partial B_{s_2}$;

- (ii) there exists a reference pair (R, x) for M which

possesses the properties of [I.3.25] and is also such that $x_{,4}$ is locally Hölder continuous on $\partial R \times \mathbb{R}$, i.e., for each compact $\bar{K} \subset \mathbb{R}$, $x_{,4}|_{\partial R \times \bar{K}}$ is Hölder continuous;

(iii) $(P, \bar{Y}, \bar{s}) \mapsto \phi_{(\bar{Y}, \bar{s})}(P)$ is a function in $C(\partial R \times \partial B)$ which is also locally Hölder continuous on ∂B , uniformly in its ∂R -argument, i.e., whenever $\bar{K} \subset \mathbb{R}$ is compact, there exist $\tilde{\kappa} > 0$ and $\tilde{\beta} \in (0, 1]$, depending on ϕ and (perhaps) $\bar{K}$, for which

$$|\phi_{(Y_2, s_2)}(P) - \phi_{(Y_1, s_1)}(P)| \leq \tilde{\kappa} |(Y_2, s_2) - (Y_1, s_1)|_4^{\tilde{\beta}} \quad (2)$$

for each $P \in \partial R$, $s_1, s_2 \in \bar{K}$, $Y_1 \in \partial B_{s_1}$, and $Y_2 \in \partial B_{s_2}$.

Then the function $w_1^*\{\phi\}: \partial B \rightarrow \mathbb{R}$, constructed from ϕ and x as in [IV.20], is locally Hölder continuous on ∂B : whenever $K \subset \mathbb{R}$ is compact, there exist $\varepsilon_K > 0$ and $\lambda_K \in (0, 1]$, depending on ϕ , M , and (perhaps) K , such that

$$|w_1^*\{\phi\}(Y_2, s_2) - w_1^*\{\phi\}(Y_1, s_1)| \leq \varepsilon_K |(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K} \quad (3)$$

whenever $s_1, s_2 \in K$, $Y_1 \in \partial B_{s_1}$, and $Y_2 \in \partial B_{s_2}$.

If it is assumed, moreover, that

(iv) whenever $\bar{K}$ is compact in $\mathbb{R}$, then $\hat{J}x(P, \cdot)|_{\bar{K}}$ is Hölder continuous, uniformly in $P \in \partial R$,

and

(v) $\mu: \partial B \rightarrow K$ is locally Hölder continuous (whenever $\bar{K}$ is a compact subset of $\mathbb{R}$, then $\mu|_{\cup_{\zeta \in \bar{K}} \{\partial B_{\zeta} \times \{\zeta\}\}}$ is Hölder continuous),

then hypothesis (iii) is fulfilled by taking either $\phi = \Lambda_{11}^{-1}\{\mu\}$ or $\phi = \tilde{\Lambda}_1^{-1}\{\mu\}$ (cf., (IV.14.33) and (IV.14.46)), with the numbers $\tilde{\epsilon}$ and $\tilde{\delta}$ depending on μ , M , and (perhaps) $\bar{K}$, whence $\omega_1^*\{\Lambda_{11}^{-1}\{\mu\}\}$ and $\omega_1^*\{\tilde{\Lambda}_1^{-1}\{\mu\}\}$ are locally Hölder continuous on ∂B .

Finally, if $M \in \mathbf{M}(2)$, then hypotheses (i), (ii), and (iv) are fulfilled.

P R O O F. Observe that $\omega_1^*\{\phi\}$ is indeed defined on ∂B , since B_{ζ}^0 is a Lyapunov domain for each $\zeta \in \mathbb{R}$, by (i). Choose a compact interval $K = [t_1, t_2]$; it suffices to prove the first conclusion of the theorem for such a compact set. According to hypothesis (i), $\nu|_{\cup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}}$ is Hölder continuous, with coefficient a_K and exponent α_K ; in particular, $\{B_{\zeta}^0\}_{\zeta \in K}$ is a uniformly Lyapunov family, for which a set of Lyapunov constants is (a_K, α_K, d_K) , wherein $d_K > 0$ and $a_K d_K^{\alpha_K} < 1/2$.

Now, to verify that the first claim of the theorem is correct, it is enough to prove the existence of $\varrho'_K > 0$ and $\lambda_K \in (0, 1]$, depending on only ϕ , M , and K , such that

$$|\omega_1^*\{\phi\}(Y_2, s_2) - \omega_1^*\{\phi\}(Y_1, s_1)| \leq \varrho'_K |(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K}$$

whenever $s_1, s_2 \in K$, $Y_1 \in \partial B_{s_1}$, $Y_2 \in \partial B_{s_2}$, (4)

and $|(Y_2, s_2) - (Y_1, s_1)|_4 < \Delta$,

for some positive number Δ . For, once this has been established, the continuity of $\omega_1^*\{\phi\}$ on the compact set $\bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ follows, with which

$$\begin{aligned} & |\omega_1^*\{\phi\}(Y_2, s_2) - \omega_1^*\{\phi\}(Y_1, s_1)| \\ &= \frac{|\omega_1^*\{\phi\}(Y_2, s_2) - \omega_1^*\{\phi\}(Y_1, s_1)|}{|(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K}} \cdot |(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K} \\ &\leq \frac{2M_K^*}{\Delta^{\lambda_K}} \cdot |(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K} \end{aligned}$$

whenever $s_1, s_2 \in K$, $Y_1 \in \partial B_{s_1}$, $Y_2 \in \partial B_{s_2}$,

and $|(Y_2, s_2) - (Y_1, s_1)|_4 \geq \Delta$,

having written

$$M_K^* := \max \{|\omega_1^*\{\phi\}(\tilde{Y}, \tilde{s})| \mid \tilde{s} \in K, \tilde{Y} \in \partial B_{\tilde{s}}\};$$

then, (3) results, with $\ell_K := \max \{\ell_K^*, 2M_K^*/\Delta^{\lambda_K}\}$. Our aim is to secure an inequality of the form of (4), with

$$\Delta = d_K/4\iota,$$

where

$$\iota := \{1 + (c^*)^2\}^{1/2}. \quad (5)$$

To begin, fix any $d \in \left[\frac{1}{2} d_K, \frac{7}{12} d_K\right]$. Throughout, (Y, s) and $(\hat{Y}, \hat{s})$ denote points of $\bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ (so $s, \hat{s} \in K$, $Y \in \partial B_s$, and $\hat{Y} \in \partial B_{\hat{s}}$); we write

$$\delta := |(\hat{Y}, \hat{s}) - (Y, s)|_4,$$

and suppose always that

$$0 < 4\delta < d_K. \quad (6)$$

Thus,

$$0 < 2\delta < \frac{1}{2} d_K < d < \frac{7}{12} d_K < \frac{7}{9} d_K,$$

implying that $C_{2\delta}^{\partial}(Y, s)$ and $C_d^{\partial}(Y, s)$ are defined, the former lying within the latter.

We shall employ the 1-embedding $\chi_{s\hat{s}} := \chi_{\hat{s}} \circ \chi_s^{-1}: \partial B_s \rightarrow \mathbb{R}^3$, carrying ∂B_s onto $\partial B_{\hat{s}}$. Obviously,

$$\chi_{\hat{s}}^{-1} \circ \chi_{s\hat{s}} = \chi_s^{-1}, \quad (7)$$

and (cf., [I.2.17.v] and [I.2.19.iii])

$$\begin{aligned} (J\chi_{\hat{s}}^{-1}) \circ \chi_{s\hat{s}} \cdot J\chi_{s\hat{s}} &= \{(J\chi_{\hat{s}}) \circ \chi_{\hat{s}}^{-1}\}^{-1} \circ \chi_{s\hat{s}} \cdot \{(J\chi_{\hat{s}}) \circ \chi_{\hat{s}}^{-1}\} \cdot J\chi_s^{-1} \\ &= \{(J\chi_{\hat{s}}) \circ \chi_{\hat{s}}^{-1}\}^{-1} \cdot \{(J\chi_{\hat{s}}) \circ \chi_{\hat{s}}^{-1}\} \cdot J\chi_s^{-1} = J\chi_s^{-1}. \end{aligned} \quad (8)$$

Recalling the transformation formula of [I.2.25], we can write

$\omega_1^*(\phi)(\hat{Y}, \hat{s})$ as an integral over ∂B_s , then appeal to (7) and (8), producing

$$\omega_1^*(\phi)(\hat{Y}, \hat{s}) = \frac{1}{4\pi} \int_{\partial B_{\hat{s}}} \frac{1}{2} \cdot r_{\hat{Y}}^k \cdot r_{\hat{Y}}^{\partial B_{\hat{s}}} \cdot r_{\hat{Y}}^{03}(\hat{Y}, \hat{s}) \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_{\hat{s}}}$$

$$\begin{aligned}
 &= \frac{1}{4\pi} \int_{\partial B_s} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot \phi(\hat{Y},\hat{s})^{\circ\chi_s^{-1}} \cdot J\chi_s^{-1} \right\}^{\circ\chi_{s\hat{s}}} \\
 &\quad \cdot J\chi_{s\hat{s}} \, d\lambda_{\partial B_s} \\
 &= \frac{1}{4\pi} \int_{\partial B_s} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\}^{\circ\chi_{s\hat{s}}} \cdot \phi(\hat{Y},\hat{s})^{\circ\chi_s^{-1}} \\
 &\quad \cdot J\chi_s^{-1} \, d\lambda_{\partial B_s}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &|w_1^*(\phi)(\hat{Y},\hat{s}) - w_1^*(\phi)(Y,s)| \\
 &= \frac{1}{4\pi} \left| \int_{\partial B_s} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\}^{\circ\chi_{s\hat{s}}} \cdot \phi(\hat{Y},\hat{s})^{\circ\chi_s^{-1}} \cdot J\chi_s^{-1} \, d\lambda_{\partial B_s} \right. \\
 &\quad \left. - \int_{\partial B_s} \frac{1}{r_Y^2} \cdot r_{Y,k}^{\nu k} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi(Y,s)^{\circ\chi_s^{-1}} \cdot J\chi_s^{-1} \, d\lambda_{\partial B_s} \right| \\
 &\leq \frac{1}{4\pi} \left| \int_{\partial B_s \cap C_d^3(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\}^{\circ\chi_{s\hat{s}}} \cdot \phi(\hat{Y},\hat{s})^{\circ\chi_s^{-1}} \right. \\
 &\quad \left. - \frac{1}{r_Y^2} \cdot r_{Y,k}^{\nu k} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi(Y,s)^{\circ\chi_s^{-1}} \right\} J\chi_s^{-1} \, d\lambda_{\partial B_s} \Big| \\
 &\quad + \frac{1}{4\pi} \left| \int_{C_d^3(Y,s) \cap C_{2,\delta}^3(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\}^{\circ\chi_{s\hat{s}}} \cdot \phi(\hat{Y},\hat{s})^{\circ\chi_s^{-1}} \right. \\
 &\quad \left. - \frac{1}{r_Y^2} \cdot r_{Y,k}^{\nu k} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi(Y,s)^{\circ\chi_s^{-1}} \right\} J\chi_s^{-1} \, d\lambda_{\partial B_s} \Big|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4\pi} \left| \int_{C_{2,1\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{\hat{S}}^{03}(\hat{Y},\hat{s}) \right\} \circ \chi_{s\hat{s}} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial\hat{B}_s} \right| \\
 & + \frac{1}{4\pi} \left| \int_{C_{2,1\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,k}^{\nu k} \cdot \Gamma_s^{03}(Y,s) \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \\
 & \leq \frac{1}{4\pi} \sum_{j=1}^8 I_j(Y,s;\hat{Y},\hat{s}), \tag{9}
 \end{aligned}$$

in which the $I_j(Y,s;\hat{Y},\hat{s}) := I_j$, $j = 1, \dots, 8$, are given by

$$\begin{aligned}
 I_1 := & \left| \int_{\partial\hat{B}_s \cap C_d^\partial(Y,s)} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{\hat{S}}^{03}(\hat{Y},\hat{s}) \{ \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1} \} \right. \\
 & \left. \cdot J\chi_s^{-1} d\lambda_{\partial\hat{B}_s} \right|, \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 I_2 := & \left| \int_{\partial\hat{B}_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,k}^{\nu k} \{ \Gamma_{\hat{S}}^{03}(\hat{Y},\hat{s}) \circ \chi_{s\hat{s}} - \Gamma_s^{03}(Y,s) \} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \right. \\
 & \left. \cdot J\chi_s^{-1} d\lambda_{\partial\hat{B}_s} \right|, \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 I_3 := & \left| \int_{\partial\hat{B}_s \cap C_d^\partial(Y,s)} \left\{ \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},k}^{\nu k} \right\} \circ \chi_{s\hat{s}} - \frac{1}{r_Y} \cdot r_{Y,k}^{\nu k} \right\} \right. \\
 & \left. \cdot \Gamma_{\hat{S}}^{03}(\hat{Y},\hat{s}) \circ \chi_{s\hat{s}} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial\hat{B}_s} \right|, \tag{12}
 \end{aligned}$$

$$\begin{aligned}
 I_4 := & \left| \int_{C_d^\partial(Y,s) \cap C_{2,1\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,k}^{\nu k} \cdot \Gamma_s^{03}(Y,s) \cdot \{ \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \right. \\
 & \left. - \phi(Y,s) \circ \chi_s^{-1} \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \tag{13}
 \end{aligned}$$

$$I_5 := \left| \int_{C_d^\partial(Y,s) \cap C_{2,\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,k}^v \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot \phi_{ss}^{-1} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi_{(\hat{Y},\hat{s})} \cdot \phi_{ss}^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (14)$$

$$I_6 := \left| \int_{C_d^\partial(Y,s) \cap C_{2,\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},k}^v \cdot \Gamma_{\partial B_{\hat{s}}}^{03} \right\} \phi_{ss}^{-1} \cdot \frac{1}{r_Y} \cdot r_{Y,k}^v \cdot \Gamma_{\partial B_s}^{03} \cdot \phi_{(\hat{Y},\hat{s})} \cdot \phi_{ss}^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (15)$$

$$I_7 := \left| \int_{C_{2,\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},k}^v \cdot \Gamma_{\partial B_{\hat{s}}}^{03} \right\} \phi_{ss}^{-1} \cdot \phi_{(\hat{Y},\hat{s})} \cdot \phi_{ss}^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (16)$$

and

$$I_8 := \left| \int_{C_{2,\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,k}^v \cdot \Gamma_{\partial B_s}^{03} \cdot \phi_{(Y,s)} \cdot \phi_{ss}^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|. \quad (17)$$

Now, in view of the remarks made previously, in order to prove the first assertion of the theorem, we need only show that there exist positive numbers $\ell(j)$ and numbers $\lambda(j) \in (0,1]$, for $j = 1, \dots, 8$, depending upon only ϕ , M , and K , such that

$$I_j(Y,s;\hat{Y},\hat{s}) \leq \ell(j) \cdot |(\hat{Y},\hat{s}) - (Y,s)|_4^{\lambda(j)}$$

$$\text{whenever } s, \hat{s} \in K, \quad Y \in \partial B_s, \quad \text{and } \hat{Y} \in \partial B_{\hat{s}}, \quad (18)$$

$$\text{with } |(\hat{Y},\hat{s}) - (Y,s)|_4 < \frac{d_K}{4}.$$

For, once this has been accomplished, an inequality of the form of (4) will follow directly. We proceed to the estimation of the expressions given in (10)-(17).

I₁: By hypothesis (iii), we know that there exist $\tilde{\kappa} > 0$ and $\beta \in (0,1]$, depending upon only ϕ and K , for which

$$|\phi_{(\hat{Y}, \hat{s})}(P) - \phi_{(Y, s)}(P)| \leq \tilde{\kappa} \delta^\beta, \quad \text{for each } P \in \partial R. \quad (19)$$

For the continuous function $(Z, \tilde{s}) \mapsto J\chi_{\tilde{s}}^{-1}(Z)$ on ∂B , set

$$M_j^K := \max \{J\chi_{\tilde{s}}^{-1}(Z) \mid \tilde{s} \in K, Z \in \partial B_{\tilde{s}}\}.$$

Write

$$M^{03} := \left\{1 + \left(\frac{c^*}{c}\right)^2\right\}^3 \cdot \left\{1 - \frac{c^*}{c}\right\}^{-6};$$

by (IV.14.59), M^{03} is an upper bound for the positive function τ^{03} .

Since we clearly have $r_Y(Z) \geq d$ if $Z \in \partial B_s \cap C_d^{\partial}(Y, s)'$, and

$$|r_{Y, k}(Z) \vee_{\partial B_s}^k(Z)| \leq 1 \quad \text{if } Z \in \partial B_s \cap \{Y\}', \quad (20)$$

(10) leads, with (19), to the inequality

$$\begin{aligned} I_1 &\leq \frac{1}{d^2} \cdot M^{03} M_j^K \cdot \tilde{\kappa} \delta^\beta \int_{\partial B_s \cap C_d^{\partial}(Y, s)'} d\lambda_{\partial B_s} \\ &\leq \left\{ \frac{1}{d^2} \cdot M^{03} \cdot M_j^K \cdot \tilde{\kappa} \cdot \max_{\tilde{s} \in K} \lambda_{\partial B_{\tilde{s}}}(\partial B_{\tilde{s}}) \right\} \delta^\beta, \end{aligned} \quad (21)$$

a relation of the form of (18), for $j = 1$.

I₂: Here, we shall first develop a Hölder-type estimate for

$|\Gamma_{(\hat{Y}, \hat{s})}^{03} \circ x_{\hat{s}\hat{s}}(Z) - \Gamma_{(Y, s)}^{03}(Z)|$, for $Z \in \partial B_s \cap \{Y\}' \cap \{x_{\hat{s}\hat{s}}(\hat{Y})\}'$, which is also to be used in the examination of I_5 ; note that if $Z \in \partial B_s$ but $Z \neq x_{\hat{s}\hat{s}}(\hat{Y})$, then $x_{\hat{s}\hat{s}}(Z) \neq \hat{Y}$ (since $x_{\hat{s}\hat{s}}^{-1} = x_{\hat{s}\hat{s}}$), so $\Gamma_{(\hat{Y}, \hat{s})}^{03} \circ x_{\hat{s}\hat{s}}(Z)$ is defined. By retracing the steps in the derivation of (IV.22.40), *mutatis mutandis*, it is routine to check that

$$\begin{aligned} & |\Gamma_{(\hat{Y}, \hat{s})}^{03}(x_{\hat{s}\hat{s}}(Z)) - \Gamma_{(Y, s)}^{03}(Z)| \\ & \leq c_3^* | [x_{,4}^c]_{(\hat{Y}, \hat{s})} \circ x_{\hat{s}}^{-1}(x_{\hat{s}\hat{s}}(Z)) - [x_{,4}^c]_{(Y, s)} \circ x_s^{-1}(Z) |_3 \\ & \quad + c_4^* | v_{(\hat{Y}, \hat{s})}^c(x_{\hat{s}\hat{s}}(Z)) - v_{(Y, s)}^c(Z) |_3 \\ & \quad + c_5^* | \text{grad } r_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)) - \text{grad } r_Y(Z) |_3, \end{aligned} \tag{22}$$

for each $Z \in \partial B_s \cap \{Y\}' \cap \{x_{\hat{s}\hat{s}}(\hat{Y})\}'$,

c_3^* , c_4^* , and c_5^* depending on only the ratio c^*/c , i.e., on M . We now investigate each of the differences appearing on the right in (22).

Define

$$t_0 := \frac{1}{c-c^*} \left\{ \frac{d_K}{4} + \max_{\hat{s} \in K} \text{diam } B_{\hat{s}} \right\},$$

and

$$\hat{K} := [t_1 - t_0, t_2].$$

Now, by (ii), $x_{,4}^c|_{\partial R \times \hat{K}}$ is Hölder continuous: there exist $\hat{A} > 0$ and $\hat{\alpha} \in (0, 1]$, depending on only M and K , with

$$|x_{,4}^c(P_2, s_2) - x_{,4}^c(P_1, s_1)|_3 \leq \hat{A} |(P_2, s_2) - (P_1, s_1)|_4^{\hat{\alpha}} \quad (23)$$

$$\text{for } s_1, s_2 \in \hat{K} \quad \text{and} \quad P_1, P_2 \in \partial R.$$

Whenever $Z \in \partial B_s$, [I.3.16.iv] allows us to write

$$\begin{aligned} \tau(x_s^{-1}(Z); \hat{Y}, \hat{s}) &= |\tau(x_s^{-1}(Z); \hat{Y}, \hat{s}) - \tau(x_s^{-1}(Z); Z, s)| \\ &\leq \frac{1}{c-c^*} \{ |Z - \hat{Y}|_3 + c^* |s - \hat{s}| \} \\ &\leq \frac{1}{c-c^*} \{ |Z - Y|_3 + |Y - \hat{Y}|_3 + c^* |s - \hat{s}| \} \\ &\leq \frac{1}{c-c^*} \{ \text{diam } B_s + \delta \} \\ &\leq \frac{1}{c-c^*} \left\{ \max_{\hat{s} \in K} \text{diam } B_{\hat{s}} + \frac{1}{4} d_K \right\} = t_0, \end{aligned} \quad (24)$$

and, similarly,

$$\tau(x_s^{-1}(Z); Y, s) \leq \frac{1}{c-c^*} |Z - Y|_3 < t_0, \quad (25)$$

so that $\hat{s} - \tau(x_s^{-1}(Z); \hat{Y}, \hat{s})$ and $s - \tau(x_s^{-1}(Z); Y, s)$ lie in $\hat{K}$. This fact, along with (23), implies that

$$\begin{aligned} &| [x_{,4}^c]_{(\hat{Y}, \hat{s})} \circ x_{\hat{s}}^{-1}(x_{\hat{s}}^{-1}(Z)) - [x_{,4}^c]_{(Y, s)} \circ x_s^{-1}(Z) |_3 \\ &= | [x_{,4}^c]_{(\hat{Y}, \hat{s})} \circ x_s^{-1}(Z) - [x_{,4}^c]_{(Y, s)} \circ x_s^{-1}(Z) |_3 \\ &= | x_{,4}^c(x_s^{-1}(Z), \hat{s} - \tau(x_s^{-1}(Z); \hat{Y}, \hat{s})) - x_{,4}^c(x_s^{-1}(Z), s - \tau(x_s^{-1}(Z); Y, s)) |_3 \\ &\leq \hat{A} \cdot |(\hat{s} - s) + (\tau(x_s^{-1}(Z); Y, s) - \tau(x_s^{-1}(Z); \hat{Y}, \hat{s}))|_4^{\hat{\alpha}} \\ &\leq \hat{A} \{ |\hat{s} - s| + \frac{1}{c-c^*} \{ |\hat{Y} - Y|_3 + c^* |\hat{s} - s| \} \}^{\hat{\alpha}} \end{aligned}$$

$$= \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \{ |\hat{Y}-Y|_3 + c |\hat{s}-s| \}^{\hat{\alpha}} \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \{ 1+c^2 \}^{\hat{\alpha}/2} \cdot \hat{\varepsilon}^{\hat{\alpha}} \quad (26)$$

for each $Z \in \partial B_s$.

Suppose that $Z \in \partial B_s \cap \{Y\}' \cap \{X_{\hat{s}s}(\hat{Y})\}'$: (IV.14.11) gives

$$\begin{aligned} v_{(\hat{Y}, \hat{s})}^c(X_{\hat{s}s}(Z)) &= \frac{1}{\tau(X_s^{-1}(X_{\hat{s}s}(Z)); \hat{Y}, \hat{s})} \{X_{\hat{s}s}^c(Z) - [X^c]_{(\hat{Y}, \hat{s})} \circ X_{\hat{s}}^{-1}(X_{\hat{s}s}(Z))\} \\ &= \frac{1}{\tau(X_s^{-1}(Z); \hat{Y}, \hat{s})} \{X^c(X_s^{-1}(Z), \hat{s}) - [X^c]_{(\hat{Y}, \hat{s})} \circ X_s^{-1}(Z)\}, \end{aligned} \quad (27)$$

and

$$v_{(Y, s)}^c(Z) = \frac{1}{\tau(X_s^{-1}(Z); Y, s)} \{X^c(X_s^{-1}(Z), s) - [X^c]_{(Y, s)} \circ X_s^{-1}(Z)\}. \quad (28)$$

For brevity, let us temporarily write $Z_s := X_s^{-1}(Z)$, $\tau_{(Y, s)} := \tau(X_s^{-1}(Z); Y, s)$, and $\tau_{(\hat{Y}, \hat{s})} := \tau(X_s^{-1}(Z); \hat{Y}, \hat{s})$. Then

$$\begin{aligned} &|v_{(\hat{Y}, \hat{s})}^c(X_{\hat{s}s}(Z)) - v_{(Y, s)}^c(Z)|_3 \\ &= \left| \frac{1}{\tau_{(\hat{Y}, \hat{s})}} \{X^c(Z_s, \hat{s}) - X^c(Z_s, \hat{s} - \tau_{(\hat{Y}, \hat{s})})\} \right. \\ &\quad \left. - \frac{1}{\tau_{(Y, s)}} \{X^c(Z_s, s) - X^c(Z_s, s - \tau_{(Y, s)})\} \right|_3 \\ &= \left| \frac{1}{\tau_{(\hat{Y}, \hat{s})}} \int_{\hat{s} - \tau_{(\hat{Y}, \hat{s})}}^{\hat{s}} \chi_{,4}^c(Z_s, \sigma) d\sigma - \frac{1}{\tau_{(Y, s)}} \int_{s - \tau_{(Y, s)}}^s \chi_{,4}^c(Z_s, \sigma) d\sigma \right|_3 \\ &= \left| \int_0^1 \{ \chi_{,4}^c(Z_s, \hat{s} + (\sigma-1)\tau_{(\hat{Y}, \hat{s})}) - \chi_{,4}^c(Z_s, s + (\sigma-1)\tau_{(Y, s)}) \} d\sigma \right|_3 \end{aligned}$$

$$\begin{aligned}
 & \leq \hat{A} \int_0^1 |(\hat{s}-s) + (\tau(\hat{Y}, \hat{s}) - \tau(Y, s))(\sigma-1)|^{\hat{\alpha}} d\sigma \\
 & \leq \hat{A} \cdot \{|\hat{s}-s| + |\tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}) - \tau(\chi_s^{-1}(Z); Y, s)|\}^{\hat{\alpha}} \\
 & \leq \hat{A} \cdot \{|\hat{s}-s| + \frac{1}{c-c^*} \{|\hat{Y}-Y|_3 + c^*|\hat{s}-s|\}\}^{\hat{\alpha}} \\
 & = \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \{|\hat{Y}-Y|_3 + c|\hat{s}-s|\}^{\hat{\alpha}} \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \{1+c^2\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}}
 \end{aligned} \tag{29}$$

for each $Z \in \partial B_s$;

in this computation, we have used (23) and the obvious inclusions $\hat{s} + (\sigma-1)\tau(\hat{Y}, \hat{s}) \in \hat{K}$, $s + (\sigma-1)\tau(Y, s) \in \hat{K}$, for each $\sigma \in [0, 1]$. While originally derived under the assumption that $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{ss}(\hat{Y})\}'$, (29) must hold for every $Z \in \partial B_s$, in view of the continuity of V .

Observe that

$$|\chi_{ss}(Z) - Z|_3 = |\chi(\chi_s^{-1}(Z), \hat{s}) - \chi(\chi_s^{-1}(Z), s)|_3 \leq c^*|\hat{s}-s|, \text{ if } Z \in \partial B_s. \tag{30}$$

Consequently,

$$\begin{aligned}
 & |\text{grad } r_{\hat{Y}}(\chi_{ss}(Z)) - \text{grad } r_Y(Z)|_3 \\
 & = \left| \frac{\chi_{ss}(Z) - \hat{Y}}{r_{\hat{Y}}(\chi_{ss}(Z))} - \frac{Z - Y}{r_Y(Z)} \right|_3 \\
 & = \left| \frac{1}{r_Y(Z)} \{(\chi_{ss}(Z) - \hat{Y}) - (Z - Y)\} + \left\{ \frac{1}{r_{\hat{Y}}(\chi_{ss}(Z))} - \frac{1}{r_Y(Z)} \right\} \cdot (\chi_{ss}(Z) - \hat{Y}) \right|_3 \\
 & \leq \frac{1}{r_Y(Z)} \{|\hat{Y}-Y|_3 + |\chi_{ss}(Z) - Z|_3\} + \frac{1}{r_Y(Z)} \cdot |r_{\hat{Y}}(\chi_{ss}(Z)) - r_Y(Z)|
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{2}{r_Y(Z)} \{ |\hat{Y}-Y|_3 + |x_{\hat{s}s}(Z)-Z|_3 \} \leq \frac{2}{r_Y(Z)} \{ |\hat{Y}-Y|_3 + c^* |\hat{s}-s| \} \\ &\leq \frac{2}{r_Y(Z)} \cdot \{1+(c^*)^2\}^{1/2} \cdot \{ |\hat{Y}-Y|_3^2 + |\hat{s}-s|^2 \}^{1/2} = \frac{2\delta}{r_Y(Z)}, \end{aligned} \quad (31)$$

$$Z \in \partial B_s \cap \{Y\}' \cap \{x_{\hat{s}s}(\hat{Y})\}'.$$

Returning to (22), we can state, with (26), (29), and (31), that there exist $k_1 > 0$ and $k_2 > 0$, depending on M and K alone, for which

$$|\Gamma_{(\hat{Y}, \hat{s})}^{03}(x_{\hat{s}s}(Z)) - \Gamma_{(Y, s)}^{03}(Z)| \leq k_1 \delta^{\hat{\alpha}} + \frac{k_2}{r_Y(Z)} \delta \quad (32)$$

$$\text{whenever } Z \in \partial B_s \cap \{Y\}' \cap \{x_{\hat{s}s}(\hat{Y})\}'.$$

Having (32), it is easy to obtain for I_2 an inequality of the desired form: setting

$$M_\phi^K := \max \{ |\phi_{(\tilde{Y}, \tilde{s})}(P)| \mid P \in \partial R, \quad (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\} \},$$

from (11) we find

$$\begin{aligned} I_2 &\leq \frac{1}{d^2} \cdot M_\phi^K \cdot M_J^K \cdot \left\{ k_1 \delta^{\hat{\alpha}} + \frac{k_2}{d} \delta \right\} \cdot \int_{\partial B_s \cap C_d^{\partial}(Y, s)'} d\lambda_{\partial B_s} \\ &\leq \{k_3 \max_{\hat{s} \in K} \lambda_{\partial B_{\hat{s}}}(\partial B_{\hat{s}})\} \delta^{\hat{\alpha}}, \end{aligned} \quad (33)$$

wherein k_3 depends upon only ϕ , M , and K .

I₃: To develop a suitable estimate for the difference appearing in the integrand in (12), we appeal first to hypothesis (i), according to which

$$|v(Y_2, s_2) - v(Y_1, s_1)|_3 \leq a_K |(Y_2, s_2) - (Y_1, s_1)|_4^{\alpha_K} \quad (34)$$

whenever $(Y_1, s_1), (Y_2, s_2) \in \cup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$,

$a_K > 0$ and $\alpha_K \in (0, 1]$ dependent upon M and K only. Thus, recalling (30),

$$\begin{aligned} |v(\chi_{s\hat{s}}(Z), \hat{s}) - v(Z, s)|_3 &\leq a_K |(\chi_{s\hat{s}}(Z), \hat{s}) - (Z, s)|_4^{\alpha_K} \\ &= a_K \{|\chi_{s\hat{s}}(Z) - Z|_3^2 + |\hat{s} - s|^2\}^{\alpha_K/2} \\ &\leq a_K \{(c^*|\hat{s} - s|)^2 + |\hat{s} - s|^2\}^{\alpha_K/2} \\ &= a_K \cdot (1 + (c^*)^2)^{\alpha_K/2} \cdot |\hat{s} - s|^{\alpha_K} \\ &\leq k_3 \delta^{\alpha_K} \quad \text{for each } Z \in \partial B_s, \end{aligned} \quad (35)$$

k_3 depending only on M and K .

Suppose next that $Z \in \partial B_s \cap C_{2\delta}^\partial(Y, s)'$, so that $r_Y(Z) \geq 2\delta$, or $\delta/r_Y(Z) \leq 1/2$: since, again using (30),

$$\begin{aligned} |r_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - r_Y(Z)| &\leq |(Y - \hat{Y}) + (\chi_{s\hat{s}}(Z) - Z)|_3 \leq |Y - \hat{Y}|_3 + c^*|\hat{s} - s| \\ &\leq (1 + (c^*)^2)^{1/2} \cdot \{|\hat{Y} - Y|_3^2 + |\hat{s} - s|^2\}^{1/2} = \delta, \end{aligned} \quad (36)$$

it follows that

$$1 - \frac{\delta}{r_Y(Z)} \leq \frac{r_{\hat{Y}}(\chi_{s\hat{s}}(Z))}{r_Y(Z)} \leq 1 + \frac{\delta}{r_Y(Z)},$$

and so

$$\frac{1}{2} \leq \frac{r_Y(\chi_{ss}(Z))}{r_Y(Z)} \leq \frac{3}{2} \quad \text{for each } Z \in \partial B_s \cap C_{2, \delta}^{\partial}(Y, s)'. \quad (37)$$

Now, whenever $Z \in \partial B_s \cap C_d^{\partial}(Y, s)'$, (30), (37), and the inequality $r_Y(Z) \geq d$ show that (note that $r_Y(\chi_{ss}(Z)) \geq \frac{1}{2} r_Y(Z) \geq \frac{1}{2} d > 0$, by (37))

$$\begin{aligned} & \left| \frac{\chi_{ss}(Z) - \hat{Y}}{r_Y(\chi_{ss}(Z))} - \frac{Z - Y}{r_Y(Z)} \right|_3 \\ &= \left| \frac{1}{r_Y^3(Z)} \{ (\chi_{ss}(Z) - \hat{Y}) - (Z - Y) \} + \left\{ \frac{1}{r_Y^3(\chi_{ss}(Z))} - \frac{1}{r_Y^3(Z)} \right\} \cdot (\chi_{ss}(Z) - Y) \right|_3 \\ &\leq \frac{1}{r_Y^3(Z)} \{ |\hat{Y} - Y|_3 + |\chi_{ss}(Z) - Z|_3 \} \\ &\quad + \frac{1}{r_Y^3(Z)} |r_Y(Z) - r_Y(\chi_{ss}(Z))| \cdot \left\{ 1 + \frac{r_Y(Z)}{r_Y(\chi_{ss}(Z))} + \frac{r_Y^2(Z)}{r_Y^2(\chi_{ss}(Z))} \right\} \\ &\leq \frac{8}{r_Y^3(Z)} \cdot \{ |\hat{Y} - Y|_3 + |\chi_{ss}(Z) - Z|_3 \} \leq \frac{8}{r_Y^3(Z)} \{ |\hat{Y} - Y|_3 + c^* |\hat{s} - s| \} \\ &\leq \frac{8_1}{d^3} \delta. \end{aligned} \quad (38)$$

Accounting for (35) and (38), we finally arrive at

$$\begin{aligned} & \left| \left\{ \frac{1}{2} r_{\hat{Y}, k}^{\nu^k} \right\} \circ \chi_{ss}(Z) - \frac{1}{r_Y^2(Z)} r_{Y, k}(Z) \nu_{\partial B_s}^k(Z) \right| \\ &\leq \frac{1}{r_Y^2(Z)} |r_{Y, k}(Z) \{ \nu^k(\chi_{ss}(Z), \hat{s}) - \nu^k(Z, s) \}| \\ &\quad + \left| \left\{ \frac{1}{2} r_{\hat{Y}, k} \right\} \circ \chi_{ss}(Z) - \frac{1}{r_Y^2(Z)} r_{Y, k}(Z) \right\} \nu^k(\chi_{ss}(Z), \hat{s}) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{r_Y^2(Z)} \cdot |v(x_{\hat{s}\hat{s}}(Z), \hat{s}) - v(Z, s)|_3 \\
 &\quad + \left| \frac{1}{r_Y^3(x_{\hat{s}\hat{s}}(Z))} \{x_{\hat{s}\hat{s}}(Z) - \hat{Y}\} - \frac{1}{r_Y^3(Z)} \{Z - Y\} \right|_3 \\
 &\leq \frac{k_3}{d^2} \delta^{\alpha_K} + \frac{8_1}{d^3} \delta, \quad \text{for each } Z \in \partial B_s \cap C_d^\partial(Y, s)'.
 \end{aligned} \tag{39}$$

Directly from (39), I_3 can be estimated in the required manner:

$$\begin{aligned}
 I_3 &\leq M_{\phi}^{03, K, K} \cdot \left\{ \frac{k_3}{d^2} \delta^{\alpha_K} + \frac{8_1}{d^3} \delta \right\} \cdot \int_{\partial B_s \cap C_d^\partial(Y, s)'} d\lambda_{\partial B_s} \\
 &\leq \{k_4 \cdot \max_{\hat{s} \in K} \lambda_{\partial B_{\hat{s}}}(\partial B_{\hat{s}})\} \cdot \delta^{\alpha_K},
 \end{aligned} \tag{40}$$

with k_4 dependent upon ϕ , M , and K only.

I_4 : Recalling (19) and [I.2.37.iii.4], and manipulating the integral in the usual manner (cf., (IV.22.52)), from (13) we find

$$\begin{aligned}
 I_4 &\leq M_{\hat{K}}^{03, K, \hat{\kappa} \delta^\beta \cdot \hat{a}_K} \cdot \int_{C_d^\partial(Y, s) \cap C_{2\delta}^\partial(Y, s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &\leq 2^{3/2} \pi \hat{\kappa} \hat{a}_K M_{\hat{K}}^{03, K, \delta^\beta} \cdot \int_{2\delta}^d \frac{1}{\zeta^{1-\alpha_K}} d\zeta \\
 &\leq \frac{d^{\alpha_K}}{\alpha_K} \cdot 2^{3/2} \pi \hat{\kappa} \hat{a}_K M_{\hat{K}}^{03, K, \delta^\beta};
 \end{aligned} \tag{41}$$

the positive number $\hat{a}_K$ can be found from [I.2.37.iii.4], in terms of a_K . The coefficient of δ^β on the right in (41) clearly depends upon only ϕ , M , and K .

I₅: We can use here the previously prepared inequality (32). Proceeding from (14),

$$I_5 \leq M_{\phi}^{KK} M_J^{KK} \cdot \hat{a}_K \cdot \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)'} \frac{1}{r_Y^{2-\alpha_K}} \left\{ k_1 \delta^{\hat{\alpha}} + \frac{k_2}{r_Y} \delta \right\} d\lambda_{\partial B_s} \quad (42)$$

$$\leq M_{\phi}^{KK} M_J^{KK} \cdot \hat{a}_K \cdot 2^{3/2} \pi \left\{ k_1 \frac{d^{\alpha_K}}{\alpha_K} \cdot \delta^{\hat{\alpha}} + k_2 \delta \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_K}} d\zeta \right\};$$

now, if $\alpha_K \in (0,1)$,

$$\delta \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_K}} d\zeta = \frac{\delta}{1-\alpha_K} \cdot \left\{ \frac{1}{(2\delta)^{1-\alpha_K}} - \frac{1}{d^{1-\alpha_K}} \right\}$$

$$< \frac{1}{(1-\alpha_K)(2\delta)^{1-\alpha_K}} \cdot \delta^{\alpha_K},$$

whereas $\alpha_K = 1$ implies that

$$\delta \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_K}} d\zeta = \delta \ln \frac{d}{2\delta} = \delta^{\alpha'_K} \cdot \delta^{1-\alpha'_K} \ln \frac{d}{2\delta} \leq M_{\alpha'_K} \delta^{\alpha'_K},$$

with α'_K chosen in $(0,1)$, and $M_{\alpha'_K} > 0$ such that

$$\zeta^{1-\alpha'_K} \ln \frac{d}{2\zeta} \leq M_{\alpha'_K} \quad \text{for each } \zeta \in (0, d_K/4).$$

In either case, (42) leads to an inequality of the required form for I_5 .

I₆: Now, we must be especially careful in developing a Hölder-type inequality for the difference appearing in the integrand on the right in (15). Suppose that $Z \in \partial B_s \cap C_{2\delta}^\partial(Y,s)'$; (37) shows that $r_Y(x_{ss}(Z)) \geq \frac{1}{2} r_Y(Z) \geq \delta > 0$. We write

$$\begin{aligned}
 & \left| \left\{ \frac{1}{r_Y^2} r_{\hat{Y},k} v_{\partial \hat{B}_s}^k \right\} \circ \chi_{ss}(Z) - \frac{1}{r_Y^2(Z)} r_{Y,k}(Z) v_{\partial \hat{B}_s}^k(Z) \right| \\
 &= \left| \frac{1}{r_{\hat{Y}}^3(\chi_{ss}(Z))} \{ \chi_{ss}^k(Z) - \hat{Y}^k \} v^k(\chi_{ss}(Z), \hat{s}) - \frac{1}{r_Y^3(Z)} (Z^k - Y^k) v^k(Z, s) \right| \\
 &= \left| \frac{1}{r_Y^3(Z)} \{ \{ \chi_{ss}^k(Z) - \hat{Y}^k \} v^k(\chi_{ss}(Z), \hat{s}) - \{ Z^k - Y^k \} v^k(Z, s) \} \right. \\
 &\quad \left. + \left\{ \frac{1}{r_{\hat{Y}}^3(\chi_{ss}(Z))} - \frac{1}{r_Y^3(Z)} \right\} \cdot \{ \chi_{ss}^k(Z) - \hat{Y}^k \} v^k(\chi_{ss}(Z), \hat{s}) \right| \\
 &= \left| \frac{1}{r_Y^3(Z)} \{ (Z^k - Y^k) \{ v^k(\chi_{ss}(Z), \hat{s}) - v^k(Z, s) \} \right. \\
 &\quad \left. + \{ (\chi_{ss}^k(Z) - Z^k) - (\chi_{ss}^k(Y) - Y^k) \} v^k(\chi_{ss}(Z), \hat{s}) \right. \\
 &\quad \left. + (\hat{Y}^k - \chi_{ss}^k(Y)) \{ v^k(\chi_{ss}(Y), \hat{s}) - v^k(\chi_{ss}(Z), \hat{s}) \} - (\hat{Y}^k - \chi_{ss}^k(Y)) \cdot v^k(\chi_{ss}(Y), \hat{s}) \} \right. \\
 &\quad \left. + \frac{1}{r_Y^3(Z) \cdot r_{\hat{Y}}(\chi_{ss}(Z))} \cdot \{ (\chi_{ss}^k(Z) - \hat{Y}^k) \cdot v^k(\chi_{ss}(Z), \hat{s}) \} \cdot \{ r_Y(Z) - r_{\hat{Y}}(\chi_{ss}(Z)) \} \right. \\
 &\quad \left. \cdot \left\{ 1 + \frac{r_Y(Z)}{r_{\hat{Y}}(\chi_{ss}(Z))} + \frac{r_Y^2(Z)}{r_{\hat{Y}}^2(\chi_{ss}(Z))} \right\} \right| \\
 &\leq \frac{1}{r_Y^3(Z)} \cdot \{ r_Y(Z) \cdot |v(\chi_{ss}(Z), \hat{s}) - v(Z, s)|_3 + |(\chi_{ss}(Z) - Z) - (\chi_{ss}(Y) - Y)|_3 \\
 &\quad + |\hat{Y} - \chi_{ss}(Y)|_3 \cdot |v(\chi_{ss}(Y), \hat{s}) - v(\chi_{ss}(Z), \hat{s})|_3 + |(\chi_{ss}^k(Y) - \hat{Y}^k) \cdot v^k(\chi_{ss}(Y), \hat{s})| \\
 &\quad + 7 \cdot |r_{\hat{Y},k}(\chi_{ss}(Z)) \cdot v^k(\chi_{ss}(Z), \hat{s})| \cdot |r_Y(Z) - r_{\hat{Y}}(\chi_{ss}(Z))| \},
 \end{aligned}
 \tag{43}$$

having used (37). We shall estimate the various expressions on the right in (43).

Observing that $\chi|_{\partial R \times K}$ is Lipschitz continuous, along with

the map $(\tilde{Y}, \tilde{s}) \mapsto \chi_{\tilde{s}}^{-1}(\tilde{Y})$ on $\cup_{\zeta \in K} \{\partial \tilde{B}_{\zeta} \times \{\zeta\}\}$, we know that there are positive numbers a_1 and a_2 , depending upon M and K alone, for which

$$\begin{aligned} |\chi(P_2, \zeta_2) - \chi(P_1, \zeta_1)|_3 &\leq a_1 |(P_2, \zeta_2) - (P_1, \zeta_1)|_4 \\ \text{whenever } \zeta_1, \zeta_2 &\in K \quad \text{and} \quad P_1, P_2 \in \partial R, \end{aligned} \quad (44)$$

and

$$\begin{aligned} |\chi_{s_2}^{-1}(Y_2) - \chi_{s_1}^{-1}(Y_1)|_3 &\leq a_2 |(Y_2, s_2) - (Y_1, s_1)|_4 \\ \text{whenever } (Y_1, s_1) \quad \text{and} \quad (Y_2, s_2) &\in \cup_{\zeta \in K} \{\partial \tilde{B}_{\zeta} \times \{\zeta\}\}. \end{aligned} \quad (45)$$

These produce, in combination with (34),

$$\begin{aligned} |\nu(\chi_{\hat{s}\hat{s}}(Y), \hat{s}) - \nu(\chi_{\hat{s}\hat{s}}(Z), \hat{s})|_3 &\leq a_K |\chi_{\hat{s}\hat{s}}(Y) - \chi_{\hat{s}\hat{s}}(Z)|_3^{\alpha_K} \\ &= a_K |\chi(\chi_s^{-1}(Y), \hat{s}) - \chi(\chi_s^{-1}(Z), \hat{s})|_3^{\alpha_K} \\ &\leq a_K a_1^{\alpha_K} |\chi_s^{-1}(Y) - \chi_s^{-1}(Z)|_3^{\alpha_K} \\ &\leq a_K a_1^{\alpha_K} a_2^{\alpha_K} |Z - Y|_3^{\alpha_K} \end{aligned} \quad (46)$$

for each $Z \in \partial \tilde{B}_{\hat{s}}$.

Next, from (23) and (45),

$$\begin{aligned} &|(\chi_{\hat{s}\hat{s}}(Z) - Z) - (\chi_{\hat{s}\hat{s}}(Y) - Y)|_3 \\ &= |\{\chi(\chi_s^{-1}(Z), \hat{s}) - \chi(\chi_s^{-1}(Z), s)\} - \{\chi(\chi_s^{-1}(Y), \hat{s}) - \chi(\chi_s^{-1}(Y), s)\}|_3 \\ &= \left| \int_s^{\hat{s}} \{\chi_{,4}(\chi_s^{-1}(Z), \sigma) - \chi_{,4}(\chi_s^{-1}(Y), \sigma)\} d\sigma \right|_3 \end{aligned}$$

$$\leq \left| \int_{\hat{s}}^{\hat{s}} |\chi_{,4}(\chi_s^{-1}(Z), \sigma) - \chi_{,4}(\chi_s^{-1}(Y), \sigma)|_3 d\sigma \right| \quad (47)$$

$$\leq c\hat{A} \cdot |\chi_s^{-1}(Z) - \chi_s^{-1}(Y)|_3^{\hat{\alpha}} \cdot |\hat{s} - s| \leq c\hat{A} \cdot a_2^{\hat{\alpha}} \cdot |Z - Y|_3^{\hat{\alpha}} \cdot \delta$$

for each $Z \in \partial B_s$.

Obviously,

$$\begin{aligned} |\hat{Y} - \chi_{s\hat{s}}(Y)|_3 &\leq |\hat{Y} - Y|_3 + |Y - \chi_{s\hat{s}}(Y)|_3 \\ &= |\hat{Y} - Y|_3 + |\chi(\chi_s^{-1}(Y), s) - \chi(\chi_s^{-1}(Y), \hat{s})|_3 \\ &\leq |\hat{Y} - Y|_3 + c^* |\hat{s} - s| \leq 1\delta, \end{aligned} \quad (48)$$

and

$$\begin{aligned} |r_Y(Z) - r_{\hat{Y}}(\chi_{s\hat{s}}(Z))| &\leq |(Y - \hat{Y}) + (\chi_{s\hat{s}}(Z) - Z)|_3 \\ &\leq |Y - \hat{Y}|_3 + c^* |s - \hat{s}| \leq 1\delta, \quad \text{for each } Z \in \partial B_s. \end{aligned} \quad (49)$$

We have arranged for the inclusion

$$\chi_{s\hat{s}}(C_d^{\partial}(Y, s)) \subset \partial B_{\hat{s}} \cap B_{d_K}^3(\hat{Y}) \quad (50)$$

to hold. To see that this is so, let $Z \in C_d^{\partial}(Y, s)$: by (I.2.37.8), we then have $r_Y(Z) < \frac{9}{7} r_Y(\Pi_Y(Z)) < \frac{9}{7} d$, so

$$\begin{aligned} |\chi_{s\hat{s}}(Z) - \hat{Y}|_3 &\leq |Y - \hat{Y}|_3 + |\chi_{s\hat{s}}(Z) - Z|_3 + |Z - Y|_3 \\ &\leq |Y - \hat{Y}|_3 + c^* |s - \hat{s}| + |Z - Y|_3 < 1\delta + \frac{9}{7} d \\ &< \frac{1}{4} d_K + \frac{9}{7} \cdot \frac{7}{12} d_K = d_K. \end{aligned}$$

i.e., $\chi_{s\hat{s}}(Z) \in \partial B_{\hat{s}} \cap B_{d_K}^3(\hat{Y})$. Since $\{B_{\zeta}^0\}_{\zeta \in K}$ is a uniformly Lyapunov family with uniform constants (a_K, α_K, d_K) , (50) allows us to write, recalling [I.2.37.iii.4], and using (37),

$$|r_{\hat{Y},k}(\chi_{s\hat{s}}(Z)) \cdot v^k(\chi_{s\hat{s}}(Z), \hat{s})| < \hat{a}_K r_{\hat{Y}}^{\alpha_K}(\chi_{s\hat{s}}(Z)) \leq \hat{a}_K \cdot (3/2)^{\alpha_K} \cdot r_Y^{\alpha_K}(Z), \quad (51)$$

for each $Z \in C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'$.

Finally, suppose that $\hat{Y} \neq \chi_{s\hat{s}}(Y)$: since (48) clearly implies that $\chi_{s\hat{s}}(Y) \in \partial B_{\hat{s}} \cap B_{d_K}^3(\hat{Y})$, we may reason as in the derivation of (51), obtaining, with (48),

$$\begin{aligned} |(\chi_{s\hat{s}}^k(Y) - \hat{Y}^k) \cdot v^k(\chi_{s\hat{s}}(Y), \hat{s})| &= |\chi_{s\hat{s}}(Y) - \hat{Y}|_3 \cdot |r_{\hat{Y},k}(\chi_{s\hat{s}}(Y)) \cdot v^k(\chi_{s\hat{s}}(Y), \hat{s})| \\ &\leq \delta \cdot \hat{a}_K \cdot r_{\hat{Y}}^{\alpha_K}(\chi_{s\hat{s}}(Y)) \\ &\leq \hat{a}_K \cdot \delta^{1+\alpha_K} \cdot \delta^{1+\alpha_K}. \end{aligned} \quad (52)$$

The resultant inequality is certainly true even if $\hat{Y} = \chi_{s\hat{s}}(Y)$.

Collecting up the results (46), (47), (48), (49), (51), and (52), and using (35) as well, (43) leads to the inequality

$$\begin{aligned} &\left| \left\{ \frac{1}{r_{\hat{Y}}} r_{\hat{Y},k} v^k \right\} \circ \chi_{s\hat{s}}(Z) - \frac{1}{r_Y(Z)} r_{Y,k}(Z) v^k(Z) \right| \\ &\leq \frac{1}{r_Y(Z)} \{ k_3 \delta^{\alpha_K} \cdot r_Y(Z) + c\hat{A} \cdot \hat{a}_2 \cdot \delta \cdot r_Y(Z) + a_K a_1 a_2^{\alpha_K} \cdot \delta \cdot r_Y^{\alpha_K}(Z) \\ &\quad + \hat{a}_K \cdot \delta^{1+\alpha_K} \cdot \delta^{1+\alpha_K} + 7 \cdot \hat{a}_K \cdot (3/2)^{\alpha_K} \cdot \delta \cdot r_Y^{\alpha_K}(Z) \} \end{aligned}$$

$$\leq k_3 \frac{\delta^{\alpha_K}}{r_Y^2(Z)} + k_4 \frac{\delta}{r_Y^{3-\alpha'}(Z)} + k_5 \frac{\delta^{1+\alpha_K}}{r_Y^3(Z)} \quad (53)$$

for each $Z \in C_d^\partial(Y, s) \cap C_{21\delta}^\partial(Y, s)'$,

wherein $\alpha' := \min\{\alpha_K, \hat{\alpha}\}$, and the positive k_4 and k_5 depend (with k_3) upon M and K alone. In turn, (53) can be used to estimate I_6 :

$$\begin{aligned} I_6 &\leq M^{03} M_{\phi}^{K K} \int_{C_d^\partial(Y, s) \cap C_{21\delta}^\partial(Y, s)'} \left\{ \frac{k_3 \delta^{\alpha_K}}{r_Y^2} + \frac{k_4 \delta}{r_Y^{3-\alpha'}} + \frac{k_5 \delta^{1+\alpha_K}}{r_Y^3} \right\} d\lambda_{\partial B_s} \\ &\leq 2^{3/2} \pi \cdot M^{03} M_{\phi}^{K K} \left\{ k_3 \delta^{\alpha_K} \int_{21\delta}^d \frac{1}{\zeta} d\zeta + k_4 \delta \int_{21\delta}^d \frac{1}{\zeta^{2-\alpha'}} d\zeta \right. \\ &\quad \left. + k_5 \delta^{1+\alpha_K} \int_{21\delta}^d \frac{1}{\zeta^2} d\zeta \right\}. \end{aligned} \quad (54)$$

The first two terms within the brackets on the right in (54) were essentially examined during the analysis of I_5 ; for the third term, we have simply

$$\delta^{1+\alpha_K} \int_{21\delta}^d \frac{1}{\zeta^2} d\zeta = \delta^{1+\alpha_K} \left\{ \frac{1}{21\delta} - \frac{1}{d} \right\} < \frac{1}{21} \delta^{\alpha_K}.$$

On the basis of these computations, we can assert that I_6 satisfies an inequality of the required form described at (18).

I_7 : To study this term, it is probably easiest to transform back to an integral over a subset of ∂B_s : this is easily effected via (7),

(8), [I.2.26.a], and the fact that $\chi_{ss}^{-1} = \chi_{ss}$. Indeed,

$$\begin{aligned}
 I_7 &= \left| \int_{C_{21\delta}^\partial(Y,s)} \left\{ \frac{1}{r_Y} r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{\partial B_{\hat{s}}}^{03}(\hat{Y},\hat{s}) \cdot \phi(\hat{Y},\hat{s})^{\circ X_{\hat{s}}^{-1} \cdot JX_{\hat{s}}^{-1}} \right\} \circ X_{\hat{s}\hat{s}} \cdot JX_{\hat{s}\hat{s}} d\lambda_{\partial B_{\hat{s}}} \right| \\
 &= \left| \int_{X_{\hat{s}\hat{s}}(C_{21\delta}^\partial(Y,s))} \frac{1}{r_{\hat{Y}}} r_{\hat{Y},k}^{\nu k} \cdot \Gamma_{\partial B_{\hat{s}}}^{03}(\hat{Y},\hat{s}) \cdot \phi(\hat{Y},\hat{s})^{\circ X_{\hat{s}}^{-1} \cdot JX_{\hat{s}}^{-1}} d\lambda_{\partial B_{\hat{s}}} \right| \quad (55) \\
 &\leq M_{\phi}^{03,K,K} \cdot \int_{X_{\hat{s}\hat{s}}(C_{21\delta}^\partial(Y,s))} \frac{1}{r_{\hat{Y}}} |r_{\hat{Y},k}^{\nu k}| d\lambda_{\partial B_{\hat{s}}}.
 \end{aligned}$$

We claim that

$$X_{\hat{s}\hat{s}}(C_{21\delta}^\partial(Y,s)) \subset \partial B_{\hat{s}} \cap B_{41\delta}^3(\hat{Y}) \subset \partial B_{\hat{s}} \cap B_{d_K}^3(\hat{Y}). \quad (56)$$

The second inclusion here follows from (6), of course. To verify the first inclusion, let $Z \in C_{21\delta}^\partial(Y,s)$: then $r_Y(Z) < \frac{9}{7} r_Y(\pi_Y(Z)) < \frac{9}{7} \cdot 21\delta$, so

$$\begin{aligned}
 |X_{\hat{s}\hat{s}}(Z) - \hat{Y}|_3 &\leq |Y - \hat{Y}|_3 + |X_{\hat{s}\hat{s}}(Z) - Z|_3 + |Z - Y|_3 \\
 &\leq |Y - \hat{Y}|_3 + c^* |s - \hat{s}| + |Z - Y|_3 < 1\delta + \frac{9}{7} \cdot 21\delta < 41\delta,
 \end{aligned}$$

whence $X_{\hat{s}\hat{s}}(Z) \in \partial B_{\hat{s}} \cap B_{41\delta}^3(\hat{Y})$. We can therefore proceed further with the computation begun in (55), in view of (56) and the uniform Lyapunov condition on $\{B_{\zeta}^0\}_{\zeta \in K}$:

$$I_7 \leq M_{\phi}^{03,K,K} \cdot \hat{a}_K \int_{\partial B_{\hat{s}} \cap B_{41\delta}^3(\hat{Y})} \frac{1}{r_{\hat{Y}}^{2-\alpha_K}} d\lambda_{\partial B_{\hat{s}}}, \quad (57)$$

while

$$\begin{aligned}
 \int_{\partial B_{\hat{s}} \cap B_{4\delta}^3(\hat{Y})} \frac{1}{r_{\hat{Y}}^{2-\alpha_K}} d\lambda_{\partial B_{\hat{s}}} &\leq \int_{\partial B_{\hat{s}} \cap B_{4\delta}^3(\hat{Y})} \frac{1}{(r_{\hat{Y}} \circ \Pi_{\hat{Y}})^{2-\alpha_K}} d\lambda_{\partial B_{\hat{s}}} \\
 &= \int_{\hat{\mathcal{H}}_{\hat{Y}} \circ \Pi_{\hat{Y}}(\partial B_{\hat{s}} \cap B_{4\delta}^3(\hat{Y}))} \frac{1}{(r_{\hat{Y}} \circ \hat{\mathcal{H}}_{\hat{Y}}^{-1})^{2-\alpha_K}} Jh_{\hat{Y}}^{-1} d\lambda_2 \\
 &\leq 2^{1/2} \int_{\hat{\mathcal{H}}_{\hat{Y}}(C_{4\delta}^{\Pi}(\hat{Y}, \hat{s}))} \frac{1}{(r_{\hat{Y}} \circ \hat{\mathcal{H}}_{\hat{Y}}^{-1})^{2-\alpha_K}} d\lambda_2 \quad (58) \\
 &= 2^{1/2} \int_{B_{4\delta}^2(0)} \frac{1}{|\xi|^{2-\alpha_K}} d\lambda_2(\xi) \\
 &= 2^{3/2} \pi \int_0^{4\delta} \frac{1}{\zeta^{1-\alpha_K}} d\zeta \\
 &= \frac{2^{3/2} \pi}{\alpha_K} (4\delta)^{\alpha_K} \delta^{\alpha_K}.
 \end{aligned}$$

Clearly, (57) and (58) show that I_7 possesses an estimate of the requisite form.

I_8 : From (17), we easily find that

$$\begin{aligned}
 I_8 &\leq M_{\phi}^{03} M_J^{K,K} \int_{C_{2\delta}^{\partial}(Y,s)} \frac{1}{r_{\hat{Y}}^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &\leq M_{\phi}^{03} M_J^{K,K} \cdot 2^{3/2} \pi \int_0^{2\delta} \frac{1}{\zeta^{1-\alpha_K}} d\zeta \\
 &= \frac{2^{3/2} \pi}{\alpha_K} (2\delta)^{\alpha_K} M_{\phi}^{03} M_J^{K,K} \delta^{\alpha_K},
 \end{aligned}$$

i.e., a relation of the form of (18) is fulfilled by I_8 .

As we have pointed out, these computations effectively complete the proof of the first assertion of the theorem, viz., that $W_1^*(\phi)$ is locally Hölder continuous on ∂B .

Let us suppose now that hypotheses (iv) and (v) are in force: we shall prove that (iii) is true when $\phi = \Lambda_{11}^1\{u\}$, the verification of the corresponding statement with $\phi = \tilde{\Lambda}_1^1\{u\}$ being quite similar. Recall from (IV.14.33) that

$$\begin{aligned} \Lambda_{11}^1\{u\}(X,t)(P) &:= \{ |1 - | [X_{,4}^c]_{(X,t)}(P) |^2 \} \cdot v^1(x_t(P), t) \\ &\quad + \{ v^j(x_t(P), t) \cdot [X_{,4}^j]_{(X,t)}(P) \} \cdot [X_{,4}^i]_{(X,t)}(P) \} \\ &\quad \cdot \left\{ \frac{1 - |v_{(X,t)}^c(x_t(P))|^2}{1 - v_{(X,t)}^k(x_t(P)) \cdot [X_{,4}^k]_{(X,t)}(P)} \right\} \cdot [u \cdot jx]_{(X,t)}(P), \end{aligned} \quad (59)$$

for each $(X,t) \in \mathbb{R}^4$, $P \in \partial R$.

It does no harm to suppose that the compact subset $\tilde{K} \subset \mathbb{R}$ is again an interval $[t_1, t_2]$, as we shall. Set

$$t'_0 := \frac{1}{c-c^*} \max_{\tilde{s} \in \tilde{K}} \text{diam } B_{\tilde{s}},$$

and

$$\tilde{K} := [t_1 - t'_0, t_2].$$

Then, whenever $P \in \partial R$, $\tilde{s} \in \tilde{K}$, and $\tilde{Y} \in \partial B_{\tilde{s}}$,

$$\begin{aligned} \tau(P; \tilde{Y}, \tilde{s}) &= |\tau(P; \tilde{Y}, \tilde{s}) - \tau(P; \chi_{\tilde{s}}(P), \tilde{s})| \leq \frac{1}{c-c^*} \cdot |\chi_{\tilde{s}}(P) - \tilde{Y}|_3 \\ &\leq \frac{1}{c-c^*} \text{diam } B_{\tilde{s}} \leq t'_0. \end{aligned} \quad (60)$$

This shows that

$$\tilde{s} - \tau(P; \tilde{Y}, \tilde{s}) \in \tilde{K} \quad \text{whenever} \quad P \in \partial R, \quad \tilde{s} \in \tilde{K}, \quad \text{and} \quad \tilde{Y} \in \partial B_{\tilde{s}}. \quad (61)$$

Now, by (ii), $\chi_{\tilde{s}}^c|_{\partial R \times \tilde{K}}$ is Hölder continuous; by (iv), $\hat{J}\chi(P, \cdot)|_{\tilde{K}}$ is Hölder continuous, uniformly in $P \in \partial R$. $\chi^*|_{\partial R \times \tilde{K}}$ is Lipschitz continuous, while $\mu|_{\chi^*(\partial R \times \tilde{K})}$ is Hölder continuous, so $\hat{\mu}|_{\partial R \times \tilde{K}} = (\mu \circ \chi^*)|_{\partial R \times \tilde{K}}$ is Hölder continuous. Thus, there exist positive numbers $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, and $\tilde{\kappa}_3$, and numbers $\tilde{\beta}_1$, $\tilde{\beta}_2$, and $\tilde{\beta}_3 \in (0, 1]$ for which, in particular,

$$|\chi_{\tilde{s}_2}^c(P, s_2) - \chi_{\tilde{s}_1}^c(P, s_1)|_3 \leq \tilde{\kappa}_1 \cdot |s_2 - s_1|^{\tilde{\beta}_1}, \quad (62)$$

$$|\hat{J}\chi(P, s_2) - \hat{J}\chi(P, s_1)| \leq \tilde{\kappa}_2 \cdot |s_2 - s_1|^{\tilde{\beta}_2}, \quad (63)$$

$$|\hat{\mu}(P, s_2) - \hat{\mu}(P, s_1)| \leq \tilde{\kappa}_3 \cdot |s_2 - s_1|^{\tilde{\beta}_3}, \quad (64)$$

$$\text{whenever} \quad P \in \partial R \quad \text{and} \quad s_1, s_2 \in \tilde{K},$$

$\tilde{\kappa}_1$, $\tilde{\kappa}_2$, $\tilde{\beta}_1$, and $\tilde{\beta}_2$ depending on only M and $\tilde{K}$, $\tilde{\kappa}_3$ and $\tilde{\beta}_3$ depending only on μ , M , and $\tilde{K}$. Certainly, $\chi|_{\partial R \times \tilde{K}}$ is Lipschitz continuous, and (i) says that $\nu|_{\cup_{\zeta \in \tilde{K}} \{\partial B_{\zeta} \times \{\zeta\}\}}$ is Hölder continuous, whence

$$|\chi_{s_2}(P) - \chi_{s_1}(P)|_3 \leq \tilde{\kappa}_4 |s_2 - s_1| \quad \text{for} \quad P \in \partial R, \quad \text{and} \quad s_1, s_2 \in \tilde{K}, \quad (65)$$

and

$$|v(Y_2, s_2) - v(Y_1, s_1)|_3 \leq a_{\tilde{K}} |Y_2, s_2 - Y_1, s_1|_4^{\alpha_{\tilde{K}}} \quad (66)$$

$$\text{for } (Y_1, s_1), (Y_2, s_2) \in \bigcup_{\zeta \in \tilde{K}} \{\partial B_{\zeta} \times \{\zeta\}\},$$

$\tilde{\kappa}_4 > 0$, $a_{\tilde{K}} > 0$, and $\alpha_{\tilde{K}} \in (0, 1]$ depending upon M and $\tilde{K}$.

Let us also establish an estimate for the expression

$|v_{(Y_2, s_2)}^c(x_{s_2}(P)) - v_{(Y_1, s_1)}^c(x_{s_1}(P))|_3$, supposing that $s_1, s_2 \in \tilde{K}$, $Y_1 \in \partial B_{s_1}$, $Y_2 \in \partial B_{s_2}$, and $P \in \partial R$, proceeding as in the derivation of (29): temporarily write $\tau_1 := \tau(P; Y_1, s_1)$ and $\tau_2 := \tau(P; Y_2, s_2)$, and suppose first that $Y_1 \neq x(P, s_1)$ and $Y_2 \neq x(P, s_2)$. From (IV.14.11),

$$\begin{aligned} & |v_{(Y_2, s_2)}^c(x_{s_2}(P)) - v_{(Y_1, s_1)}^c(x_{s_1}(P))|_3 \\ &= \left| \frac{1}{\tau(P; Y_2, s_2)} \{x^c(P, s_2) - x^c(P, s_2 - \tau(P; Y_2, s_2))\} \right. \\ & \quad \left. - \frac{1}{\tau(P; Y_1, s_1)} \{x^c(P, s_1) - x^c(P, s_1 - \tau(P; Y_1, s_1))\} \right| \\ &= \left| \frac{1}{\tau_2} \int_{s_2 - \tau_2}^{s_2} x_{,4}^c(P, \sigma) d\sigma - \frac{1}{\tau_1} \int_{s_1 - \tau_1}^{s_1} x_{,4}^c(P, \sigma) d\sigma \right| \\ &= \left| \int_0^1 \{x_{,4}^c(P, s_2 + (\sigma-1)\tau_2) - x_{,4}^c(P, s_1 + (\sigma-1)\tau_1)\} d\sigma \right| \\ &\leq \tilde{\kappa}_1 \int_0^1 |(s_2 - s_1) + (\tau_2 - \tau_1)(\sigma-1)|^{\tilde{\beta}_1} d\sigma \\ &\leq \tilde{\kappa}_1 \{|s_2 - s_1| + |\tau(P; Y_2, s_2) - \tau(P; Y_1, s_1)|\}^{\tilde{\beta}_1} \\ &\leq \tilde{\kappa}_1 \{|s_2 - s_1| + \frac{1}{c-c^*} \{|Y_2 - Y_1|_3 + c^* |s_2 - s_1|\}\}^{\tilde{\beta}_1} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\tilde{\kappa}_1}{(c-c^*)^{\tilde{\beta}_1}} \{ |Y_2 - Y_1|_3 + c |s_2 - s_1| \}^{\tilde{\beta}_1} \\
 &\leq \frac{\tilde{\kappa}_1}{(c-c^*)^{\tilde{\beta}_1}} \cdot (1+c^2)^{\tilde{\beta}_1/2} \cdot |(Y_2, s_2) - (Y_1, s_1)|_4^{\tilde{\beta}_1}, \quad (67)
 \end{aligned}$$

having appealed to (62) and the inclusions $s_2 + (\sigma-1)\tau_2 \in \tilde{K}$, $s_1 + (\sigma-1)\tau_1 \in \tilde{K}$, for each $\sigma \in [0,1]$. From the continuity of V , it is clear that (67) is true even if $Y_1 = \chi(P, s_1)$, or $Y_2 = \chi(P, s_2)$, or both.

Now, $\tilde{\mu}$ and $\hat{J}\chi$ are bounded on $\partial R \times \tilde{K}$, while $|\chi^c_{,4}|_3 \leq c^*/c$ and $|V^c|_3 \leq c^*/c$. Consequently, it is easy to see from (59) that there exist positive numbers $k'_1, \dots, k'_5$, depending on μ , M , and $\tilde{K}$, such that

$$\begin{aligned}
 &|\Lambda_{11}^1\{\mu\}(Y_2, s_2)^{(P)} - \Lambda_{11}^1\{\mu\}(Y_1, s_1)^{(P)}| \\
 &\leq k'_1 |[\chi^c_{,4}](Y_2, s_2)^{(P)} - [\chi^c_{,4}](Y_1, s_1)^{(P)}|_3 + k'_2 |v(\chi_{s_2}(P), s_2) - v(\chi_{s_1}(P), s_1)|_3 \\
 &\quad + k'_3 |V^c_{(Y_2, s_2)}(\chi_{s_2}(P)) - V^c_{(Y_1, s_1)}(\chi_{s_1}(P))|_3 \\
 &\quad + k'_4 |[\hat{J}\chi](Y_2, s_2)^{(P)} - [\hat{J}\chi](Y_1, s_1)^{(P)}|_3 + k'_5 |[\tilde{\mu}](Y_2, s_2)^{(P)} - [\tilde{\mu}](Y_1, s_1)^{(P)}|_3
 \end{aligned}$$

for $P \in \partial R$, $s_1, s_2 \in \tilde{K}$, $Y_1 \in \partial \mathcal{B}_{s_1}$, and $Y_2 \in \partial \mathcal{B}_{s_2}$;

noting (61), and using (62)-(67), the latter expression is

$$\begin{aligned}
 &\leq k'_1 \cdot \tilde{\kappa}_1 \cdot |\tau(P; Y_2, s_2) - \tau(P; Y_1, s_1)|^{\tilde{\beta}_1} + k'_2 \cdot a_{\tilde{K}} |(\chi_{s_2}(P), s_2) - (\chi_{s_1}(P), s_1)|_4^{\alpha_{\tilde{K}}} \\
 &\quad + k'_3 \frac{\tilde{\kappa}_1}{(c-c^*)^{\tilde{\beta}_1}} \cdot (1+c^2)^{\tilde{\beta}_1/2} \cdot |(Y_2, s_2) - (Y_1, s_1)|_4^{\tilde{\beta}_1}
 \end{aligned}$$

$$+k'_4 \cdot \tilde{\kappa}_2 \cdot |\tau(P; Y_2, s_2) - \tau(P; Y_1, s_1)|^{\tilde{\beta}_2} + k'_5 \cdot \tilde{\kappa}_3 \cdot |\tau(P; Y_2, s_2) - \tau(P; Y_1, s_1)|^{\tilde{\beta}_3}, \quad (68)$$

while

$$\begin{aligned} |\tau(P; Y_2, s_2) - \tau(P; Y_1, s_1)| &\leq \frac{1}{c-c^*} \{ |Y_2 - Y_1|_3 + c^* |s_2 - s_1| \} \\ &\leq \frac{1}{c-c^*} \cdot |(Y_2, s_2) - (Y_1, s_1)|_4, \end{aligned} \quad (69)$$

and

$$\begin{aligned} |(x_{s_2}(P), s_2) - (x_{s_1}(P), s_1)|_4 &= \{ |x_{s_2}(P) - x_{s_1}(P)|_3^2 + |s_2 - s_1|^2 \}^{1/2} \\ &\leq \{ 1 + \tilde{\kappa}_4^2 \}^{1/2} \cdot |s_2 - s_1| \\ &\leq \{ 1 + \tilde{\kappa}_4^2 \}^{1/2} \cdot |(Y_2, s_2) - (Y_1, s_1)|_4. \end{aligned} \quad (70)$$

Upon combining (68)-(70), it follows easily that $\phi = \Lambda_{11}^1\{u\}$ possesses the property demanded in hypothesis (iii).

Finally, suppose that $M \in M(2)$: the proof of Theorem [IV.22] contains the reasoning required to show that M fulfills hypotheses (i), (ii), and (iv); cf., also, Remark [IV.23.a]. $\square$.

[IV.25] R E M A R K. Let M be a motion satisfying the hypotheses of [IV.24]. If $(P, Y, s) \mapsto \phi_{(Y, s)}(P)$ is merely continuous on $\partial R \times R^4$, then the proof of [IV.24] can easily be modified to show that $\omega_1^*\{\phi\}$ is continuous on ∂B .

The facts to be secured in the next statement will be crucial in the analysis of the functions $\omega_{21}\{\phi\}$: the first part of this statement will allow us to define a kind of "direct value" for such

functions, while the second part of the statement plays a role which is similar to that performed by [I.2.44] in the examination of the functions $\omega_1\{\phi\}$.

[IV.26] L E M M A. Let M be a motion in $M(1)$.

- (i) Suppose that, for some $s \in \mathbb{R}$, B_s^0 is a Lyapunov domain, with Lyapunov constants (a, α, d) . Suppose that $\psi \in \mathbb{R}^3$ with $|\psi|_3 < 1$, and $\omega \in \mathbb{R}^3$. Let $Y \in \partial B_s$. Then, whenever $0 < \Delta_1 < \Delta_2 < \frac{7}{9}d$, it follows that

$$\int_{C_{\Delta_2}^{\partial}(Y,s) \cap C_{\Delta_1}^{\partial}(Y,s)} \left(\frac{1}{r_Y^2} r_{Y,q} \cdot r_Y\{\psi\} \right) \circ \Pi_Y \cdot \omega^q \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j d\lambda_{\partial B_s} = 0. \quad (1)$$

- (ii) Suppose that K is a compact subset of $\mathbb{R}$, and $\{B_\zeta^0\}_{\zeta \in K}$ is a uniformly Lyapunov family of domains, with uniform Lyapunov constants (a_K, α_K, d_K) . Let ψ and T denote functions on $\cup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ with values in $\mathbb{R}^3$ such that

$$|\psi(Y,s)|_3 \leq \lambda^* < 1 \quad \text{whenever} \quad s \in K \quad \text{and} \quad Y \in \partial B_s, \quad (2)$$

while T is bounded, with

$$T^j(Y,s) \nu^j(Y,s) = 0 \quad \text{whenever} \quad s \in K \quad \text{and} \quad Y \in \partial B_s, \quad (3)$$

i.e., $T(Y,s) \in T_{\partial B_s}(Y)$ for $s \in K$ and $Y \in \partial B_s$.

If $\psi(Y,s) \neq 0$ for some such s and Y , let $\phi_\psi(Y,s)$ denote the angle in $[0, \pi]$ formed by $\psi(Y,s)$ and

$v(Y,s)$; if then also $\Theta_\psi(Y,s) \notin \{0,\pi\}$ and $T(Y,s) \neq 0$, let $\Theta_T(Y,s)$ denote the angle in $[0,\pi]$ which is formed by $T(Y,s)$ and the orthogonal projection of $\psi(Y,s)$ onto $T_{\partial B_s}(Y)$. With these notations, define $R(Y,s)$ for each $s \in K$ and $Y \in \partial B_s$ by

$$R(Y,s) := \begin{cases} 0, & \text{if } \psi(Y,s) = 0, \quad T(Y,s) = 0, \quad \text{or } \Theta_\psi(Y,s) \in \{0,\pi\}, \\ \frac{1}{2} \cdot \frac{|T(Y,s)|_3 \cos \Theta_T(Y,s) \cdot |\psi(Y,s)|_3^2 \sin \Theta_\psi(Y,s) \cdot \cos \Theta_\psi(Y,s)}{(1-|\psi(Y,s)|_3^2) \cdot (1-|\psi(Y,s)|_3^2 \cos^2 \Theta_\psi(Y,s))}, & \\ \text{otherwise.} & \end{cases} \quad (4)$$

Then, given $n > 0$ and $\Delta \in (0, (7/9)d_K)$, there exists a positive number $\kappa_0(\Delta, n)$, depending also on λ^* , d_K , and the number $\sup \{|T(Y,s)|_3 \mid s \in K, Y \in \partial B_s\}$, such that, whenever $s \in K$, $Y \in \partial B_s$, and $X \in L_V^+(Y,s)$ [$X \in L_V^-(Y,s)$] with $(0 <) |X-Y|_3 < n$,

$$\left| \frac{1}{4\pi} \int_{C_\Delta^3(Y,s)} \left(\frac{1}{2} r_{X,q} \cdot r_X^{\{\psi(Y,s)\}} \right) \circ \Pi_Y \cdot T^q(Y,s) \cdot v_{\partial B_s}^j(Y) v_{\partial B_s}^j d\lambda_{\partial B_s} \right. \quad (5)$$

$$\left. - \{ \left[\begin{smallmatrix} + \\ - \end{smallmatrix} \right] R(Y,s) \} \right| \leq \kappa_0(\Delta, n) \cdot |X-Y|_3,$$

so that, uniformly for $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$,

$$\lim_{\substack{X \rightarrow Y \\ X \in L_V^+(Y,s) \\ [X \in L_V^-(Y,s)]}} \frac{1}{4\pi} \int_{C_\Delta^3(Y,s)} \left(\frac{1}{r_X} r_{X,q} \cdot r_X \{\psi(Y,s)\} \right) \circ \pi_Y \cdot T^q(Y,s) \quad (6)$$

$$\cdot v_{\partial B_s}^j(Y) \cdot v_{\partial B_s}^j d\lambda_{\partial B_s} = [-]^+ R(Y,s).$$

If (R,X) is a reference pair for M as in [I.3.25], these assertions are valid when $\psi(Y,s) = \chi_{,4}^c(\chi_s^{-1}(Y),s)$ and, for some $i \in \{1,2,3\}$, $T^q(Y,s) = \tilde{T}_s^{iq}(Y)$, for $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$; for these choices, it follows that

$$R(Y,s) = \frac{1}{2} \cdot \frac{v^c(Y,s) \{v^c(Y,s) v^i(Y,s) - \chi_{,4}^i(\chi_s^{-1}(Y),s)\}}{\{1 - |\chi_{,4}^c(\chi_s^{-1}(Y),s)|^2\} \{1 - (v^c(Y,s))^2\}}, \quad (7)$$

for each $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$.

P R O O F. We begin by developing certain results to be used in the proofs of both (i) and (ii). We have $M \in \mathcal{M}(1)$. Suppose that $s \in \mathbb{R}$ and B_s^0 is a Lyapunov domain, with constants (a,α,d) . Choose $Y \in \partial B_s$. Let $0 \leq \Delta_1 < \Delta_2 < \frac{7}{9}d$, and suppose that g is, say, continuous and bounded on $C_{\Delta_2}^\Pi(Y,s) \cap C_{\Delta_1}^\Pi(Y,s)' \subset Y + T_{\partial B_s}(Y)$ (where $C_{\Delta_1}^\Pi(Y,s) := \emptyset$ if $\Delta_1 = 0$). According to [I.2.37.ii], we have

$$Jh_Y^{-1} = \{v_{\partial B_s}^j(Y) \cdot v_{\partial B_s}^j \circ h_Y^{-1}\}^{-1} \quad \text{on} \quad h_Y(\partial B_s \cap B_d^3(Y)),$$

so that

$$\begin{aligned}
 & \int_{C_{\Delta_2}^{\partial}(Y,s) \cap C_{\Delta_1}^{\partial}(Y,s)} \text{go} \pi_Y \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j d\lambda_{\partial B_s} \\
 &= \int_{h_Y\{C_{\Delta_2}^{\partial}(Y,s) \cap C_{\Delta_1}^{\partial}(Y,s)\}} \text{go} \pi_Y \circ h_Y^{-1} \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j \circ h_Y^{-1} \cdot Jh_Y^{-1} d\lambda_2 \quad (8) \\
 &= \int_{B_{\Delta_2}^2(0) \cap B_{\Delta_1}^2(0)} \text{go} \hat{\mathcal{H}}_Y^{-1} d\lambda_2.
 \end{aligned}$$

Next, suppose that $\tilde{X} \in L_{\tilde{Y}}(Y,s)$, $\Delta \in (0, (7/9)d)$, and $\hat{\xi} \in B_{\Delta}^2(0)$:
 since $\hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) \in Y + T_{\partial B_s}(Y)$, we have

$$r_{\tilde{X}}^2 \circ \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) = r_{\tilde{X}}^2(Y) + r_Y^2 \circ \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) = r_{\tilde{X}}^2(Y) + |\hat{\xi}|_2^2. \quad (9)$$

Further, assuming that $\hat{\xi} \neq 0$ in case $\tilde{X} = Y$, whenever $\tilde{\omega} \in \mathbb{R}^3$ we can compute, using (9),

$$r_{\tilde{X},q}^2 \circ \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) \cdot \tilde{\omega}^q = \frac{1}{\{r_{\tilde{X}}^2(Y) + |\hat{\xi}|_2^2\}^{1/2}} (\hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) - \tilde{X})^q \tilde{\omega}^q. \quad (10)$$

Since $\hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) - \tilde{X} = \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) - Y - (\tilde{X} - Y) = \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) - Y - \nu_{\partial B_s}(Y)$, where

$$\nu := \begin{cases} -1, & \text{if } \tilde{X} \in L_{\tilde{Y}}^-(Y,s), \\ +1, & \text{if } \tilde{X} \in L_{\tilde{Y}}^+(Y,s) \cup \{Y\}, \end{cases} \quad (11)$$

and we know that $\hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) - Y = \mathcal{H}_Y^{-1}(\hat{\xi}, 0) - Y = A_Y^{-1}(\hat{\xi}, 0)$, while $A_Y \nu_{\partial B_s}(Y) = e_3^{(3)}$, and A_Y preserves inner products, equality (10) can be rewritten as

$$\begin{aligned}
 r_{\tilde{X},q} \circ \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) \cdot \omega^q &= \frac{1}{\{r_{\tilde{X}}^2(Y) + |\hat{\xi}|_2^2\}^{1/2}} \{A_Y^{-1}(\hat{\xi}, 0) - r_{\tilde{X}}(Y) \cdot \nu_{\partial B_S}(Y)\}^q \omega^q \\
 &= \frac{1}{\{r_{\tilde{X}}^2(Y) + |\hat{\xi}|_2^2\}^{1/2}} \{(\hat{\xi}, 0) - r_{\tilde{X}}(Y) e_3^{(3)}\}^q (A_Y \tilde{\omega})^q \\
 &= \frac{1}{\{r_{\tilde{X}}^2(Y) + |\hat{\xi}|_2^2\}^{1/2}} \{\hat{\xi}^1 (A_Y \tilde{\omega})^1 + \hat{\xi}^2 (A_Y \tilde{\omega})^2 - r_{\tilde{X}}(Y) \cdot (A_Y \tilde{\omega})^3\}.
 \end{aligned} \tag{12}$$

Finally, recall that, whenever $\tilde{\psi} \in \mathbb{R}^3$ with $|\tilde{\psi}|_3 < 1$, we defined

$$\Gamma_{\tilde{X}}(\tilde{\psi}) := \{(1 - |\tilde{\psi}|_3^2) + (\tilde{\psi}^\ell r_{\tilde{X},\ell})^2\}^{-3/2} \quad \text{on } \mathbb{R}^3 \cap \{\tilde{X}\}',$$

so that

$$\Gamma_{\tilde{X}}(\tilde{\psi}) \circ \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}) = \{(1 - |\tilde{\psi}|_3^2) + (\tilde{\psi}^\ell r_{\tilde{X},\ell} \circ \hat{\mathcal{H}}_Y^{-1}(\hat{\xi}))^2\}^{-3/2}, \tag{13}$$

where $\hat{\xi} \neq 0$ if $\tilde{X} = Y$.

We now proceed to prove the assertions made.

(i) With notation as set down in the hypotheses, we can use (8) and (13) to write first

$$\begin{aligned}
 &\int_{C_{\Delta_2}^\partial(Y,s) \cap C_{\Delta_1}^\partial(Y,s)} \left\{ \frac{1}{r_Y} r_{Y,q} \Gamma_Y(\psi) \right\} \circ \pi_Y \cdot \omega^q \nu_{\partial B_S}^j(Y) \nu_{\partial B_S}^j d\lambda_{\partial B_S} \\
 &= \int_{B_{\Delta_2}^2(0) \cap B_{\Delta_1}^2(0)} \frac{\omega^q \cdot r_{Y,q} \circ \hat{\mathcal{H}}_Y^{-1}}{r_Y^2 \circ \hat{\mathcal{H}}_Y^{-1} \cdot \{(1 - |\tilde{\psi}|_3^2) + (\tilde{\psi}^\ell \cdot r_{Y,\ell} \circ \hat{\mathcal{H}}_Y^{-1})^2\}^{3/2}} d\lambda_2.
 \end{aligned} \tag{14}$$

Recall that we required A_Y to be a linear isometry taking $\nu_{\partial B_S}(Y)$ to $e_3^{(3)}$, $\hat{e}_1$ to $e_1^{(3)}$, and $\hat{e}_2$ to $e_2^{(3)}$, where $\{\hat{e}_1, \hat{e}_2\}$ is

some selected orthonormal basis for $T_{\partial B_s}(Y)$. Clearly, we can choose such a basis for which $\psi = (\psi \cdot \hat{e}_1) \hat{e}_1 + (\psi \cdot \nu_{\partial B_s}(Y)) \nu_{\partial B_s}(Y)$, i.e., for which $\psi \cdot \hat{e}_2 = 0$, so that, for the corresponding A_Y , we have $(A_Y \psi)^2 = (A_Y \psi) \cdot e_2^{(3)} = (A_Y \psi) \cdot (A_Y \hat{e}_2) = \psi \cdot \hat{e}_2 = 0$. Then, taking $\tilde{X} = Y$ and, successively, $\tilde{\omega} = \omega$ and $\tilde{\omega} = \psi$ in (12), and using (9) with $\tilde{X} = Y$, (14) produces

$$\begin{aligned} & \int_{C_{\Delta_2}^3(Y,s) \cap C_{\Delta_1}^3(Y,s)} \left(\frac{1}{r_Y} r_{Y,q} \Gamma_Y(\psi) \right) \circ \pi_Y \cdot \omega^q \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j d\lambda_{\partial B_s} \\ &= \int_{B_{\Delta_2}^2(0) \cap B_{\Delta_1}^2(0)} \frac{|\hat{\xi}|_2^{-1} \{ \hat{\xi}^1 (A_Y \omega)^1 + \hat{\xi}^2 (A_Y \omega)^2 \}}{|\hat{\xi}|_2^2 \{ (1-|\psi|_3^2) + (|\hat{\xi}|_2^{-1} \cdot \hat{\xi}^1 \cdot (A_Y \psi)^1)^2 \}^{3/2}} d\lambda_2(\hat{\xi}) \\ &= \int_{\Delta_1}^{\Delta_2} \int_0^{2\pi} \frac{\{ (A_Y \omega)^1 \cos \theta + (A_Y \omega)^2 \sin \theta \}}{\rho \{ (1-|\psi|_3^2) + ((A_Y \psi)^1)^2 \cos^2 \theta \}^{3/2}} d\theta d\rho = 0, \end{aligned} \quad (15)$$

the latter equality holding because, as it is easy to check,

$$\int_0^{2\pi} \frac{\sin \theta}{\{a+b \cos^2 \theta\}^{3/2}} d\theta = \int_0^{2\pi} \frac{\cos \theta}{\{a+b \cos^2 \theta\}^{3/2}} d\theta = 0, \quad (16)$$

for $a > 0$ and $b \geq 0$. This completes the proof of (i).

(ii) Let $n > 0$ and $\Delta \in (0, (7/9)d_K)$. Select first $s \in K$, then $Y \in \partial B_s$, then $X \in L_V^+(Y,s) [X \in L_V^-(Y,s)]$, with $|X-Y|_3 < n$ (note that $X \neq Y$; in fact, $X \notin C_{\Delta}^3(Y,s)$). For brevity, let us write

$$I_{\Delta}(Y,s;X) := \frac{1}{4\pi} \int_{C_{\Delta}^{\partial}(Y,s)} \left[\frac{1}{2} r_{X,q} \cdot r_X \{ \psi(Y,s) \} \right] \circ \pi_Y \cdot T^q(Y,s) \cdot \nu_{\partial B_s}^j(Y) \cdot \nu_{\partial B_s}^j d\lambda_{\partial B_s}. \quad (17)$$

Then, again using (8) and (13),

$$I_{\Delta}(Y,s;X) = \frac{1}{4\pi} \int_{B_{\Delta}^2(0)} \frac{T^q(Y,s) \cdot r_{X,q} \circ \hat{K}_Y^{-1}}{r_X^2 \circ \hat{K}_Y^{-1} \cdot \{ (1 - |\psi(Y,s)|_3^2) + (\psi^{\ell}(Y,s) \cdot r_{X,\ell} \circ \hat{K}_Y^{-1})^2 \}^{3/2}} d\lambda_2. \quad (18)$$

Now, because $T(Y,s) \in T_{\partial B_s}(Y)$, it is clear from the properties of A_Y that

$$\begin{aligned} (A_Y T(Y,s))^3 &= (A_Y T(Y,s)) \circ e_3^{(3)} = (A_Y T(Y,s)) \circ (A_Y \nu_{\partial B_s}(Y)) \\ &= T(Y,s) \circ \nu_{\partial B_s}(Y) = 0, \end{aligned}$$

so, using (12),

$$r_{X,q} \circ \hat{K}_Y^{-1}(\xi) \cdot T^q(Y,s) = \frac{1}{\{\delta^2 + |\xi|_2^2\}^{1/2}} \{ \xi^1 (A_Y T(Y,s))^1 + \xi^2 (A_Y T(Y,s))^2 \}, \quad (19)$$

having written

$$\delta := |X-Y|_3.$$

Now, obviously, we may, and shall, suppose that $T(Y,s) \neq 0$, for otherwise, $I_{\Delta}(Y,s;X) = R(Y,s) = 0$. Assume next that $\psi(Y,s) = \{\psi(Y,s) \circ \nu(Y,s)\} \nu(Y,s)$ (which includes the possibility that $\psi(Y,s) = 0$). Then $A_Y \psi(Y,s) = \{\psi(Y,s) \circ \nu(Y,s)\} e_3^{(3)}$, whence, with (12),

$$r_{X,q} \hat{\psi}_Y^{-1}(\hat{\xi}) \cdot \psi^q(Y,s) = \frac{1}{\{\delta^2 + |\hat{\xi}|_2^2\}^{1/2}} \{ [\bar{+}] \delta \cdot \psi(Y,s) \bullet v(Y,s) \}, \quad \hat{\xi} \in B_\Delta^2(0).$$

Using the latter equality, (9), (18), and (19), we compute

$$\begin{aligned} I_\Delta(Y,s;X) &= \frac{1}{4\pi} \int_{B_\Delta^2(0)} \frac{\{\hat{\xi}^1(A_Y T(Y,s))^1 + \hat{\xi}^2(A_Y T(Y,s))^2\}}{\{\delta^2 + |\hat{\xi}|_2^2\}^{3/2} \{(1 - |\psi(Y,s)|_3^2) + \frac{1}{\delta^2 + |\hat{\xi}|_2^2} \cdot \{\delta \cdot \psi(Y,s) \bullet v(Y,s)\}^2\}^{3/2}} d\lambda_2(\hat{\xi}) \\ &= \frac{1}{4\pi} \int_0^\Delta \frac{\rho^2}{\{(1 - |\psi(Y,s)|_3^2)(\delta^2 + \rho^2) + (\delta \cdot \psi(Y,s) \bullet v(Y,s))^2\}^{3/2}} d\rho \\ &\quad \cdot \int_0^{2\pi} \{(A_Y T(Y,s))^1 \cos \theta + (A_Y T(Y,s))^2 \sin \theta\} d\theta = 0, \end{aligned}$$

so that $I_\Delta(Y,s;X)$ again vanishes along with $R(Y,s)$.

Finally, we consider the case in which $\psi(Y,s) \neq \{\psi(Y,s) \bullet v(Y,s)\} v(Y,s)$ (and, of course, $T(Y,s) \neq 0$). Then, also, $\psi(Y,s) \neq 0$, and the angle $\theta_\psi(Y,s) \in [0, \pi]$ which is determined by the requirement

$$\cos \theta_\psi(Y,s) = \frac{\psi(Y,s) \bullet v(Y,s)}{|\psi(Y,s)|_3} \quad (20)$$

is neither 0 nor π , so $R(Y,s)$ is to be computed from the quotient appearing in (4). In the latter, $\theta_T(Y,s) \in [0, \pi]$ is, by hypothesis, determined by the condition

$$\cos \theta_T(Y,s) = \frac{T(Y,s) \bullet \hat{e}_1(Y,s)}{|T(Y,s)|_3}, \quad (21)$$

where

$$\hat{e}_1(Y, s) \quad (22)$$

$$:= \{\psi(Y, s) - \{\psi(Y, s) \bullet v(Y, s)\} v(Y, s)\} / \|\psi(Y, s) - \{\psi(Y, s) \bullet v(Y, s)\} v(Y, s)\|_3,$$

$\psi(Y, s) - \{\psi(Y, s) \bullet v(Y, s)\} v(Y, s)$ being the orthogonal projection of $\psi(Y, s)$ onto $T_{\partial B_s}(Y)$. Now, setting

$$\begin{aligned} \hat{e}_2(Y, s) &:= \{v_{\partial B_s}(Y) \times \hat{e}_1(Y, s)\} / \|v_{\partial B_s}(Y) \times \hat{e}_1(Y, s)\|_3 \\ &= \{v_{\partial B_s}(Y) \times \psi(Y, s)\} / \|v_{\partial B_s}(Y) \times \psi(Y, s)\|_3, \end{aligned} \quad (23)$$

it is clear that $\{\hat{e}_1(Y, s), \hat{e}_2(Y, s)\}$ is an orthonormal basis for $T_{\partial B_s}(Y)$, and we shall suppose, as we may without loss, that

$$A_Y \hat{e}_i(Y, s) = e_i^{(3)}, \quad \text{for } i = 1 \text{ and } 2. \quad (24)$$

We can write

$$\begin{aligned} \psi(Y, s) &= \{\psi(Y, s) \bullet v(Y, s)\} v(Y, s) + \{\psi(Y, s) - \{\psi(Y, s) \bullet v(Y, s)\} v(Y, s)\} \\ &= \|\psi(Y, s)\|_3 \cos \Theta_\psi(Y, s) v(Y, s) + \|\psi(Y, s) - \{\psi(Y, s) \bullet v(Y, s)\} \\ &\quad \cdot v(Y, s)\|_3 \hat{e}_1(Y, s), \end{aligned}$$

and

$$\begin{aligned} \|\psi(Y, s) - \{\psi(Y, s) \bullet v(Y, s)\} v(Y, s)\|_3 &= \{\|\psi(Y, s)\|_3^2 - \{\psi(Y, s) \bullet v(Y, s)\}^2\}^{1/2} \\ &= \|\psi(Y, s)\|_3 \{1 - \cos^2 \Theta_\psi(Y, s)\}^{1/2} \quad (25) \\ &= \|\psi(Y, s)\|_3 \sin \Theta_\psi(Y, s), \end{aligned}$$

so

$$\psi(Y,s) = |\psi(Y,s)|_3 \{ \cos \Theta_\psi(Y,s) v(Y,s) + \sin \Theta_\psi(Y,s) \hat{e}_1(Y,s) \},$$

producing

$$A_Y \psi(Y,s) = |\psi(Y,s)|_3 \{ \cos \Theta_\psi(Y,s) e_3^{(3)} + \sin \Theta_\psi(Y,s) e_1^{(3)} \}.$$

With this, (12) implies that

$$\begin{aligned} \psi^q(Y,s) r_{X,q} \circ \hat{\kappa}_Y^{-1}(\hat{\xi}) &= \frac{|\psi(Y,s)|_3}{\{\delta^2 + |\hat{\xi}|_2^2\}^{1/2}} \{ \hat{\xi}^1 \sin \Theta_\psi(Y,s) [\bar{+}]^\delta \cos \Theta_\psi(Y,s) \}, \\ &\text{for each } \hat{\xi} \in B_\Delta^2(0). \end{aligned} \quad (26)$$

Since $T(Y,s) \in T_{\partial B_s}(Y)$,

$$\begin{aligned} T(Y,s) &= \{T(Y,s) \bullet \hat{e}_1(Y,s)\} \hat{e}_1(Y,s) + \{T(Y,s) \bullet \hat{e}_2(Y,s)\} \hat{e}_2(Y,s) \\ &= |T(Y,s)|_3 \{ \cos \Theta_T(Y,s) \hat{e}_1(Y,s) + \sin \Theta_T(Y,s) \hat{e}_2(Y,s) \}, \end{aligned}$$

with $|v_T| = 1$, whence

$$A_Y T(Y,s) = |T(Y,s)|_3 \{ \cos \Theta_T(Y,s) e_1^{(3)} + \sin \Theta_T(Y,s) e_2^{(3)} \};$$

once more using (12), it follows that

$$\begin{aligned} T^q(Y,s) r_{X,q} \circ \hat{\kappa}_Y^{-1}(\hat{\xi}) &= \frac{|T(Y,s)|_3}{\{\delta^2 + |\hat{\xi}|_2^2\}^{1/2}} \{ \hat{\xi}^1 \cos \Theta_T(Y,s) + \hat{\xi}^2 v_T \sin \Theta_T(Y,s) \}, \\ &\text{for each } \hat{\xi} \in B_\Delta^2(0). \end{aligned} \quad (27)$$

Having (26) and (27), (18) becomes

$$I_{\Delta}(Y,s;X) = \frac{1}{4\pi} \int_{B_{\Delta}^2(0)} \frac{|T|_3 \cdot \{\xi^1 \cos \Theta_T + \xi^2 i_T \sin \Theta_T\}}{(\delta^2 + |\xi|_2^2)^{\frac{3}{2}}} \left\{ (1 - |\psi|_3^2) + \frac{|\psi|_3}{(\delta^2 + |\xi|_2^2)^{\frac{1}{2}}} (\xi^1 \sin \Theta_{\psi} [\bar{+}] \delta \cos \Theta_{\psi}) \right\}^{\frac{3}{2}} d\lambda_2(\xi)$$

$$= \frac{|T|_3}{4\pi} \int_0^{\Delta} \int_0^{2\pi} \frac{\rho^2 \{\cos \theta \cos \Theta_T + i_T \sin \theta \sin \Theta_T\}}{\{(1 - |\psi|_3^2)(\delta^2 + \rho^2) + |\psi|_3^2 (\rho \cos \theta \sin \Theta_{\psi} [\bar{+}] \delta \cos \Theta_{\psi})^2\}^{\frac{3}{2}}} d\theta d\rho, \quad (28)$$

in which we have omitted the arguments (Y,s) throughout, as we shall usually do. Since, for α, β , and $\gamma \in \mathbb{R}$, with $\alpha \neq 0$,

$$\int_0^{2\pi} \frac{\sin \theta}{\{\alpha + (\beta \cos \theta + \gamma)^2\}^{3/2}} d\theta = - \int_{-\pi}^{\pi} \frac{\sin \theta}{\{\alpha + (\beta \cos \theta - \gamma)^2\}^{3/2}} d\theta = 0,$$

and

$$\int_0^{2\pi} \frac{\cos \theta}{\{\alpha + (\beta \cos \theta + \gamma)^2\}^{3/2}} d\theta = \int_0^{\pi} \frac{\cos \theta}{\{\alpha + (\beta \cos \theta + \gamma)^2\}^{3/2}} d\theta$$

$$- \int_0^{\pi} \frac{\cos \theta}{\{\alpha + (\beta \cos \theta - \gamma)^2\}^{3/2}} d\theta$$

$$\begin{aligned}
 &= -2 \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{\alpha + (\beta \sin \theta - \gamma)^2\}^{3/2}} d\theta \\
 &= 2 \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{\alpha + (\beta \sin \theta + \gamma)^2\}^{3/2}} d\theta,
 \end{aligned}$$

(28) can be rewritten as

$$\begin{aligned}
 I_{\Delta}(Y, s; X) &= [\bar{+}] \frac{1}{2\pi} |T|_3 \cos \Theta_T \int_{-\pi/2}^{\pi/2} \int_0^{\Delta} \\
 &\quad \frac{\rho^2 \sin \theta}{\{(1 - |\psi|_3^2)(\delta^2 + \rho^2) + |\psi|_3^2(\rho \sin \Theta_{\psi} \sin \theta + \delta \cos \Theta_{\psi})^2\}^{3/2}} d\theta d\rho.
 \end{aligned} \tag{29}$$

The inner integral in (29) can be evaluated in an elementary manner:

introducing the notation

$$\hat{\mu}_0^2(Y, s) := \frac{|\psi(Y, s)|_3^2 \sin^2 \Theta_{\psi}(Y, s)}{1 - |\psi(Y, s)|_3^2 \cos^2 \Theta_{\psi}(Y, s)}, \quad \text{with } \hat{\mu}_0(Y, s) \geq 0, \tag{30}$$

$$\alpha_0(\theta; Y, s) := \{1 - |\psi(Y, s)|_3^2\} \{1 - |\psi(Y, s)|_3^2 \sin^2 \Theta_{\psi}(Y, s) \cdot \cos^2 \theta\}, \tag{31}$$

$$\beta_0(\theta; Y, s) := |\psi(Y, s)|_3^2 \sin \Theta_{\psi}(Y, s) \cdot \cos \Theta_{\psi}(Y, s) \cdot \sin \theta, \tag{32}$$

$$\gamma_0(\theta; Y, s) := \{1 - |\psi(Y, s)|_3^2 \cos^2 \Theta_{\psi}(Y, s)\} \{1 - \hat{\mu}_0^2(Y, s) \cos^2 \theta\}, \tag{33}$$

and once again omitting the arguments (Y, s) , the integrand in (29)

can be rearranged to give

$$\begin{aligned}
 I_{\Delta}(Y,s;X) &= [\bar{+}] \frac{1}{2\pi} |T|_3 \cos \Theta_T \cdot \{1-|\psi|_3^2 \cos^2 \Theta_{\psi}\}^{3/2} \\
 &\cdot \int_{-\pi/2}^{\pi/2} \sin \theta \cdot (1-\hat{\mu}_0^2 \cos^2 \theta)^{3/2} \\
 &\cdot \int_0^{\Delta} \frac{\rho^2 d\rho}{\{\alpha_0(\theta) \cdot \delta^2 + (\gamma_0(\theta)\rho + \beta_0(\theta)\delta)^2\}^{3/2}} d\theta.
 \end{aligned} \tag{34}$$

Now, for $\alpha > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$, (IV.A.1) implies that

$$\begin{aligned}
 \int_0^{\Delta} \frac{\rho^2 d\rho}{\{\alpha + (\gamma\rho + \beta)^2\}^{3/2}} &= \frac{1}{\gamma^3} \left\{ \ln \frac{(\gamma\Delta + \beta) + \sqrt{\alpha + (\gamma\Delta + \beta)^2}}{\beta + \sqrt{\alpha + \beta^2}} \right. \\
 &\quad \left. + \frac{\left(\frac{\beta^2}{\alpha} - 1\right)(\gamma\Delta + \beta) + 2\beta}{\sqrt{\alpha + (\gamma\Delta + \beta)^2}} - \frac{\beta}{\alpha} \cdot \sqrt{\alpha + \beta^2} \right\}.
 \end{aligned} \tag{35}$$

Using inequality (2), it is clear that

$$\{1 - (\lambda^*)^2\}^2 \leq \alpha_0 \leq 1 \tag{36}$$

and

$$\hat{\mu}_0^2 = |\psi|_3^2 \frac{1 - \cos^2 \Theta_{\psi}}{1 - |\psi|_3^2 \cos^2 \Theta_{\psi}} < (\lambda^*)^2, \tag{37}$$

so also

$$\{1 - (\lambda^*)^2\}^2 \leq \gamma_0 \leq 1. \tag{38}$$

Since $\delta > 0$ and $\lambda^* < 1$, (36) and (38) show that we may use (35) to evaluate the inner integral in (34), producing

$$I_{\Delta}(Y, s; X) = [\bar{+}] \frac{|T(Y, s)|_3 \cos \varphi_T(Y, s)}{2\pi \{1 - |\psi(Y, s)|_3^2 \cos^2 \varphi_{\psi}(Y, s)\}^{3/2}} \quad (39)$$

$$\cdot \{I_1(Y, s) + I_2(Y, s) + I_3(Y, X) + I'_{\Delta}(Y, s; X) + I''_{\Delta}(Y, s; X)\},$$

wherein

$$I_1(Y, s) := - \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{1 - \hat{\mu}_0^2 \cos^2 \theta\}^{3/2}} \cdot \ln \{s_0(\theta) + \sqrt{\alpha_0(\theta) + s_0^2(\theta)}\} d\theta, \quad (40)$$

$$I_2(Y, s) := - \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{1 - \hat{\mu}_0^2 \cos^2 \theta\}^{3/2}} \cdot \frac{s_0(\theta)}{\alpha_0(\theta)} \cdot \sqrt{\alpha_0(\theta) + s_0^2(\theta)} d\theta, \quad (41)$$

$$I_3(Y, X) := -\ln \delta \cdot \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{1 - \hat{\mu}_0^2 \cos^2 \theta\}^{3/2}} d\theta, \quad (42)$$

$$I'_{\Delta}(Y, s; X)$$

$$:= \int_{-\pi/2}^{\pi/2} \frac{\sin \theta \cdot \ln \{(\gamma_0(\theta)\Delta + s_0(\theta)\delta) + \sqrt{\alpha_0(\theta)\delta^2 + (\gamma_0(\theta)\Delta + s_0(\theta)\delta)^2}\}}{\{1 - \hat{\mu}_0^2 \cos^2 \theta\}^{3/2}} d\theta, \quad (43)$$

and

$$I''_{\Delta}(Y, s; X)$$

$$:= \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{1 - \hat{\mu}_0^2 \cos^2 \theta\}^{3/2}} \cdot \frac{\left\{ \frac{s_0^2(\theta)}{\alpha_0(\theta)} - 1 \right\} \{ \gamma_0(\theta)\Delta + s_0(\theta)\delta \} + 2s_0(\theta)\delta}{\sqrt{\alpha_0(\theta)\delta^2 + (\gamma_0(\theta)\Delta + s_0(\theta)\delta)^2}} d\theta. \quad (44)$$

Obviously, $I_3(Y, X) = 0$. Moreover, $I_1(Y, s)$ and $I_2(Y, s)$ can be explicitly evaluated, as follows: first, a short manipulation shows

that

$$\alpha_0(\theta; Y, s) + \beta_0^2(\theta; Y, s) = \mu_1^2(Y, s) \cdot \mu_2^2(Y, s) \cdot \{1 - \mu_0^2(Y, s) \cos^2 \theta\}, \quad (45)$$

in which

$$\mu_1^2(Y, s) := 1 - |\psi(Y, s)|_3^2 \sin^2 \Theta_\psi(Y, s) \quad (\mu_1(Y, s) > 0) \quad (46)$$

and

$$\mu_2^2(Y, s) := 1 - |\psi(Y, s)|_3^2 \cos^2 \Theta_\psi(Y, s) \quad (\mu_2(Y, s) > 0). \quad (47)$$

Then, with

$$\mu_3(Y, s) := |\psi(Y, s)|_3^2 \cdot \sin \Theta_\psi(Y, s) \cdot \cos \Theta_\psi(Y, s), \quad (48)$$

we find that

$$\begin{aligned} & \frac{\beta_0(\theta; Y, s)}{\alpha_0(\theta; Y, s)} \cdot \sqrt{\alpha_0(\theta; Y, s) + \beta_0^2(\theta; Y, s)} \\ &= \frac{\mu_1(Y, s) \cdot \mu_2(Y, s) \cdot \mu_3(Y, s)}{\{1 - |\psi(Y, s)|_3^2\}} \cdot \frac{\sin \theta \cdot \{1 - \mu_0^2(Y, s) \cos^2 \theta\}^{1/2}}{1 - |\psi(Y, s)|_3^2 \sin^2 \Theta_\psi(Y, s) \cdot \cos^2 \theta} \end{aligned} \quad (49)$$

and

$$\begin{aligned} \beta_0(\theta; Y, s) + \sqrt{\alpha_0(\theta; Y, s) + \beta_0^2(\theta; Y, s)} &= \mu_3(Y, s) \cdot \sin \theta + \mu_1(Y, s) \cdot \mu_2(Y, s) \\ &\quad \cdot \{1 - \mu_0^2(Y, s) \cos^2 \theta\}^{1/2}. \end{aligned} \quad (50)$$

Thus, from (40) and (41), respectively,

$$I_1(Y, s) = - \int_{-\pi/2}^{\pi/2} \frac{\sin \theta \cdot \ln \{ \mu_3 \sin \theta + \mu_1 \cdot \mu_2 (1 - \mu_0^2 \cos^2 \theta)^{1/2} \}}{\{1 - \mu_0^2 \cos^2 \theta\}^{3/2}} d\theta, \quad (51)$$

and

$$I_2(Y,s) = - \frac{\mu_1 \cdot \mu_2 \cdot \mu_3}{1 - |\psi|_3^2} \cdot \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \theta}{\{1 - \mu_0^2 \cos^2 \theta\} \{1 - |\psi|_3^2 \sin^2 \Theta_\psi \cdot \cos^2 \theta\}} d\theta. \quad (52)$$

Now, if $\Theta_\psi(Y,s) = \pi/2$, it is clear that $I_1(Y,s) = I_2(Y,s) = 0$, since then $\mu_3(Y,s) = 0$. Otherwise, i.e., for $\Theta_\psi(Y,s) \neq \pi/2$,[†] the integrals appearing in (51) and (52) can be obtained from (IV.A.22) and (IV.A.21), respectively, whence

$$\begin{aligned} I_1(Y,s) &= -\pi \cdot \frac{\mu_1 \cdot \mu_2}{\mu_3} \cdot \left\{ \frac{\mu_2}{(1 - |\psi|_3^2)^{1/2}} - \frac{1}{\mu_1} \right\} \\ &= -\pi \cdot \frac{\mu_2}{\mu_3 (1 - |\psi|_3^2)} \cdot \{ \mu_1 \cdot \mu_2 \cdot (1 - |\psi|_3^2)^{1/2} - (1 - |\psi|_3^2) \}, \end{aligned}$$

and

$$\begin{aligned} I_2(Y,s) &= - \frac{\mu_1 \cdot \mu_2 \cdot \mu_3}{(1 - |\psi|_3^2)} \cdot \frac{\pi}{\mu_3} \cdot \{ \mu_1 \cdot \mu_2^2 - (1 - |\psi|_3^2)^{1/2} \cdot \mu_2 \} \\ &= -\pi \cdot \frac{\mu_2}{\mu_3 (1 - |\psi|_3^2)} \cdot \{ \mu_1^2 \cdot \mu_2^2 - (1 - |\psi|_3^2) \cdot \mu_1 \cdot \mu_2 \}, \end{aligned}$$

so

[†] Recall that we have already supposed that $|\psi(Y,s)|_3 \neq 0$ and $\Theta_\psi(Y,s) \notin \{0, \pi\}$.

$$\begin{aligned} I_1(Y,s) + I_2(Y,s) &= -\pi \cdot \frac{\mu_2}{\mu_3 \cdot (1-|\psi|_3^2)} \cdot \{\mu_1^2 \cdot \mu_2^2 - 1 + |\psi|_3^2\} \\ &= -\pi \cdot \frac{\mu_2 \cdot \mu_3}{1-|\psi|_3^2}, \end{aligned}$$

having noted that $\mu_1^2 \cdot \mu_2^2 = 1 - |\psi|_3^2 + \mu_3^2$. The latter equality gives

$$\begin{aligned} &\frac{|T(Y,s)|_3 \cos \Theta_T(Y,s)}{2\pi \{1-|\psi(Y,s)|_3^2 \cos^2 \Theta_\psi(Y,s)\}^{3/2}} \{I_1(Y,s) + I_2(Y,s)\} \\ &= -\frac{1}{2} \cdot \frac{|T(Y,s)|_3 \cos \Theta_T(Y,s) \cdot |\psi(Y,s)|_3^2 \sin \Theta_\psi(Y,s) \cos \Theta_\psi(Y,s)}{\{1-|\psi(Y,s)|_3^2 \cos^2 \Theta_\psi(Y,s)\} \cdot \{1-|\psi(Y,s)|_3^2\}} \\ &= -R(Y,s). \end{aligned}$$

Thus, from (39)

$$\begin{aligned} &|I_\Delta(Y,s;X) - \{I_\Delta^+(Y,s)\}| \\ &\leq \frac{M_T}{2\pi \{1-(\lambda^*)^2\}^{3/2}} \cdot \{|I_\Delta'(Y,s;X)| + |I_\Delta''(Y,s;X)|\}, \end{aligned} \tag{53}$$

where

$$M_T := \sup \{|T(\tilde{Y}, \tilde{s})|_3 \mid \tilde{s} \in K, \tilde{Y} \in \partial B_{\tilde{s}}\}.$$

Next, for each $\theta \in [-\pi/2, \pi/2]$ and $(\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$, define $f_\Delta(\cdot; \theta, \tilde{Y}, \tilde{s})$ and $g_\Delta(\cdot; \theta, \tilde{Y}, \tilde{s})$ on $\mathbb{R}$ by

$$f_{\Delta}(\rho; \theta, \tilde{Y}, \tilde{s}) := \frac{\left\{ \frac{\beta_0^2(\theta; \tilde{Y}, \tilde{s})}{\alpha_0(\theta; \tilde{Y}, \tilde{s})} - 1 \right\} \{ \gamma_0(\theta; \tilde{Y}, \tilde{s}) \Delta + \beta_0(\theta; \tilde{Y}, \tilde{s}) \rho \} + 2 \beta_0(\theta; \tilde{Y}, \tilde{s}) \rho}{\{ \alpha_0(\theta; \tilde{Y}, \tilde{s}) \rho^2 + \{ \gamma_0(\theta; \tilde{Y}, \tilde{s}) \Delta + \beta_0(\theta; \tilde{Y}, \tilde{s}) \rho \}^2 \}^{1/2}}, \quad (54)$$

and

$$g_{\Delta}(\rho; \theta, \tilde{Y}, \tilde{s}) := \ln \tilde{g}_{\Delta}(\rho; \theta, \tilde{Y}, \tilde{s}), \quad (55)$$

wherein

$$\begin{aligned} \tilde{g}_{\Delta}(\rho; \theta, \tilde{Y}, \tilde{s}) &:= \{ \gamma_0(\theta; \tilde{Y}, \tilde{s}) \Delta + \beta_0(\theta; \tilde{Y}, \tilde{s}) \rho \} \\ &\quad + \{ \alpha_0(\theta; \tilde{Y}, \tilde{s}) \rho^2 + \{ \gamma_0(\theta; \tilde{Y}, \tilde{s}) \Delta + \beta_0(\theta; \tilde{Y}, \tilde{s}) \rho \}^2 \}^{1/2}. \end{aligned} \quad (56)$$

We are using here the fact that α_0 and γ_0 are bounded below by the positive number $\{1 - (\lambda^*)^2\}^2$ (cf., (36) and (38)), so that the denominator in (54) and the function $\tilde{g}_{\Delta}$ are positive for all values of the variables involved. Indeed, simple reasoning shows that the inequalities

$$\begin{aligned} \alpha_0 \rho^2 + (\gamma_0 \Delta + \beta_0 \rho)^2 &\geq \alpha_0 \left(\frac{\beta_0 \gamma_0 \Delta}{\alpha_0 + \beta_0^2} \right)^2 + \left(\gamma_0 \Delta + \beta_0 \frac{\beta_0 \gamma_0 \Delta}{\alpha_0 + \beta_0^2} \right)^2 \\ &= \left\{ \frac{\alpha_0 \beta_0^2 + (\alpha_0 + 2\beta_0^2)^2}{(\alpha_0 + \beta_0^2)^2} \right\} \cdot \gamma_0^2 \Delta^2 \\ &\geq \{1 + (\lambda^*)^4\}^{-2} \cdot \{1 - (\lambda^*)^2\}^8 \Delta^2 > 0, \end{aligned} \quad (57)$$

and

$$\begin{aligned}
 \tilde{g}_\Delta(\rho) &\geq \tilde{g}_\Delta \left(-\frac{2\beta_0\gamma_0\Delta}{\alpha_0+\beta_0^2} \right) \\
 &= \gamma_0\Delta - \frac{2\beta_0^2\gamma_0\Delta}{\alpha_0+\beta_0^2} + \left\{ \alpha_0 \left(\frac{2\beta_0\gamma_0\Delta}{\alpha_0+\beta_0^2} \right)^2 + \left\{ \gamma_0\Delta - \frac{2\beta_0^2\gamma_0\Delta}{\alpha_0+\beta_0^2} \right\}^2 \right\}^{1/2} \\
 &= \gamma_0\Delta - \frac{2\beta_0^2\gamma_0\Delta}{\alpha_0+\beta_0^2} + \gamma_0\Delta = \frac{2\alpha_0\gamma_0}{\alpha_0+\beta_0^2} \cdot \Delta \\
 &\geq 2\{1+(\lambda^*)^4\}^{-1} \cdot \{1-(\lambda^*)^2\}^4 \cdot \Delta > 0,
 \end{aligned} \tag{58}$$

hold for each $\rho \in \mathbb{R}$ and all pertinent values of θ , $\tilde{Y}$, and $\tilde{s}$ (which have not been explicitly displayed); in these derivations, we have made use of (36), (38), and the obvious bound

$$|\beta_0| \leq (\lambda^*)^2. \tag{59}$$

Upon computing the derivatives of f_Δ and g_Δ (with respect to their principal arguments) and recalling the positive η , we obtain the estimates

$$\begin{aligned}
 |f'_\Delta(\rho; \theta, \tilde{Y}, \tilde{s})| &= \left| \frac{2\beta_0\gamma_0\Delta + (\beta_0^2 + \alpha_0)\rho}{\{\alpha_0\rho^2 + (\gamma_0\Delta + \beta_0\rho)^2\}^{3/2}} \cdot \gamma_0\Delta \right| \\
 &\leq \{1+(\lambda^*)^4\}^3 \cdot \{1-(\lambda^*)^2\}^{-12} \cdot \frac{1}{\Delta^3} \cdot \{2(\lambda^*)^2 d_K + (1+(\lambda^*)^4)\eta\} \cdot \Delta \tag{60} \\
 &= \kappa_1(\eta, \lambda^*, d_K) \cdot \frac{1}{\Delta^2},
 \end{aligned}$$

and

$$\begin{aligned}
 |g'_\Delta(\rho; \theta, \tilde{Y}, \tilde{s})| &= \frac{1}{\tilde{g}_\Delta(\rho; \theta, \tilde{Y}, \tilde{s})} \cdot |\tilde{g}'_\Delta(\rho; \theta, \tilde{Y}, \tilde{s})| \\
 &= \frac{1}{\tilde{g}_\Delta(\rho; \theta, \tilde{Y}, \tilde{s})} \cdot \left| \beta_0 + \frac{\alpha_0 \rho + \beta_0 (\gamma_0 \Delta + \beta_0 \rho)}{\{\alpha_0 \rho^2 + (\gamma_0 \Delta + \beta_0 \rho)^2\}^{1/2}} \right| \\
 &\leq \frac{1}{2\Delta} \cdot \{1 + (\lambda^*)^4\} \cdot \{1 - (\lambda^*)^2\}^{-4} \\
 &\quad \cdot \{(\lambda^*)^2 + \{1 + (\lambda^*)^4\} \cdot \{1 - (\lambda^*)^2\}^{-4} \cdot \frac{1}{\Delta} \cdot \{\eta + (\lambda^*)^2 \{d_K + (\lambda^*)^2 \eta\}\}\} \\
 &\leq \kappa_2(\eta, \lambda^*, d_K) \cdot \frac{1}{\Delta^2},
 \end{aligned} \tag{61}$$

for $|\rho| \leq \eta$, $\theta \in [-\pi/2, \pi/2]$, and

$$(\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\},$$

having applied (36), (38), and (57)-(59).

We can now derive an estimate of the required form for the right-hand side of (53). Recall that we have imposed the requirement $\delta := |X - Y|_3 < \eta$. Upon referring to (31)-(33), it is clear that $f_\Delta(0; \theta, Y, s) = \{\beta_0^2(\theta; Y, s) / \alpha_0(\theta; Y, s)\} - 1$ and $g_\Delta(0; \theta, Y, s) = \ln \{2\Delta \cdot \gamma_0(\theta; Y, s)\}$ are even in the variable θ , so that, from (43) and (44), applying the mean-value theorem and using (60) and (61), we can write

$$\begin{aligned}
 |I'_\Delta(Y, s; X)| &= \left| \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{1 - \beta_0^2 \cos^2 \theta\}^{3/2}} \{g_\Delta(\delta; \theta, Y, s) - g_\Delta(0; \theta, Y, s)\} d\theta \right| \\
 &\leq \tilde{I} \cdot \delta \cdot \sup \{|g'_\Delta(\rho; \theta, \tilde{Y}, \tilde{s})| \mid \rho \in [0, \eta], \theta \in [-\pi/2, \pi/2], \\
 &\quad (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}\}
 \end{aligned}$$

$$\leq \tilde{I} \cdot \kappa_2(n, \lambda^*, d_K) \cdot \frac{1}{\Delta^2} \cdot \delta, \quad (62)$$

and

$$\begin{aligned} |I''_{\Delta}(Y, s; X)| &= \left| \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{\{1 - \hat{u}_0^2 \cos^2 \theta\}^{3/2}} \{f_{\Delta}(\delta; \theta, Y, s) - f_{\Delta}(0; \theta, Y, s)\} d\theta \right| \\ &\leq \tilde{I} \cdot \delta \cdot \sup \{ |f'_{\Delta}(\rho; \theta, Y, s)| \mid \rho \in [0, n], \theta \in [-\pi/2, \pi/2], \\ &\quad (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\} \} \\ &\leq \tilde{I} \cdot \kappa_1(n, \lambda^*, d_K) \cdot \frac{1}{\Delta^2} \cdot \delta, \end{aligned} \quad (63)$$

in which

$$\tilde{I} := \int_{-\pi/2}^{\pi/2} \frac{|\sin \theta|}{\{1 - (\lambda^*)^2 \cos^2 \theta\}^{3/2}} d\theta$$

(recall that $\hat{u}_0^2 < (\lambda^*)^2$; cf., (37)). Combining (62), (63), and (57),

$$\begin{aligned} |I_{\Delta}(Y, s; X) - \begin{bmatrix} + \\ - \end{bmatrix} R(Y, s)| &\leq \frac{M_T \tilde{I}}{2\pi \{1 - (\lambda^*)^2\}^{3/2}} \{\kappa_1(n, \lambda^*, d_K) + \kappa_2(n, \lambda^*, d_K)\} \\ &\quad \cdot \frac{r_X(Y)}{\Delta^2}. \end{aligned}$$

Now, the latter inequality clearly holds not only in the present case, but also if either $T(Y, s) = 0$ or $\psi(Y, s) = \{\psi(Y, s) \vee \psi(Y, s)\} \vee \psi(Y, s)$, when, as we have seen, $I_{\Delta}(Y, s; X) = R(Y, s) = 0$. Consequently, the first assertion of (ii), viz., inequality (5), has clearly been proven.

The existence of the limit in (6), uniformly for $(Y, s) \in$

$\bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$, follows directly from (5).

Finally, consider the choices $\psi(Y,s) = x_{,4}^c(x_s^{-1}(Y),s)$ and, for some $i \in \{1,2,3\}$, $T^q(Y,s) = T_s^{iq}(Y) = v^i(Y,s)v^q(Y,s) - \delta^{iq}$, for each $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$: since

$$|x_{,4}^c|_3 \leq \frac{c^*}{c} < 1,$$

and

$$T_s^{iq}(Y)v^q(Y,s) = v^i(Y,s)v^i(Y,s) = 0,$$

it is easy to see that the hypotheses of (ii) are fulfilled with these selections, so (5) and (6) hold, as well. We wish to show that R is now given by (7). Then, fix $s \in \mathbb{R}$ and $Y \in \partial B_s$. We assume first that $T(Y,s) \neq 0$. In case we have either $x_{,4}^c(x_s^{-1}(Y),s) = 0$ or $\Theta_\psi(Y,s) \in \{0,\pi\}$, then

$$\begin{aligned} x_{,4}^c(x_s^{-1}(Y),s) &= \{x_{,4}^c(x_s^{-1}(Y),s) \bullet v(Y,s)\} v(Y,s) \\ &= v^c(Y,s) v(Y,s), \end{aligned}$$

whence it is clear that the expression on the right in (7) vanishes along with $R(Y,s)$, so the equality in (7) is valid in these instances. Suppose next that $x_{,4}^c(x_s^{-1}(Y),s) \neq 0$ and $\Theta_\psi(Y,s) \in (0,\pi)$, which imply that

$$x_{,4}^c(x_s^{-1}(Y),s) \neq v^c(Y,s) v(Y,s).$$

By (20),

$$|\psi(Y,s)|_3 \cos \Theta_\psi(Y,s) = x_{,4}^c(x_s^{-1}(Y),s) \bullet v(Y,s) = v^c(Y,s), \quad (64)$$

and, by (21), (22), and (25),

$$\begin{aligned}
 & |T(Y,s)|_3 \cos \Theta_T(Y,s) \\
 &= T(Y,s) \bullet \hat{e}_1(Y,s) \\
 &= \frac{\{v^1(Y,s)v^q(Y,s)-\delta^{1q}\}\{x_{,4}^q(\chi_s^{-1}(Y),s)-v^c(Y,s)v^q(Y,s)\}}{| \psi(Y,s)-\{\psi(Y,s)\bullet v(Y,s)\}v(Y,s)|_3} \\
 &= \frac{1}{| \psi(Y,s)|_3 \sin \Theta_\psi(Y,s)} \{v^c(Y,s)v^1(Y,s)-v^c(Y,s)v^1(Y,s) \\
 &\quad -x_{,4}^1(\chi_s^{-1}(Y),s)+v^c(Y,s)v^1(Y,s)\} \\
 &= \frac{1}{| \psi(Y,s)|_3 \sin \Theta_\psi(Y,s)} \{v^c(Y,s)v^1(Y,s)-x_{,4}^1(\chi_s^{-1}(Y),s)\}.
 \end{aligned} \tag{65}$$

Upon using these equalities (64) and (65) in (4), we find

$$R(Y,s) = \frac{1}{2} \cdot \frac{v^c(Y,s)\{v^c(Y,s)v^1(Y,s)-x_{,4}^1(\chi_s^{-1}(Y),s)\}}{\{1-|x_{,4}^c(\chi_s^{-1}(Y),s)|_3^2\}\{1-(v^c(Y,s))^2\}},$$

which, again, is just (7).

Finally, we must account for the possibility that $T(Y,s) = 0$:
but, if this should hold,

$$\begin{aligned}
 0 &= \{v^1(Y,s)v^q(Y,s)-\delta^{1q}\}\{x_{,4}^q(\chi_s^{-1}(Y),s)-v^c(Y,s)v^q(Y,s)\} \\
 &= \{v^c(Y,s)v^1(Y,s)-x_{,4}^1(\chi_s^{-1}(Y),s)\},
 \end{aligned}$$

which clearly shows that (7) is true, since $R(Y,s) = 0$. $\square$.

When we attempt to define a "direct value" for a function

$\omega_{2i}\{\phi\}$ as in [IV.15.ii], we encounter a difficulty, because the integral appearing in (IV.15.2) fails to exist, in general, for $(X,t) \in \partial B$, even under the conditions imposed on the motion and ϕ in [IV.24]. We find that we must define this direct value in terms of a "Cauchy principal-value" integral. The next statement provides sufficient conditions under which this is permissible.

[IV.27] P R O P O S I T I O N. Let M be a motion in $M(1)$.

Suppose further that

- (i) $\{B_{\zeta}^0\}_{\zeta \in \mathbb{R}}$ is locally uniformly Lyapunov: whenever $\tilde{K} \subset \mathbb{R}$ is compact, then $\{B_{\zeta}^0\}_{\zeta \in \tilde{K}}$ is uniformly Lyapunov;
- (ii) there exists a reference pair (R,X) for M which possesses the properties of [I.3.25] and is also such that $x_{,4}$ and $\hat{J}x$ are locally Hölder continuous on $\partial R \times \mathbb{R}$: whenever $\tilde{K}$ is compact in $\mathbb{R}$, then $x_{,4}|_{\partial R \times \tilde{K}}$ and $\hat{J}x|_{\partial R \times \tilde{K}}$ are Hölder continuous;
- (iii) $(P, \tilde{Y}, \tilde{s}) \mapsto \phi_{(\tilde{Y}, \tilde{s})}(P)$ is a function in $C(\partial R \times \partial B)$ which also satisfies the following local Hölder condition: whenever $\tilde{K} \subset \mathbb{R}$ is compact, there exist $\tilde{\kappa} > 0$ and $\tilde{\beta} \in (0,1]$, depending on ϕ , $\tilde{K}$, and M , for which

$$|\phi_{(\tilde{Y}, \tilde{s})} \circ x_{\tilde{s}}^{-1}(Z) - \phi_{(\tilde{Y}, \tilde{s})} \circ x_{\tilde{s}}^{-1}(\tilde{Y})| \leq \tilde{\kappa} |Z - \tilde{Y}|_{\tilde{B}}^{\tilde{\beta}},$$

(1)

whenever $\tilde{s} \in \tilde{K}$, $\tilde{Y} \in \partial \tilde{B}_{\tilde{s}}$, and $Z \in \partial \tilde{B}_{\tilde{s}}$.

Then, whenever $K \subset \mathbb{R}$ is compact, the limit

$$\begin{aligned} \omega_{21}^*\{\phi\}(Y,s) &:= \frac{1}{4\pi} \int_{\partial B_s}^* \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y,q} \cdot \phi(Y,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \\ &:= \lim_{\Delta \rightarrow 0^+} \frac{1}{4\pi} \int_{\partial B_s \cap C_\Delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y,q} \cdot \phi(Y,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \end{aligned} \quad (2)$$

exists uniformly for $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$; in fact, there exist $a > 0$, $\Delta^* > 0$, and $\lambda \in (0,1]$, depending on ϕ , K , and M , such that

$$\begin{aligned} \left| \omega_{21}^*\{\phi\}(Y,s) - \frac{1}{4\pi} \int_{\partial B_s \cap C_\Delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y,q} \cdot \phi(Y,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| &\leq a\Delta^\lambda \end{aligned} \quad (3)$$

whenever $0 < \Delta < \Delta^*$, $s \in K$, and $Y \in \partial B_s$.

Assume, moreover, that

- (iv) $\mu: \partial B \rightarrow K$ is locally Hölder continuous: whenever $\bar{K} \subset \mathbb{R}$ is compact, then $\mu|_{\bigcup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}}$ is Hölder continuous.

Then hypothesis (iii) is fulfilled by taking ϕ to be $\Lambda_{2j}^1(u)$, $\Lambda_{3jk}^1(u)$, or $\tilde{\Lambda}_{2j}^1(u)$, whence the assertions made above hold for any of these

choices for ϕ .

Finally, if $M \in \mathcal{M}(2)$, then M satisfies requirements (i) and (ii).

P R O O F. It suffices to provide the proof for the case in which $K = [t_1, t_2]$, a compact interval in $\mathbb{R}$, which we shall do. By (i), $\{B_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov, and we denote by (a_K, α_K, d_K) a set of Lyapunov constants for B_ζ^0 , for each $\zeta \in K$. Choose Δ and δ in $(0, (7/9)d_K)$, with $\delta < \Delta$. Select any $s \in K$, then $Y \in \partial B_s$, and write

$$\begin{aligned} & \left| \frac{1}{4\pi} \int_{\partial B_s \cap C_\Delta^\partial(Y, s)} \frac{1}{r_Y} \cdot \Gamma_{(Y, s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y, q} \cdot \phi(Y, s) \circ \chi_s^{-1} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \right. \\ & \quad \left. - \frac{1}{4\pi} \int_{\partial B_s \cap C_\delta^\partial(Y, s)} \frac{1}{r_Y} \cdot \Gamma_{(Y, s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y, q} \cdot \phi(Y, s) \circ \chi_s^{-1} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \right| \\ & = \frac{1}{4\pi} \left| \int_{C_\Delta^\partial(Y, s) \cap C_\delta^\partial(Y, s)} \frac{1}{r_Y} \cdot \Gamma_{(Y, s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y, q} \cdot \phi(Y, s) \circ \chi_s^{-1} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \right| \\ & = \frac{1}{4\pi} \left| \int_{C_\Delta^\partial(Y, s) \cap C_\delta^\partial(Y, s)} \frac{1}{r_Y} \cdot \Gamma_{(Y, s)}^{03} \cdot \tilde{T}_s^{1q} r_{Y, q} \right. \\ & \quad \cdot \{ \phi(Y, s) \circ \chi_s^{-1} \cdot J \chi_s^{-1} - \phi(Y, s) \circ \chi_s^{-1}(Y) \cdot J \chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \\ & \quad + \phi(Y, s) \circ \chi_s^{-1}(Y) \cdot J \chi_s^{-1}(Y) \cdot \int_{C_\Delta^\partial(Y, s) \cap C_\delta^\partial(Y, s)} \frac{1}{r_Y} \cdot \Gamma_{(Y, s)}^{03} \\ & \quad \cdot \{ \tilde{T}_s^{1q} - \tilde{T}_s^{1q}(Y) \} r_{Y, q} d\lambda_{\partial B_s} \end{aligned}$$

$$\begin{aligned}
 & + \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \cdot \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \frac{\gamma_{1q}}{T_s}(Y) r_{Y,q} \quad (4) \\
 & \cdot \{ \Gamma_{(Y,s)}^{03} - \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \} d\lambda_{\partial B_s} \\
 & + \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \cdot \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{\gamma_{1q}}{T_s}(Y) \\
 & \cdot \left\{ \frac{1}{r_Y} r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} - \left[\frac{1}{r_Y} r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right] \circ \Pi_Y \right. \\
 & \left. \cdot \frac{\gamma_{1q}}{T_s}(Y) \cdot \frac{\gamma_{1q}}{T_s}(Y) \right\} d\lambda_{\partial B_s} \Big| \\
 & \leq \frac{1}{4\pi} \left\{ I_1 + M_\phi^K \cdot \sum_{\ell=2}^4 I_\ell \right\},
 \end{aligned}$$

wherein

$$M_\phi^K := \max \{ |\phi_{(\tilde{Y}, \tilde{s})} \circ \chi_{\tilde{s}}^{-1}(\tilde{Y}) \cdot J\chi_{\tilde{s}}^{-1}(\tilde{Y})| \mid \tilde{s} \in K, \tilde{Y} \in \partial B_{\tilde{s}} \}, \quad (5)$$

$$\begin{aligned}
 I_1 := & \left| \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot \frac{\gamma_{1q}}{T_s}(Y) r_{Y,q} \right. \\
 & \left. \cdot \{ \phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} - \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right|, \quad (6)
 \end{aligned}$$

$$I_2 := \left| \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot \{ \frac{\gamma_{1q}}{T_s} - \frac{\gamma_{1q}}{T_s}(Y) \} r_{Y,q} d\lambda_{\partial B_s} \right|, \quad (7)$$

$$I_3 := \left| \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y} \cdot T_s^{1q}(Y) r_{Y,q} \cdot \{ \Gamma_{(Y,s)}^{03} - \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \} d\lambda_{\partial B_s} \right|, \quad (8)$$

$$I_4 := \left| \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} T_s^{1q}(Y) \cdot \left\{ \frac{1}{r_Y} r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} - \left(\frac{1}{r_Y^2} r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right) \circ \Pi_Y \cdot v_{\partial B_s}^j(Y) v_{\partial B_s}^j \right\} d\lambda_{\partial B_s} \right|, \quad (9)$$

and we have used the equality

$$\int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \left(\frac{1}{r_Y} r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right) \circ \Pi_Y \cdot T_s^{1q}(Y) \cdot v_{\partial B_s}^j(Y) v_{\partial B_s}^j d\lambda_{\partial B_s} = 0, \quad (10)$$

which clearly follows from [IV.26.1]. We now examine, in turn, I_1 , I_2 , I_3 , and I_4 , obtaining for each an appropriate estimate which is uniform for $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$.

I_1 : Let M denote the upper bound for the positive function Γ^{03} which is obtained from (IV.14.59). Using (IV.15.3), we have

$$\begin{aligned} |T_s^{1q}(Z) r_{Y,q}(Z)| &= |v^1(Z,s) v^q(Z,s) r_{Y,q}(Z) - r_{Y,i}(Z)| \\ &\leq |v^1(Z,s)| \cdot |v^q(Z,s) r_{Y,q}(Z)| + |r_{Y,i}(Z)| \leq 2, \quad (11) \end{aligned}$$

whenever $Z \in \partial B_s \cap \{Y\}'$.

Now, the relations (IV.22.45, 60, 61, and 66) clearly remain valid in the present setting, the exponent $\hat{\alpha}$, the bound m_j^K , and the positive coefficients A_K and $\hat{A}$ depending only on K and M . Thus, (IV.22.67) is also valid: there exists a positive a_1 , depending only on K and M , such that

$$|J\chi_s^{-1}(Z) - J\chi_s^{-1}(Y)| \leq a_1 |Z - Y|^{\hat{\alpha}}, \quad \text{for } Z \in \partial B_s. \quad (12)$$

Using (1), (12), and (IV.22.62),

$$\begin{aligned} & |\phi_{(Y,s)} \circ \chi_s^{-1}(Z) \cdot J\chi_s^{-1}(Z) - \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)| \\ & \leq J\chi_s^{-1}(Z) |\phi_{(Y,s)} \circ \chi_s^{-1}(Z) - \phi_{(Y,s)} \circ \chi_s^{-1}(Y)| + |\phi_{(Y,s)} \circ \chi_s^{-1}(Y)| \cdot |J\chi_s^{-1}(Z) - J\chi_s^{-1}(Y)| \\ & \leq (m_j^K)^{-1} \cdot \tilde{\kappa} r_Y^\beta(Z) + M_\phi^K \cdot a_1 r_Y^{\hat{\alpha}}(Z), \quad \text{for } Z \in \partial B_s, \end{aligned} \quad (13)$$

where $\tilde{\kappa} > 0$ and $\beta \in (0, 1]$ depend only on ϕ , K , and M . Let $\hat{\beta} := \min\{\beta, \hat{\alpha}\}$. Then, if $Z \in \partial B_s$ and $r_Y(Z) < 1$, we have $r_Y^\beta(Z)$ and $r_Y^{\hat{\alpha}}(Z)$ each $\leq r_Y^{\hat{\beta}}(Z)$, while if $r_Y(Z) \geq 1$, then

$$r_Y^\beta(Z) = \{r_Y^\beta(Z)/r_Y^{\hat{\beta}}(Z)\} r_Y^{\hat{\beta}}(Z) \leq \left\{ \max_{\zeta \in K} \text{diam } B_\zeta \right\} \cdot r_Y^{\hat{\beta}}(Z),$$

with a similar inequality for $r_Y^{\hat{\alpha}}(Z)$. It follows then from (13) that there exists a positive a_2 , depending only on ϕ , K , and M , such that

$$\begin{aligned} & |\phi_{(Y,s)} \circ \chi_s^{-1}(Z) \cdot J\chi_s^{-1}(Z) - \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)| \leq a_2 \cdot r_Y^{\hat{\beta}}(Z), \\ & \text{for } Z \in \partial B_s. \end{aligned} \quad (14)$$

In view of (11), (14), and a calculation like that of (IV.22.52), we obtain the estimate

$$\begin{aligned}
 I_1 &\leq 2Ma_2 \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y^{2-\beta}} d\lambda_{\partial B_s} \\
 &\leq 2^{3/2} \pi Ma_2 \cdot \int_\delta^\Delta \frac{1}{\zeta^{1-\beta}} d\zeta = 2^{3/2} \pi Ma_2 \cdot \hat{\beta}^{-1} \{\Delta^{\hat{\beta}} - \delta^{\hat{\beta}}\} \\
 &\leq 2^{3/2} \pi Ma_2 \hat{\beta}^{-1} \cdot \Delta^{\hat{\beta}}.
 \end{aligned} \tag{15}$$

I₂: Using the fact that $\{\beta_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov, with Lyapunov constants (a_K, α_K, d_K) , we see first that

$$\begin{aligned}
 |\{T_s^{v^1q}(Z) - T_s^{v^1q}(Y)\} r_{Y,q}(Z)| &= |\{v^1(Z,s)v^q(Z,s) - v^1(Y,s)v^q(Y,s)\} r_{Y,q}(Z)| \\
 &\leq |v^1(Z,s)\{v^q(Z,s) - v^q(Y,s)\} r_{Y,q}(Z)| \\
 &\quad + |\{v^1(Z,s) - v^1(Y,s)\} v^q(Y,s) r_{Y,q}(Z)| \\
 &\leq 2|v(Z,s) - v(Y,s)|_3 \\
 &\leq 2a_K r_Y^{\alpha_K}(Z), \quad \text{for } Z \in \partial B_s \cap \{Y\}'.
 \end{aligned} \tag{16}$$

Thus,

$$\begin{aligned}
 I_2 &\leq 2Ma_K \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &\leq 2^{3/2} \pi Ma_K \cdot \alpha_K^{-1} \{\Delta^{\alpha_K} - \delta^{\alpha_K}\} \\
 &< 2^{3/2} \pi Ma_K \cdot \alpha_K^{-1} \cdot \Delta^{\alpha_K}.
 \end{aligned} \tag{17}$$

I₃: An estimation almost identical with that carried out in (11) gives

$$|\frac{v^{iq}}{T_s}(Y)r_{Y,q}(Z)| \leq 2 \quad \text{for} \quad Z \in \partial B_s \cap \{Y\}^c. \quad (18)$$

Further, it is an easy exercise to check that the derivation of (IV.22.48), with $\hat{X} = Y$ in that inequality, can be effected here as well, since the hypotheses on M in the present case are the same as those imposed in [IV.22]. Thus, there exist a positive a_3 and $\tilde{\alpha} \in (0,1)$, dependent upon only K and M , for which

$$|\Gamma_{(Y,s)}^{03}(Z) - \Gamma_Y\{X_{,4}^c(X_s^{-1}(Y),s)\}(Z)| \leq a_3 r_Y^{\tilde{\alpha}}(Z) \quad \text{for} \quad Z \in \partial B_s \cap \{Y\}^c. \quad (19)$$

Using (18) and (19) with (8),

$$I_3 \leq 2a_3 \int_{C_{\Delta}^{\partial}(Y,s) \cap C_0^{\partial}(Y,s)} \frac{1}{r_Y^{2-\tilde{\alpha}}} d\lambda_{\partial B_s} \leq 2^{3/2} \pi a_3 \cdot \tilde{\alpha}^{-1} \cdot \Delta^{\tilde{\alpha}}. \quad (20)$$

I₄: Observe that, whenever $\xi \in \mathbb{R}^3$,

$$|\frac{v^{iq}}{T_s}(Y)\xi^q| = |v^i(Y,s)v^q(Y,s)\xi^q - \xi^i| \leq 2 \cdot |\xi|_3, \quad (21)$$

so

$$\begin{aligned} & \left| \frac{v^{iq}}{T_s}(Y) \left\{ \frac{1}{r_Y^2(Z)} r_{Y,q}(Z) \cdot \Gamma_Y\{X_{,4}^c(X_s^{-1}(Y),s)\}(Z) \right. \right. \\ & \quad \left. \left. - \left[\frac{1}{r_Y^2} r_{Y,q} \cdot \Gamma_Y\{X_{,4}^c(X_s^{-1}(Y),s)\} \right] \circ \Pi_Y(Z) \cdot v^j(Y,s)v^j(Z,s) \right\} \right| \\ & \leq 2 |1 - v^j(Y,s)v^j(Z,s)| \cdot \frac{1}{r_Y^2(Z)} \Gamma_Y\{X_{,4}^c(X_s^{-1}(Y),s)\}(Z) \end{aligned}$$

$$\begin{aligned}
 & +2|v^j(Y,s)v^j(Z,s)| \cdot \left| \frac{Z-Y}{r_Y^3(Z)} \cdot \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\}(Z) \right. \\
 & \left. - \frac{\pi_Y(Z)-Y}{r_Y^3(\pi_Y(Z))} \cdot \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\} \circ \pi_Y(Z) \right|_3 \\
 & \leq 2|1-v^j(Y,s)v^j(Z,s)| \cdot \frac{1}{r_Y^2(Z)} \cdot \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\}(Z) \\
 & +2 \cdot \frac{1}{r_Y^2(Z)} \cdot |\Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\}(Z) - \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\} \circ \pi_Y(Z)| \\
 & +2 \cdot \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\} \circ \pi_Y(Z) \cdot \left| \frac{Z-Y}{r_Y^3(Z)} - \frac{\pi_Y(Z)-Y}{r_Y^3(\pi_Y(Z))} \right|_3, \\
 & \text{for each } Z \in \partial \delta_s \cap B_{d_K}^3(Y) \cap \{Y\}'.
 \end{aligned} \tag{22}$$

Let us examine the differences appearing in the latter expression.

First,

$$\begin{aligned}
 |1-v^j(Z,s)v^j(Y,s)| &= |(v^j(Y,s)-v^j(Z,s))v^j(Y,s)| \\
 &\leq |v(Z,s)-v(Y,s)|_3 \leq a_K r_Y^{\alpha_K}(Z), \text{ for } Z \in \partial \delta_s.
 \end{aligned} \tag{23}$$

Next, since $|x_{,4}^c(x_s^{-1}(Y),s)|_3 \leq c^*/c$, an argument which is quite similar to that which produced (IV.22.24) results in

$$\begin{aligned}
 & |\Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\}(Z) - \Gamma_Y\{x_{,4}^c(x_s^{-1}(Y),s)\} \circ \pi_Y(Z)| \\
 &= | \{ \{ x_{,4}^c(x_s^{-1}(Y),s) \cdot r_{Y,\ell}(Z) \}^2 + (1 - |x_{,4}^c(x_s^{-1}(Y),s)|_3^2) \}^{-3/2} \\
 & - \{ \{ x_{,4}^c(x_s^{-1}(Y),s) \cdot r_{Y,\ell}(\pi_Y(Z)) \}^2 + (1 - |x_{,4}^c(x_s^{-1}(Y),s)|_3^2) \}^{-3/2} |
 \end{aligned}$$

$$\leq 3 \cdot \left(\frac{c^*}{c}\right)^2 \left\{1 - \left(\frac{c^*}{c}\right)^2\right\}^{-7/2} \cdot |\text{grad } r_Y(Z) - \text{grad } r_Y(\pi_Y(Z))|_3, \quad (24)$$

whenever $Z \in \partial S_s \cap B_{d_K}^3(Y) \cap \{Y\}'$.

Meanwhile, appealing to [I.2.37.iii.3],

$$\begin{aligned} & |\text{grad } r_Y(Z) - \text{grad } r_Y(\pi_Y(Z))|_3 \\ &= \left| \frac{Z-Y}{r_Y(Z)} - \frac{\pi_Y(Z)-Y}{r_Y(\pi_Y(Z))} \right|_3 \\ &= \left| \frac{1}{r_Y(Z)} (Z - \pi_Y(Z)) + \left\{ \frac{1}{r_Y(Z)} - \frac{1}{r_Y(\pi_Y(Z))} \right\} (\pi_Y(Z) - Y) \right|_3 \\ &\leq \frac{1}{r_Y(Z)} |Z - \pi_Y(Z)|_3 + \frac{1}{r_Y(Z)} |r_Y(Z) - r_Y(\pi_Y(Z))| \\ &\leq \frac{2}{r_Y(Z)} \cdot |Z - \pi_Y(Z)|_3 \leq \frac{2\tilde{a}_K}{r_Y(Z)} \cdot r_Y^{1+\alpha_K}(Z) = 2\tilde{a}_K \cdot r_Y^{\alpha_K}(Z), \end{aligned} \quad (25)$$

for $Z \in \partial S_s \cap B_{d_K}^3(Y) \cap \{Y\}'$,

where $\tilde{a}_K := \frac{8}{7} \cdot a_K \cdot \left(\frac{65}{49}\right)^{\alpha_K/2} \cdot (1+\alpha_K)^{-1}$. Thus, (24) and (25) give

$$\begin{aligned} & |\Gamma_Y\{X_{,4}^c(X_s^{-1}(Y), s)\}(Z) - \Gamma_Y\{X_{,4}^c(X_s^{-1}(Y), s)\} \circ \Gamma_Y(Z)| \\ &\leq 6\tilde{a}_K \cdot (c^*/c)^2 \cdot \{1 - (c^*/c)^2\}^{-7/2} \cdot r_Y^{\alpha_K}(Z), \end{aligned} \quad (26)$$

for each $Z \in \partial S_s \cap B_{d_K}^3(Y) \cap \{Y\}'$.

Finally, again using [I.2.37.iii.3], and also noting that

$$\frac{r_Y(Z)}{r_Y(\pi_Y(Z))} < \frac{9}{7} < 2 \quad \text{for each } Z \in \partial S_s \cap B_{d_K}^3(Y) \cap \{Y\}', \quad (27)$$

which follows from [I.2.37.iii.5], we obtain the estimate

$$\begin{aligned}
 & \left| \frac{1}{r_Y^3(Z)} (Z-Y) - \frac{1}{r_Y^3(\Pi_Y(Z))} (\Pi_Y(Z)-Y) \right|_3 \\
 &= \left| \frac{1}{r_Y^3(Z)} (Z-\Pi_Y(Z)) + \left\{ \frac{1}{r_Y^3(Z)} - \frac{1}{r_Y^3(\Pi_Y(Z))} \right\} \cdot (\Pi_Y(Z)-Y) \right|_3 \\
 &\leq \frac{1}{r_Y^3(Z)} |Z-\Pi_Y(Z)|_3 \\
 &\quad + \frac{1}{r_Y^3(Z)} |r_Y(Z)-r_Y(\Pi_Y(Z))| \cdot \left\{ 1 + \frac{r_Y(Z)}{r_Y(\Pi_Y(Z))} + \frac{r_Y^2(Z)}{r_Y^2(\Pi_Y(Z))} \right\} \\
 &\leq \frac{8}{r_Y^3(Z)} \cdot |Z-\Pi_Y(Z)|_3 \leq 8\tilde{a}_K \cdot \frac{1}{r_Y^{2-\alpha_K}(Z)},
 \end{aligned} \tag{28}$$

whenever $Z \in \partial B_s \cap B_{d_K}^3(Y) \cap \{Y\}'$.

Since

$$|\Gamma_Y\{x_{,4}^c(x_s^{-1}(Y), s)\}| \leq \{1-(c^*/c)^2\}^{-3/2} \quad \text{on} \quad \mathbb{R}^3 \cap \{Y\}'$$

(cf., (VI.69.4)), (22), (23), (26), and (28) can be combined to give

$$\begin{aligned}
 I_4 &\leq 2a_K \cdot \{1-(c^*/c)^2\}^{-3/2} \cdot \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &\quad + 12\tilde{a}_K \cdot (c^*/c)^2 \cdot \{1-(c^*/c)^2\}^{-7/2} \cdot \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &\quad + 16\tilde{a}_K \cdot \{1-(c^*/c)^2\}^{-3/2} \cdot \int_{C_\Delta^\partial(Y,s) \cap C_\delta^\partial(Y,s)'} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 &\leq a_4 \Delta^{\alpha_K},
 \end{aligned} \tag{29}$$

the positive coefficient a_4 depending upon only K and M .

Collecting up our results in (4), (15), (17), (20), and (29), it is clear that there exist $a > 0$ and $\lambda \in (0,1]$ (in fact $\lambda = \min \{\beta, \bar{\alpha}, \alpha_K\}$ will do), depending on only ϕ , K , and M , such that

$$\begin{aligned} & \left| \frac{1}{4\pi} \int_{\partial B_s \cap C_\Delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot T_s^{1q} r_{Y,q} \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \right. \\ & \quad \left. - \frac{1}{4\pi} \int_{\partial B_s \cap C_\delta^\partial(Y,s)} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot T_s^{1q} r_{Y,q} \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \right| \\ & \leq a \Delta^\lambda \end{aligned} \quad (30)$$

whenever $s \in K$, $Y \in \partial B_s$, and $0 < \delta < \Delta < (7/9)d_K$.

By the Cauchy criterion, it follows immediately from (30) that the limit indicated in (2) exists for each $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$. Moreover, choosing $s \in K$, $Y \in \partial B_s$, and $\Delta \in (0, (7/9)d_K)$, we can allow $\delta \rightarrow 0^+$ in (30): this produces the inequality (3) (with $\Delta^* = (7/9)d_K$), which, in turn, obviously verifies that the limit in (2) exists uniformly with respect to $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$.

Now, suppose that $\mu: \partial B \rightarrow K$ is locally Hölder continuous, in the sense specified in hypothesis (iv). Observing that hypotheses (i) and (ii) here are the same as [IV.22.i] and [IV.22.ii], respectively, and noting the similarity in form of the functions $\Lambda_2^1\{\mu\}$, $\Lambda_{3jk}^1\{\mu\}$,

$\tilde{\Lambda}_{2j}^1\{\mu\}$ considered here ((IV.14.34, 35, and 47)) and $\Lambda_{1i}^1\{\mu\}$, $\tilde{\Lambda}_1^1\{\mu\}$ considered in [IV.22] ((IV.14.33 and 46)), it should be clear that we can prove that (iii) holds when ϕ is replaced by any one of $\Lambda_2^1\{\mu\}$, $\Lambda_{3jk}^1\{\mu\}$, or $\tilde{\Lambda}_{2j}^1\{\mu\}$ by the same reasoning used to show that (IV.22.2) is true when ϕ is replaced there by either $\Lambda_{1i}^1\{\mu\}$ or $\tilde{\Lambda}_1^1\{\mu\}$. We omit the details.

Finally, if $M \in \mathcal{M}(2)$, it was shown in the proof of [IV.22] that M satisfies (i) and (ii); cf., also, [IV.23.a]. $\square$.

The next theorem is, for the functions $\omega_{2i}\{\phi\}$, the counterpart of [IV.22], which concerned the functions $\omega_1\{\phi\}$. That is, we now examine the limiting values of $\omega_{2i}^0\{\phi\}(\cdot, s)$ and $\omega_{2i}^I\{\phi\}(\cdot, s)$, for $s \in \mathbb{R}$, as their arguments approach a point $Y \in \partial\mathcal{S}_s$ along $L_V^+(Y, s)$ and $L_V^-(Y, s)$, respectively.

[IV.28] T H E O R E M. Let M be a motion in $\mathcal{M}(1)$. Suppose further that

- (i) $\{B_\zeta^0\}_{\zeta \in \mathbb{R}}$ is locally uniformly Lyapunov: whenever $\tilde{K}$ is compact in $\mathbb{R}$, then $\{B_\zeta^0\}_{\zeta \in \tilde{K}}$ is uniformly Lyapunov;
- (ii) there exists a reference pair (R, x) for M which possesses the properties of [I.3.25] and is also such that $x_{,4}$ and $\hat{J}x$ are locally Hölder continuous on $\partial R \times \mathbb{R}$: for each compact $\tilde{K} \subset \mathbb{R}$, $x_{,4}|_{\partial R \times \tilde{K}}$ and $\hat{J}x|_{\partial R \times \tilde{K}}$ are Hölder continuous;

(iii) $(P, X, t) \mapsto \phi_{(X, t)}(P)$ is a function in $C(\partial R \times \mathbb{R}^4)$ which also satisfies the following local Hölder conditions: whenever $\bar{K} \subset \mathbb{R}$ is compact, there exist positive numbers $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3$, and $\eta_{\bar{K}}$, and numbers $\bar{\beta}_1, \bar{\beta}_2$, and $\bar{\beta}_3$ in $(0, 1]$, depending on ϕ, M , and $\bar{K}$, such that

$$|\phi_{(\bar{X}, \bar{s})} \circ \chi_{\bar{s}}^{-1}(\bar{Z}) - \phi_{(\bar{Y}, \bar{s})} \circ \chi_{\bar{s}}^{-1}(\bar{Z})| \leq \bar{\kappa}_1 |\bar{X} - \bar{Y}|_3^{\bar{\beta}_1} \quad (1)$$

and

$$|\phi_{(\bar{X}, \bar{s})} \circ \chi_{\bar{s}}^{-1}(\bar{Z}) - \phi_{(\bar{Y}, \bar{s})} \circ \chi_{\bar{s}}^{-1}(\bar{Y})| \leq \bar{\kappa}_2 |\bar{Z} - \bar{Y}|_3^{\bar{\beta}_2} + \bar{\kappa}_3 |\bar{Z} - \bar{X}|_3^{\bar{\beta}_3}, \quad (2)$$

whenever $\bar{s} \in \bar{K}, \quad \bar{Y} \in \partial B_{\bar{s}}, \quad \bar{Z} \in \partial B_{\bar{s}},$

and $\bar{X} \in L_{\nu}(\bar{Y}, \bar{s}) \cap B_{\eta_{\bar{K}}}^3(\bar{Y}).$

Define $R^1: \partial B \rightarrow \mathbb{R}$ by

$$R^1(Y, s) := \frac{u^c(Y, s) \{u^c(Y, s) v^1(Y, s) - \chi_{s, 4}^{1c}(\chi_s^{-1}(Y), s)\}}{2 \{1 - |\chi_{s, 4}^c(\chi_s^{-1}(Y), s)|_3^2\} \{1 - (u^c(Y, s))^2\}}, \quad (3)$$

for $(Y, s) \in \partial B.$

Then, whenever $K \subset \mathbb{R}$ is compact,

$$\lim_{\substack{X \rightarrow Y \\ X \in L_{\nu}^+(Y, s) \\ [X \in L_{\nu}^-(Y, s)]}} \omega_{21}^{[1]}(\phi)(X, s) = \begin{bmatrix} + \\ - \end{bmatrix} R^1(Y, s) \cdot \phi_{(Y, s)} \circ \chi_s^{-1}(Y) \cdot J \chi_s^{-1}(Y) \quad (4)$$

$$+ \omega_{21}^*(\phi)(Y, s)$$

uniformly for $(Y, s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$. That is, given $\epsilon > 0$, there exists $\delta_0(\epsilon) > 0$, depending also on ϕ , K , and M , such that

$$|w_{21}^{[I]}(\phi)(X, s) - \{ \begin{smallmatrix} + \\ - \end{smallmatrix} R^1(Y, s) \cdot \phi(Y, s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) + w_{21}^*(\phi)(Y, s) \}| < \epsilon$$

whenever $s \in K$, $Y \in \partial B_s$, and (5)

$$X \in L_V^+(Y, s) \quad [X \in L_V^-(Y, s)] \quad \text{with} \quad |X - Y|_3 < \delta_0(\epsilon).$$

Assume further that

(iv) $\mu: \partial B \rightarrow K$ is locally Hölder continuous: for each compact $\bar{K} \subset K$, $\mu|_{\bigcup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}}$ is Hölder continuous. Then hypothesis (iii) is fulfilled when ϕ is replaced by $\Lambda_2^1\{\mu\}$, $\Lambda_{3jk}^1\{\mu\}$, or $\tilde{\Lambda}_{2j}^1\{\mu\}$ (cf., (IV.14.34, 35, and 47), respectively), so that the conclusion above holds for any of these choices for ϕ .

In this connection, we have

$$\Lambda_2^1\{\mu\}(Y, s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) = \{1 - |\chi_{,4}^c(\chi_s^{-1}(Y), s)|_3^2\} \cdot \mu(Y, s), \quad (6)$$

$$\Lambda_{3jk}^1\{\mu\}(Y, s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) = \chi_{,4}^{jc}(\chi_s^{-1}(Y), s) \cdot \chi_{,4}^{kc}(\chi_s^{-1}(Y), s) \cdot \mu(Y, s), \quad (7)$$

and

$$\tilde{\Lambda}_{2j}^1\{\mu\}(Y, s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) = \chi_{,4}^{jc}(\chi_s^{-1}(Y), s) \cdot \mu(Y, s), \quad (8)$$

for each $(Y, s) \in \partial B$.

Finally, if $M \in M(2)$, then M satisfies requirements (i)

and (ii).

P R O O F. We observe at the outset that the hypotheses of Proposition [IV.27] are fulfilled here by M and ϕ , [IV.27.iii] clearly following from the present hypothesis (iii). Thus, the limit function $\omega_{21}^*\{\phi\}: \partial B \rightarrow K$ exists (locally uniformly). We shall suppose that the compact set $K \subset \mathbb{R}$ is an interval $[t_1, t_2]$, as we can without loss of generality. By (i), $\{\mathcal{B}_\zeta^0\}_{\zeta \in K}$ is uniformly Lyapunov; let (a_K, α_K, d_K) be Lyapunov constants for $\mathcal{B}_\zeta^0$, for each $\zeta \in K$. As in the proof of [IV.22], we shall suppose that the positive d_K is so small that $d_K \leq \eta_K$, and there exists a number $\gamma_K \in (0, 1)$, depending only upon a_K , α_K , and d_K , and such that

$$\gamma_K < \frac{r_{\tilde{X}}(Z)}{r_{\tilde{X}}(\Pi_{\tilde{Y}}(Z))} < \frac{1}{\gamma_K}$$

$$\text{whenever } \tilde{s} \in K, \quad \tilde{Y} \in \partial \mathcal{B}_{\tilde{s}}, \quad \tilde{X} \in L_{\tilde{Y}}(\tilde{Y}, \tilde{s}), \quad (9)$$

$$\text{and } Z \in \partial \mathcal{B}_{\tilde{s}} \cap \mathcal{B}_{d_K}^3(\tilde{Y}) \cap \{\tilde{X}\},$$

(cf., also, Lemma [I.2.38]). Let Δ denote a number such that $2\Delta \in (0, (7/9)d_K)$; Δ is to be chosen more precisely later. Throughout, we suppose that $s \in K$, $Y \in \partial \mathcal{B}_s$, and $X \in L_Y^+(Y, s)$ [$X \in L_Y^-(Y, s)$], with $|X - Y|_3 < \Delta$. We begin with the inequalities

$$\begin{aligned} & \left| \omega_{21}^{[I]} \{\phi\}(X, s) - \left\{ \begin{bmatrix} + \\ - \end{bmatrix} R^1(Y, s) \cdot \phi(Y, s) \circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y) + \omega_{21}^*\{\phi\}(Y, s) \right\} \right| \\ & \leq \left| \omega_{21}^{[I]} \{\phi\}(X, s) - \left\{ \begin{bmatrix} + \\ - \end{bmatrix} R^1(Y, s) \cdot \phi(Y, s) \circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y) \right\} \right| \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{4\pi} \left| \int_{\partial B_s \cap C_{2\Delta}^\partial(Y,s)} \frac{1}{r_Y} r_{Y,q} \gamma_{\Gamma(Y,s)}^{03} \cdot \phi(Y,s)^{\circ X_s^{-1} \cdot J X_s^{-1}} d\lambda_{\partial B_s} \right| \\
 & + \left| \frac{1}{4\pi} \int_{\partial B_s \cap C_{2\Delta}^\partial(Y,s)} \frac{1}{r_Y} r_{Y,q} \gamma_{\Gamma(Y,s)}^{03} \cdot \phi(Y,s)^{\circ X_s^{-1} \cdot J X_s^{-1}} d\lambda_{\partial B_s} \right. \\
 & \left. - \omega_{21}^* \{ \phi \} (Y,s) \right| \\
 & \leq \frac{1}{4\pi} \left| \int_{\partial B_s \cap C_{2\Delta}^\partial(Y,s)} \left\{ \frac{1}{r_X} r_{X,q} \gamma_{\Gamma(X,s)}^{03} \cdot \phi(X,s)^{\circ X_s^{-1}} \right. \right. \\
 & \left. \left. - \frac{1}{r_Y} r_{Y,q} \gamma_{\Gamma(Y,s)}^{03} \cdot \phi(Y,s)^{\circ X_s^{-1}} \right\} J X_s^{-1} d\lambda_{\partial B_s} \right| \\
 & + \left| \frac{1}{4\pi} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X} r_{X,q} \gamma_{\Gamma(X,s)}^{03} \cdot \phi(X,s)^{\circ X_s^{-1} \cdot J X_s^{-1}} d\lambda_{\partial B_s} \right. \\
 & \left. - \{ [^+] R^1(Y,s) \cdot \phi(Y,s)^{\circ X_s^{-1}(Y) \cdot J X_s^{-1}(Y)} \} \right| \\
 & + \left| \frac{1}{4\pi} \int_{\partial B_s \cap C_{2\Delta}^\partial(Y,s)} \frac{1}{r_Y} r_{Y,q} \gamma_{\Gamma(Y,s)}^{03} \cdot \phi(Y,s)^{\circ X_s^{-1} \cdot J X_s^{-1}} d\lambda_{\partial B_s} \right. \\
 & \left. - \omega_{21}^* \{ \phi \} (Y,s) \right| \\
 & \leq \frac{1}{4\pi} \sum_{j=1}^5 I_j(Y,s,X;\Delta) + I_6(Y,s,X;\Delta) + I_7(Y,s,X;\Delta),
 \end{aligned}
 \tag{10}$$

in which the $I_j(Y,s,X;\Delta) = I_j$ are given by

$$I_1 := \left| \int_{\partial B_s \cap C_{2\Delta}^\partial(Y,s)} \left\{ \frac{1}{r_X^2} r_{X,q} \tilde{T}_s^{1q} \cdot \Gamma_{(X,s)}^{03} \cdot \phi(X,s) \circ \chi_s^{-1} \right. \right. \\ \left. \left. - \frac{1}{r_Y^2} r_{Y,q} \tilde{T}_s^{1q} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi(Y,s) \circ \chi_s^{-1} \right\} J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (11)$$

$$I_2 := \left| \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X^2} r_{X,q} \{ \tilde{T}_s^{1q} - \tilde{T}_s^{1q}(Y) \} \cdot \Gamma_{(X,s)}^{03} \cdot \phi(X,s) \circ \chi_s^{-1} \right. \\ \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (12)$$

$$I_3 := \left| \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X^2} r_{X,q} \tilde{T}_s^{1q}(Y) \cdot \Gamma_{(X,s)}^{03} \right. \\ \left. \cdot \{ \phi(X,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right|, \quad (13)$$

$$I_4 := \left| \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \cdot \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X^2} r_{X,q} \tilde{T}_s^{1q}(Y) \right. \\ \left. \cdot \{ \Gamma_{(X,s)}^{03} - \Gamma_X^c(\chi_s^{-1}(Y), s) \} d\lambda_{\partial B_s} \right|, \quad (14)$$

$$I_5 := \left| \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \right. \\ \left. \cdot \int_{C_{2\Delta}^\partial(Y,s)} \tilde{T}_s^{1q}(Y) \left\{ \frac{1}{r_X^2} r_{X,q} \cdot \Gamma_X^c(\chi_s^{-1}(Y), s) \right\} \right.$$

$$-\left\{ \frac{1}{2} r_{X,q} \cdot \Gamma_X \{ \chi^c, 4(\chi_s^{-1}(Y), s) \} \right\} \circ \pi_Y \vee_{\partial B_s}^j(Y) \vee_{\partial B_s}^j \left\} d\lambda_{\partial B_s} \right|, \quad (15)$$

$$\begin{aligned} I_6 := & \left| \phi(Y,s) \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y) \right| \\ & \cdot \left| \frac{1}{4\pi} \int_{C_{2\Delta}^\partial(Y,s)} \left\{ \frac{1}{2} r_{X,q} \cdot \Gamma_X \{ \chi^c, 4(\chi_s^{-1}(Y), s) \} \right\} \circ \pi_Y \cdot \tilde{T}_s^{iq}(Y) \right. \\ & \left. \cdot \vee_{\partial B_s}^j(Y) \vee_{\partial B_s}^j d\lambda_{\partial B_s} - \left\{ \begin{bmatrix} + \\ - \end{bmatrix} R^1(Y,s) \right\} \right|, \end{aligned} \quad (16)$$

and

$$\begin{aligned} I_7 := & \left| \frac{1}{4\pi} \int_{\partial B_s \cap C_{2\Delta}^\partial(Y,s)} \frac{1}{2} r_{Y,q} \tilde{T}_s^{iq} \cdot \Gamma^{03}(Y,s) \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right. \\ & \left. - u_{21}^* \{ \phi \}(Y,s) \right|. \end{aligned} \quad (17)$$

Now, let $\varepsilon > 0$. In view of (10), in order to prove the first assertion of the theorem, it suffices to show that there exist numbers $\Delta(\varepsilon)$ and $\delta(\varepsilon)$, with $2\Delta(\varepsilon) \in (0, (7/9)d_K)$ and $\delta(\varepsilon) \in (0, \Delta(\varepsilon))$, each independent of $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ and $X \in L_Y^+(Y,s)$ [$X \in L_Y^-(Y,s)$], but perhaps depending upon some or all of ϕ , K , and M , as well as ε , such that

$$\begin{aligned} I_j(Y,s,X;\Delta(\varepsilon)) &< \varepsilon \quad \text{if} \quad |X-Y|_3 < \delta(\varepsilon), \\ &\text{for each } j \in \{1, \dots, 7\}. \end{aligned} \quad (18)$$

We shall carry this out presently.

Let us summarize various estimates which we shall require in the subsequent reasoning. For the most part, the derivations of these inequalities are effected by proceeding as in the proof of [IV.22], to which we shall refer wherever possible.

Let $Z \in \partial B_s$, with $Z \neq Y$: then $Z \neq X$, and

$$\begin{aligned} |\text{grad } r_X(Z) - \text{grad } r_Y(Z)|_3 &\leq \frac{1}{r_Y(Z)} |X-Y|_3 + \frac{1}{r_Y(Z)} |r_X(Z) - r_Y(Z)| \\ &\leq \frac{2}{r_Y(Z)} |X-Y|_3: \end{aligned} \quad (19)$$

cf., (IV.22.21). Then, just as (IV.22.25) was derived, we find that there exists a $k_1 > 0$, depending only on c^*/c , such that

$$\begin{aligned} |\Gamma_X\{X_{,4}^c(X_s^{-1}(Y), s)\}(Z) - \Gamma_Y\{X_{,4}^c(X_s^{-1}(Y), s)\}(Z)| &\leq \frac{k_1}{r_Y(Z)} |X-Y|_3, \\ \text{for each } Z \in \partial B_s \cap \{Y\}'. \end{aligned} \quad (20)$$

Suppose next that $Z \in \partial B_s \cap C_{2\Delta}^\partial(Y, s)'$. Then it is easy to see that $r_Y(Z) \geq 2\Delta > 2|X-Y|_3$, so $|X-Y|_3/r_Y(Z) < 1/2$. The inequality $|r_X(Z) - r_Y(Z)| \leq |X-Y|_3$ then leads to the estimates

$$\frac{1}{2} < \frac{r_X(Z)}{r_Y(Z)} < \frac{3}{2}, \quad \text{for each } Z \in \partial B_s \cap C_{2\Delta}^\partial(Y, s)'. \quad (21)$$

Thus, if $T \in \mathbb{R}^3$, we find (cf., the derivation of (IV.22.27)),

$$\begin{aligned} &\left| \left\{ \frac{1}{r_X^2(Z)} r_{X,q}(Z) - \frac{1}{r_Y^2(Z)} r_{Y,q}(Z) \right\} T^q \right| \\ &\leq |T|_3 \cdot \left| \frac{(Z-X)}{r_X^3(Z)} - \frac{(Z-Y)}{r_Y^3(Z)} \right|_3 \end{aligned}$$

$$\begin{aligned} &\leq |T|_3 \left\{ \frac{|X-Y|_3}{r_Y^3(Z)} + \frac{1}{r_Y^3(Z)} |r_Y(Z) - r_X(Z)| \cdot \left\{ 1 + \frac{r_Y(Z)}{r_X(Z)} + \frac{r_Y^2(Z)}{r_X^2(Z)} \right\} \right\} \\ &\leq \frac{8|T|_3}{r_Y^3(Z)} \cdot |X-Y|_3, \quad \text{for each } Z \in \partial B_s \cap C_{2\Delta}^3(Y, s). \end{aligned} \quad (22)$$

Writing

$$t_0 := \frac{1}{c-c^*} \left\{ \frac{d}{2} + \max_{t_1 \leq \zeta \leq t_2} \text{diam } B_\zeta \right\},$$

and

$$\tilde{K} := [t_1 - t_0, t_2],$$

it is easy to check that $s^{-1}(X_s^{-1}(Z); X, s)$ and $s^{-1}(X_s^{-1}(Z); Y, s)$ lie in $\tilde{K}$ whenever $Z \in \partial B_s$ (cf., inequalities (IV.22.29) and (IV.22.30)).

Now, by hypothesis (ii), $X_{,4}^c|_{\partial R \times \tilde{K}}$ is Hölder continuous: there exist $\tilde{A} > 0$ and $\tilde{\alpha} \in (0, 1]$ such that

$$\begin{aligned} |X_{,4}^c(P_2, \zeta_2) - X_{,4}^c(P_1, \zeta_1)|_3 &\leq \tilde{A} |(P_2, \zeta_2) - (P_1, \zeta_1)|_4^{\tilde{\alpha}} \\ \text{whenever } P_1, P_2 &\in \partial R \quad \text{and} \quad \zeta_1, \zeta_2 \in \tilde{K}. \end{aligned} \quad (23)$$

Therefore, just as in (IV.22.32),

$$\begin{aligned} |[X_{,4}^c]_{(X,s)} \circ X_s^{-1}(Z) - [X_{,4}^c]_{(Y,s)} \circ X_s^{-1}(Z)|_3 &\leq \frac{\tilde{A}}{(c-c^*)^{\tilde{\alpha}}} \cdot |X-Y|_3^{\tilde{\alpha}}, \\ \text{for each } Z &\in \partial B_s. \end{aligned} \quad (24)$$

Having (23), the reasoning used to obtain (IV.22.34) can be carried through here, whence

$$|v_{(X,s)}^c(Z) - v_{(Y,s)}^c(Z)|_3 \leq \frac{\tilde{A}}{(1+\tilde{\alpha})(c-c^*)^{\tilde{\alpha}}} \cdot |X-Y|_3^{\tilde{\alpha}}, \quad (25)$$

for each $Z \in \partial B_s$.

Proceeding to examine $|\Gamma_{(X,s)}^{03}(Z) - \Gamma_{(Y,s)}^{03}(Z)|$ for $Z \in \partial B_s \cap \{Y\}'$, it is easy to check that the inequality (IV.22.40) remains valid, so that, using (19), (24), and (25), we can assert that there exist $k_2 > 0$ and $k_3 > 0$, k_2 depending only upon M and K , k_3 depending only upon M , for which

$$|\Gamma_{(X,s)}^{03}(Z) - \Gamma_{(Y,s)}^{03}(Z)| \leq k_2 |X-Y|_3^{\tilde{\alpha}} + \frac{k_3}{r_Y(Z)} |X-Y|_3, \quad (26)$$

for each $Z \in \partial B_s \cap \{Y\}'$.

Finally, consider $|\Gamma_{(X,s)}^{03}(Z) - \Gamma_{X\{X_s^c(X_s^{-1}(Y),s)\}}^{03}(Z)|$, where $Z \in \partial B_s$: since $(\tilde{Z}, \tilde{s}) \mapsto X_s^{-1}(\tilde{Z})$ is in $C^1(\partial B; \mathbb{R}^3)$, it is Lipschitz continuous on the compact subset $X^*(\partial R \times K)$ of ∂B :

$$|X_{s_2}^{-1}(Z_2) - X_{s_1}^{-1}(Z_1)|_3 \leq \hat{A} |(Z_2, s_2) - (Z_1, s_1)|_4, \quad (27)$$

whenever $s_1, s_2 \in K$, $Z_1 \in \partial B_{s_1}$, and $Z_2 \in \partial B_{s_2}$,

with $\hat{A} > 0$ depending only on K . Therefore, in view of (23),

$$\begin{aligned} |X_{s_2}^c(X_s^{-1}(Z), s) - X_{s_1}^c(X_s^{-1}(Y), s)|_3 &\leq \tilde{A} |X_s^{-1}(Z) - X_s^{-1}(Y)|_3^{\tilde{\alpha}} \\ &\leq \tilde{A} \hat{A} |Z-Y|_3^{\tilde{\alpha}}, \quad \text{for each } Z \in \partial B_s. \end{aligned} \quad (28)$$

In fact, all of the computations required to prove (IV.22.48) can be carried over to the present setting. In particular, taking $\tilde{X} = X$

in (IV.22.48), we have

$$|\Gamma_{(X,s)}^{03}(Z) - \Gamma_X\{\chi_{,4}^c(\chi_s^{-1}(Z), s)\}(Z)| \leq k_4 |Z-X|_3^{\tilde{\alpha}} + k_5 |Z-Y|_3^{\tilde{\alpha}},$$

for each $Z \in \partial B_s$,

(29)

for certain positive numbers k_4 and k_5 , depending upon only M and K .

We return now to the analysis of the $I_j(Y, s, X; \Delta)$, for $j = 1, \dots, 7$, developing various estimates prior to showing that we can produce $\Delta(\epsilon)$ and $\delta(\epsilon)$ with the desired properties.

I_1 : From (11), we begin with the estimate

$$\begin{aligned} I_1 \leq & \left| \int_{\partial B_s \cap C_{2\Delta}^\partial(Y, s)} \left\{ \frac{1}{r_X} r_{X,q} - \frac{1}{r_Y} r_{Y,q} \right\} T_s^{Yiq} \cdot \Gamma_{(X,s)}^{03} \cdot \phi(X, s) \circ \chi_s^{-1} \right. \\ & \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Big| + \left| \int_{\partial B_s \cap C_{2\Delta}^\partial(Y, s)} \frac{1}{r_Y} r_{Y,q} T_s^{Yiq} \cdot \{ \Gamma_{(X,s)}^{03} - \Gamma_{(Y,s)}^{03} \} \right. \\ & \cdot \phi(X, s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Big| + \left| \int_{\partial B_s \cap C_{2\Delta}^\partial(Y, s)} \frac{1}{r_Y} r_{Y,q} T_s^{Yiq} \cdot \Gamma_{(Y,s)}^{03} \right. \\ & \cdot \{ \phi(X, s) \circ \chi_s^{-1} - \phi(Y, s) \circ \chi_s^{-1} \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Big|. \end{aligned}$$
(30)

Let

$$M^{03} := \{1 + (c^*/c)^2\}^3 \cdot \{1 - (c^*/c)\}^{-6};$$

then, by (IV.14.59), we have $\Gamma^{03} \leq M^{03}$. Since $(Z, \tilde{s}) \mapsto J\chi_{\tilde{s}}^{-1}(Z)$ is

continuous on ∂B , and $\bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\} \subset \partial B$ is compact,

$$M_J^K := \max \{J\chi_s^{-1}(Z) \mid (Z, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}\} < \infty.$$

$(P, \tilde{X}, \tilde{s}) \mapsto \phi_{(\tilde{X}, \tilde{s})}(P)$ being continuous on $\partial R \times \mathbb{R}^4$, it is clear that

$(Z, \tilde{s}, \tilde{X}) \mapsto \phi_{(\tilde{X}, \tilde{s})} \circ \chi_s^{-1}(Z)$ is continuous on $\partial B \times \mathbb{R}^3$. Then

$$M_\phi^K := \max \{|\phi_{(\tilde{X}, \tilde{s})} \circ \chi_s^{-1}(Z)| \mid (Z, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}, \text{ dist } (\tilde{X}, \partial B_{\tilde{s}}) \leq d\}$$

is finite, since, as it is easy to check, the set

$$\{(Z, \tilde{s}, \tilde{X}) \mid (Z, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}, \text{ dist } (\tilde{X}, \partial B_{\tilde{s}}) \leq d\}$$

is compact. Recall that

$$\forall i q \quad T_{\tilde{s}}^{iq}(Z) = v^i(Z, \tilde{s}) v^q(Z, \tilde{s}) - \delta^{iq}, \quad \text{for each } (Z, \tilde{s}) \in \partial B,$$

so, if $\xi \in \mathbb{R}^3$,

$$|\xi^q T_{\tilde{s}}^{iq}(Z)| \leq |\xi|_3 + |\xi^i| \leq 2|\xi|_3, \quad \text{whenever } (Z, \tilde{s}) \in \partial B. \quad (31)$$

For example,

$$|r_{\tilde{Y}, q}(Z) T_{\tilde{s}}^{iq}(Z)| \leq 2 \quad \text{whenever } (\tilde{Y}, \tilde{s}) \in \partial B \quad \text{and } Z \in \partial B_{\tilde{s}} \cap \{\tilde{Y}\}'. \quad (32)$$

Suppose that $Z \in \partial B_s \cap C_{2\Delta}^\partial(Y, s)$: then estimate (22) is valid.

Moreover, it is easy to see that $r_Y(Z) \geq 2\Delta$, since $\partial B_s \cap B_{2\Delta}^3(Y) \subset C_{2\Delta}^\partial(Y, s)$.

Using hypothesis (ii), we know that there exist $\kappa_1 > 0$ and

$\beta_1 \in (0, 1]$ such that

$$|\phi_{(\tilde{X}, \tilde{s})} \circ x_{\tilde{s}}^{-1}(Z) - \phi_{(\tilde{Y}, \tilde{s})} \circ x_{\tilde{s}}^{-1}(Z)| \leq \kappa_1 |\tilde{X} - \tilde{Y}|_3^{\beta_1},$$

$$\text{whenever } \tilde{s} \in K, \quad \tilde{Y} \in \partial B_{\tilde{s}}, \quad Z \in \partial B_{\tilde{s}}, \quad \text{and} \quad (33)$$

$$\tilde{X} \in L_{\nu}(\tilde{Y}, \tilde{s}) \cap B_{\eta_K}^3(\tilde{Y});$$

since we demanded that $d_K \leq \eta_K$, the latter obviously holds if $\tilde{X} \in L_{\nu}(\tilde{Y}, \tilde{s}) \cap B_{d_K}^3(\tilde{Y})$.

Upon using all of these facts, along with (26), in (30), we find that

$$\begin{aligned} I_1 &\leq M_{\phi}^{03, K, K} \int_{\partial B_s \cap C_{2\Delta}^{\partial}(Y, s)} \left| \left\{ \frac{1}{r_X} r_{X, q} - \frac{1}{r_Y} r_{Y, q} \right\} T_s^{v, q} \right| d\lambda_{\partial B_s} \\ &\quad + 2M_{\phi}^{K, K} \int_{\partial B_s \cap C_{2\Delta}^{\partial}(Y, s)} \frac{1}{r_Y} |\Gamma_{(X, s)}^{03} - \Gamma_{(Y, s)}^{03}| d\lambda_{\partial B_s} \\ &\quad + 2M_{\phi}^{03, K} \int_{\partial B_s \cap C_{2\Delta}^{\partial}(Y, s)} \frac{1}{r_Y} |\phi_{(X, s)} \circ x_s^{-1} - \phi_{(Y, s)} \circ x_s^{-1}| d\lambda_{\partial B_s} \\ &\leq M_{\phi}^{03, K, K} \cdot \frac{16}{(2\Delta)^3} \cdot |X - Y|_3 \cdot \lambda_{\partial B_s}(\partial B_s) \\ &\quad + 2M_{\phi}^{K, K} \cdot \frac{1}{(2\Delta)^2} \{k_2 |X - Y|_3^{\alpha_2} + \frac{k_3}{2\Delta} |X - Y|_3\} \cdot \lambda_{\partial B_s}(\partial B_s) \\ &\quad + 2M_{\phi}^{03, K} \cdot \frac{1}{(2\Delta)^2} \cdot \kappa_1 |X - Y|_3^{\beta_1} \cdot \lambda_{\partial B_s}(\partial B_s) \\ &\leq \frac{M_1}{\Delta^3} \cdot \max_{\tilde{s} \in K} \lambda_{\partial B_{\tilde{s}}}(\partial B_{\tilde{s}}) \cdot |X - Y|_3^{\alpha'}, \end{aligned} \quad (34)$$

where $\alpha' := \min \{\tilde{\alpha}, \beta_1\}$ and M_1 depends upon only ϕ , M , K , and d_K .

I₂: Suppose that $Z \in \partial B_s$: then

$$\begin{aligned} |r_{X,q}(Z) \{T_s^{v1q}(Z) - T_s^{v1q}(Y)\}| &= |r_{X,q}(Z) \{v^1(Z,s)v^q(Z,s) - v^1(Y,s)v^q(Y,s)\}| \\ &\leq |v^1(Z,s)v(Z,s) - v^1(Y,s)v(Y,s)|_3 \quad (35) \\ &\leq 2|v(Z,s) - v(Y,s)|_3 \leq 2a_K|Z-Y|_3^{\alpha_K}, \end{aligned}$$

the latter inequality following, of course, from the uniform Lyapunov condition on $\{B_\zeta^0\}_{\zeta \in K}$.

Whenever $Z \in C_{2\Delta}^\partial(Y,s) \cap \{Y\}'$, then $Z \in \partial B_s \cap B_{d_K}^3(Y) \cap \{Y\}'$, and two applications of (9) give

$$\frac{1}{r_X(Z)} < \frac{1}{\gamma_K} \cdot \frac{1}{r_X(\Pi_Y(Z))} < \frac{1}{\gamma_K} \cdot \frac{1}{r_Y(\Pi_Y(Z))} \quad (36)$$

(it is obvious that $r_X(\Pi_Y(Z)) > r_Y(\Pi_Y(Z))$), and

$$r_Y(Z) < \frac{1}{\gamma_K} r_Y(\Pi_Y(Z)). \quad (37)$$

Thus, using (35)-(37) in (12), and estimating the resultant integral as in the computations (IV.22.52 and 57),

$$\begin{aligned} I_2 &\leq M_{\phi}^{03} M_{M_J}^{K,K} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X} |r_{X,q} \{T_s^{v1q} - T_s^{v1q}(Y)\}| d\lambda_{\partial B_s} \\ &\leq 2a_K M_{\phi}^{03} M_{M_J}^{K,K} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X} r_Y^{\alpha_K} d\lambda_{\partial B_s} \end{aligned}$$

$$\begin{aligned}
 & < 2a_K^{03} M_\phi^{K,K} \cdot \frac{1}{\gamma_K^{2+\alpha_K}} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{(r_Y \circ \pi_Y)^{2-\alpha_K}} d\lambda_{\partial B_s} \\
 & \leq 2^{3/2} \pi a_K^{03} M_\phi^{K,K} \cdot \frac{1}{\gamma_K^{2+\alpha_K}} \cdot \int_0^{2\Delta} \frac{d\zeta}{1-\alpha_K} = M_2 \Delta^{\alpha_K},
 \end{aligned} \tag{38}$$

the positive M_2 depending only on ϕ , M , and K .

I₃: According to hypothesis (iii),

$$|\phi(\tilde{X}, \tilde{s})^{\circ X_{\tilde{s}}^{-1}}(Z) - \phi(\tilde{Y}, \tilde{s})^{\circ X_{\tilde{s}}^{-1}}(\tilde{Y})| \leq \kappa_2 |Z - \tilde{Y}|_3^{\beta_2 + \kappa_3} |Z - \tilde{X}|_3^{\beta_3}$$

whenever $\tilde{s} \in K$, $\tilde{Y} \in \partial B_{\tilde{s}}$, $Z \in \partial B_{\tilde{s}}$, and

$$\tilde{X} \in L_{\vee}(\tilde{Y}, \tilde{s}) \cap B_{\eta_K}^3(\tilde{Y}),$$

where $\kappa_2 > 0$, $\kappa_3 > 0$, $\eta_K > 0$, $\beta_2 \in (0,1]$, and $\beta_3 \in (0,1]$ depend only on ϕ , M , and K . Further, by (ii), $\hat{J}\chi|_{\partial R \times K}$ is Hölder continuous, so that there exist $\hat{A} > 0$, $\hat{\alpha} \in (0,1]$, depending only on M and K such that, in particular,

$$|\hat{J}\chi(P_2, \zeta) - \hat{J}\chi(P_1, \zeta)| \leq \hat{A} |P_2 - P_1|_3^{\hat{\alpha}} \quad \text{for } P_1, P_2 \in \partial R \quad \text{and } \zeta \in K. \tag{40}$$

Now, having (39) and (40), it is a simple matter to check that the derivation of (IV.22.69) can be carried through here, whence there exist positive numbers k_6 and k_7 , depending only on ϕ , M , and K , for which

$$|\phi_{(X,s)} \circ \chi_s^{-1}(Z) \cdot J\chi_s^{-1}(Z) - \phi_{(Y,s)} \circ \chi_s^{-1}(Y) \cdot J\chi_s^{-1}(Y)| \leq k_6 r_X^{\beta_3}(Z) + k_7 r_Y^{\beta'}(Z) \quad (41)$$

for each $Z \in \partial B_s$,

in which $\beta' := \min \{\hat{\alpha}, \beta_2\}$. With (41), recalling (32), (36), and (37), (13) leads to

$$\begin{aligned} I_3 &\leq 2M^{03} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X} \{k_6 r_X^{\beta_3} + k_7 r_Y^{\beta'}\} d\lambda_{\partial B_s} \\ &< 2M^{03} \left\{ \frac{k_6}{\gamma_K^{2-\beta_3}} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^{2-\beta_3}} d\lambda_{\partial B_s} \right. \\ &\quad \left. + \frac{k_7}{\gamma_K^{2+\beta'}} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^{2-\beta'}} d\lambda_{\partial B_s} \right\} \quad (42) \\ &\leq 2^{3/2} \pi M^{03} \left\{ \frac{k_6}{\gamma_K^{2-\beta_3}} \int_0^{2\Delta} \frac{d\zeta}{\zeta^{2-\beta_3}} + \frac{k_7}{\gamma_K^{2+\beta'}} \int_0^{2\Delta} \frac{d\zeta}{\zeta^{2-\beta'}} \right\} \\ &\leq M_3 \Delta^{\beta''}, \end{aligned}$$

where $\beta'' := \min \{\beta_3, \beta'\}$ and M_3 depends only upon ϕ , M , and K .

I₄: Having prepared (29), we can write directly from (14), once more taking note of (36) and (37),

$$I_4 \leq 2M_{\phi M J}^{K,K} \int_{C_{2\Delta}^\partial(Y,s)} \frac{1}{r_X} \{k_4 r_X^{\tilde{\alpha}} + k_5 r_Y^{\tilde{\alpha}}\} d\lambda_{\partial B_s}$$

$$\begin{aligned}
 & < 2M_{\phi}^{K,K} \left\{ \frac{k_4}{\gamma_K^{2-\tilde{\alpha}}} + \frac{k_5}{\gamma_K^{2+\tilde{\alpha}}} \right\} \int_{C_{2\Delta}^{\partial}(Y,s)} \frac{1}{(r_Y \circ \Pi_Y)^{2-\tilde{\alpha}}} d\lambda_{\partial B_s} \\
 & \leq M_4 \Delta^{\tilde{\alpha}},
 \end{aligned} \tag{43}$$

M_4 depending, as usual, only upon ϕ , M , and K .

I₅: Here, we must first develop an appropriate estimate for the difference appearing in the integrand in (15). For this, we employ a variant of the argument carried out in analyzing a similar difference in the proof of Proposition [IV.27]: as in (IV.27.22), we find, for each $z \in \partial B_s \cap B_K^3(Y)$,

$$\begin{aligned}
 & \left| \frac{v^i q(Y)}{r_X^2(Z)} \left\{ \frac{1}{r_X^2(Z)} r_{X,q}(Z) \cdot \Gamma_X \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} (Z) \right. \right. \\
 & \quad \left. \left. - \left(\frac{1}{r_X^2} r_{X,q} \Gamma_X \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right) \circ \Pi_Y(Z) \cdot v^j(Y,s) v^j(Z,s) \right\} \right| \\
 & \leq 2 |1 - v^j(Y,s) v^j(Z,s)| \cdot \frac{1}{r_X^2(Z)} \cdot \Gamma_X \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} (Z) \\
 & \quad + 2 \cdot \frac{1}{r_X^2(Z)} \cdot | \Gamma_X \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} (Z) - \Gamma_X \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \circ \Pi_Y(Z) | \\
 & \quad + 2 \cdot \Gamma_X \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \circ \Pi_Y(Z) \cdot \left| \frac{Z-X}{r_X^3(Z)} - \frac{\Pi_Y(Z)-X}{r_X^3(\Pi_Y(Z))} \right|_3.
 \end{aligned} \tag{44}$$

Obviously, (IV.27.23) remains valid here:

$$|1 - v^j(Z,s) v^j(Y,s)| \leq a_K r_Y^{\alpha_K}(Z), \quad \text{for each } z \in \partial B_s. \tag{45}$$

We indicated that (IV.27.24) is obtained in a manner quite similar to

that producing (IV.22.24); with no essential modification, we can also show that

$$\begin{aligned} & |\Gamma_X\{X, {}^c_4(X_s^{-1}(Y), s)\}(Z) - \Gamma_X\{X, {}^c_4(X_s^{-1}(Y), s)\} \circ \Pi_Y(Z)| \\ & \leq k_8 |\text{grad } r_X(Z) - \text{grad } r_X(\Pi_Y(Z))|_3, \end{aligned} \quad (46)$$

$$\text{whenever } Z \in \partial B_s \cap B_{d_K}^3(Y),$$

having written $k_8 := 3 \cdot (c^*/c)^2 \{1 - (c^*/c)^2\}^{-7/2}$. Further, recalling inequality [I.2.37.iii.3], and using (36),

$$\begin{aligned} & |\text{grad } r_X(Z) - \text{grad } r_X(\Pi_Y(Z))|_3 \\ &= \left| \frac{Z-X}{r_X(Z)} - \frac{\Pi_Y(Z)-X}{r_X(\Pi_Y(Z))} \right|_3 \\ &= \left| \frac{1}{r_X(Z)} (Z - \Pi_Y(Z)) + \left\{ \frac{1}{r_X(Z)} - \frac{1}{r_X(\Pi_Y(Z))} \right\} (\Pi_Y(Z) - X) \right|_3 \\ &\leq \frac{1}{r_X(Z)} |Z - \Pi_Y(Z)|_3 + \frac{1}{r_X(Z)} |r_X(Z) - r_X(\Pi_Y(Z))| \\ &\leq \frac{2}{r_X(Z)} \cdot |Z - \Pi_Y(Z)|_3 \\ &< \frac{2}{\gamma_K} \cdot \frac{1}{r_Y(\Pi_Y(Z))} \cdot \bar{a}_K r_Y^{1+\alpha_K}(\Pi_Y(Z)) \\ &= \frac{2\bar{a}_K}{\gamma_K} \cdot r_Y^{\alpha_K}(\Pi_Y(Z)), \quad \text{for each } Z \in \partial B_s \cap B_{d_K}^3(Y), \end{aligned} \quad (47)$$

where $\bar{a}_K := \frac{8}{7} \cdot a_K \cdot \left(\frac{65}{49}\right)^{\alpha_K/2} \cdot (1+\alpha_K)^{-1}$. Combining (46) and (47),

$$|\Gamma_X\{X_{,4}^c(X_s^{-1}(Y), s)\}(Z) - \Gamma_X\{X_{,4}^c(X_s^{-1}(Y), s)\} \circ \Pi_Y(Z)| < \frac{2k_8 \bar{a}_K}{\gamma_K} \cdot r_Y^{\alpha_K}(\Pi_Y(Z)), \quad (48)$$

for each $Z \in \partial B_s \cap B_{d_K}^3(Y)$.

Proceeding as in the derivation of (47), and using (9),

$$\begin{aligned} & \left| \frac{Z-X}{r_X^3(Z)} - \frac{\Pi_Y(Z)-X}{r_X^3(\Pi_Y(Z))} \right|_3 \\ &= \left| \frac{1}{r_X^3(Z)} (Z - \Pi_Y(Z)) + \left\{ \frac{1}{r_X^3(Z)} - \frac{1}{r_X^3(\Pi_Y(Z))} \right\} (\Pi_Y(Z) - X) \right|_3 \\ &\leq \frac{1}{r_X^3(Z)} |Z - \Pi_Y(Z)|_3 + \frac{1}{r_X^3(Z)} \cdot |r_X(Z) - r_X(\Pi_Y(Z))| \\ &\quad \cdot \left\{ 1 + \frac{r_X(Z)}{r_X(\Pi_Y(Z))} + \frac{r_X^2(Z)}{r_X^2(\Pi_Y(Z))} \right\} \quad (49) \\ &< \left\{ 2 + \frac{1}{\gamma_K} + \frac{1}{\gamma_K^2} \right\} \cdot \frac{1}{r_X^3(Z)} \cdot |Z - \Pi_Y(Z)|_3 \\ &< \left\{ 2 + \frac{1}{\gamma_K} + \frac{1}{\gamma_K^2} \right\} \cdot \frac{\bar{a}_K}{\gamma_K^3} \cdot \frac{1}{r_Y^{2-\alpha_K}(\Pi_Y(Z))}, \end{aligned}$$

for each $Z \in \partial B_s \cap B_{d_K}^3(Y) \cap \{Y\}^c$.

Upon using (36), (37), (45), (48), and (49) with (44), and observing that

$$|\Gamma_X\{X_{,4}^c(X_s^{-1}(Y), s)\}| \leq M_T := \{1 - (c^*/c)^2\}^{-3/2} \quad \text{on } \mathbb{R}^3 \cap \{X\}^c$$

(cf., (VI.69.4)), the definition (15) leads to

$$\begin{aligned}
 I_5 &\leq 2M_{\phi}^{KK} \cdot \int_{C_{2\Delta}^{\partial}(Y,s)} \left\{ \frac{a_K M_{\Gamma}}{r_X^2} \cdot r_Y^{\alpha_K} + \frac{2k_8 \bar{a}_K}{r_K} \cdot \frac{1}{r_X^2} \cdot r_Y^{\alpha_K} \right. \\
 &\quad \left. + M_{\Gamma} \cdot \left(2 + \frac{1}{r_K} + \frac{1}{r_K^2} \right) \cdot \frac{\bar{a}_K}{r_K^3} \cdot \frac{1}{r_Y^{2-\alpha_K} \Pi_Y} \right\} d\lambda_{\partial B_s} \\
 &\leq 2M_{\phi}^{KK} \cdot \left\{ \frac{a_K M_{\Gamma}}{r_K^{2+\alpha_K}} + \frac{2k_8 \bar{a}_K}{r_K^3} + \left(2 + \frac{1}{r_K} + \frac{1}{r_K^2} \right) \cdot \frac{M_{\Gamma} \bar{a}_K}{r_K^3} \right\} \\
 &\quad \cdot \int_{C_{2\Delta}^{\partial}(Y,s)} \frac{1}{r_Y^{2-\alpha_K} \Pi_Y} d\lambda_{\partial B_s} \\
 &\leq M_5 \Delta^{\alpha_K},
 \end{aligned} \tag{50}$$

M_5 depending solely upon ϕ , M , and K .

I₆: Here, we can appeal to Lemma [IV.26]. It is easy to check that the hypotheses of [IV.26.ii] are fulfilled in the present setting; as noted, we may, and shall, take $\psi(Y,s) = \chi_{\Delta}^c(\chi_s^{-1}(Y),s)$ and $T^q(Y,s) = \chi_s^{1q}(Y)$. Then, comparing (IV.26.7) with (3), we can assert that there exists a positive $\kappa(\Delta)$, depending only on Δ , c^*/c , and d_K , i.e., only on Δ , M , and K , such that[†]

$$I_6 \leq M_{\phi}^{KK} \cdot \kappa(\Delta) \cdot |X-Y|_3. \tag{51}$$

I₇: Proposition [IV.27] provides an estimate for this term. Obviously, the present hypotheses imply that those of [IV.27] are fulfilled. Since an inspection of the proof of [IV.27] reveals that we may set

[†]In the statement of [IV.26.ii], we have chosen $\eta = \Delta$.

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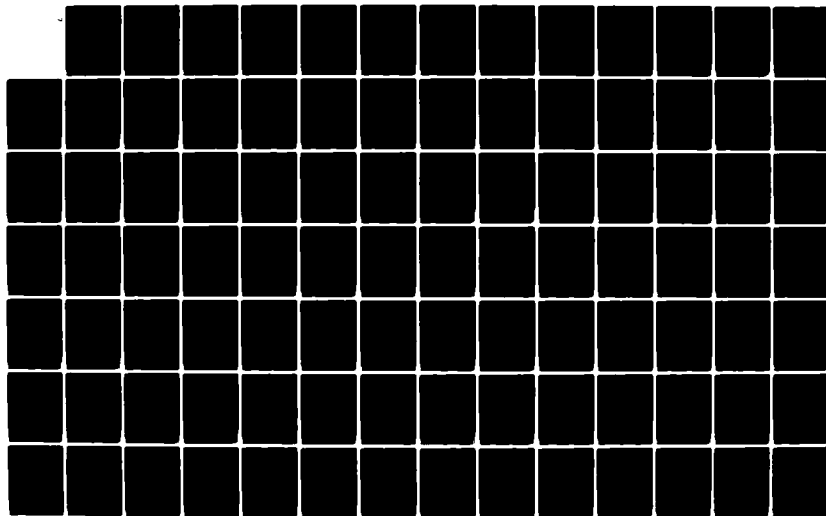
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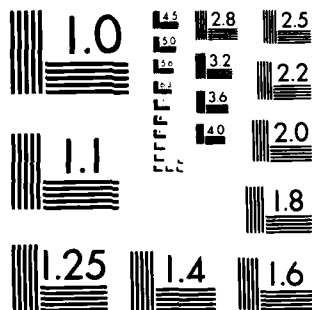
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$\Delta^* = (7/9)d_K$ in the statement of that proposition, we can immediately conclude that there exist $a > 0$ and $\lambda \in (0,1]$, depending only on ϕ , M , and K , for which

$$I_j \leq a\Delta^\lambda. \quad (52)$$

For emphasis, we point out that the inequalities (34), (38), (42), (43), (50), (51), and (52), involving $I_j(Y,s,X;\Delta)$ for $j = 1, \dots, 7$, respectively, are valid whenever $2\Delta \in (0, (7/9)d_K)$, $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$, and $X \in I_V^+(Y,s)$ [$X \in L_V^-(Y,s)$] with $|X-Y|_3 < \Delta$. Having these estimates, it is quite simple to demonstrate the existence of $\Delta(\epsilon)$ and $\delta(\epsilon)$ with the properties described at (18): first, from (38), (42), (43), (50), and (52), we can surely find $\Delta(\epsilon)$, with $2\Delta(\epsilon) \in (0, (7/9)d_K)$, depending only upon ϵ , ϕ , M , and K , and such that

$$I_j(Y,s,X;\Delta(\epsilon)) < \epsilon \quad \text{for } j = 2, 3, 4, 5, \text{ and } 7,$$

$$\text{whenever } (Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\} \quad \text{and} \quad (53)$$

$$X \in L_V^+(Y,s) \quad [X \in L_V^-(Y,s)], \quad \text{with } |X-Y|_3 < \Delta(\epsilon).$$

Now, fix $\Delta = \Delta(\epsilon)$ in (34) and (51). From the two resultant inequalities, it is clear that we can find $\delta(\epsilon) \in (0, \Delta(\epsilon))$, depending only on ϵ , ϕ , M , and K , for which

$$I_j(Y,s,X;\Delta(\epsilon)) < \epsilon \quad \text{for } j = 1 \text{ and } 6,$$

$$\text{whenever } (Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\} \quad \text{and} \quad (54)$$

$$X \in L_V^+(Y, s) \quad [X \in L_V^-(Y, s)], \quad \text{with} \quad |X-Y|_3 < \delta(\epsilon).$$

Because of (10), the first assertion of the theorem is implied by (53) and (54) (obviously, (53) holds if $|X-Y|_3 < \delta(\epsilon)$), for we can replace Δ by $\Delta(\epsilon)$ in (10).

Next, suppose that $\mu: \partial B \rightarrow K$ is locally Hölder continuous, as in hypothesis (iv). Because of the similarities in form exhibited by $\Lambda_2^1\{\mu\}$, $\Lambda_{3jk}^1\{\mu\}$, $\tilde{\Lambda}_{2j}^1\{\mu\}$ and $\Lambda_{1i}^1\{\mu\}$, $\tilde{\Lambda}_1^1\{\mu\}$, it is plain that the proof of the contention that hypothesis (iii) is fulfilled by taking ϕ to be any one of the former triple of functions differs in no essential from that of the corresponding claim in Theorem [IV.22], concerning the latter pair of functions. Consequently, we shall omit the details of the proof of this statement.

To see that (6), (7), and (8) are correct, fix $s \in \mathbb{R}$, then $Y \in \partial B_s$. Now, $\tau(X_s^{-1}(Y); Y, s) = 0$, so $[X_{,4}^c]_{(Y,s)} \circ X_s^{-1}(Y) = X_{,4}^c(X_s^{-1}(Y), s) = V_{(Y,s)}^c(Y)$. Thus, (IV.14.34, 35, and 47) lead, respectively, to

$$\begin{aligned} & \Lambda_2^1\{\mu\}_{(Y,s)} \circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y) \\ &= \{1 - |X_{,4}^c(X_s^{-1}(Y), s)|_3^2\} \cdot \mu(X_s^{-1}(Y), s) \cdot \hat{J}X(X_s^{-1}(Y), s) \cdot JX_s^{-1}(Y) \\ &= \{1 - |X_{,4}^c(X_s^{-1}(Y), s)|_3^2\} \cdot \mu(X_s^*(X_s^{-1}(Y), s)) \cdot JX_s(X_s^{-1}(Y)) \cdot JX_s^{-1}(Y), \end{aligned}$$

$$\begin{aligned} & \Lambda_{3jk}^1\{\mu\}_{(Y,s)} \circ X_s^{-1}(Y) \cdot JX_s^{-1}(Y) \\ &= X_{,4}^{jc}(X_s^{-1}(Y), s) \cdot X_{,4}^{kc}(X_s^{-1}(Y), s) \cdot \mu(X_s^{-1}(Y), s) \cdot \hat{J}X(X_s^{-1}(Y), s) \cdot JX_s^{-1}(Y) \end{aligned}$$

$$= x_{,4}^{j,c}(x_s^{-1}(Y),s) \cdot x_{,4}^{k,c}(x_s^{-1}(Y),s) \cdot \mu(x^*(x_s^{-1}(Y),s)) \cdot Jx_s(x_s^{-1}(Y)) \cdot Jx_s^{-1}(Y),$$

and

$$\begin{aligned} & \tilde{\Lambda}_{2j}^1\{\mu\}(Y,s) \circ x_s^{-1}(Y) \cdot Jx_s^{-1}(Y) \\ &= x_{,4}^{j,c}(x_s^{-1}(Y),s) \cdot \mu(x_s^{-1}(Y),s) \cdot Jx(x_s^{-1}(Y),s) \cdot Jx_s^{-1}(Y) \\ &= x_{,4}^{j,c}(x_s^{-1}(Y),s) \cdot \mu(x^*(x_s^{-1}(Y),s)) \cdot Jx_s(x_s^{-1}(Y)) \cdot Jx_s^{-1}(Y). \end{aligned}$$

Since $x^*(x_s^{-1}(Y),s) = (Y,s)$ (recall that $x^{*-1}(Y,s) = (x_s^{-1}(Y),s)$), and $Jx_s(x_s^{-1}(Y)) \cdot Jx_s^{-1}(Y) = 1$ (cf., [I.2.17.iv]), the latter three equalities give (6), (7), and (8), respectively.

Finally, let $M \in M(2)$: the proof that M satisfies hypotheses (i) and (ii) is contained in [IV.22]; cf., also, [IV.23.a]. $\square$.

Under hypotheses on ϕ and M of the same sort as those already imposed in Theorem [IV.24] in order to obtain a statement of local Hölder continuity for $w_1^*\{\phi\}$ on ∂B , we can prove a similar result for $w_{21}^*\{\phi\}$. Unfortunately, the verification is significantly more involved.

[IV.29] T H E O R E M. Let M be a motion in $M(1)$. Suppose further that

- (i) $\{B_\zeta^0\}_{\zeta \in \mathbb{R}}$ is strongly locally uniformly Lyapunov, i.e., whenever $\bar{K} \subset \mathbb{R}$ is compact, $\forall \mid \cup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}$

is Hölder continuous, so that there exist $a_{\tilde{K}} > 0$ and $\alpha_{\tilde{K}} \in (0,1]$, depending on M and (perhaps) $\tilde{K}$, such that

$$|v(Y_2, s_2) - v(Y_1, s_1)|_3 \leq a_{\tilde{K}} |(Y_2, s_2) - (Y_1, s_1)|_4^{\alpha_{\tilde{K}}} \quad (1)$$

whenever $s_1, s_2 \in \tilde{K}$, $Y_1 \in \partial B_{s_1}$, and $Y_2 \in \partial B_{s_2}$;

(ii) there exists a reference pair (R, X) for M which possesses the properties of [I.3.25] and is also such that $x_{,4}$ and $\hat{J}x$ are locally Hölder continuous on $\partial R \times R$, i.e., for each compact $\tilde{K} \subset R$, $x_{,4}|_{\partial R \times \tilde{K}}$ and $\hat{J}x|_{\partial R \times \tilde{K}}$ are Hölder continuous;

(iii) $(P, \tilde{Y}, \tilde{s}) \mapsto \phi(\tilde{Y}, \tilde{s})(P)$ is a function in $C(\partial R \times \partial B)$ which also satisfies the following local Hölder-type conditions: whenever $\tilde{K}$ is a compact subset of R , there exist positive numbers $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, and $\tilde{\kappa}_3$ and numbers $\tilde{\beta}_1$, $\tilde{\beta}_2$, and $\tilde{\beta}_3 \in (0,1]$, depending on ϕ and perhaps on $\tilde{K}$ and M , such that

$$|\phi(Y_1, s_1) \circ x_{s_1}^{-1}(Z) - \phi(Y_1, s_1) \circ x_{s_1}^{-1}(Y_1)| \leq \tilde{\kappa}_1 |Z - Y_1|_3^{\tilde{\beta}_1}, \quad (2)$$

$$|\phi(Y_2, s_2) \circ x_{s_1}^{-1}(Z) - \phi(Y_1, s_1) \circ x_{s_1}^{-1}(Z)| \leq \tilde{\kappa}_2 |(Y_2, s_2) - (Y_1, s_1)|_4^{\tilde{\beta}_2}, \quad (3)$$

and

$$|\phi_{(Y_2, s_2)} \circ \chi_{s_2}^{-1}(Y_2) - \phi_{(Y_1, s_1)} \circ \chi_{s_1}^{-1}(Y_1)| \leq \tilde{\kappa}_3 |(Y_2, s_2) - (Y_1, s_1)|_4^{\tilde{\beta}_3}, \quad (4)$$

whenever $s_1, s_2 \in \tilde{K}$, $Y_1 \in \partial B_{s_1}$, $Y_2 \in \partial B_{s_2}$,

and $Z \in \partial B_{s_1}$.

Then the function $w_{21}^*\{\phi\}: \partial B \rightarrow K$, constructed from ϕ and χ as in [IV.27], is locally Hölder continuous on ∂B : whenever $K \subset \mathbb{R}$ is compact, there exist $\ell_K > 0$ and $\lambda_K \in (0, 1]$, depending on ϕ , M , and (perhaps) K , such that

$$|w_{21}^*\{\phi\}(Y_2, s_2) - w_{21}^*\{\phi\}(Y_1, s_1)| \leq \ell_K |(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K} \quad (5)$$

whenever $s_1, s_2 \in K$, $Y_1 \in \partial B_{s_1}$, and $Y_2 \in \partial B_{s_2}$.

If it is assumed, moreover, that

(iv) $\mu: \partial B \rightarrow K$ is locally Hölder continuous, i.e., for each compact subset $\tilde{K}$ of $\mathbb{R}$, $\mu|_{\cup_{\zeta \in \tilde{K}} \{\partial B_\zeta \times \{\zeta\}\}}$ is Hölder continuous,

then hypothesis (iii) is fulfilled by taking $\phi = \Lambda_2^1\{\mu\}$, $\phi = \Lambda_{3jk}^1\{\mu\}$, or $\phi = \tilde{\Lambda}_{2j}^1\{\mu\}$ (cf., (IV.14.34, 35, and 47)), whence $w_{21}^*\{\Lambda_2^1\{\mu\}\}$, $w_{21}^*\{\Lambda_{3jk}^1\{\mu\}\}$, and $w_{21}^*\{\tilde{\Lambda}_{2j}^1\{\mu\}\}$ are locally Hölder continuous on ∂B .

Finally, if $M \in M(2)$, then hypotheses (i) and (ii) are fulfilled.

P R O O F. The present hypotheses (i), (ii), and (2) clearly imply

that those of Proposition [IV.27] are satisfied. In particular, it follows that $\omega_{21}^*\{\phi\}$ is defined on ∂B by (IV.27.2). It is enough to prove the first assertion of the theorem for $K = [t_1, t_2]$, a compact interval, which we shall do. By (1), we know that

$$|v(Y_2, s_2) - v(Y_1, s_1)|_3 \leq a_K |(Y_2, s_2) - (Y_1, s_1)|_4^{\alpha_K} \quad (6)$$

for $s_1, s_2 \in K$, $Y_1 \in \partial B_{s_1}$, and $Y_2 \in \partial B_{s_2}$,

$a_K > 0$ and $\alpha_K \in (0, 1]$ depending on M and K . Choosing $d_K > 0$ such that $a_K d_K^{\alpha_K} < 1/2$, it is evident that $\{\partial B_\zeta^0\}_{\zeta \in K}$ is a uniformly Lyapunov family of domains in $\mathbb{R}^3$, with uniform constants (a_K, α_K, d_K) .

To demonstrate that $\omega_{21}^*\{\phi\} \mid \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ is Hölder continuous, it suffices to choose a positive Δ and show that there can be found $\ell'_K > 0$ and $\lambda_K \in (0, 1]$, depending on ϕ , M , and K , with

$$|\omega_{21}^*\{\phi\}(Y_2, s_2) - \omega_{21}^*\{\phi\}(Y_1, s_1)| \leq \ell'_K |(Y_2, s_2) - (Y_1, s_1)|_4^{\lambda_K} \quad (7)$$

whenever $s_1, s_2 \in K$, $Y_1 \in \partial B_{s_1}$, $Y_2 \in \partial B_{s_2}$,

$$\text{and } |(Y_2, s_2) - (Y_1, s_1)|_4 < \Delta;$$

the existence of an $\ell'_K > 0$ with the required properties and for which (5) holds can be subsequently deduced easily, as in the proof of [IV.24]. Fixing a number d satisfying

$$0 < d < \min \left\{ \frac{1}{3} d_K, \frac{7}{18} a_K \right\}, \quad (8)$$

we shall provide reasoning which leads to an inequality of the form of (7), with

$$\Delta = \min \{1, (d/61)^{1/\alpha_K}, \Delta_K\},$$

in which

$$1 := \{1+(c^*)^2\}^{1/2},$$

and Δ_K is a positive number specified in (152), *infra*. Throughout the argument, (Y,s) and $(\hat{Y},\hat{s})$ denote points of $\bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ (so that $s, \hat{s} \in K$, $Y \in \partial B_s$, and $\hat{Y} \in \partial B_{\hat{s}}$), we write

$$\delta := |(\hat{Y},\hat{s}) - (Y,s)|_4,$$

and suppose that

$$0 < \delta < \min \{1, (d/61)^{1/\alpha_K}, \Delta_K\}. \quad (10)$$

Clearly, we have $31\delta \leq 31\delta^{\alpha_K} < d < (7/9)d_K$, so $C_{31\delta}^\partial(Y,s)$ and $C_d^\partial(Y,s)$ are well-defined subsets of ∂B_s , the former lying within the latter.

Select and fix any number η such that

$$0 < \eta < 1\delta. \quad (11)$$

We then begin by writing

$$\begin{aligned}
 & |w_{21}^*\{\phi\}(\hat{Y}, \hat{s}) - w_{21}^*\{\phi\}(Y, s)| \\
 & \leq \left| w_{21}^*\{\phi\}(\hat{Y}, \hat{s}) - \frac{1}{4\pi} \int_{\partial B_{\hat{s}} \cap C_{\eta}^{\partial}(Y, s)} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q}^{\vee 1 q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \right. \\
 & \quad \cdot J\chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \left| + \left| w_{21}^*\{\phi\}(Y, s) - \frac{1}{4\pi} \int_{\partial B_s \cap C_{\eta}^{\partial}(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q}^{\vee 1 q} \cdot \Gamma_{(Y, s)}^{03} \right. \right. \\
 & \quad \cdot \phi(Y, s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \left| + \frac{1}{4\pi} \left| \int_{\partial B_{\hat{s}} \cap C_{\eta}^{\partial}(\hat{Y}, \hat{s})} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q}^{\vee 1 q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \right. \right. \\
 & \quad \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \cdot J\chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} - \int_{\partial B_s \cap C_{\eta}^{\partial}(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q}^{\vee 1 q} \cdot \Gamma_{(Y, s)}^{03} \\
 & \quad \cdot \phi(Y, s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \left| \right. .
 \end{aligned} \tag{12}$$

Now, recalling Proposition [IV.27], there exist $a' > 0$ and $\lambda' \in (0, 1]$, depending upon ϕ , M , and K only, such that[†]

$$\begin{aligned}
 & \left| w_{21}^*\{\phi\}(\tilde{Y}, \tilde{s}) - \frac{1}{4\pi} \int_{\partial B_{\tilde{s}} \cap C_{\eta}^{\partial}(\tilde{Y}, \tilde{s})} \frac{1}{r_{\tilde{Y}}} \cdot r_{\tilde{Y}, q}^{\vee 1 q} \cdot \Gamma_{(\tilde{Y}, \tilde{s})}^{03} \right. \\
 & \quad \cdot \phi(\tilde{Y}, \tilde{s}) \circ \chi_{\tilde{s}}^{-1} \cdot J\chi_{\tilde{s}}^{-1} d\lambda_{\partial B_{\tilde{s}}} \left| \leq a' \eta^{\lambda'} < a' \delta^{\lambda'} \right.
 \end{aligned} \tag{13}$$

for $\tilde{s} \in K$ and $\tilde{Y} \in \partial B_{\tilde{s}}$.

Thus, each of the first two terms on the right-hand side of (12) is

[†]Note that we can take $\Delta^* = (7/9)d_K$ in the statement of [IV.27], as an inspection of the proof reveals; here, $\eta < \delta < (7/9)d_K$.

majorized by $a'_{1\lambda'} \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4^{\lambda'}$.

Turning to the third term on the right in (12), we use the 1-embedding $\chi_{\hat{s}\hat{s}} := \chi_{\hat{s}} \circ \chi_s^{-1} : \partial B_s \rightarrow \mathbb{R}^3$, taking ∂B_s onto $\partial B_{\hat{s}}$, and the relations

$$\chi_{\hat{s}}^{-1} \circ \chi_{\hat{s}\hat{s}} = \chi_s^{-1}, \quad (14)$$

$$(J\chi_{\hat{s}}^{-1}) \circ \chi_{\hat{s}\hat{s}} \cdot J\chi_{\hat{s}\hat{s}} = J\chi_s^{-1} \quad (15)$$

(cf., (IV.24.7 and 8)), in conjunction with [I.2.26.a] and the equality

$$\chi_{\hat{s}\hat{s}}(\partial B_{\hat{s}} \cap C_n^{\partial}(\hat{Y}, \hat{s}))' = \partial B_s \cap \{\chi_{\hat{s}\hat{s}}(C_n^{\partial}(\hat{Y}, \hat{s}))\}',$$

to produce

$$\begin{aligned} & \left| \int_{\partial B_s \cap C_n^{\partial}(\hat{Y}, \hat{s})'} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q}^{\vee i q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \cdot J\chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \right. \\ & - \left. \int_{\partial B_s \cap C_n^{\partial}(Y, s)'} \frac{1}{r_Y} \cdot r_{Y, q}^{\vee i q} \cdot \Gamma_{(Y, s)}^{03} \cdot \phi(Y, s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \\ & = \left| \int_{\partial B_s \cap \{\chi_{\hat{s}\hat{s}}(C_n^{\partial}(\hat{Y}, \hat{s}))\}'} \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q}^{\vee i q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} \right. \\ & \quad \cdot J\chi_s^{-1} d\lambda_{\partial B_s} - \left. \int_{\partial B_s \cap C_n^{\partial}(Y, s)'} \frac{1}{r_Y} \cdot r_{Y, q}^{\vee i q} \cdot \Gamma_{(Y, s)}^{03} \cdot \phi(Y, s) \circ \chi_s^{-1} \right. \\ & \quad \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \end{aligned}$$

$$\begin{aligned}
 & \leq \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q}^{\gamma 1q} \cdot \Gamma_{(Y,s)}^{03} \right\} \circ \chi_{s\hat{s}}^{-1} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \right. \\
 & \quad \left. - \frac{1}{r_Y^2} \cdot r_{Y,q}^{\gamma 1q} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi(Y,s) \circ \chi_s^{-1} \right\} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_d^\partial(Y,s) \cap \{ \chi_{\hat{s}\hat{s}}(C_n^\partial(\hat{Y},\hat{s})) \}} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q}^{\gamma 1q} \cdot \Gamma_{(Y,s)}^{03} \right\} \circ \chi_{s\hat{s}}^{-1} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \right. \\
 & \quad \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} - \int_{C_d^\partial(Y,s) \cap C_n^\partial(Y,s)} \frac{1}{r_Y^2} \cdot r_{Y,q}^{\gamma 1q} \cdot \Gamma_{(Y,s)}^{03} \cdot \phi(Y,s) \circ \chi_s^{-1} \right. \\
 & \quad \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \\
 & \leq \sum_{j=1}^4 I_j(Y,s;\hat{Y},\hat{s}) \\
 & + \left| \int_{C_d^\partial(Y,s) \cap \{ \chi_{\hat{s}\hat{s}}(C_n^\partial(\hat{Y},\hat{s})) \}} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q}^{\gamma 1q} \cdot \Gamma_{(Y,s)}^{03} \right\} \circ \chi_{s\hat{s}}^{-1} \right. \\
 & \quad \left. \cdot \{ \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} - \phi(\hat{Y},\hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right. \\
 & \quad \left. - \int_{C_d^\partial(Y,s) \cap C_n^\partial(Y,s)} \frac{1}{r_Y^2} \cdot r_{Y,q}^{\gamma 1q} \cdot \Gamma_{(Y,s)}^{03} \cdot \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \right. \\
 & \quad \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) \cdot \int_{C_d^\partial(Y, s) \cap \{\chi_{\hat{s}s}(C_\eta^\partial(\hat{Y}, \hat{s}))\}} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y}, q} \cdot \frac{\gamma^{1q}}{\hat{s}} \cdot \Gamma^{03}(\hat{Y}, \hat{s}) \right\} \circ \chi_{s\hat{s}} \right. \\
 & J\chi_s^{-1} d\lambda_{\partial B_s} - \phi(Y, s) \circ \chi_s^{-1}(Y) \\
 & \cdot \left| \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \frac{1}{r_Y^2} \cdot r_{Y, q} \cdot \frac{\gamma^{1q}}{\hat{s}} \cdot \Gamma^{03}(Y, s) \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \\
 & \leq \sum_{j=1}^4 I_j(Y, s; \hat{Y}, \hat{s}) \\
 & + \left| \int_{C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)} \left\{ \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y}, q} \cdot \frac{\gamma^{1q}}{\hat{s}} \cdot \Gamma^{03}(\hat{Y}, \hat{s}) \right\} \circ \chi_{s\hat{s}} \right. \right. \\
 & \cdot \{ \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} - \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) \} - \frac{1}{r_Y^2} \cdot r_{Y, q} \cdot \frac{\gamma^{1q}}{\hat{s}} \cdot \Gamma^{03}(Y, s) \\
 & \cdot \{ \phi(Y, s) \circ \chi_s^{-1} - \phi(Y, s) \circ \chi_s^{-1}(Y) \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_{3, \delta}^\partial(Y, s) \cap \{\chi_{\hat{s}s}(C_\eta^\partial(\hat{Y}, \hat{s}))\}} \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y}, q} \cdot \frac{\gamma^{1q}}{\hat{s}} \cdot \Gamma^{03}(\hat{Y}, \hat{s}) \right\} \circ \chi_{s\hat{s}} \right. \\
 & \cdot \{ \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} - \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Big|
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \int_{C_{3, \delta}^{\partial}(Y, s) \cap C_n^{\partial}(Y, s)} \frac{1}{r_Y^2} \cdot r_{Y, q}^{\gamma i q} \cdot \Gamma_{(Y, s)}^{03} \cdot \{\phi_{(Y, s)}^{\circ X_s^{-1}} - \phi_{(Y, s)}^{\circ X_s^{-1}(Y)}\} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| \\
 & + |\phi_{(\hat{Y}, \hat{s})}^{\circ X_{\hat{s}}^{-1}(\hat{Y})} - \phi_{(Y, s)}^{\circ X_s^{-1}(Y)}| \\
 & \cdot \left| \int_{C_d^{\partial}(Y, s) \cap C_n^{\partial}(Y, s)} \frac{1}{r_Y^2} \cdot r_{Y, q}^{\gamma i q} \cdot \Gamma_{(Y, s)}^{03} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| \\
 & + |\phi_{(\hat{Y}, \hat{s})}^{\circ X_{\hat{s}}^{-1}(\hat{Y})}| \\
 & \cdot \left| \int_{C_d^{\partial}(Y, s) \cap \{X_{\hat{s}}(C_n^{\partial}(\hat{Y}, \hat{s}))\}} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y}, q}^{\gamma i q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \right\} \circ X_{\hat{s}\hat{s}} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right. \\
 & \left. - \int_{C_d^{\partial}(Y, s) \cap C_n^{\partial}(Y, s)} \frac{1}{r_Y^2} \cdot r_{Y, q}^{\gamma i q} \cdot \Gamma_{(Y, s)}^{03} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|, \quad (16)
 \end{aligned}$$

having defined $I_j = I_j(Y, s; \hat{Y}, \hat{s})$, $j = 1, \dots, 4$, by

$$\begin{aligned}
 I_1 := & \left| \int_{\partial B_s \cap C_d^{\partial}(Y, s)} \frac{1}{r_Y^2} \cdot r_{Y, q}^{\gamma i q} \cdot \Gamma_{(Y, s)}^{03} \cdot \{\phi_{(\hat{Y}, \hat{s})}^{\circ X_s^{-1}} - \phi_{(Y, s)}^{\circ X_s^{-1}}\} \right. \\
 & \left. \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|, \quad (17)
 \end{aligned}$$

$$I_2 := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,q} \cdot \tilde{T}_s^{1q} \cdot \{ \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{ss}^{-1} \Gamma_{(Y,s)}^{03} \} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (18)$$

$$I_3 := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,q} \cdot \{ \tilde{T}_s^{1q} \circ \chi_{ss} - \tilde{T}_s^{1q} \} \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{ss} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (19)$$

and

$$I_4 := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \left\{ \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},q} \right\} \circ \chi_{ss} - \frac{1}{r_Y} \cdot r_{Y,q} \right\} \cdot \tilde{T}_s^{1q} \circ \chi_{ss} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{ss} \cdot \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|. \quad (20)$$

The inclusion

$$\chi_{ss}(C_n^\partial(\hat{Y},\hat{s})) \subset C_{3\delta}^\partial(Y,s) \quad (21)$$

has been used in the derivation of (16), and is quite easy to verify:

if $Z \in C_n^\partial(\hat{Y},\hat{s})$, then $r_{\hat{Y}}(Z) < \frac{9}{7} r_{\hat{Y}}(\Pi_{\hat{Y}}(Z)) < \frac{9}{7} n < \frac{9}{7} \delta$, so

$$\begin{aligned} r_Y(\Pi_Y(\chi_{ss}(Z))) &\leq r_Y(\chi_{ss}(Z)) \leq |Y-\hat{Y}|_3 + |\chi_{ss}(Z)-Z|_3 + |Z-\hat{Y}|_3 \\ &\leq |Y-\hat{Y}|_3 + c^*|s-\hat{s}| + |Z-\hat{Y}|_3 < \delta + \frac{9}{7} \delta < 3\delta, \end{aligned} \quad (22)$$

since

$$|\chi_{\hat{s}\hat{s}}(Z)-Z|_3 = |\chi(\chi_{\hat{s}}^{-1}(Z),s)-\chi(\chi_{\hat{s}}^{-1}(Z),\hat{s})|_3 \leq c^*|s-\hat{s}|. \quad (23)$$

The result (22) implies (21).

Now, to estimate the fifth term on the right of inequality (16), we can write

$$\begin{aligned} & \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \frac{\gamma_{\hat{s}}^{\gamma_{\hat{s}}}}{r_{\hat{s}}^{\gamma_{\hat{s}}}} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} \right. \\ & \quad \cdot \{ \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} - \phi(\hat{Y},\hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) \} - \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \frac{\gamma_s^{\gamma_s}}{r_s^{\gamma_s}} \cdot \Gamma_{(Y,s)}^{03} \\ & \quad \cdot \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \Big| \\ & \leq \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \frac{\gamma_{\hat{s}}^{\gamma_{\hat{s}}}}{r_{\hat{s}}^{\gamma_{\hat{s}}}} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} \right. \\ & \quad \cdot \{ \{ \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} - \phi(\hat{Y},\hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) \} - \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \Big| \\ & \quad + \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \frac{\gamma_{\hat{s}}^{\gamma_{\hat{s}}}}{r_{\hat{s}}^{\gamma_{\hat{s}}}} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} - \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \frac{\gamma_s^{\gamma_s}}{r_s^{\gamma_s}} \cdot \Gamma_{(Y,s)}^{03} \right\} \right. \\ & \quad \cdot \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \Big| \\ & \leq \sum_{j=5}^8 I_j(Y,s;\hat{Y},\hat{s}), \end{aligned} \quad (24)$$

wherein the $I_j = I_j(Y,s;\hat{Y},\hat{s})$, $j = 5, \dots, 8$, are given by

$$I_5 := \left| \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \left\{ \frac{1}{r_Y} \cdot r_{\hat{Y},q} \cdot \frac{\gamma^{iq}}{\hat{s}} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{s\hat{s}} \right. \\ \left. \cdot \{ \{ \phi(\hat{Y},\hat{s}) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1} \} \right. \\ \left. - \{ \phi(\hat{Y},\hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \} \cdot J \chi_s^{-1} d\lambda_{\partial \mathcal{B}_s} \right|, \quad (25)$$

$$I_6 := \left| \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,q} \cdot \frac{\gamma^{iq}}{\hat{s}} \{ \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} - \Gamma_{(Y,s)}^{03} \} \right. \\ \left. \cdot \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \cdot J \chi_s^{-1} d\lambda_{\partial \mathcal{B}_s} \right|, \quad (26)$$

$$I_7 := \left| \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \frac{1}{r_Y} \cdot r_{Y,q} \{ \frac{\gamma^{iq}}{\hat{s}} \circ \chi_{s\hat{s}} - \frac{\gamma^{iq}}{\hat{s}} \} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} \right. \\ \left. \cdot \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \cdot J \chi_s^{-1} d\lambda_{\partial \mathcal{B}_s} \right|, \quad (27)$$

and

$$I_8 := \left| \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \left\{ \left\{ \frac{1}{r_Y} \cdot r_{\hat{Y},q} \right\} \circ \chi_{s\hat{s}} - \frac{1}{r_Y} \cdot r_{Y,q} \right\} \right. \\ \left. \cdot \frac{\gamma^{iq}}{\hat{s}} \circ \chi_{s\hat{s}} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} \cdot \{ \phi(Y,s) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1}(Y) \} \cdot J \chi_s^{-1} d\lambda_{\partial \mathcal{B}_s} \right|. \quad (28)$$

To obtain an appropriate bound for the eighth term on the right-hand side of (16), we shall make use of Lemma [IV.26], which allows us to assert that, whenever $\hat{s} \in K$, $\hat{Y} \in \partial \mathcal{B}_{\hat{s}}$, and

$$\tilde{d} \in (\eta, (7/9)d_K),$$

$$\int_{C_d^{\partial}(\tilde{Y}, \tilde{s}) \cap C_n^{\partial}(\tilde{Y}, \tilde{s})} \left(\frac{1}{r_Y} \cdot r_{\tilde{Y}, q} \cdot \Gamma_{\tilde{Y}} \{ \chi_{,4}^c(\chi_s^{-1}(\tilde{Y}), \tilde{s}) \} \right) \circ \Pi_{\tilde{Y}} \cdot \tilde{T}_s^{1q}(\tilde{Y}) \quad (29)$$

$$\cdot \nu_{\partial B_s}^j(\tilde{Y}) \nu_{\partial B_s}^j d\lambda_{\partial B_s} = 0,$$

whence

$$\begin{aligned} & | \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1}(\hat{Y}) - \phi(Y, s) \circ \chi_s^{-1}(Y) | \\ & \cdot \left| \int_{C_d^{\partial}(Y, s) \cap C_n^{\partial}(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q} \cdot \tilde{T}_s^{1q} \cdot \Gamma_{(Y, s)}^{03} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \\ & \leq | \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1}(\hat{Y}) - \phi(Y, s) \circ \chi_s^{-1}(Y) | \\ & \cdot \left\{ \left| \int_{C_d^{\partial}(Y, s) \cap C_n^{\partial}(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q} \cdot \{ \tilde{T}_s^{1q} \cdot \Gamma_{(Y, s)}^{03} \cdot J\chi_s^{-1} \right. \right. \\ & \quad \left. \left. - \tilde{T}_s^{1q}(Y) \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \cdot J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right| \right. \\ & \quad \left. + J\chi_s^{-1}(Y) \cdot \left| \int_{C_d^{\partial}(Y, s) \cap C_n^{\partial}(Y, s)} \left\{ \frac{1}{r_Y} \cdot r_{Y, q} \cdot \tilde{T}_s^{1q}(Y) \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right. \right. \right. \\ & \quad \left. \left. - \left(\frac{1}{r_Y} \cdot r_{Y, q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right) \circ \Pi_Y \cdot \tilde{T}_s^{1q}(Y) \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j \right\} d\lambda_{\partial B_s} \right| \right\} \\ & \leq \sum_{j=11}^{14} I_j(Y, s; \hat{Y}, \hat{s}), \end{aligned} \quad (30)$$

with $I_j = I_j(Y, s; \hat{Y}, \hat{s})$, $j = 11, \dots, 14$, being given by

$$I_{11} := |\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(\hat{Y}) - \phi_{(Y, s)} \circ \chi_s^{-1}(Y)| \quad (31)$$

$$\cdot \left| \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q} \{ \tilde{T}_s^{1q} - \tilde{T}_s^{1q}(Y) \} \Gamma_{(Y, s)}^{03} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| ,$$

$$I_{12} := |\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(\hat{Y}) - \phi_{(Y, s)} \circ \chi_s^{-1}(Y)|$$

$$\cdot \left| \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q} \tilde{T}_s^{1q}(Y) \right. \quad (32)$$

$$\left. \cdot \{ \Gamma_{(Y, s)}^{03} - \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| ,$$

$$I_{13} := |\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(\hat{Y}) - \phi_{(Y, s)} \circ \chi_s^{-1}(Y)|$$

$$\cdot \left| \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \frac{1}{r_Y} \cdot r_{Y, q} \tilde{T}_s^{1q}(Y) \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right. \quad (33)$$

$$\left. \cdot \{ J\chi_s^{-1} - J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right| ,$$

and

$$I_{14} := |\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(\hat{Y}) - \phi_{(Y, s)} \circ \chi_s^{-1}(Y)| \cdot J\chi_s^{-1}(Y)$$

$$\cdot \left| \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \left\{ \frac{1}{r_Y} \cdot r_{Y, q} \tilde{T}_s^{1q}(Y) \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right. \right.$$

$$\begin{aligned}
 & - \left\{ \frac{1}{2} r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \right\} \circ \pi_Y \cdot \Gamma_s^{1q}(Y) \\
 & \cdot \left\{ \frac{1}{\partial B_s} (Y) \frac{1}{\partial B_s} \right\} d\lambda_{\partial B_s} \Big| .
 \end{aligned} \tag{34}$$

For the estimation of the ninth term on the right-hand side of (16), we require a quite delicate (and extensive) decomposition.

Set

$$M_\phi^K := \max \{ |\phi_{(\tilde{Y}, \tilde{s})}(P)| \mid (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}, P \in \partial R \}.$$

Then, once again employing the transformation formula of [I.2.26.a] and (29) (with $\tilde{s} = \hat{s}$, $\tilde{Y} = \hat{Y}$, and $\tilde{d} = d$),

$$\begin{aligned}
 & |\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(\hat{Y})| \\
 & \cdot \left| \int_{C_d^\partial(Y, s) \cap (\chi_{\hat{s}\hat{s}}^\partial(C_\eta^\partial(\hat{Y}, \hat{s})))} \left\{ \frac{1}{2} r_{\hat{Y}, q} \cdot \Gamma_{\hat{s}}^{1q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right. \\
 & \quad \left. - \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \frac{1}{2} r_{Y, q} \cdot \Gamma_s^{1q} \cdot \Gamma_{(Y, s)}^{03} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right| \\
 & = |\phi_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}}^{-1}(\hat{Y})| \\
 & \cdot \left| \int_{\chi_{\hat{s}\hat{s}}(C_d^\partial(Y, s) \cap C_\eta^\partial(\hat{Y}, \hat{s}))} \frac{1}{2} r_{\hat{Y}, q} \cdot \Gamma_{\hat{s}}^{1q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \cdot J\chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \right. \\
 & \quad \left. - \int_{C_d^\partial(Y, s) \cap C_\eta^\partial(Y, s)} \frac{1}{2} r_{Y, q} \cdot \Gamma_s^{1q} \cdot \Gamma_{(Y, s)}^{03} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|
 \end{aligned}$$

$$\begin{aligned}
 & \leq M_{\phi}^K \cdot \left| \int_{\chi_{s\hat{s}}(C_d^{\partial}(Y,s)) \cap C_{\eta}^{\partial}(\hat{Y},\hat{s})} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \tilde{T}_{\hat{s}}^{1q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot J\chi_{\hat{s}}^{-1} \right. \right. \\
 & \quad \left. \left. - \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}}\{\chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s})\} \right) \circ \Pi_{\hat{Y}} \cdot \tilde{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot v_{\partial B_{\hat{s}}}^j(\hat{Y}) v_{\partial B_{\hat{s}}}^j \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \right\} d\lambda_{\partial B_{\hat{s}}} \right. \\
 & \quad - \int_{C_d^{\partial}(Y,s) \cap C_{\eta}^{\partial}(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \tilde{T}_s^{1q} \cdot \Gamma_{(Y,s)}^{03} \cdot J\chi_s^{-1} \right. \\
 & \quad \left. - \left(\frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_Y\{\chi_{,4}^c(\chi_s^{-1}(Y),s)\} \right) \circ \Pi_Y \cdot \tilde{T}_s^{1q}(Y) \cdot v_{\partial B_s}^j(Y) v_{\partial B_s}^j \cdot J\chi_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \Big| \\
 & \quad + M_{\phi}^K \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \cdot \left| \int_{\chi_{s\hat{s}}(C_d^{\partial}(Y,s)) \cap C_{\eta}^{\partial}(\hat{Y},\hat{s})} \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}}\{\chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s})\} \right) \circ \Pi_{\hat{Y}} \right. \\
 & \quad \left. \cdot \tilde{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot v_{\partial B_{\hat{s}}}^j(\hat{Y}) v_{\partial B_{\hat{s}}}^j d\lambda_{\partial B_{\hat{s}}} \right| \\
 & \leq M_{\phi}^K \cdot \left| \int_{\chi_{s\hat{s}}(C_d^{\partial}(Y,s)) \cap \{\chi_{s\hat{s}}(C_{3,6}^{\partial}(Y,s))\}} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \tilde{T}_{\hat{s}}^{1q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot J\chi_{\hat{s}}^{-1} \right. \right. \\
 & \quad \left. \left. - \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}}\{\chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s})\} \right) \circ \Pi_{\hat{Y}} \cdot \tilde{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot v_{\partial B_{\hat{s}}}^j(\hat{Y}) v_{\partial B_{\hat{s}}}^j \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \right\} d\lambda_{\partial B_{\hat{s}}} \right. \\
 & \quad - \int_{C_d^{\partial}(Y,s) \cap C_{3,6}^{\partial}(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \tilde{T}_s^{1q} \cdot \Gamma_{(Y,s)}^{03} \cdot J\chi_s^{-1} \right. \\
 & \quad \left. - \left(\frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_Y\{\chi_{,4}^c(\chi_s^{-1}(Y),s)\} \right) \circ \Pi_Y \cdot \tilde{T}_s^{1q}(Y) \cdot v_{\partial B_s}^j(Y) v_{\partial B_s}^j \cdot J\chi_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \Big|
 \end{aligned}$$

$$\begin{aligned}
 & +M_{\phi}^K \cdot \left| \int_{\chi_{ss}(C_{31\delta}^{\partial}(Y,s)) \cap C_n^{\partial}(\hat{Y},\hat{s})} \left\{ \frac{1}{2} \cdot r_{\hat{Y},q} \cdot \tilde{r}_{\hat{s}}^{1q} \cdot \tilde{r}_{(\hat{Y},\hat{s})}^{03} \cdot J\chi_{\hat{s}}^{-1} \right. \right. \\
 & \left. - \left(\frac{1}{2} \cdot r_{\hat{Y},q} \cdot \tilde{r}_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s}) \} \right) \circ \Pi_{\hat{Y}} \cdot \tilde{r}_{\hat{s}}^{1q}(\hat{Y}) \cdot \nu_{\partial B_{\hat{s}}}^j(\hat{Y}) \nu_{\partial B_{\hat{s}}}^j \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \right\} d\lambda_{\partial B_{\hat{s}}} \Big| \\
 & +M_{\phi}^K \cdot \left| \int_{C_{31\delta}^{\partial}(Y,s) \cap C_n^{\partial}(Y,s)} \left\{ \frac{1}{2} \cdot r_{Y,q} \cdot \tilde{r}_s^{1q} \cdot \tilde{r}_{(Y,s)}^{03} \cdot J\chi_s^{-1} \right. \right. \\
 & \left. - \left(\frac{1}{2} \cdot r_{Y,q} \cdot \tilde{r}_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y),s) \} \right) \circ \Pi_Y \cdot \tilde{r}_s^{1q}(Y) \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j \cdot J\chi_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \Big| \\
 & +M_{\phi}^K \cdot I_{15}(Y,s;\hat{Y},\hat{s}), \tag{35}
 \end{aligned}$$

in which $I_{15} = I_{15}(Y,s;\hat{Y},\hat{s})$ is given by

$$\begin{aligned}
 I_{15} & := J\chi_{\hat{s}}^{-1}(\hat{Y}) \\
 & \cdot \left| \int_{\chi_{ss}(C_d^{\partial}(Y,s)) \cap C_n^{\partial}(\hat{Y},\hat{s})} \left\{ \frac{1}{2} \cdot r_{\hat{Y},q} \cdot \tilde{r}_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \right. \tag{36} \\
 & \left. \cdot \tilde{r}_{\hat{s}}^{1q}(\hat{Y}) \cdot \nu_{\partial B_{\hat{s}}}^j(\hat{Y}) \nu_{\partial B_{\hat{s}}}^j d\lambda_{\partial B_{\hat{s}}} \right|.
 \end{aligned}$$

The inclusion

$$\chi_{ss}(C_d^{\partial}(Y,s)) \subset \partial B_{\hat{s}} \cap B_{d_K}^3(\hat{Y}) \tag{37}$$

obtains, implying that $\Pi_{\hat{Y}}$ is defined on $\chi_{ss}(C_d^{\partial}(Y,s))$, hence that the reasoning producing (35) is legitimate, along with the definition (36). To see that (37) is correct, suppose that $Z \in C_d^{\partial}(Y,s)$: then $r_Y(Z) < \frac{9}{7} r_Y(\Pi_Y(Z)) < \frac{9}{7} d$, and so, with (23),

$$\begin{aligned}
 |\chi_{s\hat{s}}(Z) - \hat{Y}|_3 &\leq |Y - \hat{Y}|_3 + |Z - \chi_{s\hat{s}}(Z)|_3 + |Z - Y|_3 \\
 &< |Y - \hat{Y}|_3 + c^* |s - \hat{s}| + \frac{9}{7} d \\
 &\leq 1\delta + \frac{9}{7} d \\
 &< \frac{1}{6} d + \frac{9}{7} d < \left(\frac{1}{18} + \frac{3}{7}\right) d_K < d_K.
 \end{aligned}
 \tag{38}$$

(37) follows from the latter inequality.

Now, to examine the second term on the right in (35), we write

$$\begin{aligned}
 &M_\phi^K \cdot \left| \int_{\chi_{s\hat{s}}(C_{3,1\delta}^\partial(Y,s)) \cap C_\eta^\partial(\hat{Y},\hat{s})} \left\{ \frac{1}{2} \cdot r_{\hat{Y},q} \cdot T_{\hat{s}}^{1q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot J\chi_{\hat{s}}^{-1} \right. \right. \\
 &\quad \left. \left. - \left(\frac{1}{2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s}) \} \right) \cap \Pi_{\hat{Y}} \cdot T_{\hat{s}}^{1q}(\hat{Y}) \cdot \nu_{\partial B_{\hat{s}}}^j(\hat{Y}) \cdot \nu_{\partial B_{\hat{s}}}^j \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \right\} d\lambda_{\partial B_{\hat{s}}} \right| \\
 &\leq M_\phi^K \cdot \left| \int_{\chi_{s\hat{s}}(C_{3,1\delta}^\partial(Y,s)) \cap C_\eta^\partial(\hat{Y},\hat{s})} \frac{1}{2} \cdot r_{\hat{Y},q} \cdot \{ T_{\hat{s}}^{1q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot J\chi_{\hat{s}}^{-1} \right. \\
 &\quad \left. - T_{\hat{s}}^{1q}(\hat{Y}) \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s}) \} \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \} d\lambda_{\partial B_{\hat{s}}} \right| \\
 &\quad + M_\phi^K \cdot \left| \int_{\chi_{s\hat{s}}(C_{3,1\delta}^\partial(Y,s)) \cap C_\eta^\partial(\hat{Y},\hat{s})} \left\{ \frac{1}{2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s}) \} \right. \right. \\
 &\quad \left. \left. - \left(\frac{1}{2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}),\hat{s}) \} \right) \cap \Pi_{\hat{Y}} \cdot \nu_{\partial B_{\hat{s}}}^j(\hat{Y}) \cdot \nu_{\partial B_{\hat{s}}}^j \cdot T_{\hat{s}}^{1q}(\hat{Y}) \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \right\} d\lambda_{\partial B_{\hat{s}}} \right| \\
 &\leq M_\phi^K \cdot \sum_{j=16}^{19} I_j(Y,s;\hat{Y},\hat{s}),
 \end{aligned}
 \tag{39}$$

wherein the $I_j = I_j(Y, s; \hat{Y}, \hat{s})$, $j = 16, \dots, 19$, are obtained from

$$I_{16} := \left| \int_{X_{\hat{s}\hat{s}}(C_{31\delta}^\partial(Y, s)) \cap C_\eta^\partial(\hat{Y}, \hat{s})} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q} \hat{T}_{\hat{s}}^{1q} \cdot \Gamma^{03}(\hat{Y}, \hat{s}) \cdot \{JX_{\hat{s}}^{-1} - JX_{\hat{s}}^{-1}(\hat{Y})\} d\lambda_{\partial B_{\hat{s}}} \right|, \quad (40)$$

$$I_{17} := \left| \int_{X_{\hat{s}\hat{s}}(C_{31\delta}^\partial(Y, s)) \cap C_\eta^\partial(\hat{Y}, \hat{s})} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q} \{\hat{T}_{\hat{s}}^{1q} - \hat{T}_{\hat{s}}^{1q}(\hat{Y})\} \cdot \Gamma^{03}(\hat{Y}, \hat{s}) \cdot JX_{\hat{s}}^{-1}(\hat{Y}) d\lambda_{\partial B_{\hat{s}}} \right|, \quad (41)$$

$$I_{18} := \left| \int_{X_{\hat{s}\hat{s}}(C_{31\delta}^\partial(Y, s)) \cap C_\eta^\partial(\hat{Y}, \hat{s})} \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q} \hat{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot JX_{\hat{s}}^{-1}(\hat{Y}) \cdot \{\Gamma_{(\hat{Y}, \hat{s})}^{03} - \Gamma_{\hat{Y}}\{X_{\hat{s}}^c(X_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\}\} d\lambda_{\partial B_{\hat{s}}} \right|, \quad (42)$$

and

$$I_{19} := \left| \int_{X_{\hat{s}\hat{s}}(C_{31\delta}^\partial(Y, s)) \cap C_\eta^\partial(\hat{Y}, \hat{s})} \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q} \cdot \Gamma_{\hat{Y}}\{X_{\hat{s}}^c(X_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\} - \left[\frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q} \cdot \Gamma_{\hat{Y}}\{X_{\hat{s}}^c(X_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\} \right] \circ \pi_{\hat{Y}} \cdot \nu_{\partial B_{\hat{s}}}^j(\hat{Y}) \nu_{\partial B_{\hat{s}}}^j \right\} \cdot \hat{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot JX_{\hat{s}}^{-1}(\hat{Y}) d\lambda_{\partial B_{\hat{s}}} \right|. \quad (43)$$

A similar manipulation of the third term on the right in (35) leads to

$$\begin{aligned}
 & M_{\phi}^K \cdot \left| \int_{C_{31\delta}^{\partial}(Y,s) \cap C_{\eta}^{\partial}(Y,s)} \left\{ \frac{1}{2} \cdot r_{Y,q} \cdot \overset{\vee}{T}_s^{1q} \cdot r^{03}(Y,s) \cdot J\chi_s^{-1} \right. \right. \\
 & \left. \left. - \left(\frac{1}{2} \cdot r_{Y,q} \cdot r_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right) \circ \Pi_Y \cdot \overset{\vee}{T}_s^{1q}(Y) \cdot \underset{\partial B_s}{\vee}^j(Y) \underset{\partial B_s}{\vee}^j(Y) \cdot J\chi_s^{-1}(Y) \right\} d\lambda_{\partial B_s} \right| \\
 & \leq M_{\phi}^K \cdot \sum_{j=20}^{23} I_j(Y,s; \hat{Y}, \hat{s}),
 \end{aligned} \tag{44}$$

in which the $I_j = I_j(Y,s; \hat{Y}, \hat{s})$, $j = 20, \dots, 23$, have been defined by

$$\begin{aligned}
 I_{20} := & \left| \int_{C_{31\delta}^{\partial}(Y,s) \cap C_{\eta}^{\partial}(Y,s)} \frac{1}{2} \cdot r_{Y,q} \cdot \overset{\vee}{T}_s^{1q} \cdot r^{03}(Y,s) \right. \\
 & \left. \cdot \{ J\chi_s^{-1} - J\chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right|,
 \end{aligned} \tag{45}$$

$$\begin{aligned}
 I_{21} := & \left| \int_{C_{31\delta}^{\partial}(Y,s) \cap C_{\eta}^{\partial}(Y,s)} \frac{1}{2} \cdot r_{Y,q} \cdot \{ \overset{\vee}{T}_s^{1q} - \overset{\vee}{T}_s^{1q}(Y) \} \right. \\
 & \left. \cdot r^{03}(Y,s) \cdot J\chi_s^{-1}(Y) d\lambda_{\partial B_s} \right|,
 \end{aligned} \tag{46}$$

$$\begin{aligned}
 I_{22} := & \left| \int_{C_{31\delta}^{\partial}(Y,s) \cap C_{\eta}^{\partial}(Y,s)} \frac{1}{2} \cdot r_{Y,q} \cdot \overset{\vee}{T}_s^{1q}(Y) \cdot J\chi_s^{-1}(Y) \right. \\
 & \left. \cdot \{ r^{03}(Y,s) - r_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \} d\lambda_{\partial B_s} \right|,
 \end{aligned} \tag{47}$$

and

$$I_{23} := \left| \int_{C_{3,\delta}^{\partial}(Y,s) \cap C_n^{\partial}(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right. \right. \\ \left. \left. - \left(\frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \right) \circ \Pi_Y \cdot v_{\partial B_s}^j(Y) v_{\partial B_s}^j \right\} \right. \\ \left. \cdot T_s^{1q}(Y) \cdot J \chi_s^{-1}(Y) d\lambda_{\partial B_s} \right|. \quad (48)$$

For the examination of the first term appearing on the right-hand side of (35), it is convenient to introduce another auxiliary function: we define

$$\begin{aligned} \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03}(Z) &:= \left\{ W_{(\hat{Y},\hat{s})}^k(Z) \cdot r_{\chi_{s\hat{s}}}(Y),k(Z) \right. \\ &\quad \left. + \left\{ (r_{\chi_{s\hat{s}}}(Y),\ell(Z) \cdot v_{(\hat{Y},\hat{s})}^{\ell c}(Z))^2 + \left\{ 1 - |v_{(\hat{Y},\hat{s})}^c(Z)|_3^2 \right\}^{1/2} \right\}^{-3} \right\} \\ &\text{for each } Z \in \partial B_{\hat{s}} \cap \{ \chi_{s\hat{s}}(Y) \}. \end{aligned} \quad (49)$$

We shall presently verify the legitimacy of this definition (cf., the analysis of I_{29} , *infra*). Now, we convert the first integral in the first term on the right in (35) to its form involving integration over $C_d^{\partial}(Y,s) \cap C_{3,\delta}^{\partial}(Y,s)$ and estimate the result, recalling (15):

$$\left| \int_{C_d^{\partial}(Y,s) \cap C_{3,\delta}^{\partial}(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot T_s^{1q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \cdot J \chi_s^{-1} \right. \right. \\ \left. \left. - \left(\frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \right) \circ \Pi_{\hat{Y}} \cdot T_{\hat{s}}^{1q}(\hat{Y}) \right. \right. \\ \left. \left. \cdot v_{\partial B_{\hat{s}}}^j(\hat{Y}) v_{\partial B_{\hat{s}}}^j \cdot J \chi_{\hat{s}}^{-1}(\hat{Y}) \right\} \circ \chi_{s\hat{s}} \cdot J \chi_{s\hat{s}} - \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot T_s^{1q} \cdot \Gamma_{(Y,s)}^{03} \cdot J \chi_s^{-1} \right. \right.$$

$$\begin{aligned}
 & - \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_Y \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \right\} \circ \Pi_Y \cdot \overset{v}{T}_s^{iq}(Y) \cdot \underset{\partial B_s}{v}^j(Y) \underset{\partial B_s}{v}^j \cdot J \chi_s^{-1}(Y) \Big\} d\lambda_{\partial B_s} \\
 & \leq \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \right\} \circ \chi_{s\hat{s}} \cdot \{ \overset{v}{T}_{\hat{s}}^{iq} \circ \chi_{s\hat{s}} - \overset{v}{T}_s^{iq} \} \right. \\
 & \quad \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \right. \\
 & \quad \cdot \{ \overset{v}{T}_s^{iq}(Y) \cdot \underset{\partial B_s}{v}^j(Y) \underset{\partial B_s}{v}^j \cdot J \chi_s^{-1}(Y) - \overset{v}{T}_{\hat{s}}^{iq}(\hat{Y}) \cdot \underset{\partial B_{\hat{s}}}{v}^j(\hat{Y}) \underset{\partial B_{\hat{s}}}{v}^j \circ \chi_{s\hat{s}} \\
 & \quad \cdot J \chi_{\hat{s}}^{-1}(\hat{Y}) \cdot J \chi_{s\hat{s}} \} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \frac{1}{r_Y^2} \cdot r_{Y,q} \overset{v}{T}_s^{iq} \cdot \{ \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} - \Gamma_{(Y,s)}^{03} \} \right. \\
 & \quad \cdot J \chi_s^{-1} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \right\} \circ \Pi_Y \cdot \overset{v}{T}_s^{iq}(Y) \cdot \underset{\partial B_s}{v}^j(Y) \underset{\partial B_s}{v}^j \cdot J \chi_s^{-1}(Y) \right. \\
 & \quad \cdot \{ \Gamma_Y \{ \chi_{,4}^c (\chi_s^{-1}(Y), s) \} \circ \Pi_Y - \Gamma_{\chi_{s\hat{s}}}(Y) \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)} \left\{ \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{s\hat{s}} \cdot \overset{v}{T}_s^{iq} \cdot J \chi_s^{-1} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{1}{r_Y^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{ss} \cdot \overset{\vee}{T}_s^{1q}(Y) \cdot \underset{\partial B_s}{v}^j(Y) \underset{\partial B_s}{v}^j \cdot J \chi_s^{-1}(Y) \Big\} \\
& - \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \overset{!}{\Gamma}^{03}(Y, s; \hat{Y}, \hat{s}) \circ \chi_{ss} \cdot \overset{\vee}{T}_s^{1q} \cdot J \chi_s^{-1} \right. \\
& - \left. \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \right\} \circ \Pi_Y \cdot \Gamma_{\chi_{ss}}(Y) \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{ss} \cdot \overset{\vee}{T}_s^{1q}(Y) \right. \\
& \left. \cdot \underset{\partial B_s}{v}^j(Y) \underset{\partial B_s}{v}^j \cdot J \chi_s^{-1}(Y) \Big\} \Big\} d\lambda_{\partial B_s} \Big| \\
& \leq \sum_{j=24}^{30} I_j(Y, s; \hat{Y}, \hat{s}) \\
& + \left| \int_{C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)} \left\{ \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \overset{!}{\Gamma}^{03}(\hat{Y}, \hat{s}) \right\} \circ \chi_{ss} - \frac{1}{r_Y^2} r_{Y,q} \cdot \overset{!}{\Gamma}^{03}(Y, s; \hat{Y}, \hat{s}) \circ \chi_{ss} \right\} \right. \\
& \left. \cdot \{ \overset{\vee}{T}_s^{1q} \cdot J \chi_s^{-1} - \overset{\vee}{T}_s^{1q}(Y) \cdot J \chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right| \\
& + \left| \int_{C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)} \left\{ \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{ss} \right. \right. \\
& \left. \left. - \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \right\} \circ \Pi_Y \cdot \Gamma_{\chi_{ss}}(Y) \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{ss} \right\} \right. \\
& \left. \cdot \overset{\vee}{T}_s^{1q}(Y) \cdot \{ 1 - \underset{\partial B_s}{v}^j(Y) \underset{\partial B_s}{v}^j \} \cdot J \chi_s^{-1}(Y) d\lambda_{\partial B_s} \right| \\
& + \left| \int_{C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)} \left\{ \left\{ \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \overset{!}{\Gamma}^{03}(\hat{Y}, \hat{s}) \right\} \circ \chi_{ss} \right. \right. \right. \\
& \left. \left. - \left\{ \frac{1}{r_Y^2} \cdot r_{Y,q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{ss} \right\} \right. \\
& \left. \cdot \{ \overset{\vee}{T}_s^{1q} \cdot J \chi_s^{-1} - \overset{\vee}{T}_s^{1q}(Y) \cdot J \chi_s^{-1}(Y) \} d\lambda_{\partial B_s} \right|
\end{aligned}$$

$$\begin{aligned}
 & - \left\{ \frac{1}{r_Y} \cdot r_{Y,q} \cdot \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ \chi_{\hat{s}\hat{s}} \right. \\
 & \left. - \left\{ \frac{1}{r_Y} \cdot r_{Y,q} \right\} \circ \Pi_Y \cdot \Gamma_{\chi_{\hat{s}\hat{s}}(Y)} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}} \right\} \cdot \hat{T}_s^{iq}(Y) \cdot J\chi_s^{-1}(Y) \, d\lambda_{\partial B_s} \Big| \\
 & \leq \sum_{j=24}^{35} I_j(Y,s;\hat{Y},\hat{s}) \\
 & + \left| \int_{C_d^{\partial}(Y,s) \cap C_{3,\delta}^{\partial}(Y,s)} \left\{ \left\{ \frac{1}{r_{\chi_{\hat{s}\hat{s}}(Y)}} \cdot r_{\chi_{\hat{s}\hat{s}}(Y),q} \right\} \circ \chi_{\hat{s}\hat{s}} - \frac{1}{r_Y} \cdot r_{Y,q} \right\} \cdot \hat{T}_s^{iq}(Y) \right. \\
 & \left. \cdot \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ \chi_{\hat{s}\hat{s}} \cdot J\chi_s^{-1}(Y) \, d\lambda_{\partial B_s} \right| \\
 & + \left| \int_{C_d^{\partial}(Y,s) \cap C_{3,\delta}^{\partial}(Y,s)} \left\{ \left\{ \frac{1}{r_Y} \cdot r_{Y,q} \right\} \circ \Pi_Y - \left\{ \frac{1}{r_{\chi_{\hat{s}\hat{s}}(Y)}} \cdot r_{\chi_{\hat{s}\hat{s}}(Y),q} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}} \right\} \right. \\
 & \left. \cdot \hat{T}_s^{iq}(Y) \cdot \Gamma_{\chi_{\hat{s}\hat{s}}(Y)} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}} \cdot J\chi_s^{-1}(Y) \, d\lambda_{\partial B_s} \right| \\
 & + \left| \int_{C_d^{\partial}(Y,s) \cap C_{3,\delta}^{\partial}(Y,s)} \left\{ \left\{ \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} \right. \right. \right. \\
 & \left. \left. - \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}} \right\} \right. \\
 & \left. - \left\{ \left\{ \frac{1}{r_{\chi_{\hat{s}\hat{s}}(Y)}} \cdot r_{\chi_{\hat{s}\hat{s}}(Y),q} \cdot \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \right\} \circ \chi_{\hat{s}\hat{s}} \right. \right. \\
 & \left. \left. - \left\{ \frac{1}{r_{\chi_{\hat{s}\hat{s}}(Y)}} \cdot r_{\chi_{\hat{s}\hat{s}}(Y),q} \cdot \Gamma_{\chi_{\hat{s}\hat{s}}(Y)} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}} \right\} \cdot \hat{T}_s^{iq}(Y) \right. \\
 & \left. \cdot J\chi_s^{-1}(Y) \, d\lambda_{\partial B_s} \right|
 \end{aligned}$$

$$= \sum_{j=24}^{37} I_j(Y, s; \hat{Y}, \hat{s}), \quad (50)$$

having introduced $I_j(Y, s; \hat{Y}, \hat{s}) = I_j, \quad j = 24, \dots, 37, \quad \text{via}$

$$I_{24} := \left| \int_{C_d^\partial(Y, s) \cap C_{3,1,6}^\partial(Y, s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y}, q} \right\} \circ \chi_{s\hat{s}} \cdot \{ \hat{T}_{\hat{s}}^{1q} \circ \chi_{s\hat{s}} - \hat{T}_s^{1q} \} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \circ \chi_{s\hat{s}} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (51)$$

$$I_{25} := \left| \int_{C_d^\partial(Y, s) \cap C_{3,1,6}^\partial(Y, s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y}, q} \cdot \Gamma_{\hat{Y}}\{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \hat{\Pi}_{\hat{Y}} \circ \chi_{s\hat{s}} \cdot \{ \hat{T}_s^{1q}(Y) - \hat{T}_{\hat{s}}^{1q}(\hat{Y}) \} \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j \cdot J\chi_s^{-1}(Y) d\lambda_{\partial B_s} \right|, \quad (52)$$

$$I_{26} := \left| \int_{C_d^\partial(Y, s) \cap C_{3,1,6}^\partial(Y, s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y}, q} \cdot \Gamma_{\hat{Y}}\{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \hat{\Pi}_{\hat{Y}} \circ \chi_{s\hat{s}} \cdot \hat{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot \{ \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j - \nu_{\partial B_s}^j(\hat{Y}) \nu_{\partial B_s}^j \} \circ \chi_{s\hat{s}} \cdot J\chi_s^{-1}(Y) d\lambda_{\partial B_s} \right|, \quad (53)$$

$$I_{27} := \left| \int_{C_d^\partial(Y, s) \cap C_{3,1,6}^\partial(Y, s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y}, q} \cdot \Gamma_{\hat{Y}}\{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \hat{\Pi}_{\hat{Y}} \circ \chi_{s\hat{s}} \cdot \hat{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot \nu_{\partial B_s}^j(\hat{Y}) \nu_{\partial B_s}^j \circ \chi_{s\hat{s}} \cdot \{ J\chi_s^{-1}(Y) - J\chi_{\hat{s}}^{-1}(\hat{Y}) \} d\lambda_{\partial B_s} \right|, \quad (54)$$

$$I_{28} := \left| \int_{C_d^\partial(Y, s) \cap C_{3,1,6}^\partial(Y, s)} \left\{ \frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y}, q} \cdot \Gamma_{\hat{Y}}\{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \hat{\Pi}_{\hat{Y}} \circ \chi_{s\hat{s}} \cdot \hat{T}_{\hat{s}}^{1q}(\hat{Y}) \cdot \nu_{\partial B_s}^j(\hat{Y}) \nu_{\partial B_s}^j \circ \chi_{s\hat{s}} \cdot J\chi_{\hat{s}}^{-1}(\hat{Y}) \cdot \{ 1 - J\chi_{s\hat{s}} \} d\lambda_{\partial B_s} \right|, \quad (55)$$

$$I_{29} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \frac{1}{r_Y} \cdot r_{Y,q} \cdot \frac{\nu^{1q}}{T_s} \cdot \{ \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} - \Gamma_{(Y,s)}^{03} \} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|, \quad (56)$$

$$I_{30} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left\{ \frac{1}{r_Y} \cdot r_{Y,q} \right\} \circ \Pi_Y \cdot \frac{\nu^{1q}}{T_s}(Y) \cdot \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j \cdot JX_s^{-1}(Y) \cdot \{ \Gamma_Y \{ \chi_{,4}^c(\chi_s^{-1}(Y), s) \} \circ \Pi_Y \right. \\ \left. - \Gamma_{\chi_{s\hat{s}}}(Y) \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \} d\lambda_{\partial B_s} \right|, \quad (57)$$

$$I_{31} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left\{ \left(\frac{1}{r_Y} \cdot r_{\hat{Y},q} \right) \circ \chi_{s\hat{s}} - \frac{1}{r_Y} \cdot r_{Y,q} \right\} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} \cdot \left(\frac{\nu^{1q}}{T_s} \cdot JX_s^{-1} - \frac{\nu^{1q}}{T_s}(Y) \cdot JX_s^{-1}(Y) \right) d\lambda_{\partial B_s} \right|, \quad (58)$$

$$I_{32} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \frac{1}{r_Y} \cdot r_{Y,q} \cdot \{ \Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} - \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} \} \cdot \left(\frac{\nu^{1q}}{T_s} \cdot JX_s^{-1} - \frac{\nu^{1q}}{T_s}(Y) \cdot JX_s^{-1}(Y) \right) d\lambda_{\partial B_s} \right|, \quad (59)$$

$$I_{33} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left\{ \left(\frac{1}{r_Y} \cdot r_{\hat{Y},q} \right) \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} - \left(\frac{1}{r_Y} \cdot r_{Y,q} \right) \circ \Pi_Y \right\} \cdot \frac{\nu^{1q}}{T_s}(Y) \cdot JX_s^{-1}(Y) \cdot \Gamma_{\hat{Y}} \{ \chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \right. \\ \left. \cdot \{ 1 - \nu_{\partial B_s}^j(Y) \nu_{\partial B_s}^j \} d\lambda_{\partial B_s} \right|, \quad (60)$$

$$I_{34} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left(\frac{1}{r_Y^2} \cdot r_{Y,q} \right) \circ \Pi_Y \cdot T_s^{1q}(Y) \cdot JX_s^{-1}(Y) \right. \\ \left. \cdot \{1 - v_{\partial B_s}^f(Y) v_{\partial B_s}^f\} \cdot \{ \Gamma_{\hat{Y}}^c \{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \right. \\ \left. - \Gamma_{\chi_{s\hat{s}}}(Y) \{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \} d\lambda_{\partial B_s} \right|, \quad (61)$$

$$I_{35} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left\{ \left(\frac{1}{r_{\chi_{s\hat{s}}}(Y)} \cdot r_{\chi_{s\hat{s}}}(Y), q \right) \circ \chi_{s\hat{s}} - \frac{1}{r_Y^2} \cdot r_{Y,q} \right\} \right. \\ \left. \cdot T_s^{1q}(Y) \cdot \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}} \cdot JX_s^{-1}(Y) d\lambda_{\partial B_s} \right|, \quad (62)$$

$$I_{36} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left\{ \left(\frac{1}{r_Y^2} \cdot r_{Y,q} \right) \circ \Pi_Y \right. \right. \\ \left. - \left(\frac{1}{r_{\chi_{s\hat{s}}}(Y)} \cdot r_{\chi_{s\hat{s}}}(Y), q \right) \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \right\} \cdot T_s^{1q}(Y) \\ \left. \cdot \Gamma_{\chi_{s\hat{s}}}(Y) \{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \cdot JX_s^{-1}(Y) d\lambda_{\partial B_s} \right|, \quad (63)$$

and

$$I_{37} := \left| \int_{C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)} \left\{ \left\{ \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{(\hat{Y},\hat{s})}^{03} \right) \circ \chi_{s\hat{s}} \right. \right. \right. \\ \left. - \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}}^c \{ \chi_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s}) \} \right) \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}} \right\} \\ \left. - \left\{ \left(\frac{1}{r_{\chi_{s\hat{s}}}(Y)} \cdot r_{\chi_{s\hat{s}}}(Y), q \cdot \Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \right) \circ \chi_{s\hat{s}} \right. \right.$$

$$\begin{aligned}
 & - \left\{ \frac{1}{r_{\chi_{ss}}^2(Y)} \cdot r_{\chi_{ss}}(Y) \cdot \Gamma_{\chi_{ss}}(Y) \{ \chi_{ss}^c(\chi_{ss}^{-1}(\hat{Y}), \hat{s}) \} \right\} \circ \Pi_{\hat{Y}} \circ \chi_{ss} \Bigg\} \\
 & \cdot T_s^{1q}(Y) \cdot J\chi_s^{-1}(Y) d\lambda_{\partial B_s} \Bigg| . \quad (64)
 \end{aligned}$$

To summarize the computations to this point, let us denote by $I_9 = I_9(Y, s; \hat{Y}, \hat{s})$ and $I_{10} = I_{10}(Y, s; \hat{Y}, \hat{s})$, respectively, the sixth and seventh terms on the right-hand side of (16):

$$\begin{aligned}
 I_9 := & \left| \int \left\{ \frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y}, q} T_s^{1q} \cdot \Gamma_{(\hat{Y}, \hat{s})}^{03} \right\} \circ \chi_{ss} \right. \\
 & \left. C_{3,6}^{\partial}(Y, s) \cap \{ \chi_{ss}^{\partial}(C_{\eta}^{\partial}(\hat{Y}, \hat{s})) \} \right| \cdot \\
 & \cdot \{ \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} - \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1}(\hat{Y}) \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Bigg| . \quad (65)
 \end{aligned}$$

$$\begin{aligned}
 I_{10} := & \left| \int \frac{1}{r_Y} \cdot r_{Y, q} T_s^{1q} \cdot \Gamma_{(Y, s)}^{03} \right. \\
 & \left. C_{3,6}^{\partial}(Y, s) \cap C_{\eta}^{\partial}(Y, s) \right| \cdot \\
 & \cdot \{ \phi(Y, s) \circ \chi_s^{-1} - \phi(Y, s) \circ \chi_s^{-1}(Y) \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \Bigg| . \quad (66)
 \end{aligned}$$

Then, from (12), (13), (16), (24), (30), (35), (39), (44), and (50),

$$\begin{aligned}
 & |w_{21}^* \{ \phi \}(\hat{Y}, \hat{s}) - w_{21}^* \{ \phi \}(Y, s)| \\
 & \leq 2a' \lambda' \cdot \delta \lambda' + \frac{1}{4\pi} \sum_{j=1}^{14} I_j(Y, s; \hat{Y}, \hat{s}) + \frac{M^K}{4\pi} \sum_{j=15}^{37} I_j(Y, s; \hat{Y}, \hat{s}). \quad (67)
 \end{aligned}$$

Recalling the reasoning accompanying (7), to complete the proof of the first statement of the theorem, it suffices to produce positive numbers $l(j)$ and numbers $\lambda(j) \in (0, 1]$, for $j = 1, \dots, 37$,

depending only upon ϕ , M , and K , such that

$$I_j(Y, s; \hat{Y}, \hat{s}) \leq \ell(j) \cdot \delta^{\lambda(j)}, \quad (68)$$

for all (Y, s) , $(\hat{Y}, \hat{s})$ as specified,

for each j . We proceed to this task.

I₁: If $\omega \in \mathbb{R}^3$, $\hat{s} \in \mathbb{R}$, and $Z \in \partial B_{\hat{s}}$, then

$$|\omega^q \cdot \hat{T}_{\hat{s}}^{1q}(Z)| = |\nu^1(Z, \hat{s}) \nu^q(Z, \hat{s}) \omega^q - \omega^1| \leq 2|\omega|_3. \quad (69)$$

We denote by M^{03} the bound for Γ^{03} which is obtained from (IV.14.59):

$$\Gamma^{03} \leq M^{03} := \{1 + (c^*/c)^2\}^3 \cdot \{1 - (c^*/c)\}^{-6}.$$

We choose to bound the function $(Z, \hat{s}) \mapsto JX_{\hat{s}}^{-1}(Z)$ on the compact set $\cup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$ as in the proof of [IV.22]: by [I.2.17.v],

$$JX_{\hat{s}}^{-1}(Z) = \{JX_{\hat{s}}(X_{\hat{s}}^{-1}(Z))\}^{-1} = \{\hat{J}X(X_{\hat{s}}^{-1}(Z), \hat{s})\}^{-1} \quad \text{for } (Z, \hat{s}) \in \partial B, \quad (70)$$

while

$$m_j^K \leq \hat{J}X(P, \zeta) \leq M_j^K \quad \text{for each } (P, \zeta) \in \partial R \times K, \quad (71)$$

m_j^K and M_j^K being positive numbers depending on M and K . Thus,

$$(M_j^K)^{-1} \leq JX_{\hat{s}}^{-1}(Z) \leq (m_j^K)^{-1} \quad \text{whenever } \hat{s} \in K, \quad Z \in \partial B_{\hat{s}}. \quad (72)$$

According to hypothesis (iii),

$$|\phi(\hat{Y}, \hat{s}) \circ X_{\hat{s}}^{-1}(Z) - \phi(Y, s) \circ X_s^{-1}(Z)| \leq \kappa_2 \delta^{B_2}, \quad \text{for } Z \in \partial B_{\hat{s}}, \quad (73)$$

$\kappa_2 > 0$ and $\beta_2 \in (0,1]$ depending on at most ϕ , M , and K .

Since $r_Y(Z) \geq d$ for $Z \in \partial B_s \cap C_d^0(Y,s)'$, we find

$$I_1 \leq \frac{2}{d^2} \cdot M^{03} \cdot (m_j^K)^{-1} \cdot \kappa_2 \cdot \left\{ \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta) \right\} \cdot \delta^2, \quad (74)$$

and inequality of the required form (68).

I₂: Here, we must develop an appropriate estimate for the expression

$|\Gamma_{(\hat{Y},\hat{s})}^{03} \circ \chi_{s\hat{s}}(Z) - \Gamma_{(Y,s)}^{03}(Z)|$, $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{s\hat{s}}(\hat{Y})\}'$. This has, in fact, already been effected in the proof of Theorem [IV.24], and we shall recall here the major steps in the reasoning: first,

$$\begin{aligned} & |\Gamma_{(\hat{Y},\hat{s})}^{03}(\chi_{s\hat{s}}(Z)) - \Gamma_{(Y,s)}^{03}(Z)| \\ & \leq c_3^* |[\chi_{,4}^c]_{(\hat{Y},\hat{s})} \circ \chi_{s\hat{s}}^{-1}(\chi_{s\hat{s}}(Z)) - [\chi_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z)|_3 \\ & \quad + c_4^* |v_{(\hat{Y},\hat{s})}^c(\chi_{s\hat{s}}(Z)) - v_{(Y,s)}^c(Z)|_3 \\ & \quad + c_5^* |\text{grad } r_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - \text{grad } r_Y(Z)|_3, \end{aligned} \quad (75)$$

for each $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{s\hat{s}}(\hat{Y})\}'$,

where c_3^* , c_4^* , and c_5^* are certain positive numbers depending upon c^*/c alone. Defining

$$t_0 := \frac{1}{c-c^*} \left\{ \frac{d}{4} + \max_{\zeta \in K} \text{diam } B_\zeta \right\}$$

and

$$\hat{K} := [t_1 - t_0, t_2],$$

it is easy to show that $\hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s})$ and $s - \tau(\chi_s^{-1}(Z); Y, s)$ lie in $\hat{K}$. Now, by (ii), $\chi_{,4}^c|_{\partial R \times \hat{K}}$ is Hölder continuous, so that, for some $\hat{A} > 0$ and $\hat{\alpha} \in (0, 1]$, dependent upon only M and K ,

$$|\chi_{,4}^c(P_2, s_2) - \chi_{,4}^c(P_1, s_1)|_3 \leq \hat{A} |(P_2, s_2) - (P_1, s_1)|_4^{\hat{\alpha}} \quad (76)$$

for $s_1, s_2 \in \hat{K}$ and $P_1, P_2 \in \partial R$.

It follows that

$$|[\chi_{,4}^c]_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}}^{-1}(\chi_{\hat{s}\hat{s}}(Z)) - [\chi_{,4}^c]_{(Y, s)} \circ \chi_s^{-1}(Z)|_3 \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1+c^2\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}} \quad (77)$$

for $Z \in \partial B_s$;

cf., the computation (IV.24.26). Further, following the derivations of (IV.24.29) and (IV.24.31), respectively, produces

$$|v_{(\hat{Y}, \hat{s})}^c(\chi_{\hat{s}\hat{s}}(Z)) - v_{(Y, s)}^c(Z)|_3 \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1+c^2\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}}, \quad (78)$$

for each $Z \in \partial B_s$,

and

$$|\text{grad } r_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - \text{grad } r_Y(Z)|_3 \leq \frac{2\delta}{r_Y(Z)}, \quad (79)$$

for each $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{\hat{s}\hat{s}}(\hat{Y})\}'$.

Consequently, from (75) and (77)-(79),

$$|\Gamma_{(\hat{Y}, \hat{s})}^{03}(\chi_{\hat{s}\hat{s}}(Z)) - \Gamma_{(Y, s)}^{03}(Z)| \leq k_1 \delta^{\hat{\alpha}} + \frac{k_2}{r_Y(Z)} \delta, \quad (80)$$

for each $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{\hat{s}\hat{s}}(\hat{Y})\}'$,

the positive k_1 and k_2 depending on M and K alone.

With (80), an estimate for I_2 of the desired form results directly:

$$I_2 \leq \frac{2}{d^2} \cdot M_\phi^K \cdot (m_j^K)^{-1} \cdot \{k_1 \delta^{\hat{\alpha}} + \frac{k_2}{d} \delta\} \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta) \leq k_3 \delta^{\hat{\alpha}}, \quad (81)$$

k_3 depending on only ϕ , M , and K .

I_3 : Suppose that $Z \in \partial B_s \cap \{Y\}'$. Then, using (6) and (23),

$$\begin{aligned} & |r_{Y,q}(Z) \{T_{\hat{s}}^{v1q} \circ \chi_{s\hat{s}}(Z) - T_s^{v1q}(Z)\}| \\ &= |r_{Y,q}(Z) \cdot \{v^1(\chi_{s\hat{s}}(Z), \hat{s}) v^q(\chi_{s\hat{s}}(Z), \hat{s}) - v^1(Z, s) v^q(Z, s)\}| \\ &\leq |v^1(\chi_{s\hat{s}}(Z), \hat{s}) v(\chi_{s\hat{s}}(Z), \hat{s}) - v^1(Z, s) v(Z, s)|_3 \\ &\leq 2 |v(\chi_{s\hat{s}}(Z), \hat{s}) - v(Z, s)|_3 \\ &\leq 2 a_K \cdot \{| \chi_{s\hat{s}}(Z) - Z |_3^2 + |\hat{s} - s|^2\}^{\alpha_K/2} \\ &\leq 2 a_K \cdot \{1 + (c^*)^2\}^{\alpha_K/2} \cdot |\hat{s} - s|^{\alpha_K} \\ &\leq 2 a_K^{\alpha_K} \cdot \delta^{\alpha_K}. \end{aligned} \quad (82)$$

Clearly, then

$$I_3 \leq \frac{1}{d^2} \cdot M_\phi^{03K} \cdot (m_j^K)^{-1} \cdot \left\{ \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta) \right\} \cdot 2 a_K^{\alpha_K} \cdot \delta^{\alpha_K}, \quad (83)$$

an inequality of the form (68).

I_4 : We shall deduce an estimate involving the difference appearing in

(20) which is to be used in subsequent computations as well; the argument here is almost identical with that used to obtain (IV.24.37) and (IV.24.38), so we shall omit most of the details in the present case. Let $Z \in \partial B_s \cap C_{3\delta}^\partial(Y, s)'$: then $r_Y(Z) \geq 3\delta$, or $\delta/r_Y(Z) \leq 1/3$. Since (23) leads to

$$|r_Y(\chi_{ss}(Z)) - r_Y(Z)| \leq \delta,$$

it follows that

$$\frac{2}{3} \leq \frac{r_Y(\chi_{ss}(Z))}{r_Y(Z)} \leq \frac{4}{3} \quad \text{for each } Z \in \partial B_s \cap C_{3\delta}^\partial(Y, s)'. \quad (84)$$

In turn, (84) can be used to show that

$$\left| \frac{\chi_{ss}(Z) - \hat{Y}}{r_Y(\chi_{ss}(Z))} - \frac{Z - Y}{r_Y(Z)} \right|_3 \leq \frac{8\delta}{r_Y(Z)} \quad \text{for each } Z \in \partial B_s \cap C_{3\delta}^\partial(Y, s)'. \quad (85)$$

From (20), it is now easy to see that

$$I_4 \leq \frac{161}{d^3} \cdot M^{03} M_\phi^K \cdot (m_j^K)^{-1} \cdot \left\{ \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta) \right\} \cdot \delta. \quad (86)$$

I₅: In addition to (73), (iii) says that there exist $\kappa_3 > 0$ and $\beta_3 \in (0, 1]$, depending on at most ϕ , M , and K , such that

$$|\phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y}) - \phi(Y, s) \circ \chi_s^{-1}(Y)| \leq \kappa_3 \delta^{\beta_3}. \quad (87)$$

From (84),

$$\frac{1}{r_Y(\chi_{ss}(Z))} \leq \frac{3}{2} \cdot \frac{1}{r_Y(Z)} \quad \text{for } Z \in \partial B_s \cap C_{3\delta}^\partial(Y, s)',$$

so, with (73) and (87),

$$\begin{aligned}
 I_5 &\leq 2M^{03} \cdot (m_j^K)^{-1} \cdot \{\kappa_2 \delta^{\beta_2} + \kappa_3 \delta^{\beta_3}\} \cdot \frac{9}{4} \int_{C_d^\partial(Y,s) \cap C_{3\delta}^\partial(Y,s)} \frac{1}{r_Y} d\lambda_{\partial B_s} \\
 &\leq 9 \cdot 2^{1/2} \pi \cdot M^{03} \cdot (m_j^K)^{-1} \cdot \{\kappa_2 \delta^{\beta_2} + \kappa_3 \delta^{\beta_3}\} \cdot \ln(d/3\delta) \\
 &\leq k_4 \delta^{\beta_4} \cdot \ln(d/3\delta) \\
 &\leq k_4 M_1 \cdot \delta^{\beta'_4},
 \end{aligned} \tag{88}$$

wherein $\beta_4 := \min\{\beta_2, \beta_3\}$, β'_4 is any number in $(0, \beta_4)$, M_1 is a positive number such that

$$\zeta^{\beta_4 - \beta'_4} \cdot \ln(d/3\zeta) \leq M_1 \quad \text{for} \quad 0 < \zeta < d/6,$$

and k_4 is a positive number dependent upon ϕ , M , and K alone.

In (88), we have an estimate for I_5 of the desired form.

I_6 : Again from hypothesis (iii), there exist $\kappa_1 > 0$ and $\beta_1 \in (0, 1]$, depending upon at most ϕ , M , and K , for which

$$\begin{aligned}
 |\phi(\tilde{Y}, \tilde{s}) \circ \chi_{\tilde{s}}^{-1}(Z) - \phi(\tilde{Y}, \tilde{s}) \circ \chi_{\tilde{s}}^{-1}(Y)| &\leq \kappa_1 \cdot r_{\tilde{Y}}^{\beta_1}(Z), \\
 \text{for } \tilde{s} \in K, \quad \tilde{Y} \text{ and } Z &\in \partial B_{\tilde{s}}.
 \end{aligned} \tag{89}$$

Recalling (80), from (26) we therefore find

$$\begin{aligned}
 I_6 &\leq 2 \cdot (m_j^K)^{-1} \cdot \kappa_1 \cdot \int_{C_d^\partial(Y,s) \cap C_{3\delta}^\partial(Y,s)} \frac{1}{2 - \beta_1} \left\{ k_1 \delta^{\hat{\alpha}} + \frac{k_2}{r_Y} \delta \right\} d\lambda_{\partial B_s} \\
 &\leq 2^{5/2} \pi \cdot (m_j^K)^{-1} \cdot \kappa_1 \cdot \left\{ k_1 \delta^{\hat{\alpha}} \int_{3\delta}^d \frac{1}{1 - \beta_1} d\zeta + k_2 \delta \int_{3\delta}^d \frac{1}{2 - \beta_1} d\zeta \right\};
 \end{aligned} \tag{90}$$

since

$$\delta^{\hat{\alpha}} \int_{3\delta}^d \frac{1}{1-\beta_1} d\zeta = \frac{\delta^{\hat{\alpha}}}{\beta_1} \{d^{\beta_1-(3\delta)^{\beta_1}}\} < \frac{d^{\beta_1}}{\beta_1} \cdot \delta^{\hat{\alpha}},$$

and

$$\delta \int_{3\delta}^d \frac{1}{2-\beta_1} d\zeta = \begin{cases} \frac{\delta}{\beta_1-1} \cdot \{d^{\beta_1-1-(3\delta)^{\beta_1-1}}\} < \frac{(3\delta)^{\beta_1-1}}{1-\beta_1} \cdot \delta^{\beta_1}, \\ \text{if } \beta_1 \in (0,1), \\ \delta \cdot \ln(d/3\delta) \leq M_2 \delta^{\beta_1'}, & \text{if } \beta_1 = 1, \end{cases}$$

in which $\beta_1' \in (0,1)$ and $M_2 > 0$ is such that

$$\zeta^{1-\beta_1'} \cdot \ln(d/3\zeta) \leq M_2 \quad \text{for } 0 < \zeta < d/6,$$

it is easy to see from (90) that we can obtain for I_6 an inequality of the form (68).

I_7 : Having already pointed out inequalities (82) and (89), it is immediately evident that

$$\begin{aligned} I_7 &\leq M^{03} \cdot (m_j^K)^{-1} \cdot \kappa_1 \cdot 2a_{K^1}^{\alpha_K} \cdot \delta^{\alpha_K} \cdot \int_{C_d^{\partial}(Y,s) \cap C_{3\delta}^{\partial}(Y,s)} \frac{1}{2-\beta_1} d\lambda_{\partial S_s} \\ &\leq 2^{5/2} \pi \cdot a_{K^1}^{\alpha_K} \cdot \kappa_1 \cdot M^{03} (m_j^K)^{-1} \cdot \frac{d^{\beta_1}}{\beta_1} \cdot \delta^{\alpha_K}. \end{aligned} \quad (91)$$

I_8 : Here, (85) and (89) can be used with (28) to yield

$$\begin{aligned}
 I_8 &\leq 2M^{03} \cdot (m_j^K)^{-1} \cdot \kappa_1 \cdot \int_{C_d^\partial(Y,s) \cap C_{3\delta}^\partial(Y,s)} \left| \frac{\chi_{s\hat{s}}(Z) - \hat{Y}}{r_Y(\chi_{s\hat{s}}(Z))} - \frac{Z-Y}{r_Y(Z)} \right|_3 \\
 &\quad \cdot r_Y^{\beta_1}(Z) d\lambda_{\partial B_s}(Z) \\
 &\leq 16\kappa_1 \cdot M^{03} (m_j^K)^{-1} \cdot \delta \cdot \int_{C_d^\partial(Y,s) \cap C_{3\delta}^\partial(Y,s)} \frac{1}{r_Y^{3-\beta_1}} d\lambda_{\partial B_s} .
 \end{aligned} \tag{92}$$

If we estimate the integral in (92) just as we did in the computation for I_6 , considering the two cases $\beta_1 \in (0,1)$ and $\beta_1 = 1$, it is clear that we should arrive at an inequality for I_8 as in (68).

I_9 : It is most convenient to first rewrite the integral in (65) in its form involving integration over a subset of ∂B_s : keeping in mind (14) and (15),

$$\begin{aligned}
 I_9 &= \left| \int_{\chi_{s\hat{s}}(C_{3\delta}^\partial(Y,s)) \cap C_n^\partial(\hat{Y},\hat{s})} \frac{1}{r_Y^2} \cdot r_{\hat{Y},q}^{\nu_{1q}} \cdot r_{\hat{s}}^{03}(\hat{Y},\hat{s}) \right. \\
 &\quad \left. \cdot \{\phi(\hat{Y},\hat{s}) \circ \chi_{\hat{s}}^{-1} - \phi(\hat{Y},\hat{s}) \circ \chi_{\hat{s}}^{-1}(\hat{Y})\} \cdot J\chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \right| .
 \end{aligned} \tag{93}$$

Since $5\delta < \frac{5}{6}d < \frac{5}{18}d_K < \frac{7}{9}d_K$, it is clear that $C_{5\delta}^\partial(\hat{Y},\hat{s})$ is defined. Let us show that

$$\chi_{s\hat{s}}(C_{3\delta}^\partial(Y,s)) \subset C_{5\delta}^\partial(\hat{Y},\hat{s}) : \tag{94}$$

suppose that $Z \in C_{3\delta}^\partial(Y,s)$, so $r_Y(Z) < \frac{9}{7} r_Y(\pi_Y(Z)) < \frac{27}{7} \delta$. Then

$$\begin{aligned} r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))) &\leq r_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) \leq |Y-\hat{Y}|_3 + |\chi_{\hat{s}\hat{s}}(Z)-Z|_3 + |Z-Y|_3 \\ &< |Y-\hat{Y}|_3 + c^*|s-\hat{s}| + \frac{27}{7} \delta \leq \frac{34}{7} \delta < 5\delta, \end{aligned}$$

from which (94) follows. Then, appealing once again to (89), and recalling the various bounds established previously, (93) leads to

$$\begin{aligned} I_9 &\leq 2M^{03} \cdot (m_j^K)^{-1} \cdot \kappa_1 \cdot \int_{\chi_{\hat{s}\hat{s}}(C_{3\delta}^\partial(Y,s)) \cap C_\eta^\partial(\hat{Y},\hat{s})} \frac{1}{2-\beta_1} d\lambda_{\partial B_{\hat{s}}} \\ &\leq 2\kappa_1 \cdot M^{03} (m_j^K)^{-1} \cdot \int_{C_{5\delta}^\partial(\hat{Y},\hat{s}) \cap C_\eta^\partial(\hat{Y},\hat{s})} \frac{1}{2-\beta_1} d\lambda_{\partial B_{\hat{s}}} \\ &\leq 2^{5/2} \pi \cdot \kappa_1 \cdot M^{03} (m_j^K)^{-1} \cdot \int_{\eta}^{5\delta} \frac{1}{1-\beta_1} d\zeta \\ &< 2^{5/2} \pi \cdot \kappa_1 \cdot M^{03} (m_j^K)^{-1} \cdot \frac{1}{\beta_1} (5\delta)^{\beta_1} \delta^{\beta_1}. \end{aligned} \quad (95)$$

It is important to note that the coefficient of δ^{β_1} in the final estimate (95) is independent of η , as it must be, since η depends on δ .

I₁₀: We can proceed directly from (66) and (89), finding that

$$\begin{aligned} I_{10} &\leq 2M^{03} \cdot (m_j^K)^{-1} \cdot \kappa_1 \cdot \int_{C_{3\delta}^\partial(Y,s) \cap C_\eta^\partial(Y,s)} \frac{1}{2-\beta_1} d\lambda_{\partial B_s} \\ &< 2^{5/2} \pi \cdot \kappa_1 \cdot M^{03} (m_j^K)^{-1} \cdot \frac{1}{\beta_1} (3\delta)^{\beta_1} \delta^{\beta_1}. \end{aligned} \quad (96)$$

Once again, the estimate is independent of η .

I₁₁: By virtue of (6), if $\tilde{s} \in K$, $\tilde{Y} \in \partial B_{\tilde{s}}$, and $Z \in \partial B_{\tilde{s}} \cap \{\tilde{Y}\}'$, then

$$\begin{aligned} & |r_{\tilde{Y},q}(Z) \{T_{\tilde{s}}^{v1q}(Z) - T_{\tilde{s}}^{v1q}(\tilde{Y})\}| \\ &= |r_{\tilde{Y},q}(Z) \{v^1(Z, \tilde{s})v^q(Z, \tilde{s}) - v^1(\tilde{Y}, \tilde{s})v^q(\tilde{Y}, \tilde{s})\}| \\ &\leq |v^1(Z, \tilde{s})v(Z, \tilde{s}) - v^1(\tilde{Y}, \tilde{s})v(\tilde{Y}, \tilde{s})|_3 \\ &\leq 2|v(Z, \tilde{s}) - v(\tilde{Y}, \tilde{s})|_3 \leq 2a_K \cdot r_{\tilde{Y}}^{\alpha_K}(Z). \end{aligned} \quad (97)$$

With this inequality and (87), (31) gives

$$\begin{aligned} I_{11} &\leq \kappa_3 \delta^{\beta_3} \cdot 2a_K \cdot M^{03} \cdot (m_j^K)^{-1} \cdot \int \frac{1}{2-\alpha_K} d\lambda_{\partial B_{\tilde{s}}} \\ &\quad c_d^{\partial}(Y, s) \cap c_n^{\partial}(Y, s), \quad r_Y \\ &< 2^{5/2} \pi \cdot \kappa_3 a_K \cdot M^{03} \cdot (m_j^K)^{-1} \cdot \frac{1}{\alpha_K} d^{\alpha_K} \delta^{\beta_3}. \end{aligned} \quad (98)$$

I₁₂: An appropriate estimate is required for the difference appearing in the integrand of (32). By tracing the steps in the derivation of (IV.22.48), *mutatis mutandis*, it is easy to show that there exists a positive number k_5 , depending upon M and K alone, for which

$$|\Gamma_{(\tilde{Y}, \tilde{s})}^{03}(Z) - \Gamma_{\tilde{Y}}^c(\chi_{,4}^c(\chi_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s}))(Z)| \leq k_5 r_{\tilde{Y}}^{\hat{\alpha}}(Z) \quad (99)$$

whenever $\tilde{s} \in K$, $\tilde{Y} \in \partial B_{\tilde{s}}$, and $Z \in \partial B_{\tilde{s}} \cap \{\tilde{Y}\}'$.

Consequently, recalling (87),

$$I_{12} \leq \kappa_3 \delta^{\beta_3} \cdot 2k_5 \cdot (m_j^K)^{-1} \cdot \int_{C_d^\partial(Y,s) \cap C_\eta^\partial(Y,s)} \frac{1}{r_Y^{2-\tilde{\alpha}}} d\lambda_{\partial B_s} \quad (100)$$

$$< 2^{5/2} \pi \cdot \kappa_3 k_5 \cdot (m_j^K)^{-1} \cdot \frac{1}{\tilde{\alpha}} d^{\tilde{\alpha}} \cdot \delta^{\beta_3}.$$

I₁₃: We examine the difference $|J\chi_{\tilde{s}}^{-1}(Z) - J\chi_{\tilde{s}}^{-1}(\tilde{Y})|$, for $\tilde{s} \in K$, and $\tilde{Y}$ and $Z \in \partial B_{\tilde{s}}$, as in the proof of Theorem [IV.22]: recall first that $(Z, \tilde{s}) \mapsto \chi_{\tilde{s}}^{-1}(Z)$ is Lipschitz continuous on the compact set $\bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$, i.e.,

$$|\chi_{s_2}^{-1}(Z_2) - \chi_{s_1}^{-1}(Z_1)|_3 \leq A_0 |(Z_2, s_2) - (Z_1, s_1)|_4 \quad (101)$$

for $s_1, s_2 \in K$, $Z_1 \in \partial B_{s_1}$, and $Z_2 \in \partial B_{s_2}$,

for some $A_0 > 0$, depending upon M and K . Moreover, by hypothesis (ii), there exist $\tilde{A} > 0$ and $\tilde{\alpha} \in (0, 1]$, also depending on M and K , such that

$$|\hat{J}\chi(P_2, s_2) - \hat{J}\chi(P_1, s_1)| \leq \tilde{A} \cdot |(P_2, s_2) - (P_1, s_1)|_4^{\tilde{\alpha}}, \quad (102)$$

whenever $s_1, s_2 \in K$ and $P_1, P_2 \in \partial R$.

Combining (70), (71), (101), and (102),

$$\begin{aligned} |J\chi_{\tilde{s}}^{-1}(Z) - J\chi_{\tilde{s}}^{-1}(\tilde{Y})| &\leq (m_j^K)^{-2} |\hat{J}\chi(\chi_{\tilde{s}}^{-1}(Z), \tilde{s}) - \hat{J}\chi(\chi_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})| \\ &\leq (m_j^K)^{-2} \tilde{A} \cdot |\chi_{\tilde{s}}^{-1}(Z) - \chi_{\tilde{s}}^{-1}(\tilde{Y})|_3^{\tilde{\alpha}} \\ &\leq (m_j^K)^{-2} \tilde{A} \tilde{A}_0^{\tilde{\alpha}} |Z - \tilde{Y}|_3^{\tilde{\alpha}}, \end{aligned} \quad (103)$$

for $\tilde{s} \in K$, and $\tilde{Y}$ and $Z \in \partial B_{\tilde{s}}$.

With (87) and (103), and recalling that

$$\Gamma_X\{\xi\} \leq \{1-(c^*/c)^2\}^{-3/2} \quad \text{on} \quad \mathbb{R}^3 \cap \{X\}',$$

whenever $X \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ with $|\xi|_3 \leq c^*/c$ (cf., (VI.69.4)),

we come directly to the desired inequality for I_{13} :

$$\begin{aligned} I_{13} &\leq \kappa_3 \delta^3 \cdot 2 \{1-(c^*/c)^2\}^{-3/2} \cdot (\mathfrak{m}_j^K)^{-2} \cdot \bar{A} \bar{A}_0 \bar{\alpha} \cdot \int_{C_d^\partial(Y,s) \cap C_\eta^\partial(Y,s)'} \frac{1}{r_Y^{2-\bar{\alpha}}} d\lambda_{\partial B_s} \\ &< 2^{5/2} \pi \cdot \kappa_3 \cdot \bar{A} \bar{A}_0 \bar{\alpha} \cdot (\mathfrak{m}_j^K)^{-2} \cdot \{1-(c^*/c)^2\}^{-3/2} \cdot \frac{1}{\bar{\alpha}} d\bar{\alpha} \cdot \delta^3. \end{aligned} \quad (104)$$

I₁₄: We can derive an estimate for the difference in the integrand of (34) by arguing as in the proof of Proposition [IV.27]. To summarize the calculation, choose $\bar{s} \in K$, $\bar{Y} \in \partial B_{\bar{s}}$, and $Z \in \partial B_{\bar{s}} \cap B_{d_K}^3(\bar{Y}) \cap \{Y\}'$. Then

$$|1 - v^j(Z, \bar{s}) v^j(Y, \bar{s})| \leq a_K r_{\bar{Y}}^{\alpha_K}(Z) \quad (105)$$

(cf., (IV.27.23)),

$$\begin{aligned} &|\Gamma_{\bar{Y}}\{x_{,4}^c(x_{\bar{s}}^{-1}(\bar{Y}), \bar{s})\}(Z) - \Gamma_{\bar{Y}}\{x_{,4}^c(x_{\bar{s}}^{-1}(\bar{Y}), \bar{s})\} \circ \Pi_{\bar{Y}}(Z)| \\ &\leq 6\bar{a}_K \cdot (c^*/c)^2 \cdot \{1-(c^*/c)^2\}^{-7/2} \cdot r_{\bar{Y}}^{\alpha_K}(Z), \end{aligned} \quad (106)$$

where $\bar{a}_K := (8/7) \cdot (65/49)^{\alpha_K/2} \cdot a_K(1+\alpha_K)^{-1}$ (cf., (IV.27.26)), and

$$\left| \frac{1}{r_{\bar{Y}}^3(Z)} (Z - \bar{Y}) - \frac{1}{r_{\bar{Y}}^3(\Pi_{\bar{Y}}(Z))} (\Pi_{\bar{Y}}(Z) - \bar{Y}) \right|_3 \leq 8\bar{a}_K \cdot \frac{1}{r_{\bar{Y}}^{2-\alpha_K}(Z)} \quad (107)$$

(cf., (IV.27.28)). Consequently, proceeding as in (IV.27.22),

$$\begin{aligned}
 & \left| \frac{1}{r_{\tilde{Y}}^2(Z)} \cdot r_{\tilde{Y},q}(Z) \cdot \Gamma_{\tilde{Y}}\{x_{,4}^c(x_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})\}(Z) \right. \\
 & \quad \left. - \left[\frac{1}{2} \cdot r_{\tilde{Y},q} \cdot \Gamma_{\tilde{Y}}\{x_{,4}^c(x_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})\} \right] \circ \Pi_{\tilde{Y}}(Z) \cdot v^j(\tilde{Y}, \tilde{s}) v^j(Z, \tilde{s}) \right| \\
 & \leq 2 |1 - v^j(\tilde{Y}, \tilde{s}) v^j(Z, \tilde{s})| \cdot \frac{1}{r_{\tilde{Y}}^2(Z)} \cdot \Gamma_{\tilde{Y}}\{x_{,4}^c(x_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})\}(Z) \\
 & \quad + \frac{2}{r_{\tilde{Y}}^2(Z)} \cdot \left| \Gamma_{\tilde{Y}}\{x_{,4}^c(x_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})\}(Z) - \Gamma_{\tilde{Y}}\{x_{,4}^c(x_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})\} \circ \Pi_{\tilde{Y}}(Z) \right| \\
 & \quad + 2 \cdot \Gamma_{\tilde{Y}}\{x_{,4}^c(x_{\tilde{s}}^{-1}(\tilde{Y}), \tilde{s})\}(Z) \cdot \left| \frac{Z-Y}{r_{\tilde{Y}}^3(Z)} - \frac{\Pi_{\tilde{Y}}(Z)-Y}{r_{\tilde{Y}}^3(\Pi_{\tilde{Y}}(Z))} \right|_3 \\
 & \leq 2 \cdot \{1 - (c^*/c)^2\}^{-3/2} \{a_K + 6\tilde{a}_K \cdot (c^*/c)^2 \cdot \{1 - (c^*/c)^2\}^{-2} + 8\tilde{a}_K\} \cdot \frac{1}{r_{\tilde{Y}}^{2-\alpha_K}(Z)},
 \end{aligned} \tag{108}$$

$$\text{for } \tilde{s} \in K, \quad \tilde{Y} \in \partial \mathcal{B}_{\tilde{s}}, \quad Z \in \partial \mathcal{B}_{\tilde{s}} \cap \mathcal{B}_{d_K}^3(\tilde{Y}) \cap \{\tilde{Y}\}^c.$$

Denoting the coefficient on the right in (108) by k_6 , and recalling (87), we have, therefore,

$$\begin{aligned}
 I_{14} & \leq \kappa_3 \delta^{\beta_3} \cdot (m_j^K)^{-1} \cdot k_6 \cdot \int_{C_d^3(Y,s) \cap C_\eta^3(Y,s)} \frac{1}{r_Y^{2-\alpha_K}} d\lambda_{\partial \mathcal{B}_s} \\
 & < 2^{3/2} \pi \cdot \kappa_3 k_6 \cdot (m_j^K)^{-1} \cdot \frac{1}{\alpha_K} d^{\alpha_K} \cdot \delta^{\beta_3}.
 \end{aligned} \tag{109}$$

I₁₅: This term is analyzed by making use of (29). We begin by pointing out that

$$\eta < 1\delta \leq 1\delta^{\alpha_K} = 61\delta^{\alpha_K} - 51\delta^{\alpha_K} < d - 51\delta^{\alpha_K} < \frac{7}{9} d_K$$

i.e., $d - 51\delta^{\alpha_K} \in (\eta, (7/9)d_K)$. Then we can take $\tilde{s} = \hat{s}$, $\tilde{Y} = \hat{Y}$, and

$\bar{d} = d - 5\delta^{\alpha_K}$ in (29) and use the result in (36) to write

$$I_{15} = J\chi_s^{-1}(\hat{Y}) \cdot \left| \int_{\chi_{ss}(C_d^{\partial}(Y,s)) \cap C_n^{\partial}(\hat{Y},\hat{s})} \left(\frac{1}{r_Y} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}}\{\chi_{,4}^c(\chi_s^{-1}(\hat{Y}),\hat{s})\} \right) \circ \Pi_{\hat{Y}} \right. \\ \left. \cdot T_s^{1q}(\hat{Y}) \cdot v_{\partial B_s}^j(\hat{Y}) v_{\partial B_s}^j d\lambda_{\partial B_s} \right. \quad (110)$$

$$- \left. \int_{C_{d-5\delta}^{\partial}(\hat{Y},\hat{s}) \cap C_n^{\partial}(\hat{Y},\hat{s})} \left(\frac{1}{r_Y} \cdot r_{\hat{Y},q} \cdot \Gamma_{\hat{Y}}\{\chi_{,4}^c(\chi_s^{-1}(\hat{Y}),\hat{s})\} \right) \circ \Pi_{\hat{Y}} \right. \\ \left. \cdot T_s^{1q}(\hat{Y}) \cdot v_{\partial B_s}^j(\hat{Y}) v_{\partial B_s}^j d\lambda_{\partial B_s} \right|.$$

Now, the inequalities $d + 5\delta^{\alpha_K} < 2d < \frac{2}{3} d_K < \frac{7}{9} d_K$ imply that

$C_{d+5\delta}^{\partial}(\hat{Y},\hat{s})$ is defined. We intend to show that

$$C_{d-5\delta}^{\partial}(\hat{Y},\hat{s}) \subset \chi_{ss}(C_d^{\partial}(Y,s)) \subset C_{d+5\delta}^{\partial}(\hat{Y},\hat{s}); \quad (111)$$

for this, we shall first develop relations which we shall find to be

quite useful in subsequent estimations, as well. Choosing $Z \in$

$C_d^{\partial}(Y,s)$, we have $\chi_{ss}(Z) \in \partial B_s \cap B_{d_K}^3(\hat{Y})$, by (37). Then

$$\{\Pi_{\hat{Y}}(\chi_{ss}(Z)) - \chi_{ss}(Z)\} - \{\Pi_Y(Z) - Z\} \\ = \{v(\hat{Y},\hat{s}) \bullet (\hat{Y} - \chi_{ss}(Z))\} v(\hat{Y},\hat{s}) - \{v(Y,s) \bullet (Y - Z)\} v(Y,s) \\ = \{v(Y,s) \bullet (Y - Z)\} \cdot \{v(\hat{Y},\hat{s}) - v(Y,s)\} \\ + \{(v(\hat{Y},\hat{s}) - v(Y,s)) \bullet (Y - Z)\} v(\hat{Y},\hat{s}) \\ + v(\hat{Y},\hat{s}) \bullet \{(\hat{Y} - \chi_{ss}(Z)) - (Y - Z)\} v(\hat{Y},\hat{s})$$

$$\begin{aligned}
 &= \{v(Y, s) \bullet (Y-Z)\} \cdot \{v(\hat{Y}, \hat{s}) - v(Y, s)\} \\
 &\quad + \{(v(\hat{Y}, \hat{s}) - v(Y, s)) \bullet (Y-Z)\} v(\hat{Y}, \hat{s}) \\
 &\quad + v(\hat{Y}, \hat{s}) \bullet \{(Z - \chi_{s\hat{s}}(Z)) - (Y - \chi_{s\hat{s}}(Y))\} v(\hat{Y}, \hat{s}) \\
 &\quad + \{v(\hat{Y}, \hat{s}) \bullet (\hat{Y} - \chi_{s\hat{s}}(Y))\} v(\hat{Y}, \hat{s}), \tag{112}
 \end{aligned}$$

for each $Z \in C_d^\partial(Y, s)$.

In the present setting, we employ (112) along with (6), (23) and its companion result

$$|\chi_{s\hat{s}}(Y) - Y|_3 \leq c^* |s - \hat{s}|, \tag{113}$$

and the inequality $\frac{18}{7} a_K d < 1$ (from (8)) to write

$$\begin{aligned}
 &| \{ \pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - \chi_{s\hat{s}}(Z) \} - \{ \pi_Y(Z) - Z \} |_3 \\
 &\leq 2r_Y(Z) \cdot |v(\hat{Y}, \hat{s}) - v(Y, s)|_3 + \{ |Z - \chi_{s\hat{s}}(Z)|_3 + |Y - \chi_{s\hat{s}}(Y)|_3 + |\hat{Y} - \chi_{s\hat{s}}(Y)|_3 \} \\
 &\leq 2 \cdot \frac{9}{7} d \cdot a_K \delta^{\alpha_K} + 2c^* |s - \hat{s}| + |Y - \hat{Y}|_3 + |Y - \chi_{s\hat{s}}(Y)|_3 \\
 &< \delta^{\alpha_K} + 2\delta + \delta < 4\delta^{\alpha_K}, \quad \text{for each } Z \in C_d^\partial(Y, s). \tag{114}
 \end{aligned}$$

To verify the second inclusion in (111), let $Z \in C_d^\partial(Y, s)$: then $r_Y(\pi_Y(Z)) < d$, so (114) shows that

$$\begin{aligned}
 | \pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - \hat{Y} | &\leq | \pi_Y(Z) - Y |_3 + | (Y - \hat{Y}) + (\chi_{s\hat{s}}(Z) - Z) |_3 \\
 &\quad + | \{ \pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - \chi_{s\hat{s}}(Z) \} - \{ \pi_Y(Z) - Z \} |_3 \\
 &< d + \delta + 4\delta^{\alpha_K} \leq d + 5\delta^{\alpha_K}.
 \end{aligned}$$

We can conclude from this that $\chi_{\hat{s}\hat{s}}(C_d^\partial(Y,s)) \subset C_{d+5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})$.

To check the validity of the first half of (111), note that we may interchange (Y,s) and $(\hat{Y}, \hat{s})$ in each of (112) and (114) to produce statements holding for each $Z \in C_d^\partial(\hat{Y}, \hat{s})$. Consequently, supposing that $Z \in C_{d-5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})$, so $r_{\hat{Y}}(\pi_{\hat{Y}}(Z)) < d-5\delta$, we have

$$\begin{aligned} |\pi_Y(\chi_{\hat{s}\hat{s}}(Z)) - Y|_3 &\leq |\pi_{\hat{Y}}(Z) - \hat{Y}|_3 + |(\hat{Y} - Y) + (\chi_{\hat{s}\hat{s}}(Z) - Z)|_3 \\ &\quad + |(\pi_Y(\chi_{\hat{s}\hat{s}}(Z)) - \chi_{\hat{s}\hat{s}}(Z)) - (\pi_{\hat{Y}}(Z) - Z)|_3 \\ &< d-5\delta + \delta + 4\delta + \delta \\ &\leq d-5\delta + 5\delta = d. \end{aligned}$$

It now follows that $\chi_{\hat{s}\hat{s}}(C_{d-5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})) \subset C_d^\partial(Y,s)$, which is equivalent to the first inclusion in (111).

Having proven (111), we can use it with (110) to write

$$I_{15} = J\chi_{\hat{s}}^{-1}(\hat{Y})$$

$$\begin{aligned} &\cdot \left| \int_{\chi_{\hat{s}\hat{s}}(C_d^\partial(Y,s)) \cap C_{d-5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})} \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot r_{\hat{Y}}\{x_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\} \right) \circ \pi_{\hat{Y}} \right. \\ &\quad \left. \cdot T_{\hat{s}}^{1q}(\hat{Y}) \cdot v_{\partial \hat{B}_{\hat{s}}}^j(\hat{Y}) v_{\partial \hat{B}_{\hat{s}}}^j d\lambda_{\partial \hat{B}_{\hat{s}}} \right| \end{aligned}$$

We can conclude from this that $\chi_{ss}(C_d^\partial(Y,s)) \subset C_{d+5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})$.

To check the validity of the first half of (111), note that we may interchange (Y,s) and $(\hat{Y}, \hat{s})$ in each of (112) and (114) to produce statements holding for each $Z \in C_d^\partial(\hat{Y}, \hat{s})$. Consequently, supposing that $Z \in C_{d-5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})$, so $r_{\hat{Y}}(\pi_{\hat{Y}}(Z)) < d-5\delta^{\alpha_K}$, we have

$$\begin{aligned} |\pi_Y(\chi_{ss}(Z)) - Y|_3 &\leq |\pi_{\hat{Y}}(Z) - \hat{Y}|_3 + |(\hat{Y} - Y) + (\chi_{ss}(Z) - Z)|_3 \\ &\quad + |(\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Z)) - (\pi_{\hat{Y}}(Z) - Z)|_3 \\ &< d-5\delta^{\alpha_K} + \delta + 4\delta^{\alpha_K} \\ &\leq d-5\delta^{\alpha_K} + 5\delta^{\alpha_K} = d. \end{aligned}$$

It now follows that $\chi_{ss}(C_{d-5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})) \subset C_d^\partial(Y,s)$, which is equivalent to the first inclusion in (111).

Having proven (111), we can use it with (110) to write

$$\begin{aligned} I_{15} &= J\chi_s^{-1}(\hat{Y}) \\ &\cdot \left| \int_{\chi_{ss}(C_d^\partial(Y,s)) \cap C_{d-5\delta}^{\partial, \alpha_K}(\hat{Y}, \hat{s})} \left(\frac{1}{r_{\hat{Y}}^2} \cdot r_{\hat{Y},q} \cdot r_{\hat{Y}}\{x_{,4}^c(\chi_s^{-1}(\hat{Y}), \hat{s})\} \right) \circ \pi_{\hat{Y}} \right. \\ &\quad \left. \cdot T_s^{1q}(\hat{Y}) \cdot v_{\partial B_s}^j(\hat{Y}) v_{\partial B_s}^j d\lambda_{\partial B_s} \right| \end{aligned}$$

$$\begin{aligned}
 &= J\chi_{\hat{s}}^{-1}(\hat{Y}) \\
 &\cdot \left| \int_{\substack{B^2 \\ d+5\delta}}^{\substack{B^2 \\ d-5\delta}} h_{\hat{Y}}(\chi_{\hat{s}}(C_d^{\partial}(Y,s))) \cap B^2 \right. \\
 &\quad \left. \left(\frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},q} \Gamma_{\hat{Y}}\{x_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\} \right) \circ \hat{K}_{\hat{Y}}^{-1} \right. \\
 &\quad \left. \cdot \hat{T}_{\hat{s}}^{iq}(\hat{Y}) \, d\lambda_2 \right| \tag{115} \\
 &\leq (m_j^K)^{-1} \cdot \int_{\substack{B^2 \\ d+5\delta}}^{\substack{B^2 \\ d-5\delta}} \left| \left(\frac{1}{r_{\hat{Y}}} \cdot r_{\hat{Y},q} \right. \right. \\
 &\quad \left. \cdot \Gamma_{\hat{Y}}\{x_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\} \right) \circ \hat{K}_{\hat{Y}}^{-1} \cdot \hat{T}_{\hat{s}}^{iq}(\hat{Y}) \Big| \, d\lambda_2;
 \end{aligned}$$

the computation here has been developed as in the proof of Lemma [IV.26] (cf., (IV.26.8)). In fact, we can use the reasoning detailed in that proof to obtain a more explicit form for the integral appearing on the right in (115): introducing the linear isometry $A_{\hat{Y}}$ as in (IV.26.15) and, for brevity, the notations

$$\hat{T} := \hat{T}(\hat{Y}, \hat{s}) := \hat{T}_{\hat{s}}^{iq}(\hat{Y}) e_q^{(3)}$$

and

$$\hat{\psi} := \hat{\psi}(\hat{Y}, \hat{s}) := x_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}),$$

we find

$$I_{15} \leq (m_j^K)^{-1} \int_0^{2\pi} \int_{d-5\delta^K}^{d+5\delta^K} \frac{|(A_Y \hat{T})^1 \cos \omega + (A_Y \hat{T})^2 \sin \omega|}{\{(1-|\hat{\psi}|_3^2) + ((A_Y \hat{\psi})^1)^2 \cos^2 \omega\}^{3/2}} \cdot \frac{1}{\rho} d\rho d\omega \quad (116)$$

$$= (m_j^K)^{-1} \cdot I(\hat{Y}, \hat{s}) \cdot \ln \frac{d+5\delta^K}{d-5\delta^K},$$

wherein *

$$I(\hat{Y}, \hat{s}) := \int_0^{2\pi} \frac{|(A_Y \hat{T}(\hat{Y}, \hat{s}))^1 \cos \omega + (A_Y \hat{T}(\hat{Y}, \hat{s}))^2 \sin \omega|}{\{(1-|\hat{\psi}(\hat{Y}, \hat{s})|_3^2) + ((A_Y \hat{\psi}(\hat{Y}, \hat{s}))^1)^2 \cos^2 \omega\}^{3/2}} d\omega. \quad (117)$$

Since A_Y is an isometry, $|\hat{T}(\hat{Y}, \hat{s})|_3 = \{1 - (v^1(\hat{Y}, \hat{s}))^2\}^{1/2} \leq 1$, and $|\hat{\psi}(\hat{Y}, \hat{s})|_3 \leq c^*/c$, it is clear that

$$I(\hat{Y}, \hat{s}) \leq 4\pi \cdot \{1 - (c^*/c)^2\}^{-3/2}.$$

Moreover, recalling the inequality $\ln \zeta \leq \zeta - 1$ for $\zeta > 0$, and noting that $6\delta^K < d$, so $5\delta^K < \frac{5}{6}d$ and $d - 5\delta^K > \frac{1}{6}d$, the third factor in (116) can be estimated by

$$\ln \frac{d+5\delta^K}{d-5\delta^K} \leq \frac{d+5\delta^K}{d-5\delta^K} - 1 = \frac{10\delta^K}{d-5\delta^K} < \frac{60}{d} \cdot \delta^K.$$

Thus, I_{15} satisfies an inequality of the form of (68), viz.,

$$I_{15} < \frac{240\pi}{d} \cdot (m_j^K)^{-1} \cdot \{1 - (c^*/c)^2\}^{-3/2} \cdot \delta^K. \quad (118)$$

I₁₆: Keeping in mind inequality (103) and the inclusion (94), (40)

leads directly to the estimate

$$\begin{aligned}
 I_{16} &\leq 2M^{03} \cdot (m_j^K)^{-2} \tilde{A} \tilde{A}_0^{\tilde{\alpha}} \cdot \int \frac{1}{r_{\hat{Y}}^{2-\tilde{\alpha}}} d\lambda_{\partial B_{\hat{s}}} \\
 &\quad \chi_{\hat{s}\hat{s}}(C_{31\delta}^{\partial}(Y, s)) \cap C_{\eta}^{\partial}(\hat{Y}, \hat{s}), \\
 &\leq 2M^{03} \cdot (m_j^K)^{-2} \tilde{A} \tilde{A}_0^{\tilde{\alpha}} \cdot \int \frac{1}{r_{\hat{Y}}^{2-\tilde{\alpha}}} d\lambda_{\partial B_{\hat{s}}} \\
 &\quad C_{51\delta}^{\partial}(\hat{Y}, \hat{s}) \\
 &\leq 2^{5/2} \pi \cdot M^{03} \cdot (m_j^K)^{-2} \tilde{A} \tilde{A}_0^{\tilde{\alpha}} \cdot \frac{1}{\tilde{\alpha}} \cdot (51)^{\tilde{\alpha}} \cdot \delta^{\tilde{\alpha}}.
 \end{aligned} \tag{119}$$

I₁₇: With (94) and (97), it is clear that

$$\begin{aligned}
 I_{17} &\leq M^{03} \cdot (m_j^K)^{-1} \cdot 2a_K \cdot \int \frac{1}{r_{\hat{Y}}^{2-\alpha_K}} d\lambda_{\partial B_{\hat{s}}} \\
 &\quad C_{51\delta}^{\partial}(\hat{Y}, \hat{s}) \\
 &\leq 2^{5/2} \pi a_K M^{03} (m_j^K)^{-1} \cdot \frac{1}{\alpha_K} \cdot (51)^{\alpha_K} \cdot \delta^{\alpha_K}.
 \end{aligned} \tag{120}$$

I₁₈: Here, we use (94) and (99), obtaining

$$\begin{aligned}
 I_{18} &\leq 2(m_j^K)^{-1} \cdot k_5 \cdot \int \frac{1}{r_{\hat{Y}}^{2-\hat{\alpha}}} d\lambda_{\partial B_{\hat{s}}} \\
 &\quad C_{51\delta}^{\partial}(\hat{Y}, \hat{s}) \\
 &\leq 2^{5/2} \pi k_5 (m_j^K)^{-1} \cdot \frac{1}{\hat{\alpha}} \cdot (51)^{\hat{\alpha}} \cdot \delta^{\hat{\alpha}}.
 \end{aligned} \tag{121}$$

I₁₉: The inequality (108) can be employed here; appealing once more to (94), we find

$$\begin{aligned}
 I_{19} &\leq (m_j^K)^{-1} \cdot k_6 \cdot \int \frac{1}{r_{\hat{Y}}^{2-\alpha_K}} d\lambda_{\partial B_{\hat{s}}} \\
 &\quad C_{51\delta}^{\partial}(\hat{Y}, \hat{s}) \\
 &\leq 2^{3/2} \pi k_6 (m_j^K)^{-1} \cdot \frac{1}{\alpha_K} \cdot (51)^{\alpha_K} \cdot \delta^{\alpha_K}.
 \end{aligned} \tag{122}$$

$I_{20}, I_{21}, I_{22}, I_{23}$: Upon comparing (45)-(48) with (40)-(43), respectively, it is plain that we can proceed essentially as in the derivations of (119)-(122) (we need not apply (94), of course) to arrive at the inequalities

$$I_{20} \leq 2^{5/2} \pi \cdot M^{03} \cdot (m_j^K)^{-2} \cdot \tilde{A} \tilde{A}_0 \cdot \tilde{\alpha}^{-1} (3_1)^{\tilde{\alpha} \cdot \tilde{\alpha}}, \quad (123)$$

$$I_{21} \leq 2^{5/2} \pi \cdot a_K M^{03} \cdot (m_j^K)^{-1} \cdot \alpha_K^{-1} \cdot (3_1)^{\alpha_K \cdot \alpha_K}, \quad (124)$$

$$I_{22} \leq 2^{5/2} \pi \cdot k_5 \cdot (m_j^K)^{-1} \cdot \alpha_K^{-1} \cdot (3_1)^{\alpha_K \cdot \alpha_K}, \quad (125)$$

and

$$I_{23} \leq 2^{3/2} \pi \cdot k_6 \cdot (m_j^K)^{-1} \cdot \alpha_K^{-1} \cdot (3_1)^{\alpha_K \cdot \alpha_K}. \quad (126)$$

I_{24} : Obviously, the reasoning of (82) would show that

$$|r_{\hat{Y},q}(\chi_{ss}(Z)) \cdot \{\tilde{T}_{\hat{s}}^{iq}(\chi_{ss}(Z)) - \tilde{T}_s^{iq}(Z)\}| \leq 2a_{K^1} \alpha_K \cdot \alpha_K, \quad (127)$$

for each $Z \in \partial B_s \cap \{\chi_{ss}(\hat{Y})\}'$.

Having this and inequality (84), (51) gives

$$\begin{aligned} I_{24} &\leq M^{03} \cdot (m_j^K)^{-1} \cdot 2a_{K^1} \alpha_K \cdot \alpha_K \cdot \int_{C_d^\partial(Y,s) \cap C_{3_1\delta}^\partial(Y,s)} \frac{1}{r_{Y \times ss}^2} d\lambda_{\partial B_s} \\ &\leq 2a_{K^1} \alpha_K \cdot M^{03} \cdot (m_j^K)^{-1} \cdot \delta \alpha_K \cdot \frac{9}{4} \cdot \int_{C_d^\partial(Y,s) \cap C_{3_1\delta}^\partial(Y,s)} \frac{1}{r_Y} d\lambda_{\partial B_s} \quad (128) \\ &\leq 9 \cdot 2^{1/2} \pi \cdot a_{K^1} \alpha_K \cdot M^{03} \cdot (m_j^K)^{-1} \cdot \delta \alpha_K \cdot \ln \frac{d}{3_1\delta} \\ &\leq 9 \cdot 2^{1/2} \pi \cdot a_{K^1} \alpha_K \cdot M^{03} \cdot (m_j^K)^{-1} \cdot M_3 \cdot \delta \alpha_K', \end{aligned}$$

in which $\alpha'_K \in (0, \alpha_K)$ and M_3 is a positive number such that

$$\zeta^{\alpha_K - \alpha'_K} \cdot \ln \frac{d}{31\zeta} \leq M_3 \quad \text{for} \quad 0 < \zeta < \frac{d}{61}. \quad (129)$$

I₂₅: If $Z \in C_d^\partial(Y, s)$, then $r_Y(\Pi_Y(Z)) \leq r_Y(Z)$, and $r_{\hat{Y}}(\Pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) > \frac{7}{9} r_{\hat{Y}}(\chi_{s\hat{s}}(Z))$. Consequently, from (84) it can be seen that

$$\frac{r_Y(\Pi_Y(Z))}{r_{\hat{Y}}(\Pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} < \frac{9}{7} \cdot \frac{r_Y(Z)}{r_{\hat{Y}}(\chi_{s\hat{s}}(Z))} \leq \frac{9}{7} \cdot \frac{3}{2} = \frac{27}{14}, \quad (130)$$

for each $Z \in C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)'$.

Next, suppose that $\omega \in \mathbb{R}^3$: by (6),

$$\begin{aligned} |\omega^q \{T_s^{1q}(Y) - T_{\hat{s}}^{1q}(\hat{Y})\}| &= |\omega^q \{v^1(Y, s) v^q(Y, s) - v^1(\hat{Y}, \hat{s}) v^q(\hat{Y}, \hat{s})\}| \\ &\leq |\omega|_3 \cdot |v^1(Y, s) v(Y, s) - v^1(\hat{Y}, \hat{s}) v(\hat{Y}, \hat{s})|_3 \\ &\leq 2|\omega|_3 \cdot |v(Y, s) - v(\hat{Y}, \hat{s})|_3 \\ &\leq 2|\omega|_3 \cdot a_K \cdot \delta^{\alpha_K}. \end{aligned} \quad (131)$$

Further, as we have already noted, whenever $X \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ with $|\xi|_3 \leq c^*/c$,

$$\Gamma_X\{\xi\} \leq \{1 - (c^*/c)^2\}^{-3/2} \quad \text{on} \quad \mathbb{R}^3 \cap \{X\}'.$$

Upon combining these estimates, we discover from (52) that

$$I_{25} \leq (m_J^K)^{-1} \cdot \left\{1 - \left(\frac{c^*}{c}\right)^2\right\}^{-3/2} \cdot 2a_K \delta^{\alpha_K} \cdot \int_{C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)'} \frac{1}{r_{\hat{Y} \circ \Pi_{\hat{Y}} \circ \chi_{s\hat{s}}}^2} d^1_{\partial E_s}$$

$$\begin{aligned}
 &\leq \frac{27}{7} a_K \cdot (m_j^K)^{-1} \cdot \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-3/2} \delta^{\alpha_K} \cdot \int_{C_d^\partial(Y,s) \cap C_{3\delta}^\partial(Y,s)} \frac{1}{r_Y^2 \circ \Pi_Y} d\lambda_{\partial B_s} \\
 &\leq \frac{27}{7} \cdot 2^{3/2} \pi \cdot a_K \cdot (m_j^K)^{-1} \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-3/2} \cdot M_3 \cdot \delta^{\alpha'_K},
 \end{aligned} \tag{132}$$

having chosen $\alpha'_K \in (0, \alpha_K)$ and M_3 as in (129).

I₂₆: With (6) and (23), for each $Z \in \partial B_s$, we have

$$\begin{aligned}
 &|v^j(Y,s) v^j(Z,s) - v^j(\hat{Y}, \hat{s}) v^j(\chi_{s\hat{s}}(Z), \hat{s})| \\
 &\leq |v(Y,s) - v(\hat{Y}, \hat{s})|_3 + |v(Z,s) - v(\chi_{s\hat{s}}(Z), \hat{s})|_3 \\
 &\leq a_K \delta^{\alpha_K} + a_K \cdot |(Z,s) - (\chi_{s\hat{s}}(Z), \hat{s})|_4^{\alpha_K} \\
 &\leq a_K \delta^{\alpha_K} + a_K \cdot \{(c^*)^2 |s - \hat{s}|^2 + |s - \hat{s}|^2\}^{\alpha_K/2} \\
 &\leq a_K (1 + \delta^{\alpha_K}) \cdot \delta^{\alpha_K}.
 \end{aligned} \tag{133}$$

Keeping in mind (130), (53) leads to the required inequality

$$\begin{aligned}
 I_{26} &\leq 2 \cdot (m_j^K)^{-1} \cdot \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-3/2} \cdot a_K (1 + \delta^{\alpha_K}) \cdot \delta^{\alpha_K} \\
 &\quad \cdot \frac{27}{14} \int_{C_d^\partial(Y,s) \cap C_{3\delta}^\partial(Y,s)} \frac{1}{r_Y^2 \circ \Pi_Y} d\lambda_{\partial B_s} \\
 &\leq \frac{27}{7} \cdot 2^{3/2} \pi \cdot a_K \cdot (1 + \delta^{\alpha_K}) \cdot (m_j^K)^{-1} \cdot \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-3/2} \cdot M_3 \cdot \delta^{\alpha'_K},
 \end{aligned} \tag{134}$$

again with $\alpha'_K \in (0, \alpha_K)$, and M_3 as in (129).

I₂₇: We first obtain a Hölder estimate for $|J\chi_s^{-1}(Y) - J\chi_s^{-1}(\hat{Y})|$:
 recalling (70), (71), (101), and (102),

$$\begin{aligned}
 |J\chi_s^{-1}(Y) - J\chi_{\hat{s}}^{-1}(\hat{Y})| &= |\{\hat{J}\chi(\chi_s^{-1}(Y), s)\}^{-1} - \{\hat{J}\chi(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\}^{-1}| \\
 &\leq (m_j^K)^{-2} |\hat{J}\chi(\chi_s^{-1}(Y), s) - \hat{J}\chi(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})| \\
 &\leq (m_j^K)^{-2} \cdot \tilde{A} \cdot |\chi_s^{-1}(Y), s) - (\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})|_{\tilde{A}}^{\tilde{A}} \\
 &\leq (m_j^K)^{-2} \cdot \tilde{A} \cdot \{1 + A_0^2\}^{\tilde{A}/2} \cdot \delta^{\tilde{A}}.
 \end{aligned} \tag{135}$$

Thus, it is plain that (54) gives

$$\begin{aligned}
 I_{27} &\leq \frac{27}{7} \cdot \left\{1 - \left(\frac{c^*}{c}\right)^2\right\}^{-3/2} \cdot (m_j^K)^{-2} \cdot \tilde{A} \cdot \{1 + A_0^2\}^{\tilde{A}/2} \cdot \delta^{\tilde{A}} \\
 &\quad \cdot \int_{C_d^{\partial}(Y, s) \cap C_{3\delta}^{\partial}(Y, s)} \frac{1}{r_{Y \circ \Pi_Y}^2} d\lambda_{\partial \tilde{S}_s} \\
 &\leq \frac{27}{7} \cdot 2^{3/2} \pi \tilde{A} \cdot \left\{1 - \left(\frac{c^*}{c}\right)^2\right\}^{-3/2} \cdot (m_j^K)^{-2} \cdot \{1 + A_0^2\}^{\tilde{A}/2} \cdot M_4 \cdot \delta^{\tilde{A}'},
 \end{aligned} \tag{136}$$

wherein $\tilde{A}'$ is a number chosen in $(0, \tilde{A})$, and M_4 is a positive number such that

$$\zeta^{\tilde{A} - \tilde{A}'} \cdot \ln \frac{d}{3\zeta} \leq M_4 \quad \text{for} \quad 0 < \zeta < \frac{d}{6}. \tag{137}$$

I₂₈: Equality (IV.18.11) shows that

$$\begin{aligned}
 J\chi_{s\hat{s}}(Z) &= \{(J\chi_s) \circ \chi_s^{-1}(Z)\} \cdot \{(J\chi_{\hat{s}}) \circ \chi_{\hat{s}}^{-1}(Z)\}^{-1} \\
 &= \{\hat{J}\chi(\chi_s^{-1}(Z), s)\} \cdot \{\hat{J}\chi(\chi_{\hat{s}}^{-1}(Z), \hat{s})\}^{-1}, \quad \text{for } Z \in \partial \tilde{S}_s.
 \end{aligned} \tag{138}$$

Appealing once more to (71) and (102) results in

$$\begin{aligned}
 |1 - J\chi_{s\hat{s}}(Z)| &= \{\hat{J}\chi(\chi_s^{-1}(Z), s)\}^{-1} \cdot |\hat{J}\chi(\chi_s^{-1}(Z), s) - \hat{J}\chi(\chi_{\hat{s}}^{-1}(Z), \hat{s})| \\
 &\leq (m_j^K)^{-1} \cdot \tilde{A} \cdot |s - \hat{s}|^{\tilde{A}} \leq (m_j^K)^{-1} \cdot \tilde{A} \cdot \delta^{\tilde{A}}.
 \end{aligned} \tag{139}$$

With this, directly from (55) we obtain

$$\begin{aligned}
 I_{28} &\leq \frac{27}{7} \cdot (m_j^K)^{-1} \cdot \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-3/2} \cdot (m_j^K)^{-1} \cdot \tilde{A} \tilde{\alpha} \\
 &\quad \cdot \int_{C_d^\partial(Y,s) \cap C_{3,1\delta}^\partial(Y,s)} \frac{1}{r_Y^2 \circ \Pi_Y} d\lambda_{\partial B_s} \\
 &\leq \frac{27}{7} \cdot 2^{3/2} \pi \cdot \tilde{A} \cdot (m_j^K)^{-2} \cdot \left\{ 1 - \left(\frac{c^*}{c} \right)^2 \right\}^{-3/2} \cdot M_4 \cdot \delta \tilde{\alpha}',
 \end{aligned} \tag{140}$$

having introduced $\tilde{\alpha}' \in (0, \tilde{\alpha})$ and, subsequently, M_4 as in (137).

I₂₉: Let us point out certain facts which shall prove to be of use in later estimates, as well as in the present one. Let the closed line segment joining $\hat{Y}$ and $\chi_{s\hat{s}}(\hat{Y})$ in $\mathbb{R}^3$ be denoted by $[\hat{Y}, \chi_{s\hat{s}}(Y)]$; this set may, of course, consist of a single point, in case $\hat{Y} = \chi_{s\hat{s}}(Y)$. Fix any $X \in [\hat{Y}, \chi_{s\hat{s}}(Y)]$. For each $Z \in \partial B_{\hat{s}}$,

$$\begin{aligned}
 \tau(\chi_{\hat{s}}^{-1}(Z); X, \hat{s}) &= |\tau(\chi_{\hat{s}}^{-1}(Z); X, \hat{s}) - \tau(\chi_{\hat{s}}^{-1}(Z); Z, \hat{s})| \\
 &\leq \frac{1}{c-c^*} |X-Z|_3 \\
 &\leq \frac{1}{c-c^*} \{ |X-\hat{Y}|_3 + |Z-\hat{Y}|_3 \} \\
 &\leq \frac{1}{c-c^*} \{ |\chi_{s\hat{s}}(Y) - \hat{Y}|_3 + \text{diam } B_{\hat{s}} \} \\
 &\leq \frac{1}{c-c^*} \left\{ \frac{d}{6} + \text{diam } B_{\hat{s}} \right\}
 \end{aligned} \tag{141}$$

(since $|\chi_{s\hat{s}}(Y) - \hat{Y}|_3 \leq |Y - \hat{Y}|_3 + |Y - \chi_{s\hat{s}}(Y)|_3 \leq 1\delta < d/6$), and

$$\tau(\chi_{\hat{s}}^{-1}(Z); \hat{Y}, \hat{s}) \leq \frac{1}{c-c^*} |Z - \hat{Y}|_3 \leq \frac{1}{c-c^*} \text{diam } B_{\hat{s}}, \tag{142}$$

so that $\hat{s}^{-1}(\chi_s^{-1}(Z); X, \hat{s})$ and $\hat{s}^{-1}(\chi_s^{-1}(Z); \hat{Y}, \hat{s})$ lie in $\hat{K}$. Therefore, recalling (76), we not only have

$$\begin{aligned} & |[\chi_{,4}^c]_{(X,\hat{s})} \circ \chi_s^{-1}(Z) - [\chi_{,4}^c]_{(\hat{Y},\hat{s})} \circ \chi_s^{-1}(Z)|_3 \\ &= |\chi_{,4}^c(\chi_s^{-1}(Z), \hat{s}^{-1}(\chi_s^{-1}(Z); X, \hat{s})) - \chi_{,4}^c(\chi_s^{-1}(Z), \hat{s}^{-1}(\chi_s^{-1}(Z); \hat{Y}, \hat{s}))|_3 \\ &\leq \hat{A} \cdot |\tau(\chi_s^{-1}(Z); X, \hat{s}) - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s})|^{\hat{\alpha}} \quad (143) \\ &\leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot |X - \hat{Y}|_3^{\hat{\alpha}} \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot |\chi_{ss}(Y) - \hat{Y}|_3^{\hat{\alpha}} \leq \frac{\hat{A} \cdot 1}{(c-c^*)^{\hat{\alpha}}} \cdot \delta^{\hat{\alpha}} \end{aligned}$$

for each $Z \in \partial \hat{B}_{\hat{s}}$,

but we can also argue as in the derivation of (IV.22.34) to conclude that

$$\begin{aligned} & |v_{(X,\hat{s})}^c(Z) - v_{(\hat{Y},\hat{s})}^c(Z)|_3 \leq \frac{\hat{A}}{1+\hat{\alpha}} \cdot |\tau(\chi_s^{-1}(Z); X, \hat{s}) - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s})|^{\hat{\alpha}} \\ &\leq \frac{\hat{A} \cdot 1}{(1+\hat{\alpha}) \cdot (c-c^*)^{\hat{\alpha}}} \cdot \delta^{\hat{\alpha}} \quad \text{for each } Z \in \partial \hat{B}_{\hat{s}} \quad (144) \end{aligned}$$

((144) is first proven for $Z \in \partial \hat{B}_{\hat{s}} \cap \{\hat{Y}\} \cap \{X\}'$; the continuity of v then implies the result in case $Z = \hat{Y}$ or $Z = X$). Further, just as in (IV.22.39), it is easy to show that

$$\begin{aligned} & |w_{(X,\hat{s})}(Z) - w_{(\hat{Y},\hat{s})}(Z)|_3 \leq c_6^* |v_{(X,\hat{s})}^c(Z) - v_{(\hat{Y},\hat{s})}^c(Z)|_3 \\ &\quad + c_7^* |[\chi_{,4}^c]_{(X,\hat{s})} \circ \chi_s^{-1}(Z) - [\chi_{,4}^c]_{(\hat{Y},\hat{s})} \circ \chi_s^{-1}(Z)|_3 \quad (145) \end{aligned}$$

for each $Z \in \partial \hat{B}_{\hat{s}}$,

wherein

$$c_6^* := \{1 - (c^*/c)^2\}^{-1} \{1 + 6(c^*/c)^2 + (c^*/c)^4\}, \quad (146)$$

and

$$c_7^* := \{1 - (c^*/c)^2\}^{-1} \{1 + 3(c^*/c)^2 + (c^*/c)^4\}. \quad (147)$$

Also, (IV.14.57) and a short calculation lead to the bounds given by

$$\begin{aligned} c_1^* &\leq w_{(\tilde{X}, \tilde{s})}^k(Z) \cdot r_{\tilde{X}, k}(Z) + \{(r_{\tilde{X}, l}(Z) \cdot v_{(\tilde{X}, \tilde{s})}^{lc}(Z))^2 + (1 - |v_{(\tilde{X}, \tilde{s})}^c(Z)|_3^2)\}^{1/2} \\ &\leq c_2^* \quad \text{whenever} \quad (\tilde{X}, \tilde{s}) \in \mathbb{R}^4 \quad \text{and} \quad Z \in \partial \mathcal{B}_{\tilde{s}} \cap \{\tilde{X}\}', \end{aligned} \quad (148)$$

with

$$c_1^* := \{1 + (c^*/c)^2\}^{-1} \cdot \{1 - (c^*/c)^2\}, \quad (149)$$

and

$$c_2^* := 1 + \{1 - (c^*/c)^2\}^{-1} \cdot (c^*/c) \cdot \{2 + (c^*/c)^2\}. \quad (150)$$

Having these results at our disposal, and still denoting by X an arbitrary point of $[\hat{Y}, X_{s\hat{s}}(Y)]$, suppose that $Z \in \partial \mathcal{B}_{\hat{s}} \cap \{X\}'$: then

$$\begin{aligned} &|\{w_{(\hat{Y}, \hat{s})}^k(Z) \cdot r_{X, k}(Z) + \{(r_{X, l}(Z) \cdot v_{(\hat{Y}, \hat{s})}^{lc}(Z))^2 + (1 - |v_{(\hat{Y}, \hat{s})}^c(Z)|_3^2)\}^{1/2}\} \\ &- \{w_{(X, \hat{s})}^k(Z) \cdot r_{X, k}(Z) + \{(r_{X, l}(Z) \cdot v_{(X, \hat{s})}^{lc}(Z))^2 + (1 - |v_{(X, \hat{s})}^c(Z)|_3^2)\}^{1/2}\}| \\ &\leq |r_{X, k}(Z) \cdot \{w_{(\hat{Y}, \hat{s})}^k(Z) - w_{(X, \hat{s})}^k(Z)\}| \\ &+ \frac{1}{2\{1 - (c^*/c)^2\}^{1/2}} \cdot \{|(r_{X, l}(Z) \cdot v_{(\hat{Y}, \hat{s})}^{lc}(Z))^2 - (r_{X, l}(Z) \cdot v_{(X, \hat{s})}^{lc}(Z))^2| \\ &+ ||v_{(X, \hat{s})}^c(Z)|_3^2 - |v_{(\hat{Y}, \hat{s})}^c(Z)|_3^2|\} \end{aligned}$$

$$\begin{aligned}
 & \leq |W_{(\hat{Y}, \hat{s})}(Z) - W_{(X, \hat{s})}(Z)|_3 \\
 & \quad + \frac{1}{2\{1-(c^*/c)^2\}^{1/2}} \cdot \{2(c^*/c) + 2(c^*/c)\} \cdot |v_{(X, \hat{s})}^c(Z) - v_{(\hat{Y}, \hat{s})}^c(Z)|_3 \\
 & \leq c_7^* \cdot |[\chi_{,4}^c]_{(X, \hat{s})} \circ x_{\hat{s}}^{-1}(Z) - [\chi_{,4}^c]_{(\hat{Y}, \hat{s})} \circ x_{\hat{s}}^{-1}(Z)|_3 \\
 & \quad + \{c_6^* + 2(c^*/c) \cdot \{1-(c^*/c)^2\}^{-1/2}\} \cdot |v_{(X, \hat{s})}^c(Z) - v_{(\hat{Y}, \hat{s})}^c(Z)|_3 \\
 & \leq \{c_6^* + c_7^* + 2(c^*/c) \cdot \{1-(c^*/c)^2\}^{-1/2}\} \cdot \frac{\hat{A} \cdot \hat{\alpha}}{(c-c^*)^{\hat{\alpha}}} \cdot \delta^{\hat{\alpha}} \\
 & < \{c_6^* + c_7^* + 2(c^*/c) \cdot \{1-(c^*/c)^2\}^{-1/2}\} \cdot \frac{\hat{A} \cdot \hat{\alpha} \cdot \Delta_K}{(c-c^*)^{\hat{\alpha}}},
 \end{aligned} \tag{151}$$

having used (10) and (143)-(145). We may, and shall, suppose that the positive number Δ_K is chosen so that

$$\{c_6^* + c_7^* + 2(c^*/c) \cdot \{1-(c^*/c)^2\}^{-1/2}\} \cdot \frac{\hat{A} \cdot \hat{\alpha} \cdot \Delta_K}{(c-c^*)^{\hat{\alpha}}} \leq \frac{1}{2} c_1^*, \tag{152}$$

with c_6^* , c_7^* , and c_1^* given by (146), (147), and (149), respectively; clearly, Δ_K depends upon M and K alone. Using (151) and (152), and accounting for (148), in which we take $\tilde{s} = \hat{s}$ and $\tilde{X} = X$, by simply noting that $a = b + (a-b) \geq b - |a-b|$ for $a, b \in \mathbb{R}$, we find

$$W_{(\hat{Y}, \hat{s})}^k(Z) \cdot r_{X,k}(Z) + \{(r_{X,k}(Z) \cdot v_{(\hat{Y}, \hat{s})}^k(Z))^2 + (1 - |v_{(\hat{Y}, \hat{s})}^c(Z)|_3^2)\}^{1/2} \geq \frac{1}{2} c_1^* \tag{153}$$

whenever $X \in [\hat{Y}, x_{\hat{s}\hat{s}}(Y)]$ and $Z \in \mathcal{B}_{\hat{s}} \cap \{X\}'$.

In particular, choosing $X = x_{\hat{s}\hat{s}}(Y)$ in (153), we find that

$\Gamma_{(Y,s;\hat{Y},\hat{s})}^{03}$ is well-defined on $\partial B_{\hat{s}} \cap \{X_{\hat{s}\hat{s}}(Y)\}'$ by (49). An upper bound for the expression appearing in (153) is easily obtained by noting that

$$|W_{(\tilde{X},\tilde{s})}(Z)|_3 \leq \{1-(c^*/c)^2\}^{-1} \cdot (c^*/c) \cdot \{2+(c^*/c)^2\} \quad (154)$$

whenever $(\tilde{X},\tilde{s}) \in \mathbb{R}^4$ and $Z \in \partial B_{\tilde{s}},$

whence

$$W_{(\hat{Y},\hat{s})}^k(Z) \cdot r_{X,k}(Z) + \{(r_{X,\ell}(Z) \cdot v_{(\hat{Y},\hat{s})}^{\ell c}(Z))^2 + (1 - |v_{(\hat{Y},\hat{s})}^c(Z)|_3^2)^{1/2}\} \leq c_2^* \quad (155)$$

for $X \in [\hat{Y}, X_{\hat{s}\hat{s}}(Y)]$ and $Z \in \partial B_{\hat{s}} \cap \{X\}'.$

Now, to prepare an appropriate estimate for the difference appearing in the integrand in (56), we begin by using (148), (153), (154), and (155) to write, for each $Z \in \partial B_s \cap \{Y\}'$,

$$\begin{aligned} & |\Gamma_{(Y,s;\hat{Y},\hat{s})}^{03} \circ X_{\hat{s}\hat{s}}(Z) - \Gamma_{(Y,s)}^{03}(Z)| \\ &= |\{W_{(\hat{Y},\hat{s})}^k \cdot r_{X_{\hat{s}\hat{s}}}(Y), k + \{(r_{X_{\hat{s}\hat{s}}}(Y), \ell \cdot v_{(\hat{Y},\hat{s})}^{\ell c})^2 + (1 - |v_{(\hat{Y},\hat{s})}^c|_3^2)^{1/2}\}^{-3} \circ X_{\hat{s}\hat{s}}(Z) \\ &\quad - \{W_{(Y,s)}^k(Z) \cdot r_{Y,k}(Z) + \{(r_{Y,\ell}(Z) \cdot v_{(Y,s)}^{\ell c}(Z))^2 + (1 - |v_{(Y,s)}^c(Z)|_3^2)^{1/2}\}^{-3} \\ &\leq \frac{3(c_2^*)^2}{\frac{1}{8}(c_1^*)^6} \cdot |\{W_{(\hat{Y},\hat{s})}^k \cdot r_{X_{\hat{s}\hat{s}}}(Y), k + \{(r_{X_{\hat{s}\hat{s}}}(Y), \ell \cdot v_{(\hat{Y},\hat{s})}^{\ell c})^2 \\ &\quad + (1 - |v_{(\hat{Y},\hat{s})}^c|_3^2)^{1/2}\} \circ X_{\hat{s}\hat{s}}(Z) - \{W_{(Y,s)}^k(Z) \cdot r_{Y,k}(Z) \\ &\quad + \{(r_{Y,\ell}(Z) \cdot v_{(Y,s)}^{\ell c}(Z))^2 + (1 - |v_{(Y,s)}^c(Z)|_3^2)^{1/2}\} \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{24(c_2^*)^2}{(c_1^*)^6} \cdot \{ |W_{(\hat{Y}, \hat{s})}^k \cdot r_{\chi_{ss}}(Y), k \} \circ \chi_{ss}(Z) - W_{(Y, s)}^k(Z) \cdot r_{Y, k}(Z) | \\
 & + \frac{1}{2} \{ 1 - (c^*/c)^2 \}^{-1/2} \cdot \{ | (r_{\chi_{ss}}(Y), l \cdot v_{(\hat{Y}, \hat{s})}^l)^2 \circ \chi_{ss}(Z) \\
 & - (r_{Y, l}(Z) \cdot v_{(Y, s)}^l(Z))^2 | + | v_{(Y, s)}^c(Z) |_3^2 - | v_{(\hat{Y}, \hat{s})}^c \circ \chi_{ss}(Z) |_3^2 | \} \\
 & \leq \frac{24(c_2^*)^2}{(c_1^*)^2} \cdot \{ |W_{(\hat{Y}, \hat{s})} \circ \chi_{ss}(Z) - W_{(Y, s)}(Z) |_3 \\
 & + c_2^* \cdot | \text{grad } r_{\chi_{ss}}(Y)(\chi_{ss}(Z)) - \text{grad } r_Y(Z) |_3 + \frac{1}{2} \{ 1 - (c^*/c)^2 \}^{-1/2} \cdot 2(c^*/c) \\
 & \cdot \{ (c^*/c) \cdot | \text{grad } r_{\chi_{ss}}(Y)(\chi_{ss}(Z)) - \text{grad } r_Y(Z) |_3 \\
 & + 2 \cdot | v_{(\hat{Y}, \hat{s})}^c \circ \chi_{ss}(Z) - v_{(Y, s)}^c(Z) |_3 \} \\
 & \leq c_8^* \{ |W_{(\hat{Y}, \hat{s})} \circ \chi_{ss}(Z) - W_{(Y, s)}(Z) |_3 + | v_{(\hat{Y}, \hat{s})}^c \circ \chi_{ss}(Z) - v_{(Y, s)}^c(Z) |_3 \\
 & + | \text{grad } r_{\chi_{ss}}(Y)(\chi_{ss}(Z)) - \text{grad } r_Y(Z) |_3 \} ,
 \end{aligned}
 \tag{156}$$

for a certain number c_8^* depending upon only the ratio c^*/c . It is easy to show that there exist c_9^* and c_{10}^* , also depending on only c^*/c , for which

$$\begin{aligned}
 & |W_{(\hat{Y}, \hat{s})} \circ \chi_{ss}(Z) - W_{(Y, s)}(Z) |_3 \\
 & \leq c_9^* \cdot | [X_{, 4}^c]_{(\hat{Y}, \hat{s})} \circ \chi_{ss}^{-1}(\chi_{ss}(Z)) - [X_{, 4}^c]_{(Y, s)} \circ \chi_s^{-1}(Z) |_3 \\
 & + c_{10}^* \cdot | v_{(\hat{Y}, \hat{s})}^c(\chi_{ss}(Z)) - v_{(Y, s)}^c(Z) |_3 , \quad \text{for } Z \in \partial \delta_s .
 \end{aligned}
 \tag{157}$$

Following the derivation of (IV.24.47), we can use (76) and (101) to produce

$$|(\chi_{\hat{s}\hat{s}}(Z)-Z)-(\chi_{\hat{s}\hat{s}}(Y)-Y)|_3 \leq c\hat{A}\hat{A}_0^{\hat{\alpha}} \cdot r_Y^{\hat{\alpha}}(Z) \cdot \delta \quad \text{for each } Z \in \partial B_s. \quad (158)$$

In turn, (158) allows us to write

$$\begin{aligned} & |\text{grad } r_{\chi_{\hat{s}\hat{s}}(Y)}(\chi_{\hat{s}\hat{s}}(Z)) - \text{grad } r_Y(Z)|_3 \\ &= \left| \frac{1}{r_Y(Z)} \{(\chi_{\hat{s}\hat{s}}(Z)-\chi_{\hat{s}\hat{s}}(Y))-(Z-Y)\} \right. \\ & \quad \left. + \left\{ \frac{1}{r_{\chi_{\hat{s}\hat{s}}(Y)}(\chi_{\hat{s}\hat{s}}(Z))} - \frac{1}{r_Y(Z)} \right\} \cdot \{\chi_{\hat{s}\hat{s}}(Z)-\chi_{\hat{s}\hat{s}}(Y)\} \right|_3 \\ &\leq \frac{2}{r_Y(Z)} \cdot |(\chi_{\hat{s}\hat{s}}(Z)-Z)-(\chi_{\hat{s}\hat{s}}(Y)-Y)|_3 \leq \frac{2c\hat{A}\hat{A}_0^{\hat{\alpha}}}{r_Y^{1-\hat{\alpha}}(Z)} \cdot \delta \end{aligned} \quad (159)$$

$$\text{for } Z \in \partial B_s \cap \{Y\}'.$$

By combining (156), (157), and (159), and recalling inequalities (77) and (78), we obtain

$$\begin{aligned} & |\hat{r}_{(Y,s;\hat{Y},\hat{s})}^{03}(\chi_{\hat{s}\hat{s}}(Z)) - \hat{r}_{(Y,s)}^{03}(Z)| \\ &\leq c_8^* \{c_9^* \cdot |[\chi_{,4}^c]_{(\hat{Y},\hat{s})} \circ \chi_{\hat{s}}^{-1}(\chi_{\hat{s}\hat{s}}(Z)) - [\chi_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Z)|_3 \\ & \quad + (1+c_{10}^*) \cdot |v_{(\hat{Y},\hat{s})}^c(\chi_{\hat{s}\hat{s}}(Z)) - v_{(Y,s)}^c(Z)|_3 \\ & \quad + |\text{grad } r_{\chi_{\hat{s}\hat{s}}(Y)}(\chi_{\hat{s}\hat{s}}(Z)) - \text{grad } r_Y(Z)|_3\} \\ &\leq c_8^* (1+c_9^*+c_{10}^*) \cdot \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1+c^2\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}} + \frac{2c_8^* c\hat{A}\hat{A}_0^{\hat{\alpha}}}{r_Y^{1-\hat{\alpha}}(Z)} \cdot \delta \\ &= k_7 \cdot \delta^{\hat{\alpha}} + \frac{k_8}{r_Y^{1-\hat{\alpha}}(Z)} \cdot \delta \quad \text{for each } Z \in \partial B_s \cap \{Y\}', \end{aligned} \quad (160)$$

the positive numbers k_7 and k_8 depending on M and K alone.

Finally, having (160), we can derive an estimate of the desired form for I_{29} : from (56),

$$\begin{aligned}
 I_{29} &\leq 2 \cdot (m_j^K)^{-1} \cdot \int_{C_d^\partial(Y,s) \cap C_{3,1,\delta}^\partial(Y,s)'} \frac{1}{r_Y} \cdot \left\{ k_7 \delta^{\hat{\alpha}} + \frac{k_8}{r_Y^{1-\hat{\alpha}}} \cdot \delta \right\} d\lambda_{\partial B_s} \\
 &\leq 2^{5/2} \pi \cdot (m_j^K)^{-1} \cdot \left\{ k_7 \cdot M_5 \delta^{\hat{\alpha}'} + k_8 \cdot \delta \int_{3,1,\delta}^d \frac{1}{\zeta^{2-\hat{\alpha}}} d\zeta \right\}, \quad (161)
 \end{aligned}$$

wherein $\hat{\alpha}' \in (0, \hat{\alpha})$ and M_5 is a sufficiently large positive number. The remaining integral in (161) can be analyzed by considering the two cases $\hat{\alpha} \in (0, 1)$ or $\hat{\alpha} = 1$, leading to a term involving, respectively, $\delta^{\hat{\alpha}}$ or $\delta^{\alpha'}$, where $\alpha' \in (0, 1)$ (cf., the similar calculation in the examination of I_6). In any event,

$$I_{29} \leq k_9 \delta^{\hat{\alpha}'}, \quad (162)$$

k_9 depending upon only M and K .

I_{30} : To examine the difference appearing in the integrand of (57), choose $Z \in C_d^\partial(Y,s) \cap C_{3,1,\delta}^\partial(Y,s)'$, from which $|\pi_Y(Z) - Y|_3 > 0$ and

$$\begin{aligned}
 |\pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)) - x_{\hat{s}\hat{s}}(Y)|_3 &\geq |\pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)) - \hat{Y}|_3 - |\hat{Y} - x_{\hat{s}\hat{s}}(Y)|_3 \\
 &> \frac{7}{9} |x_{\hat{s}\hat{s}}(Z) - \hat{Y}|_{3-1\delta} \\
 &\geq \frac{7}{9} \{|Z - Y|_3 - |Y - \hat{Y}|_3 - |x_{\hat{s}\hat{s}}(Z) - Z|_3\}_{-1\delta} \\
 &> \frac{7}{9} \{3\delta - 1\delta\}_{-1\delta} > 0.
 \end{aligned}$$

Thus, using (76), (101), and (159),

$$\begin{aligned}
 & |\Gamma_Y\{x_{,4}^c(x_s^{-1}(Y), s)\} \circ \Pi_Y(Z) - \Gamma_{X_{s\hat{s}}(Y)}\{x_{,4}^c(x_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\} \circ \Pi_{\hat{Y}} \circ x_{s\hat{s}}(Z)| \\
 &= | \{ (r_{Y,k}(\Pi_Y(Z)) \cdot x_{,4}^c(x_s^{-1}(Y), s))^2 + (1 - |x_{,4}^c(x_s^{-1}(Y), s)|_3^2) \}^{-3/2} \\
 &\quad - \{ (r_{X_{s\hat{s}},k}(\Pi_{\hat{Y}}(x_{s\hat{s}}(Z))) \cdot x_{,4}^c(x_{\hat{s}}^{-1}(\hat{Y}), \hat{s}))^2 + (1 - |x_{,4}^c(x_{\hat{s}}^{-1}(\hat{Y}), \hat{s})|_3^2) \}^{-3/2} | \\
 &\leq \frac{3}{2} \{1 - (c^*/c)^2\}^{-7/2} \cdot \{4(c^*/c) \cdot |x_{,4}^c(x_{\hat{s}}^{-1}(\hat{Y}), \hat{s}) - x_{,4}^c(x_s^{-1}(Y), s)|_3 \\
 &\quad + 2 \cdot (c^*/c)^2 \cdot |\text{grad } r_{X_{s\hat{s}}(Y)}(x_{s\hat{s}}(Z)) - \text{grad } r_Y(Z)|_3\} \\
 &\leq 6 \cdot (c^*/c) \cdot \{1 - (c^*/c)^2\}^{-7/2} \cdot \hat{A} \{A_0^2 + 1\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}} \\
 &\quad + 3 \cdot (c^*/c)^2 \cdot \{1 - (c^*/c)^2\}^{-7/2} \cdot 2c\hat{A} \cdot A_0^{\hat{\alpha}} \cdot \frac{\delta}{r_Y^{1-\hat{\alpha}}(Z)} \\
 &= k_{10} \delta^{\hat{\alpha}} + k_{11} \cdot \frac{\delta}{r_Y^{1-\hat{\alpha}}(Z)} \quad \text{for each } Z \in C_d^{\partial}(Y, s) \cap C_{3, \delta}^{\partial}(Y, s)',
 \end{aligned} \tag{163}$$

k_{10} and k_{11} depending upon only M and K . Applying the latter relation in conjunction with (57),

$$\begin{aligned}
 I_{30} &\leq 2 \cdot (m_j^K)^{-1} \cdot \int_{C_d^{\partial}(Y, s) \cap C_{3, \delta}^{\partial}(Y, s)'} \frac{1}{r_Y^{\partial} \Pi_Y} \cdot \left\{ k_{10} \delta^{\hat{\alpha}} + k_{11} \cdot \frac{\delta}{r_Y^{1-\hat{\alpha}}} \right\} d\lambda_{\partial B_s} \\
 &\leq 2 \cdot (m_j^K)^{-1} \cdot \int_{C_d^{\partial}(Y, s) \cap C_{3, \delta}^{\partial}(Y, s)'} \left\{ \frac{k_{10} \cdot \delta^{\hat{\alpha}}}{r_Y^{\partial} \Pi_Y} + \frac{k_{11} \cdot \delta}{r_Y^{3-\hat{\alpha}} \Pi_Y} \right\} d\lambda_{\partial B_s} \\
 &\leq 2^{5/2} \pi \cdot (m_j^K)^{-1} \left\{ k_{10} \cdot M_5 \delta^{\hat{\alpha}'} + k_{11} \delta \int_{3, \delta}^d \frac{1}{\zeta^{2-\hat{\alpha}}} d\zeta \right\} \\
 &\leq k_{12} \cdot \delta^{\hat{\alpha}'},
 \end{aligned} \tag{164}$$

having selected $\hat{\alpha}' \in (0, \hat{\alpha})$ and appealed to the reasoning which

produced (162) from (161); here, k_{12} depends on only M and K , of course.

I₃₁: Here, we can write

$$\frac{\partial}{\partial s} JX_s^{-1} \cdot \frac{\partial}{\partial s} T_s^{1q}(Y) \cdot JX_s^{-1}(Y) = \left\{ \frac{\partial}{\partial s} T_s^{1q} - \frac{\partial}{\partial s} T_s^{1q}(Y) \right\} \cdot JX_s^{-1} + T_s^{1q}(Y) \cdot \{ JX_s^{-1} - JX_s^{-1}(Y) \}, \quad (165)$$

apply (85) and (103), and use the type of arguments adduced in (69) and (97) to produce

$$\begin{aligned} I_{31} &\leq (m_j^K)^{-1} \cdot M^{03} \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \left| \left\{ \left(\frac{1}{2} r_{\hat{Y},q} \right)^{\alpha_{\hat{Y},q}} \cdot \frac{1}{r_Y} r_{Y,q} \right\} \cdot \left\{ \frac{\partial}{\partial s} T_s^{1q} - \frac{\partial}{\partial s} T_s^{1q}(Y) \right\} \right| d\lambda_{\partial B_s} \\ &\quad + M^{03} \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \left| \left\{ \left(\frac{1}{2} r_{\hat{Y},q} \right)^{\alpha_{\hat{Y},q}} \cdot \frac{1}{r_Y} r_{Y,q} \right\} \cdot T_s^{1q}(Y) \right| d\lambda_{\partial B_s} \\ &\quad \cdot |JX_s^{-1} - JX_s^{-1}(Y)| d\lambda_{\partial B_s} \\ &\leq (m_j^K)^{-1} \cdot M^{03} \cdot 2a_K \cdot 8\delta \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \frac{1}{r_Y^{3-\alpha_K}} d\lambda_{\partial B_s} \\ &\quad + M^{03} \cdot (m_j^K)^{-2} \cdot \tilde{a}_0 \cdot 2 \cdot 8\delta \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)} \frac{1}{r_Y^{3-\tilde{\alpha}}} d\lambda_{\partial B_s}; \end{aligned}$$

from this point, as in the previous calculations, we can show that I_{31} satisfies an inequality of the form of (68) by considering the four possible cases based on the values of α_K and $\tilde{\alpha}$: both α_K and $\tilde{\alpha} \in (0,1)$, $\alpha_K = 1$ and $\tilde{\alpha} \in (0,1)$, etc. For example,

if $\alpha_K \in (0,1)$ and $\tilde{\alpha} = 1$, then

$$\begin{aligned}
 I_{31} &\leq 16 \cdot 2^{5/2} \pi_1 \cdot (m_j^K)^{-1} \cdot M^{03} \cdot \left\{ a_K^\delta \int_{31\delta}^d \frac{1}{2-\alpha_K} d\zeta \right. \\
 &\quad \left. + (m_j^K)^{-1} \cdot \tilde{A} \tilde{A}_0^{\tilde{\alpha}} \cdot \delta \int_{31\delta}^d \frac{1}{\zeta} d\zeta \right\} \\
 &\leq 16 \cdot 2^{5/2} \pi_1 \cdot (m_j^K)^{-1} \cdot M^{03} \cdot \left\{ \frac{a_K}{1-\alpha_K} \cdot (31)^{\alpha_K-1} \cdot \delta^{\alpha_K} + (m_j^K)^{-1} \cdot \tilde{A} \tilde{A}_0^{\tilde{\alpha}} \cdot M_6 \delta^{\alpha'} \right\} \\
 &\leq k_{13} \cdot \delta^{\alpha_K},
 \end{aligned} \tag{166}$$

wherein k_{13} depends upon M and K only, $\alpha' \in (\alpha_K, 1)$, and M_6 is sufficiently large.

I₃₂: Suppose that $Z \in \partial B_s \cap \{\hat{Y}\}' \cap \{X_{s\hat{s}}(Y)\}'$; for example, if $Z = X_{s\hat{s}}(\tilde{Z})$ for some $\tilde{Z} \in \partial B_s \cap C_{31\delta}^{\partial}(Y, s)'$, it is easy to check that this inclusion holds. Then, using (143), (150), and (153)-(155), it is evident that

$$\begin{aligned}
 &|\Gamma_{(Y,s;\hat{Y},\hat{s})}^{03}(Z) - \Gamma_{(\hat{Y},\hat{s})}^{03}(Z)| \\
 &= |\{W_{(\hat{Y},\hat{s})}^k(Z) \cdot r_{X_{s\hat{s}}(Y),k}(Z) + \{(r_{X_{s\hat{s}}(Y),\ell}(Z) \cdot v_{(\hat{Y},\hat{s})}^{\ell^c}(Z))\}^2 \\
 &\quad + (1 - |v_{(\hat{Y},\hat{s})}^c(Z)|^2)^{1/2}\}^{-3} - \{W_{(\hat{Y},\hat{s})}^k(Z) \cdot r_{\hat{Y},k}(Z) \\
 &\quad + \{(r_{\hat{Y},\ell}(Z) \cdot v_{(\hat{Y},\hat{s})}^{\ell^c}(Z))\}^2 + (1 - |v_{(\hat{Y},\hat{s})}^c(Z)|^2)^{1/2}\}^{-3}|
 \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{3(c_2^*)^2}{8(c_1^*)^6} \cdot \{|w_{(\hat{Y}, \hat{s})}^k(Z) \cdot \{r_{\chi_{ss}(Y), k}(Z) - r_{\hat{Y}, k}(Z)\}| \\
 & + \frac{1}{2} \{1 - (c^*/c)^2\}^{-1/2} \cdot |(r_{\chi_{ss}(Y), l}(Z) \cdot v_{(\hat{Y}, \hat{s})}^{lc}(Z))^2 \\
 & - (r_{\hat{Y}, l}(Z) \cdot v_{(\hat{Y}, \hat{s})}^{lc}(Z))^2| \} \\
 & \leq \frac{24(c_2^*)^2}{(c_1^*)^6} \cdot \{c_2^* + (c^*/c) \cdot \{1 - (c^*/c)^2\}^{-1/2}\} \\
 & \cdot |\text{grad } r_{\chi_{ss}(Y)}(Z) - \text{grad } r_{\hat{Y}}(Z)|_3 \\
 & = c_{11}^* \cdot |\text{grad } r_{\chi_{ss}(Y)}(Z) - \text{grad } r_{\hat{Y}}(Z)|_3.
 \end{aligned} \tag{167}$$

Reasoning in the usual manner, we find

$$\begin{aligned}
 |\text{grad } r_{\chi_{ss}(Y)}(Z) - \text{grad } r_{\hat{Y}}(Z)|_3 &= \left| \frac{1}{r_{\chi_{ss}(Y)}(Z)} (Z - \chi_{ss}(Y)) - \frac{1}{r_{\hat{Y}}(Z)} (Z - \hat{Y}) \right|_3 \\
 &\leq \frac{2}{r_{\hat{Y}}(Z)} \cdot |\hat{Y} - \chi_{ss}(Y)|_3 \leq \frac{2\delta}{r_{\hat{Y}}(Z)}
 \end{aligned} \tag{168}$$

$$\text{for } Z \in \partial B_s \cap \{\hat{Y}\}' \cap \{\chi_{ss}(Y)\}'.$$

Thus,

$$|\Gamma_{(Y, s; \hat{Y}, \hat{s})}^{03}(Z) - \Gamma_{(\hat{Y}, \hat{s})}^{03}(Z)| \leq \frac{2c_{11}^*}{r_{\hat{Y}}(Z)} \cdot \delta \tag{169}$$

$$\text{for } Z \in \partial B_s \cap \{\hat{Y}\}' \cap \{\chi_{ss}(Y)\}',$$

c_{11}^* depending on only the ratio c^*/c . Applying this result along with (84), (97), (103), and (165) in (59) leads to

$$\begin{aligned}
 I_{32} &\leq 2c_{11}^* \delta \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)'} \frac{1}{r_Y^2 \cdot r_Y^{\alpha_X} s \hat{s}} \\
 &\quad \cdot \left\{ (m_j^K)^{-1} \cdot |r_{Y,q} \{T_s^{Yiq} - T_s^{Yiq}(Y)\}| + |r_{Y,q} \cdot T_s^{Yiq}(Y)| \right. \\
 &\quad \left. \cdot |J\chi_s^{-1} - J\chi_s^{-1}(Y)| \right\} d\lambda_{\partial B_s} \quad (170) \\
 &\leq 2c_{11}^* \delta \cdot \frac{3}{2} \left\{ (m_j^K)^{-1} \cdot 2a_K \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)'} \frac{1}{r_Y^{3-\alpha_K}} d\lambda_{\partial B_s} \right. \\
 &\quad \left. + 2 \cdot (m_j^K)^{-2} \cdot \bar{A} \bar{A}_0^{\bar{\alpha}} \cdot \int_{C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)'} \frac{1}{r_Y^{3-\bar{\alpha}}} d\lambda_{\partial B_s} \right\}.
 \end{aligned}$$

From (170), it follows, just as in the examination of I_{31} , that I_{32} satisfies an inequality of the required form (68).

I_{33} : Obviously, appealing to (6) once again,

$$\begin{aligned}
 |1 - v^j(Y,s) v^j(Z,s)| &= |1 - v^j(Y,s) v^j(Z,s)| \\
 &\leq \frac{1}{2} |v(Z,s) - v(Y,s)|^2 \\
 &\leq \frac{1}{2} a_K^2 r_Y^{2\alpha_K}(Z) \quad (171)
 \end{aligned}$$

for each $Z \in \partial B_s$.

Next, suppose that $Z \in C_d^\partial(Y,s) \cap C_{3,\delta}^\partial(Y,s)'$; as in the analysis of I_{30} , it is then easy to see that $\Pi_Y(Z) \neq Y$ and $\Pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)) \neq \hat{Y}$. With (130), we find

$$\begin{aligned}
 & \left| \frac{1}{r_Y^3(\pi_Y(\chi_{ss}(Z)))} \{ \pi_Y(\chi_{ss}(Z)) - \hat{Y} \} - \frac{1}{r_Y^3(\pi_Y(Z))} \{ \pi_Y(Z) - Y \} \right|_3 \\
 & \leq \frac{1}{r_Y^3(\pi_Y(Z))} \cdot | \{ \pi_Y(\chi_{ss}(Z)) - \hat{Y} \} - \{ \pi_Y(Z) - Y \} |_3 \\
 & \quad + \left| \frac{1}{r_Y^3(\pi_Y(\chi_{ss}(Z)))} - \frac{1}{r_Y^3(\pi_Y(Z))} \right| \cdot | \pi_Y(\chi_{ss}(Z)) - \hat{Y} |_3 \\
 & \leq \left\{ 2 + \frac{27}{14} + \left(\frac{27}{14} \right)^2 \right\} \cdot \frac{1}{r_Y^3(\pi_Y(Z))} \cdot | \{ \pi_Y(\chi_{ss}(Z)) - \hat{Y} \} - \{ \pi_Y(Z) - Y \} |_3.
 \end{aligned} \tag{172}$$

To estimate the difference appearing on the right in (172), we recall (112), which gives, with (6), (I.2.37.7), and (158), supposing first that $\hat{Y} \neq \chi_{ss}(Y)$,

$$\begin{aligned}
 & | \{ \pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Z) \} - \{ \pi_Y(Z) - Z \} |_3 \\
 & \leq 2r_Y(Z) \cdot | v(\hat{Y}, \hat{s}) - v(Y, s) |_3 + | \{ \chi_{ss}(Z) - Z \} - \{ \chi_{ss}(Y) - Y \} |_3 \\
 & \quad + | \hat{Y} - \chi_{ss}(Y) |_3 \cdot | v^k(\hat{Y}, \hat{s}) \cdot r_{\chi_{ss}(Y), k}(\hat{Y}) | \\
 & \leq 2a_K r_Y(Z) \cdot \delta^{\alpha_K + c\hat{A}\hat{A}_0} \cdot r_Y^{\hat{A}}(Z) \cdot \delta + \hat{a}_K \cdot | \hat{Y} - \chi_{ss}(Y) |_3^{1+\alpha_K} \\
 & \leq 2a_K r_Y(Z) \cdot \delta^{\alpha_K + c\hat{A}\hat{A}_0} \cdot r_Y^{\hat{A}}(Z) \cdot \delta + \hat{a}_K \cdot \delta^{1+\alpha_K} \cdot \delta^{1+\alpha_K}
 \end{aligned} \tag{173}$$

for each $Z \in C_d^{\partial}(Y, s)$,

in which $\hat{a}_K$ depends on a_K and α_K only (cf., [I.2.37.iii.4]).

The application of (I.2.37.7) to estimate $| v^k(\hat{Y}, \hat{s}) \cdot r_{\chi_{ss}(Y), k}(\hat{Y}) |$ is legitimate, since $| \hat{Y} - \chi_{ss}(Y) |_3 \leq \delta < d_K$. It is clear that the resultant inequality in (173) is valid even if $\hat{Y} = \chi_{ss}(Y)$. From (173),

$$\begin{aligned}
 & \left| \{ \pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - \hat{Y} \} - \{ \pi_Y(Z) - Y \} \right|_3 \\
 & \leq \left| \{ \pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - \chi_{\hat{s}\hat{s}}(Z) \} - \{ \pi_Y(Z) - Z \} \right|_3 + \left| (Y - \hat{Y}) + (\chi_{\hat{s}\hat{s}}(Z) - Z) \right|_3 \\
 & \leq 2a_K \cdot r_Y(Z) \cdot \delta^{\alpha_K} + c_{AA} \hat{a}_0 \cdot r_Y^{\hat{a}}(Z) \cdot \delta + \hat{a}_K \cdot 1^{1+\alpha_K} \delta^{1+\alpha_K} \\
 & \leq k_{14} \{ r_Y(Z) \cdot \delta^{\alpha_K} \} \quad \text{for each } Z \in C_d^{\partial}(Y, s),
 \end{aligned} \tag{174}$$

k_{14} depending on only M and K , with which we can complete the computation begun in (172):

$$\begin{aligned}
 & \left| \frac{1}{r_Y^3(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)))} \{ \pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - \hat{Y} \} - \frac{1}{r_Y^3(\pi_Y(Z))} \{ \pi_Y(Z) - Y \} \right|_3 \\
 & \leq \frac{8k_{14}}{r_Y^3(\pi_Y(Z))} \{ r_Y(Z) \cdot \delta^{\alpha_K} \} \\
 & < 8k_{14} \left\{ \frac{9/7}{r_Y^2(\pi_Y(Z))} \cdot \delta^{\alpha_K} + \frac{1}{r_Y^3(\pi_Y(Z))} \cdot \delta \right\} \\
 & \quad \text{for } Z \in C_d^{\partial}(Y, s) \cap C_{3, \delta}^{\partial}(Y, s)'.
 \end{aligned} \tag{175}$$

Finally, returning to (60), (171) and (175) show that

$$\begin{aligned}
 I_{33} & \leq \{ 1 - (c^*/c)^2 \}^{-3/2} \cdot (m_j^K)^{-1} \cdot 2 \cdot 8k_{14} \cdot \frac{1}{2} a_K^2 \\
 & \quad \cdot \left\{ \frac{9}{7} \delta^{\alpha_K} \cdot \int_{C_d^{\partial}(Y, s) \cap C_{3, \delta}^{\partial}(Y, s)'} \frac{1}{r_Y^2 \circ \pi_Y} \cdot r_Y^{2\alpha_K} d\lambda_{\partial \mathcal{B}_s} \right. \\
 & \quad \left. + \delta \int_{C_d^{\partial}(Y, s) \cap C_{3, \delta}^{\partial}(Y, s)'} \frac{1}{r_Y^3 \circ \pi_Y} \cdot r_Y^{2\alpha_K} d\lambda_{\partial \mathcal{B}_s} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \leq k_{15} \cdot \left(\frac{9}{7}\right)^{2\alpha_K} \cdot \left\{ \frac{9}{7} \delta^{\alpha_K} \cdot \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)'} \frac{1}{r_Y^{2-2\alpha_K} \pi_Y} d\lambda_{\partial B_s} \right. \\
 & \quad \left. + \delta \int_{C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)'} \frac{1}{r_Y^{3-2\alpha_K} \pi_Y} d\lambda_{\partial B_s} \right\} \\
 & \leq k_{16} \left\{ \delta^{\alpha_K} \cdot \int_{31\delta}^d \frac{1}{1-2\alpha_K} d\zeta + \delta \int_{31\delta}^d \frac{1}{2-2\alpha_K} d\zeta \right\},
 \end{aligned} \tag{176}$$

k_{15} and k_{16} depending upon M and K alone. Clearly, we can derive from (176) an estimate for I_{33} of the form of (68).

I₃₄: Let $Z \in C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)'$: then it is easy to show that $\pi_Y(\chi_{s\hat{s}}(Z))$ can equal neither $\hat{Y}$ nor $\chi_{s\hat{s}}(Y)$ and, using (130) after reasoning in a familiar manner,

$$\begin{aligned}
 & |\text{grad } r_{\hat{Y}}(\pi_Y(\chi_{s\hat{s}}(Z))) - \text{grad } r_{\chi_{s\hat{s}}(Y)}(\pi_Y(\chi_{s\hat{s}}(Z)))|_3 \\
 & \leq \frac{2}{r_{\hat{Y}}(\pi_Y(\chi_{s\hat{s}}(Z)))} \cdot |\hat{Y} - \chi_{s\hat{s}}(Y)|_3 \\
 & < \frac{27}{7} \cdot \frac{1}{r_Y(\pi_Y(Z))} \cdot \{|\hat{Y} - Y|_3 + |Y - \chi_{s\hat{s}}(Y)|_3\} \\
 & < \frac{41\delta}{r_Y(\pi_Y(Z))}.
 \end{aligned} \tag{177}$$

Consequently, proceeding as in the derivation of (IV.22.24),

$$\begin{aligned}
 & |\Gamma_{\hat{Y}}\{\chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\}(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))) - \Gamma_{\chi_{\hat{s}\hat{s}}(Y)}\{\chi_{,4}^c(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s})\}(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)))| \\
 & \leq 3 \cdot (c^*/c)^2 \cdot \{1 - (c^*/c)^2\}^{-7/2} \\
 & \quad \cdot |\text{grad } r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))) - \text{grad } r_{\chi_{\hat{s}\hat{s}}(Y)}(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)))|_3 \\
 & < 121 \cdot (c^*/c) \cdot \{1 - (c^*/c)^2\}^{-7/2} \cdot \frac{1}{r_Y(\pi_Y(Z))} \cdot \delta \\
 & = \frac{k_{17}}{r_Y(\pi_Y(Z))} \cdot \delta
 \end{aligned} \tag{178}$$

for each $Z \in C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)'$.

Directly from (61), using (178) and recalling (171), we obtain

$$\begin{aligned}
 I_{34} & \leq 2 \cdot (m_j^K)^{-1} \cdot \frac{1}{2} \cdot a_K^2 \cdot k_{17} \cdot \delta \cdot \int_{C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)'} \frac{1}{r_Y^3 \circ \pi_Y} \cdot r_Y^{2\alpha_K} d\lambda_{\partial B_s} \\
 & < \left(\frac{9}{7}\right)^{2\alpha_K} \cdot a_K^2 \cdot k_{17} \cdot (m_j^K)^{-1} \cdot \delta \cdot \int_{C_d^\partial(Y, s) \cap C_{31\delta}^\partial(Y, s)'} \frac{1}{r_Y^{3-2\alpha_K} \circ \pi_Y} d\lambda_{\partial B_s},
 \end{aligned} \tag{179}$$

whence an inequality of the required form can be obtained for I_{34} , via computations of the sort outlined previously.

I₃₅: If $Z \in \partial B_s \cap C_{31\delta}^\partial(Y, s)'$, then $r_Y(Z) \geq 31\delta$ and

$$\begin{aligned}
 |r_{\chi_{\hat{s}\hat{s}}(Y)}(\chi_{\hat{s}\hat{s}}(Z)) - r_Y(Z)| & \leq |(\chi_{\hat{s}\hat{s}}(Z) - Z) - (\chi_{\hat{s}\hat{s}}(Y) - Y)|_3 \\
 & \leq 2c^* \cdot |\hat{s} - s| < 21\delta,
 \end{aligned}$$

whence

$$\left| \frac{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))}{r_Y(Z)} - 1 \right| < \frac{2\delta}{r_Y(Z)} \leq \frac{2\delta}{3\delta} = \frac{2}{3}.$$

This implies that

$$\frac{1}{3} < \frac{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))}{r_Y(Z)} < \frac{5}{3} \quad \text{for each } Z \in \partial B_s \cap C_{3\delta}^{\partial}(Y, s). \quad (180)$$

Employing (180) along with (158), we can write

$$\begin{aligned} & \left| \frac{1}{r_Y^3(Z)} (Z-Y) - \frac{1}{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))} (\chi_{ss}(Z) - \chi_{ss}(Y)) \right|_3 \\ & \leq \frac{1}{r_Y^3(Z)} \cdot |(\chi_{ss}(Z) - Z) - (\chi_{ss}(Y) - Y)|_3 \\ & \quad + \frac{1}{r_Y^3(Z)} \cdot |r_Y(Z) - r_{\chi_{ss}(Y)}(\chi_{ss}(Z))| \\ & \quad \cdot \left\{ 1 + \frac{r_Y(Z)}{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))} + \frac{r_Y^2(Z)}{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))} \right\} \\ & < \frac{14}{r_Y^3(Z)} \cdot |(\chi_{ss}(Z) - Z) - (\chi_{ss}(Y) - Y)|_3 \\ & \leq 14c\hat{A}\hat{A}_0^{\hat{\alpha}} \cdot \frac{1}{r_Y^{3-\hat{\alpha}}(Z)} \cdot \delta \quad \text{for each } Z \in \partial B_s \cap C_{3\delta}^{\partial}(Y, s). \end{aligned} \quad (181)$$

Now, (181) and the bound afforded for $\Gamma_{(Y,s;\hat{Y},\hat{s})}^{03}$ by (153) lead to

$$I_{35} \leq 2 \cdot \left(\frac{K}{J}\right)^{-1} \cdot \frac{8}{(c_1^*)^3} \cdot 14c\hat{A}\hat{A}_0^{\hat{\alpha}} \cdot \delta \cdot \int_{C_d^{\partial}(Y,s) \cap C_{3\delta}^{\partial}(Y,s)} \frac{1}{r_Y^{3-\hat{\alpha}}} d\lambda_{\partial B_s}, \quad (182)$$

whence an argument familiar by now shows that I_{35} fulfills a relation as required in (68).

I₃₆: If $Z \in C_d^\partial(Y, s)$, then $r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) \leq r_{\hat{Y}}(\chi_{s\hat{s}}(Z))$ and $r_Y(\pi_Y(Z)) > \frac{7}{9} r_Y(Z)$. Coupling these facts with (84) and recalling (130), we arrive at

$$\frac{7}{12} < \frac{r_Y(\pi_Y(Z))}{r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} < \frac{27}{14} \quad \text{for each } Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'. \quad (183)$$

For $Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'$, we have both

$$\begin{aligned} r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) &> \frac{7}{9} |\hat{Y} - \chi_{s\hat{s}}(Z)|_3 \geq \frac{7}{9} \{ |Y - Z|_3 - (|Y - \hat{Y}|_3 + |Z - \chi_{s\hat{s}}(Z)|_3) \} \\ &\geq \frac{7}{9} (3\delta - \delta) = \frac{14}{9} \delta \end{aligned}$$

and

$$\begin{aligned} |r_{\chi_{s\hat{s}}(Y)}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) - r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))| &\leq |\hat{Y} - \chi_{s\hat{s}}(Y)|_3 \\ &\leq |\hat{Y} - Y|_3 + |Y - \chi_{s\hat{s}}(Y)|_3 \\ &\leq \delta, \end{aligned}$$

so

$$\left| \frac{r_{\chi_{s\hat{s}}(Y)}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))}{r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} - 1 \right| \leq \frac{\delta}{r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} < \frac{9}{14}.$$

From the latter inequality,

$$\begin{aligned} \frac{5}{14} &< \frac{r_{\chi_{s\hat{s}}(Y)}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))}{r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} < \frac{23}{14} \\ &\text{for each } Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'. \end{aligned} \quad (184)$$

Using (183) and (184), we obtain the desired estimates

$$\left| \frac{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))}{r_Y(Z)} - 1 \right| < \frac{2\delta}{r_Y(Z)} \leq \frac{2\delta}{3\delta} = \frac{2}{3}.$$

This implies that

$$\frac{1}{3} < \frac{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))}{r_Y(Z)} < \frac{5}{3} \quad \text{for each } Z \in \partial B_s \cap C_{3\delta}^{\partial}(Y, s). \quad (180)$$

Employing (180) along with (158), we can write

$$\begin{aligned} & \left| \frac{1}{r_Y(Z)} (Z-Y) - \frac{1}{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))} (\chi_{ss}(Z) - \chi_{ss}(Y)) \right|_3 \\ & \leq \frac{1}{r_Y(Z)} \cdot |(\chi_{ss}(Z) - Z) - (\chi_{ss}(Y) - Y)|_3 \\ & \quad + \frac{1}{r_Y(Z)} \cdot |r_Y(Z) - r_{\chi_{ss}(Y)}(\chi_{ss}(Z))| \\ & \quad \cdot \left\{ 1 + \frac{r_Y(Z)}{r_{\chi_{ss}(Y)}(\chi_{ss}(Z))} + \frac{r_Y^2(Z)}{r_{\chi_{ss}(Y)}^2(\chi_{ss}(Z))} \right\} \\ & < \frac{14}{r_Y(Z)} \cdot |(\chi_{ss}(Z) - Z) - (\chi_{ss}(Y) - Y)|_3 \\ & \leq 14c_{AA}^{\hat{\alpha}} \cdot \frac{1}{r_Y^{3-\hat{\alpha}}(Z)} \cdot \delta \quad \text{for each } Z \in \partial B_s \cap C_{3\delta}^{\partial}(Y, s). \end{aligned} \quad (181)$$

Now, (181) and the bound afforded for $\Gamma_{(Y,s;\hat{Y},\hat{s})}^{03}$ by (153) lead to

$$I_{35} \leq 2 \cdot (m_J^K)^{-1} \cdot \frac{8}{(c_1^*)^3} \cdot 14c_{AA}^{\hat{\alpha}} \cdot \delta \cdot \int_{C_d^{\partial}(Y,s) \cap C_{3\delta}^{\partial}(Y,s)} \frac{1}{r_Y^{3-\hat{\alpha}}} d\partial B_s, \quad (182)$$

whence an argument familiar by now shows that I_{35} fulfills a relation as required in (68).

$$\frac{1}{3} < \frac{7}{12} \cdot \frac{14}{23} < \frac{r_Y(\pi_Y(Z))}{r_{\chi_{ss}(Y)}(\pi_Y(\chi_{ss}(Z)))} < \frac{27}{14} \cdot \frac{14}{5} < 6, \quad (185)$$

for each $Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'$.

As another preliminary for the investigation of the difference displayed in the integrand in (63), (158) and (173) imply that

$$\begin{aligned} & |\{\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Y)\} - \{\pi_Y(Z) - Y\}|_3 \\ & \leq |\{\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Z)\} - \{\pi_Y(Z) - Z\}|_3 + |\{\chi_{ss}(Z) - Z\} - \{\chi_{ss}(Y) - Y\}|_3 \\ & \leq 2a_K \cdot r_Y(Z) \delta^{\alpha_K + 2cAA_0 \hat{\alpha}} \cdot r_Y^{\hat{\alpha}}(Z) \delta + \hat{a}_K \cdot 1^{1+\alpha_K} \delta^{1+\alpha_K} \\ & \leq k_{18} \{r_Y(Z) \cdot \delta^{\alpha_K} + r_Y^{\hat{\alpha}}(Z) \cdot \delta + \delta^{1+\alpha_K}\} \end{aligned} \quad (186)$$

for each $Z \in C_d^\partial(Y, s)$,

in which k_{18} depends upon M and K alone. Now, proceeding initially in the usual manner, then applying (185) and (186), we find

$$\begin{aligned} & \left| \frac{1}{r_Y^3(\pi_Y(Z))} \{\pi_Y(Z) - Y\} - \frac{1}{r_{\chi_{ss}(Y)}^3(\pi_Y(\chi_{ss}(Z)))} \{\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Y)\} \right|_3 \\ & \leq \frac{1}{r_Y^3(\pi_Y(Z))} \cdot |\{\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Y)\} - \{\pi_Y(Z) - Y\}|_3 \\ & \quad + \frac{1}{r_Y^3(\pi_Y(Z))} \cdot |r_{\chi_{ss}(Y)}(\pi_Y(\chi_{ss}(Z))) - r_Y(\pi_Y(Z))| \\ & \quad \cdot \left\{ 1 + \frac{r_Y(\pi_Y(Z))}{r_{\chi_{ss}(Y)}(\pi_Y(\chi_{ss}(Z)))} + \frac{r_Y^2(\pi_Y(Z))}{r_{\chi_{ss}(Y)}^2(\pi_Y(\chi_{ss}(Z)))} \right\} \end{aligned}$$

$$< \frac{44}{r_Y^3(\pi_Y(Z))} \cdot |\{\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Y)\} - \{\pi_Y(Z) - Y\}|_3 \quad (187)$$

$$\leq \frac{44 \cdot k_{18}}{r_Y^3(\pi_Y(Z))} \cdot \{r_Y(Z) \cdot \delta^{\alpha_K} + r_Y^{\hat{\alpha}}(Z) \cdot \delta^{\hat{\alpha}} \cdot \delta^{1+\alpha_K}\}$$

$$< 44k_{18} \left\{ \frac{9/7}{r_Y^2(\pi_Y(Z))} \cdot \delta^{\alpha_K} + \frac{(9/7)^{\hat{\alpha}}}{r_Y^{3-\hat{\alpha}}(\pi_Y(Z))} \cdot \delta^{\hat{\alpha}} + \frac{1}{r_Y^3(\pi_Y(Z))} \cdot \delta^{1+\alpha_K} \right\},$$

for each $Z \in C_d^{\partial}(Y, s) \cap C_{31\delta}^{\partial}(Y, s)'$.

Turning finally to the estimation of I_{36} itself, the use of (187) with (63) yields

$$I_{36} \leq (m_J^K)^{-1} \cdot \{1 - (c^*/c)^2\}^{-3/2} \cdot 2 \cdot 44 \cdot k_{18} \cdot 2^{3/2} \pi \quad (188)$$

$$\cdot \left\{ \frac{9}{7} \delta^{\alpha_K} \cdot \int_{31\delta}^d \frac{1}{\zeta} d\zeta + \left(\frac{9}{7}\right)^{\hat{\alpha}} \delta^{\hat{\alpha}} \cdot \int_{31\delta}^d \frac{1}{\zeta^{2-\hat{\alpha}}} d\zeta + \delta^{1+\alpha_K} \cdot \int_{31\delta}^d \frac{1}{\zeta^2} d\zeta \right\}.$$

As in previous computations, the first term within the brackets on the right in (188) leads to an estimate involving $\delta^{\alpha'_K}$ for any $\alpha'_K \in (0, \alpha_K)$, while the second can be estimated in terms of either $\delta^{\hat{\alpha}}$ (if $\hat{\alpha} \in (0, 1)$) or $\delta^{\hat{\alpha}'}$ for any $\hat{\alpha}' \in (0, 1)$ (in case $\hat{\alpha} = 1$); the third term is just

$$\delta^{1+\alpha_K} \int_{31\delta}^d \frac{1}{\zeta^2} d\zeta = \delta^{1+\alpha_K} \cdot \left\{ \frac{1}{31\delta} - \frac{1}{d} \right\} < \frac{1}{31} \cdot \delta^{\alpha_K}.$$

With these remarks, it is obvious that (188) can be used to derive for I_{36} an inequality of the required form.

I₃₇: For brevity, in the analysis of this term we shall write

$$\hat{V} := V_{(\hat{Y}, \hat{s})},$$

$$\hat{W} := W_{(\hat{Y}, \hat{s})},$$

and

$$\hat{\chi}_{,4} := \chi_{,4}(\chi_{\hat{s}}^{-1}(\hat{Y}), \hat{s}).$$

Now, for each $Z \in C_d^{\partial}(Y, s) \cap C_{31\delta}^{\partial}(Y, s)'$, we shall define $f^q(\cdot; Z, \hat{Y}, \hat{s}, s)$ on the set $R^3 \cap \{\chi_{ss}(Z)\}' \cap \{\pi_{\hat{Y}}(\chi_{ss}(Z))\}'$ according to

$$\begin{aligned} & f^q(X; Z, \hat{Y}, \hat{s}, s) \\ &:= \left\{ \frac{1}{r_X^2} \cdot r_{X,q} \cdot (\hat{W}^k \cdot r_{X,k} + ((\hat{V}^l)^c \cdot r_{X,l})^2 + (1 - |\hat{V}^c|^2/3)^{1/2})^{-3} \right\} \circ \chi_{ss}(Z) \\ & - \left\{ \frac{1}{r_X^2} \cdot r_{X,q} \cdot ((\hat{\chi}_{,4}^k)^c \cdot r_{X,k})^2 + (1 - |\hat{\chi}_{,4}^c|^2/3)^{1/2} \right\}^{-3/2} \circ \pi_{\hat{Y}} \circ \chi_{ss}(Z) \\ & \text{for each } X \in R^3 \cap \{\chi_{ss}(Z)\}' \cap \{\pi_{\hat{Y}}(\chi_{ss}(Z))\}' . \end{aligned} \tag{189}$$

then it is easy to see that

$$\begin{aligned} I_{37} = & \left| \int_{C_d^{\partial}(Y, s) \cap C_{31\delta}^{\partial}(Y, s)'} \{f^q(\hat{Y}; Z, \hat{Y}, \hat{s}, s) - f^q(\chi_{ss}(Y); Z, \hat{Y}, \hat{s}, s)\} \right. \\ & \left. \cdot \hat{T}_s^{1q}(Y) \cdot J\chi_s^{-1}(Y) \, d\lambda_{\partial\delta_s}(Z) \right|. \end{aligned} \tag{190}$$

Recall that we are denoting by $[\hat{Y}, \chi_{ss}(Y)]$ the closed line segment joining $\hat{Y}$ and $\chi_{ss}(Y)$ in R^3 (which may consist of a single point). We claim that

$$[\hat{Y}, \chi_{\hat{s}\hat{s}}(Y)] \subset \mathbb{R}^3 \cap \{\chi_{\hat{s}\hat{s}}(Z)\}' \cap \{\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))\}' \quad (191)$$

for each $Z \in C_d^\partial(Y, s) \cap C_{3\delta}^\partial(Y, s)'$.

To verify this, we observe first that $|\hat{Y} - Y|_3 \leq \delta < \delta$ and $|Y - \chi_{\hat{s}\hat{s}}(Y)|_3 \leq c^* |\hat{s} - s| \leq c^* \delta < \delta$, so the inclusion

$$[\hat{Y}, \chi_{\hat{s}\hat{s}}(Y)] \subset B_{\delta}^3(Y) \quad (192)$$

must hold. On the other hand, choosing any $Z \in C_d^\partial(Y, s) \cap C_{3\delta}^\partial(Y, s)'$, we have

$$|\chi_{\hat{s}\hat{s}}(Z) - Y|_3 \geq |Z - Y|_3 - |\chi_{\hat{s}\hat{s}}(Z) - Z|_3 \geq 3\delta - c^* \delta > 3\delta - \delta > \delta,$$

and so also

$$\begin{aligned} |\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - Y|_3 &\geq |\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - \hat{Y}|_3 - |\hat{Y} - Y|_3 \\ &> \frac{7}{9} |\chi_{\hat{s}\hat{s}}(Z) - \hat{Y}|_3 - |\hat{Y} - Y|_3 \\ &\geq \frac{7}{9} (3\delta - c^* \delta) - \delta \\ &= \left\{ \frac{21}{9} - \left(1 + \frac{7}{9} c^* \right) \right\} \delta > \delta, \end{aligned}$$

the latter inequality following since

$$1 + \frac{7}{9} c^* \leq \{1 + (c^*)^2\}^{1/2} \cdot \left\{1 + \frac{49}{81}\right\}^{1/2} = 1 \cdot \frac{\sqrt{130}}{9} < \frac{12}{9} = \frac{21}{9}.$$

Thus, $\chi_{\hat{s}\hat{s}}(Z)$ and $\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))$ lie outside $B_{\delta}^3(Y)$ whenever $Z \in C_d^\partial(Y, s) \cap C_{3\delta}^\partial(Y, s)'$: coupling this fact with (192), the inclusion (191) clearly follows for each such Z . But (191) allows us to apply the mean-value theorem in the especially convenient form given

in Theorem (12.9) of Apostol [1], which shows that there exists for each $Z \in C_d^{\partial}(Y,s) \cap C_{3,6}^{\partial}(Y,s)$ a point $X(Z;Y,s,\hat{Y},\hat{s}) \in [\hat{Y},\chi_{ss}(Y)]$, depending upon Z , (Y,s) , and $(\hat{Y},\hat{s})$, such that

$$\begin{aligned} & \{f^q(\hat{Y};Z,\hat{Y},\hat{s},s) - f^q(\chi_{ss}(Y);Z,\hat{Y},\hat{s},s)\} \cdot \bar{T}_s^{1q}(Y) \\ & = f_{,j}^q(X(Z;Y,s,\hat{Y},\hat{s});Z,\hat{Y},\hat{s},s) \cdot (\hat{Y}^j - \chi_{ss}^j(Y)) \cdot \bar{T}_s^{1q}(Y). \end{aligned} \quad (193)$$

A straightforward, albeit somewhat tedious, computation, beginning with (189), produces

$$\begin{aligned} & f_{,j}^q(X;Z,\hat{Y},\hat{s},s) \\ & = \left\{ \frac{1}{r_X^3} \{ \hat{W}^k r_{X,k} + (\hat{V}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{V}^c|^2/3) \}^{1/2} \right\}^{-4} \\ & \quad \cdot \{ 3\hat{W}^j r_{X,q} - (\hat{W}^k r_{X,k}) \delta_{jq} \} \circ \chi_{ss}(Z) \\ & \quad + \left\{ \frac{1}{r_X^3} \{ \hat{W}^k r_{X,k} + (\hat{V}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{V}^c|^2/3) \}^{1/2} \right\}^{-4} \cdot \{ (\hat{V}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{V}^c|^2/3) \}^{-1/2} \\ & \quad \cdot \{ 3 \{ (1 - |\hat{V}^c|^2/3) r_{X,j} r_{X,q} + (\hat{V}^{kc} r_{X,k}) \cdot \hat{V}^{jc} r_{X,q} \} \\ & \quad - \delta_{jq} \{ (\hat{V}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{V}^c|^2/3) \} \} \circ \chi_{ss}(Z) \\ & \quad - \left\{ \frac{1}{r_X^3} \{ (\hat{\chi}_{,4}^{kc} r_{X,k})^2 + (1 - |\hat{\chi}_{,4}^c|^2/3) \} \right\}^{-5/2} \\ & \quad \cdot \{ 3 \{ (1 - |\hat{\chi}_{,4}^c|^2/3) r_{X,j} r_{X,q} + (\hat{\chi}_{,4}^{kc} r_{X,k}) \cdot \hat{\chi}_{,4}^{jc} r_{X,q} \} \\ & \quad - \delta_{jq} \{ (\hat{\chi}_{,4}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{\chi}_{,4}^c|^2/3) \} \} \circ \Pi_{\hat{Y}} \circ \chi_{ss}(Z) \end{aligned}$$

$$\begin{aligned}
 &= \left\{ \frac{1}{3} \{ \hat{w}^k r_{X,k} + (\hat{v}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{v}^c|_3^2) \}^{1/2} \right\}^{-4} \\
 &\quad \cdot \{ 3 \hat{w}^j r_{X,q} - (\hat{w}^k r_{X,k}) \delta_{jq} \} \circ \chi_{\hat{s}\hat{s}}(Z) + \left\{ \frac{1}{r_X^3(\chi_{\hat{s}\hat{s}}(Z))} - \frac{1}{r_X^3(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)))} \right\} \\
 &\quad \cdot \{ (\hat{\chi}_{,4}^{k c} r_{X,k})^2 + (1 - |\hat{\chi}_{,4}^c|_3^2) \}^{-5/2} \\
 &\quad \cdot \{ 3 \{ (1 - |\hat{\chi}_{,4}^c|_3^2) r_{X,j} + (\hat{\chi}_{,4}^{k c} r_{X,k}) \cdot \hat{\chi}_{,4}^{j c} \} r_{X,q} \\
 &\quad - \{ (\hat{\chi}_{,4}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{\chi}_{,4}^c|_3^2) \} \cdot \delta_{jq} \} \circ \pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}}(Z) \\
 &\quad + \frac{1}{r_X^3(\chi_{\hat{s}\hat{s}}(Z))} \{ (\hat{w}^k r_{X,k} + (\hat{v}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{v}^c|_3^2) \}^{1/2} \}^{-4} \\
 &\quad \cdot \{ (\hat{v}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{v}^c|_3^2) \}^{-1/2} \circ \chi_{\hat{s}\hat{s}}(Z) \\
 &\quad - \{ (\hat{\chi}_{,4}^{k c} r_{X,k})^2 + (1 - |\hat{\chi}_{,4}^c|_3^2) \}^{-5/2} \circ \pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}}(Z) \} \\
 &\quad \cdot \{ 3 \{ (1 - |\hat{\chi}_{,4}^c|_3^2) r_{X,j} + (\hat{\chi}_{,4}^{k c} r_{X,k}) \cdot \hat{\chi}_{,4}^{j c} \} r_{X,q} \\
 &\quad - \{ (\hat{\chi}_{,4}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{\chi}_{,4}^c|_3^2) \} \delta_{jq} \} \circ \pi_{\hat{Y}} \circ \chi_{\hat{s}\hat{s}}(Z) \\
 &\quad + \frac{1}{r_X^3(\chi_{\hat{s}\hat{s}}(Z))} \{ (\hat{w}^k r_{X,k} + (\hat{v}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{v}^c|_3^2) \}^{1/2} \}^{-4} \\
 &\quad \cdot \{ (\hat{v}^{\ell c} r_{X,\ell})^2 + (1 - |\hat{v}^c|_3^2) \}^{-1/2} \circ \chi_{\hat{s}\hat{s}}(Z) \\
 &\quad \cdot \{ 3 \{ (1 - |\hat{\chi}_{,4}^c|_3^2) r_{X,j} + (\hat{\chi}_{,4}^{k c} r_{X,k}) \cdot \hat{\chi}_{,4}^{j c} \} \circ \pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) \\
 &\quad \cdot \{ r_{X,q}(\chi_{\hat{s}\hat{s}}(Z)) - r_{X,q}(\pi_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))) \}
 \end{aligned}$$

$$\begin{aligned}
& +3r_{X,q}(\chi_{s\hat{s}}(Z)) \cdot \{(1-|\hat{V}^c(\chi_{s\hat{s}}(Z))|_3^2) \cdot (r_{X,j}(\chi_{s\hat{s}}(Z)) - r_{X,j}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))) \\
& + \hat{\chi}_{,4}^{j,c} \cdot \hat{V}^{k,c}(\chi_{s\hat{s}}(Z)) \cdot (r_{X,k}(\chi_{s\hat{s}}(Z)) - r_{X,k}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))) \\
& + (\hat{V}^{k,c} r_{X,k}) \circ \chi_{s\hat{s}}(Z) \cdot (\hat{V}^{j,c}(\chi_{s\hat{s}}(Z)) - \hat{\chi}_{,4}^{j,c}) \\
& + \hat{\chi}_{,4}^{j,c} \cdot r_{X,k}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) \cdot (\hat{V}^{k,c}(\chi_{s\hat{s}}(Z)) - \hat{\chi}_{,4}^{k,c}) \\
& + r_{X,j}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) \cdot (|\hat{\chi}_{,4}^c|_3 + |\hat{V}^c(\chi_{s\hat{s}}(Z))|_3) \cdot (|\hat{\chi}_{,4}^c|_3 - |\hat{V}^c(\chi_{s\hat{s}}(Z))|_3) \\
& + \{(\hat{\chi}_{,4}^{l,c} r_{X,l}) \circ \pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)) + (\hat{V}^{l,c} r_{X,l}) \circ \chi_{s\hat{s}}(Z)\} \\
& - \{\hat{\chi}_{,4}^{k,c} \cdot (r_{X,k}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) - r_{X,k}(\chi_{s\hat{s}}(Z))) \\
& + r_{X,k}(\chi_{s\hat{s}}(Z)) \cdot (\hat{\chi}_{,4}^{k,c} - \hat{V}^{k,c}(\chi_{s\hat{s}}(Z)))\} \cdot \delta_{jq} \\
& + \{|\hat{V}^c(\chi_{s\hat{s}}(Z))|_3 + |\hat{\chi}_{,4}^c|_3\} \cdot \{|\hat{V}^c(\chi_{s\hat{s}}(Z))|_3 - |\hat{\chi}_{,4}^c|_3\} \cdot \delta_{jq} \}, \tag{194}
\end{aligned}$$

for each $X \in \mathbb{R}^3 \cap \{\chi_{s\hat{s}}(Z)\}' \cap \{\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))\}'$,

for each $Z \in C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)'$.

Let us write

$$\hat{X} := X(Z; Y, s, \hat{Y}, \hat{s}),$$

for a fixed $Z \in C_d^\partial(Y,s) \cap C_{31\delta}^\partial(Y,s)'$ under consideration. Then,

using (193) and (194), and keeping in mind the bounds given in (69)

and (153), along with the inequalities $|\hat{\chi}_{,4}^c|_3 \leq c^*/c$, $|\hat{V}^c|_3 \leq c^*/c$

on ∂B_s ,

$$1 - (c^*/c)^2 \leq (\xi^2 \cdot r_{X,\ell})^2 + (1 - |\xi|_3^2) \leq 1 \quad \text{on } \mathbb{R}^3 \setminus \{X\}, \quad (195)$$

for $\xi \in \mathbb{R}^3$ with $|\xi|_3 \leq c^*/c$, and

$$|\hat{Y} - \chi_{s\hat{s}}(Y)|_3 \leq |\hat{Y} - Y|_3 + |Y - \chi_{s\hat{s}}(Y)|_3 \leq |\hat{Y} - Y|_3 + c^*|\hat{s} - s| \leq 1\delta,$$

it is found that

$$\begin{aligned} & | \{ f^q(\hat{Y}; Z, \hat{Y}, \hat{s}, s) - f^q(\chi_{s\hat{s}}(Y); Z, \hat{Y}, \hat{s}, s) \} \cdot T_s^{1q}(Y) | \\ & \leq \frac{1}{r_X^3(\chi_{s\hat{s}}(Z))} \cdot \left(\frac{1}{2} c_1^* \right)^{-4} \cdot 8 \cdot |\hat{W}(\chi_{s\hat{s}}(Z))|_3 \cdot 1\delta \\ & + \left| \frac{1}{r_X^3(\chi_{s\hat{s}}(Z))} - \frac{1}{r_X^3(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} \right| \cdot \{1 - (c^*/c)^2\}^{-5/2} \cdot 2\{4 + 3(c^*/c)^2\} \cdot 1\delta \\ & + \frac{1}{r_X^3(\chi_{s\hat{s}}(Z))} \cdot \{ (\hat{W}^k r_{\hat{X},k} + (\hat{V}^{\ell c} r_{\hat{X},\ell})^2 + (1 - |\hat{V}^c|_3^2) \}^{1/2} \}^{-4} \\ & \cdot \{ (\hat{V}^{\ell c} r_{\hat{X},\ell})^2 + (1 - |\hat{V}^c|_3^2) \}^{-1/2} \} \circ \chi_{s\hat{s}}(Z) \\ & - \{ (\hat{X}_{,4}^k r_{\hat{X},k}^c + (1 - |\hat{X}_{,4}^c|_3^2) \}^{-5/2} \} \circ \pi_{\hat{Y}} \circ \chi_{s\hat{s}}(Z) | \\ & \cdot \{1 - (c^*/c)^2\}^{-5/2} \cdot 2\{4 + 3(c^*/c)^2\} \cdot 1\delta \\ & + \frac{1}{r_X^3(\chi_{s\hat{s}}(Z))} \cdot \left(\frac{1}{2} c_1^* \right)^{-4} \cdot \{1 - (c^*/c)^2\}^{-1/2} \cdot 1\delta \\ & \cdot \{4\{3 + 4(c^*/c)^2\} \cdot |\text{grad } r_X(\chi_{s\hat{s}}(Z)) - \text{grad } r_X(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))|_3 \\ & + 32(c^*/c) \cdot |\hat{V}^c(\chi_{s\hat{s}}(Z)) - \hat{X}_{,4}^c|_3 \} \end{aligned} \quad (196)$$

for each $Z \in C_d^3(Y, s) \cap C_{3,1\delta}^3(Y, s)$.

To pursue the estimate begun in (196), we examine various expressions appearing there. First, from the definition (IV.14.28), remembering that $\chi_{\hat{s}}^{-1} \circ \chi_{s\hat{s}} = \chi_s^{-1}$, by (14), it is easy to check that

$$\begin{aligned} |w_{(\hat{Y}, \hat{s})}(\chi_{s\hat{s}}(Z))|_3 &\leq \{1+(c^*/c)^2\} \cdot \{1-(c^*/c)^2\}^{-1} \\ &\quad \cdot |v_{(\hat{Y}, \hat{s})}^c(\chi_{s\hat{s}}(Z)) - [\chi_{s\hat{s}}^c]_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(Z)|_3 \\ &= \{1+(c^*/c)^2\} \cdot \{1-(c^*/c)^2\}^{-1} \end{aligned} \quad (197)$$

$$\cdot |v_{(\hat{Y}, \hat{s})}^c(\chi_{s\hat{s}}(Z)) - \chi_{s\hat{s}}^c(\chi_s^{-1}(Z), \hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}))|_3$$

for each $Z \in \partial B_s$.

Now, again using (14), and noting that $\hat{Y} \neq \chi_{s\hat{s}}(Z)$ whenever $Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'$ (cf., (191)), directly from the definition (IV.14.11), we find

$$v_{(\hat{Y}, \hat{s})}(\chi_{s\hat{s}}(Z)) = \frac{1}{\tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s})} \{ \chi(\chi_s^{-1}(Z), \hat{s}) - \chi(\chi_s^{-1}(Z), \hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s})) \}$$

for each $Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'$.

Thus, for each $i \in \{1, 2, 3\}$, there exists some $\hat{t}^i(Z, \hat{Y}, \hat{s}, s) \in (\hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}), \hat{s})$ such that

$$v_{(\hat{Y}, \hat{s})}^{i,c}(\chi_{s\hat{s}}(Z)) = \chi_{s\hat{s}}^{i,c}(\chi_s^{-1}(Z), \hat{t}^i(Z, \hat{Y}, \hat{s}, s)) \quad (198)$$

for each $Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)'$.

Next, noting that $\tau(\chi_s^{-1}(Z); Z, s) = 0$ for $Z \in \partial B_s$, we use [I.3.16.iv] to write

$$\begin{aligned}
 \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}) &\leq |\tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}) - \tau(\chi_s^{-1}(Z); Y, s)| + \tau(\chi_s^{-1}(Z); Y, s) \\
 &= |\tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}) - \tau(\chi_s^{-1}(Z); Y, s)| \\
 &\quad + |\tau(\chi_s^{-1}(Z); Y, s) - \tau(\chi_s^{-1}(Z); Z, s)| \quad (199) \\
 &\leq \frac{1}{c-c^*} \{ |\hat{Y}-Y|_3 + c^* |\hat{s}-s| + |Z-Y|_3 \} \\
 &\leq \frac{1}{c-c^*} \{ 1\delta + |Z-Y|_3 \} \quad \text{for each } Z \in \partial B_s.
 \end{aligned}$$

The latter inequality clearly implies that $\hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s})$ and $\hat{t}^1(Z, \hat{Y}, \hat{s}, s)$ are in $\hat{K}$ for each $Z \in \partial B_s$. In view of this fact, we may apply (76) with (198) and (199), obtaining

$$\begin{aligned}
 &|v_{(\hat{Y}, \hat{s})}^{1c}(\chi_{s\hat{s}}(Z)) - \chi_{s4}^{1c}(\chi_s^{-1}(Z), \hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}))| \\
 &\leq \hat{A} \cdot |\hat{t}^1(Z, \hat{Y}, \hat{s}, s) - (\hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}))|^{\hat{\alpha}} \\
 &< \hat{A} |\hat{s} - (\hat{s} - \tau(\chi_s^{-1}(Z); \hat{Y}, \hat{s}))|^{\hat{\alpha}} \\
 &= \hat{A} \cdot \tau^{\hat{\alpha}}(\chi_s^{-1}(Z); \hat{Y}, \hat{s}) \quad (200) \\
 &\leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1\delta + |Z-Y|_3\}^{\hat{\alpha}} \\
 &\leq \frac{2^{\hat{\alpha}} \hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1^{\hat{\alpha}} \cdot \delta^{\hat{\alpha}} + r_Y^{\hat{\alpha}}(Z)\} \\
 &\text{for each } Z \in C_d^{\partial}(Y, s) \cap C_{31\delta}^{\partial}(Y, s)'.
 \end{aligned}$$

From (197), then,

$$|w_{(\hat{Y}, \hat{s})}(x_{s\hat{s}}(Z))|_3 < \{1+(c^*/c)^2\} \cdot \{1-(c^*/c)^2\} \\ \cdot \frac{3^{1/2} \cdot 2^{\hat{\alpha}} \hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1 \cdot \hat{\alpha} \cdot \delta^{\hat{\alpha}} + r_Y^{\hat{\alpha}}(Z)\} \quad (201)$$

for each $Z \in C_d^{\partial}(Y, s) \cap C_{3\delta}^{\partial}(Y, s)'$.

Moreover, we can reason as in the derivation of (201) (noting that $|\hat{t}^1(Z, \hat{Y}, \hat{s}, s) - \hat{s}| < \tau(x_s^{-1}(Z); \hat{Y}, \hat{s})$ for $Z \in \partial B_s$) and appeal to (76) and (101) to produce

$$|v_{(\hat{Y}, \hat{s})}^c(x_{s\hat{s}}(Z)) - x_{,4}^c(x_{\hat{s}}^{-1}(\hat{Y}), \hat{s})|_3 \\ \leq |v_{(\hat{Y}, \hat{s})}^c(x_{s\hat{s}}(Z)) - x_{,4}^c(x_s^{-1}(Z), \hat{s})|_3 + |x_{,4}^c(x_s^{-1}(Z), \hat{s}) - x_{,4}^c(x_s^{-1}(Y), \hat{s})|_3 \\ + |x_{,4}^c(x_s^{-1}(Y), \hat{s}) - x_{,4}^c(x_{\hat{s}}^{-1}(\hat{Y}), \hat{s})|_3 \quad (202) \\ \leq \frac{3^{1/2} \cdot 2^{\hat{\alpha}} \cdot \hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1 \cdot \hat{\alpha} \cdot \delta^{\hat{\alpha}} + r_Y^{\hat{\alpha}}(Z)\} + \hat{A} \cdot |x_s^{-1}(Z) - x_s^{-1}(Y)|_3^{\hat{\alpha}} + \hat{A} \cdot |x_s^{-1}(Y) - x_{\hat{s}}^{-1}(\hat{Y})|_3^{\hat{\alpha}} \\ \leq \left\{ \frac{3^{1/2} \cdot (2\hat{\alpha})}{(c-c^*)^{\hat{\alpha}}} + A_0^{\hat{\alpha}} \right\} \cdot \hat{A} \cdot \delta^{\hat{\alpha}} + \left\{ \frac{3^{1/2} \cdot 2^{\hat{\alpha}}}{(c-c^*)^{\hat{\alpha}}} + A_0^{\hat{\alpha}} \right\} \cdot \hat{A} \cdot r_Y^{\hat{\alpha}}(Z)$$

for each $Z \in C_d^{\partial}(Y, s) \cap C_{3\delta}^{\partial}(Y, s)'$.

Choose $X \in [\hat{Y}, x_{s\hat{s}}(Y)]$. If $Z \in \partial B_s \cap C_{3\delta}^{\partial}(Y, s)'$, then

$$r_{\hat{Y}}(x_{s\hat{s}}(Z)) \geq |Z - Y|_3 - \{|Y - \hat{Y}|_3 + |x_{s\hat{s}}(Z) - Z|_3\} \geq 3\delta - \delta = 2\delta, \quad (203)$$

and

$$|r_X(x_{s\hat{s}}(Z)) - r_{\hat{Y}}(x_{s\hat{s}}(Z))| \leq |X - \hat{Y}|_3 \leq |x_{s\hat{s}}(Y) - \hat{Y}|_3 \leq \delta,$$

so

$$\left| \frac{r_X(\chi_{s\hat{s}}(Z))}{r_{\hat{Y}}(\chi_{s\hat{s}}(Z))} - 1 \right| \leq \frac{1\delta}{r_{\hat{Y}}(\chi_{s\hat{s}}(Z))} \leq \frac{1}{2}.$$

This gives

$$\frac{1}{2} \leq \frac{r_X(\chi_{s\hat{s}}(Z))}{r_{\hat{Y}}(\chi_{s\hat{s}}(Z))} \leq \frac{3}{2} \quad \text{for } Z \in \partial B_s \cap C_{3,1\delta}^\partial(Y,s)', \quad X \in [\hat{Y}, \chi_{s\hat{s}}(Y)]. \quad (204)$$

From (84) and (203), it is clear that

$$\frac{1}{3} \leq \frac{r_X(\chi_{s\hat{s}}(Z))}{r_Y(Z)} \leq 2 \quad \text{for } Z \in \partial B_s \cap C_{3,1\delta}^\partial(Y,s)', \quad X \in [\hat{Y}, \chi_{s\hat{s}}(Y)]. \quad (205)$$

Now supposing that $X \in [\hat{Y}, \chi_{s\hat{s}}(Y)]$ and $Z \in C_d^\partial(Y,s) \cap C_{3,1\delta}^\partial(Y,s)'$, we certainly have

$$r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) > \frac{7}{9} \cdot r_{\hat{Y}}(\chi_{s\hat{s}}(Z)) \geq \frac{14}{9} 1\delta$$

(cf., (203)) and

$$|r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) - r_X(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))| \leq |X - \hat{Y}|_3 \leq 1\delta.$$

In the usual manner, it therefore follows that

$$\frac{5}{14} \leq \frac{r_X(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))}{r_{\hat{Y}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)))} \leq \frac{23}{14} \quad (206)$$

$$\text{for } Z \in C_d^\partial(Y,s) \cap C_{3,1\delta}^\partial(Y,s)', \quad X \in [\hat{Y}, \chi_{s\hat{s}}(Y)].$$

Combining (183), (205), and (206) with the inequalities

$$1 \leq \frac{r_Y(Z)}{r_Y(\pi_Y(Z))} \leq \frac{9}{7}, \quad \text{for } Z \in C_d^\partial(Y,s) \cap \{Y\}',$$

there results

$$\frac{1}{12} < \frac{1}{3} \cdot 1 \cdot \frac{7}{12} \cdot \frac{14}{23} \leq \frac{r_X(\chi_{ss}(Z))}{r_X(\pi_Y(\chi_{ss}(Z)))} \leq 2 \cdot \frac{9}{7} \cdot \frac{27}{14} \cdot \frac{14}{5} < 24 \quad (207)$$

$$\text{for } Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)', \quad X \in [\hat{Y}, \chi_{ss}(Y)].$$

As the first consequence of these estimates, specifically (205) and (207), recalling (84) and (I.2.37.6), we discover that

$$\begin{aligned} & \left| \frac{1}{r_X^3(\chi_{ss}(Z))} - \frac{1}{r_X^3(\pi_Y(\chi_{ss}(Z)))} \right| \\ &= \frac{1}{r_X^4(\chi_{ss}(Z))} \cdot |r_X(\pi_Y(\chi_{ss}(Z))) - r_X(\chi_{ss}(Z))| \cdot \frac{r_X(\chi_{ss}(Z))}{r_X(\pi_Y(\chi_{ss}(Z)))} \\ & \cdot \left\{ 1 + \frac{r_X(\chi_{ss}(Z))}{r_X(\pi_Y(\chi_{ss}(Z)))} + \frac{r_X^2(\chi_{ss}(Z))}{r_X^2(\pi_Y(\chi_{ss}(Z)))} \right\} \\ &\leq 24 \cdot \{1 + 24 + (24)^2\} \cdot \frac{1}{r_X^4(\chi_{ss}(Z))} \cdot |\pi_Y(\chi_{ss}(Z)) - \chi_{ss}(Z)|_3 \quad (208) \\ &< 24 \cdot 601 \cdot \bar{a}_K \cdot \frac{1}{r_X^4(\chi_{ss}(Z))} \cdot r_Y^{1+\alpha_K}(\chi_{ss}(Z)) \\ &< 24 \cdot 601 \cdot (3)^4 \cdot (4/3)^{1+\alpha_K} \cdot \bar{a}_K \cdot \frac{1}{r_Y^{3-\alpha_K}(Z)} \end{aligned}$$

$$\text{for each } Z \in C_d^\partial(Y, s) \cap C_{3, \delta}^\partial(Y, s)';$$

here, of course, $\bar{a}_K$ can be obtained from [I.2.37.iii.3] in terms of a_K and α_K . In a similar manner,

$$\begin{aligned}
 & \left| \text{grad } r_{\hat{X}}(x_{\hat{s}\hat{s}}(Z)) - \text{grad } r_{\hat{X}}(\Pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z))) \right|_3 \\
 & \leq \frac{1}{r_{\hat{X}}(x_{\hat{s}\hat{s}}(Z))} \cdot \left| \Pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)) - x_{\hat{s}\hat{s}}(Z) \right|_3 \\
 & \quad + \left| \frac{1}{r_{\hat{X}}(x_{\hat{s}\hat{s}}(Z))} - \frac{1}{r_{\hat{X}}(\Pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)))} \right| \cdot \left| \Pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)) - \hat{X} \right|_3 \\
 & \leq \frac{2}{r_{\hat{X}}(x_{\hat{s}\hat{s}}(Z))} \cdot \left| \Pi_{\hat{Y}}(x_{\hat{s}\hat{s}}(Z)) - x_{\hat{s}\hat{s}}(Z) \right|_3 \\
 & < \frac{2}{r_{\hat{X}}(x_{\hat{s}\hat{s}}(Z))} \cdot \tilde{a}_K \cdot r_{\hat{Y}}^{1+\alpha_K}(x_{\hat{s}\hat{s}}(Z)) \\
 & \leq 2 \cdot 3 \cdot (4/3)^{1+\alpha_K} \cdot \tilde{a}_K \cdot r_Y^{\alpha_K}(Z) \quad \text{for each } Z \in C_d^3(Y, s) \cap C_{316}^3(Y, s)'.
 \end{aligned} \tag{209}$$

We observe next that, for positive numbers a , b , and β ,

$$\begin{aligned}
 \frac{1}{(a+b)^{1/2} \cdot b^{1/2}} - \frac{1}{\beta^{5/2}} &= \frac{1}{(a+b)^{1/2} \cdot b^{1/2} \cdot \beta^{5/2}} \\
 &\quad \cdot \{(\beta^{5/2} - b^{5/2}) + (1 - (1+a \cdot b^{-1/2})^4) \cdot b^{5/2}\} \\
 &= \frac{1}{(a+b)^{1/2} \cdot b^{1/2} \cdot \beta^{5/2}} \\
 &\quad \cdot \{(\beta^{1/2} + b^{1/2}) - 1 \\
 &\quad \cdot (\beta^2 + \beta^{3/2} b^{1/2} + \beta b + \beta^{1/2} b^{3/2} + b^2) \cdot (\beta - b) \\
 &\quad - \{2 + ab^{-1/2} + (1 + ab^{-1/2})^2 + (1 + ab^{-1/2})^3\} \cdot b^2 \cdot a\}.
 \end{aligned}$$

Employing this simple fact, using the inequalities (153), (154), and (195) along with the estimates (201), (202), and (209), it is plain that there exist positive numbers c_{12}^* and c_{13}^* , depending on the ratio c^*/c alone, for which

$$\begin{aligned}
 & \cdot | \{ (\hat{W}^k r_{\hat{X},k} + \{ (\hat{V}^l r_{\hat{X},l})^2 + (1 - |\hat{V}^c|^2_3) \}^{1/2})^{-4} \\
 & \quad \cdot \{ (\hat{V}^l r_{\hat{X},l})^2 + (1 - |\hat{V}^c|^2_3) \}^{-1/2} \} \circ \chi_{s\hat{s}}(Z) \\
 & \quad - \{ (\hat{X}^k r_{\hat{X},k})^2 + (1 - |\hat{X}^c|^2_3) \}^{-5/2} \} \circ \pi_{\hat{Y}} \circ \chi_{s\hat{s}}(Z) | \\
 & \leq c_{12}^* \cdot | \{ (\hat{X}^k r_{\hat{X},k})^2 + (1 - |\hat{X}^c|^2_3) \} \circ \pi_{\hat{Y}} \circ \chi_{s\hat{s}}(Z) \\
 & \quad - \{ (\hat{V}^l r_{\hat{X},l})^2 + (1 - |\hat{V}^c|^2_3) \} \circ \chi_{s\hat{s}}(Z) | + c_{13}^* \cdot | \hat{W}^k(\chi_{s\hat{s}}(Z)) \cdot r_{\hat{X},k}(\chi_{s\hat{s}}(Z)) | \\
 & \leq c_{12}^* \left\{ 4 \cdot \frac{c^*}{c} \cdot | v_{(\hat{Y},\hat{s})}^c(\chi_{s\hat{s}}(Z)) - \chi_{\hat{s}}^{-1}(\hat{Y},\hat{s}) |_3 \right. \\
 & \quad \left. + 2 \left(\frac{c^*}{c} \right)^2 \cdot | \text{grad } r_{\hat{X}}(\chi_{s\hat{s}}(Z)) - \text{grad } r_{\hat{X}}(\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z))) |_3 \right\} \\
 & \quad + c_{13}^* \cdot | W_{(\hat{Y},\hat{s})}(\chi_{s\hat{s}}(Z)) |_3 \\
 & \leq c_{12}^* \{ k_{19} \cdot \delta^{\hat{\alpha}} + k_{20} \cdot r_{\hat{Y}}^{\hat{\alpha}}(Z) + k_{21} \cdot r_{\hat{Y}}^{\alpha_K}(Z) \} + c_{13}^* k_{22} \{ \delta^{\hat{\alpha}} + r_{\hat{Y}}^{\hat{\alpha}}(Z) \} \\
 & \leq k_{23} \{ \delta^{\hat{\alpha}} + r_{\hat{Y}}^{\hat{\alpha}}(Z) + r_{\hat{Y}}^{\alpha_K}(Z) \}
 \end{aligned} \tag{210}$$

for each $Z \in C_d^{\partial}(Y,s) \cap C_{3,\delta}^{\partial}(Y,s)'$,

wherein $k_{19}, \dots, k_{23}$ depend on only M and K .

With (201), (202), (205), (208), (209), and (210), the inequality (196) can be further developed, yielding

$$\begin{aligned}
 & | \{ f^q(\hat{Y}; Z, \hat{Y}, \hat{s}, s) - f^q(\chi_{s\hat{s}}(Y); Z, \hat{Y}, \hat{s}, s) \} \cdot T_s^{1q}(Y) | \\
 & \leq 3^3 \cdot \frac{1}{r_Y^3(Z)} \cdot \left(\frac{1}{2} c_1^* \right)^{-4} \cdot 8 \cdot \delta \cdot k_{24} \{ \delta^{\hat{\alpha}} + r_{\hat{Y}}^{\hat{\alpha}}(Z) \} \\
 & \quad + k_{25} \cdot \frac{1}{r_Y^{3-\alpha_K}(Z)} \cdot \delta + 3^3 \cdot \frac{1}{r_Y^3(Z)} \cdot k_{26} \{ \delta^{\hat{\alpha}} + r_{\hat{Y}}^{\hat{\alpha}}(Z) + r_{\hat{Y}}^{\alpha_K}(Z) \} \cdot \delta
 \end{aligned}$$

$$+3^3 \cdot \frac{1}{r_Y^3(Z)} \cdot \delta \{k_{27} \cdot r_Y^{\alpha_K}(Z) + k_{28}(\delta^{\hat{\alpha}} + r_Y^{\hat{\alpha}}(Z))\} \quad (211)$$

$$\leq k_{29} \left\{ \frac{\delta}{r_Y^{3-\hat{\alpha}}(Z)} + \frac{\delta}{r_Y^{3-\alpha_K}(Z)} + \frac{\delta^{1+\hat{\alpha}}}{r_Y^3(Z)} \right\}$$

for each $Z \in C_d^{\partial}(Y,s) \cap C_{31\delta}^{\partial}(Y,s)'$,

with $k_2, \dots, k_{29}$ dependent solely upon M and K . Directly from (190) and (211), there results

$$I_{37} \leq 2^{3/2} \cdot (m_j^K)^{-1} \cdot k_{29} \left\{ \delta \int_{31\delta}^d \frac{1}{\zeta^{2-\hat{\alpha}}} d\zeta + \delta \int_{31\delta}^d \frac{1}{\zeta^{2-\alpha_K}} d\zeta + \delta^{1+\hat{\alpha}} \int_{31\delta}^d \frac{1}{\zeta^2} d\zeta \right\},$$

whence the arguments of previous estimations lead to the conclusion that I_{37} can be shown to satisfy an inequality of the form of (68).

With the demonstration that there exist $\lambda(j) > 0$ and $\lambda(j) \in (0,1]$, depending upon ϕ , M , and K alone, such that (68) is true for $j = 1, \dots, 3L$, we have completed the proof of the first assertion of the theorem.

Now, let $\nu: \partial B \rightarrow K$ be locally Hölder continuous on ∂B , as in hypothesis (iv): we must show that (iii) is true when the function ϕ of that statement is taken to be $\Lambda_2^1\{\nu\}$, $\Lambda_{3jk}^1\{\nu\}$, or $\tilde{\Lambda}_{2j}^1\{\nu\}$. Because of the similarities in form between the latter functions and $\Lambda_{11}^1\{\nu\}$ and $\tilde{\Lambda}_1^1\{\nu\}$, it is fairly clear that the proofs of the required results here for (2) are essentially the same as those of the

corresponding statements relative to (IV.22.2) in the proof of [IV.22], while the verifications of the claims here concerning (3) do not significantly differ from those of the corresponding facts for (IV.24.2), which are contained in the proof of [IV.24]. Therefore, we shall accept the statements for (2) and (3) as having been proven. Further, we shall prove that (4) is true when $\phi = \Lambda_2^1\{\mu\}$; in the remaining two cases, the details of the proofs are very similar, and so we shall omit them. Then choose $(Y, s) \in \partial B$, i.e., $s \in \mathbb{R}$ and $Y \in \partial B_s$: since $\tau(\chi_s^{-1}(Y); Y, s) = 0$ and $\chi^{*-1}(Y, s) = (\chi_s^{-1}(Y), s)$, we find that

$$\begin{aligned} [\chi_{,4}^c]_{(Y,s)} \circ \chi_s^{-1}(Y) &= \chi_{,4}^c(\chi_s^{-1}(Y), s) = \chi_{,4}^c \circ \chi^{*-1}(Y, s), \\ v_{(Y,s)}^c \circ \chi_s(\chi_s^{-1}(Y)) &= v_{(Y,s)}^c(Y) = \chi_{,4}^c(\chi_s^{-1}(Y), s) = \chi_{,4}^c \circ \chi^{*-1}(Y, s), \end{aligned}$$

and

$$\begin{aligned} [\hat{\mu} \cdot \hat{J}\chi]_{(Y,s)} \circ \chi_s^{-1}(Y) &= (\hat{\mu} \cdot \hat{J}\chi)(\chi_s^{-1}(Y), s) \\ &= \{(\hat{\mu} \circ \chi^{*-1})(Y, s)\} \cdot (\hat{J}\chi) \circ \chi^{*-1}(Y, s) \\ &= \mu(Y, s) \cdot (\hat{J}\chi) \circ \chi^{*-1}(Y, s). \end{aligned}$$

Consequently, from (IV.14.34), we have simply

$$\begin{aligned} \Lambda_2^1\{\mu\}_{(Y,s)} \circ \chi_s^{-1}(Y) &= (\{1 - |\chi_{,4}^c|^2\}^{\frac{1}{2}} \cdot \hat{J}\chi) \circ \chi^{*-1}(Y, s) \cdot \mu(Y, s) \\ &\text{for each } (Y, s) \in \partial B. \end{aligned} \tag{212}$$

The desired result follows directly from (212): μ is locally

Hölder continuous on ∂B , χ^{*-1} is locally Lipschitz continuous on ∂B , while $\chi_{,4}$ and $J\chi$ are locally Hölder continuous on $\partial R \times R$, whence $(Y,s) \mapsto \Lambda_2^1\{\mu\}_{(Y,s)} \circ \chi_s^{-1}(Y)$ must be locally Hölder continuous on ∂B . Thus, whenever $\bar{K} \subset R$ is compact, there exist $\bar{\kappa}_3 > 0$ and $\bar{\beta}_3 \in (0,1]$ for which (4) holds with $\phi = \Lambda_2^1\{\mu\}$.

Finally, suppose that $M \in M(2)$: the proof of [IV.22] provides reasoning which shows that (i) and (ii) are fulfilled in that case; cf., also, [IV.23.a]. $\square$.

Before we can return to the examination of the partial derivatives of the kinematic single layer potential, begun in [IV.14 and 17], we must point out the most elementary properties of the class of functions $\omega_{3\beta}\{\phi\}$, as in [IV.15.iii]. Specifically, we wish to determine conditions sufficient to ensure that such a function possesses a continuous extension to R^4 which is locally Hölder continuous on ∂B , for $\beta = 1$ or $\beta = 2$.

[IV.30] DEFINITIONS. Let M be a motion in $M(1)$ which is also such that B_t^0 is a Lyapunov domain for each $t \in R$. Let (R,X) be a reference pair for M as in [I.3.25], and suppose that ϕ and Γ are as in [IV.15].

- (i) If $0 < \beta < 2$, it is clear from Example [IV.19] that $r_Y^{-\beta} \cdot \Gamma(Y,s) \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} \in L_1(\partial B_s)$ whenever $(Y,s) \in \partial B$, for then B_s^0 is a Lyapunov domain, while $\Gamma(Y,s) \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} \in L_\infty(\partial B_s)$. In this case,

we extend the definition of $w_{3\beta}(\phi)$ to all of $\mathbb{R}^4$, maintaining the same notation for the extension, by declaring that (IV.15.4) holds for each $(X,t) \in \mathbb{R}^4$.

- (ii) If, for each $(Y,s) \in \partial B$, there exist $\tilde{\kappa} > 0$, $\Delta > 0$, and $\alpha \in (0,1]$, perhaps depending on (Y,s) , such that

$$|\phi_{(Y,s)} \circ \chi_s^{-1}(Z)| \leq \tilde{\kappa} \cdot r_Y^\alpha(Z) \quad \text{for each } Z \in \partial B_s \cap B_d^3(Y), \quad (1)$$

then, again from [IV.19], $r_Y^{-2} \cdot \Gamma_{(Y,s)} \cdot \phi_{(Y,s)} \circ \chi_s^{-1} \cdot J\chi_s^{-1} \in L_1(\partial B_s)$ for each $(Y,s) \in \partial B$. In this case, we extend the definition of $w_{32}(\phi)$ to all of $\mathbb{R}^4$, maintaining the same notation for the extension, by declaring that (IV.15.4) (with $\beta = 2$) holds for each $(X,t) \in \mathbb{R}^4$. ■.

Within the settings described in [IV.30], the following theorem is concerned with the characteristics of $w_{3\beta}(\phi)|_{\partial B}$, for $0 < \beta \leq 2$.

[IV.31] T H E O R E M. Let M be a motion in $\mathcal{M}(1)$ such that $\{B_\zeta^0\}_{\zeta \in \mathbb{R}}$ is locally uniformly Lyapunov. Let (R,X) be a reference pair for M with the properties listed in [I.3.25], and ϕ and Γ be as in [IV.15], i.e., $(P,X,t) \mapsto \phi_{(X,t)}(P)$ is in $C(\partial R \times \mathbb{R}^4)$, while $(Z,X,t) \mapsto \Gamma_{(X,t)}(Z)$ is continuous and bounded on $\{(Z,X,t) \mid (X,t) \in \mathbb{R}^4, Z \in \partial B_\zeta \cap \{X\}'\}$ into $\mathbb{R}$. Suppose further that

- (i) whenever $\bar{K}$ is a compact subset of $\mathbb{R}$, there exist positive numbers $\bar{\kappa}_1$, $\bar{\kappa}'_1$, and $\bar{\Delta}_1$, and a number $\bar{\alpha}_1 \in (0,1]$ for which

$$\begin{aligned} & |\Gamma_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}\hat{s}}(Z) - \Gamma_{(Y, s)}(Z)| \\ & \leq \bar{\kappa}_1 \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4^{\bar{\alpha}_1} + \frac{\bar{\kappa}'_1}{r_Y(Z)} \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4 \\ & \text{for } (Y, s) \text{ and } (\hat{Y}, \hat{s}) \in \bigcup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\} \end{aligned} \quad (1)$$

$$\text{with } |(\hat{Y}, \hat{s}) - (Y, s)|_4 \leq \bar{\Delta}_1,$$

$$\text{and } Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{\hat{s}\hat{s}}(\hat{Y})\}'.$$

Then

- (i)' For $0 < \beta < 2$, $w_{3\beta}\{\phi\} | \partial B$ is continuous.

Now, let ϕ satisfy the following local Hölder condition:

- (ii) Whenever $\bar{K} \subset \mathbb{R}$ is compact, there exist positive numbers $\bar{\kappa}_2$ and $\bar{\Delta}_2$, and a number $\bar{\alpha}_2 \in (0,1]$ such that

$$\begin{aligned} & |\phi_{(\hat{Y}, \hat{s})}(P) - \phi_{(Y, s)}(P)| \leq \bar{\kappa}_2 \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4^{\bar{\alpha}_2} \\ & \text{for } (Y, s) \text{ and } (\hat{Y}, \hat{s}) \in \bigcup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\} \end{aligned} \quad (2)$$

$$\text{with } |(\hat{Y}, \hat{s}) - (Y, s)|_4 \leq \bar{\Delta}_2, \quad \text{and } P \in \partial R.$$

Then

(ii)' for $0 < \beta < 2$, $w_{3\beta}\{\phi\}|_{\partial B}$ is locally Hölder continuous.

If it is also known that

(iii) whenever $\bar{K} \subset \mathbb{R}$ is compact, there exist positive numbers $\bar{\kappa}_3$ and $\bar{\Delta}_3$, and a number $\bar{\alpha}_3 \in (0,1]$ such that

$$|\phi_{(Y,s)} \circ \chi_s^{-1}(Z)| \leq \bar{\kappa}_3 \cdot r_Y^{\bar{\alpha}_3}(Z) \quad (3)$$

for $(Y,s) \in \bigcup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}$ and $Z \in \partial B_s \cap B_{\bar{\Delta}_3}^3(Y)$,

then

(iii)' $w_{32}\{\phi\}|_{\partial B}$ is locally Hölder continuous.

Next, suppose that $\mu \in C(\partial B)$. We consider the application of (i)', (ii)', and (iii)' to those functions defined by certain of the terms appearing on the right in (IV.14.32, 37, 42, 45, 49, and 52): assume that, for each compact set $\bar{K} \subset \mathbb{R}$, $\chi_{,4}(P, \cdot)|_{\bar{K}}$ is Hölder continuous, uniformly for $P \in \partial R$.

(iv)' Hypothesis (i) is fulfilled if $\Gamma_{(X,t)}$ is replaced therein by $\Gamma_{(X,t)}^{01}$, $\Gamma_{(X,t)}^{12}$, $\Gamma_{(X,t)}^{13}$, $\Gamma_{(X,t)}^{23}$, $\Gamma_{(X,t)}^{02} r_{X,i}$, $\Gamma_{(X,t)}^{13} r_{X,i}$, or $\Gamma_{(X,t)}^{03} r_{X,i} r_{X,j}$ on $\partial B_t \cap \{X\}'$, for each $(X,t) \in \mathbb{R}^4$,

and

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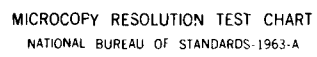
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(v)' hypothesis (iii) is satisfied by taking ϕ to be either $\Lambda_{41}^1(\mu)$ or $\tilde{\Lambda}_3^1(\mu)$ (cf., (IV.14.36 and 48), respectively).

(vi)' If μ is locally Hölder continuous on ∂B , and, for each compact $\bar{K} \subset \mathbb{R}$, $\hat{J}x(p, \cdot)|_{\bar{K}}$ is Hölder continuous, uniformly for $p \in \partial R$, then the functions given on ∂B by

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{13} \cdot \Lambda_{41}^1(\mu)_{(Y,s)} \circ x_s^{-1} \cdot Jx_s^{-1} d\lambda_{\partial B_s} \quad (4)$$

and

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{13} \cdot \tilde{\Lambda}_3^1(\mu)_{(Y,s)} \circ x_s^{-1} \cdot Jx_s^{-1} d\lambda_{\partial B_s}, \quad (5)$$

for each $(Y,s) \in \partial B$, are locally Hölder continuous.

(vii)' Suppose that $M \in M(1;0)$, and the reference pair (R,x) possesses the properties described in [IV.10], as well as those already enumerated. Then the functions given on ∂B by

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{03} \cdot r_{Y,1} r_{Y,2} \cdot \Lambda_{11}^2(\mu)_{(Y,s)} \circ x_s^{-1} \cdot Jx_s^{-1} d\lambda_{\partial B_s}, \quad (6)$$

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{13} \cdot r_{Y,2} \cdot \Lambda_{21}^2(\mu)_{(Y,s)} \circ x_s^{-1} \cdot Jx_s^{-1} d\lambda_{\partial B_s}, \quad (7)$$

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{13} \cdot r_{Y,i} \cdot \Lambda_{3\{\mu\}}^2(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (8)$$

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{23} \cdot \Lambda_{41}^2\{\mu\}(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (9)$$

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{13} \cdot r_{Y,\ell} \cdot \tilde{\Lambda}_{1\ell}^2\{\mu\}(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (10)$$

and

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{23} \cdot \tilde{\Lambda}_2^2\{\mu\}(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (11)$$

for each $(Y,s) \in \partial B$, are continuous (cf., (IV.14.38-41, 50, and 51)). If it is also known that μ is locally Hölder continuous on ∂B , and, for each compact set $\bar{K} \subset \mathbb{R}$, $\chi_{,44}(P, \cdot)|_{\bar{K}}$ and $J\chi(P, \cdot)|_{\bar{K}}$ are Hölder continuous, uniformly for $P \in \partial R$, then the functions on ∂B which are constructed using (6)-(11) are locally Hölder continuous.

(viii)' Suppose that $M \in \mathcal{M}(1;0)$, and the reference pair (R, χ) possesses the properties described in [IV.10], in addition to those already imposed. Assume, moreover, that $D_4^0 \mu \in C(\partial R \times \mathbb{R})$. Then the functions which are given on ∂B by

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{02} \cdot r_{Y,i} \cdot \Lambda_1^3\{\mu\}(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (12)$$

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{12} \cdot \Lambda_{21}^3\{\mu\}(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (13)$$

and

$$\int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma_{(Y,s)}^{12} \cdot \tilde{\Lambda}_1^3\{\mu\}(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s}, \quad (14)$$

for each $(Y,s) \in \partial B$, are continuous (cf., (IV.14.43, 44, and 53)). If it is also known that, for each compact set $\bar{K} \subset \mathbb{R}$, $D_4^0 \hat{u}(P, \cdot)|_{\bar{K}}$ and $D_4 \hat{J}X(P, \cdot)|_{\bar{K}}$ are Hölder continuous, uniformly for $P \in \partial R$, then the functions constructed on ∂B by using (12)-(14) are locally Hölder continuous.

P R O O F. Until we consider the proof of (v)', let it be understood that we are supposing hypothesis (iii) to be in force whenever we consider $w_{32}\{\phi\}$; we shall not do so when speaking of $w_{3\beta}\{\phi\}$ with $0 < \beta < 2$. With [IV.30], it follows that $w_{3\beta}\{\phi\}$ is defined on all of $\mathbb{R}^4$ by (IV.15.4) if $\beta \in (0, 2]$. Then, selecting (Y,s) and $(\hat{Y}, \hat{s}) \in \partial B$, and recalling that

$$\chi_{\hat{s}}^{-1} \circ \chi_{s\hat{s}} = \chi_s^{-1}, \quad (15)$$

$$(J\chi_{\hat{s}}^{-1}) \circ \chi_{s\hat{s}} \cdot J\chi_{s\hat{s}} = J\chi_s^{-1}, \quad (16)$$

for $\beta \in (0, 2]$ we can write

$$\begin{aligned}
 & |w_{3\beta}\{\phi\}(\hat{Y}, \hat{s}) - w_{3\beta}\{\phi\}(Y, s)| \\
 &= \left| \int_{\partial B_s} \left\{ \frac{1}{r_{\hat{Y}}^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \cdot \phi(\hat{Y}, \hat{s})^{\circ X_{\hat{s}}^{-1}} \cdot JX_{\hat{s}}^{-1} \right\} \circ X_{s\hat{s}} \cdot JX_{s\hat{s}} d\lambda_{\partial B_s} \right. \\
 &\quad \left. - \int_{\partial B_s} \frac{1}{r_Y^\beta} \cdot \Gamma(Y, s) \cdot \phi(Y, s)^{\circ X_s^{-1}} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| \quad (17) \\
 &= \left| \int_{\partial B_s} \left\{ \left\{ \frac{1}{r_{\hat{Y}}^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \right\} \circ X_{s\hat{s}} \cdot \phi(\hat{Y}, \hat{s})^{\circ X_s^{-1}} - \frac{1}{r_Y^\beta} \cdot \Gamma(Y, s) \cdot \phi(Y, s)^{\circ X_s^{-1}} \right\} \right. \\
 &\quad \left. \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|.
 \end{aligned}$$

Now let $K = [t_1, t_2]$ denote a compact interval such that s and $\hat{s}$ lie in (t_1, t_2) : $\{B_c^0\}_{c \in K}$ is a uniformly Lyapunov family, and we let (a_K, α_K, d_K) denote a uniform set of Lyapunov constants for this collection. Fix d so that

$$0 < d < \frac{7}{9} d_K. \quad (18)$$

Setting

$$\delta := |(\hat{Y}, \hat{s}) - (Y, s)|_4,$$

we assume further that

$$0 < \delta < \frac{1}{21} d. \quad (19)$$

Clearly, $C_d^\partial(Y, s)$ and $C_{2, \delta}^\partial(Y, s)$ are defined, with $C_{2, \delta}^\partial(Y, s) \subset C_d^\partial(Y, s)$. Continuing from (17), therefore,

$$\begin{aligned}
 & |w_{3\beta}\{\phi\}(\hat{Y}, \hat{s}) - w_{3\beta}\{\phi\}(Y, s)| \\
 & \leq \left| \int_{\partial B_s \cap C_d^\partial(Y, s)} \left\{ \frac{1}{r_Y^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \right\}^{\circ X_{ss}} \cdot \phi(\hat{Y}, \hat{s})^{\circ X_s^{-1}} \right. \\
 & \quad \left. - \frac{1}{r_Y^\beta} \cdot \Gamma(Y, s) \cdot \phi(Y, s)^{\circ X_s^{-1}} \right\} JX_s^{-1} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_d^\partial(Y, s) \cap C_{2,1\delta}^\partial(Y, s)} \left\{ \frac{1}{r_Y^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \right\}^{\circ X_{ss}} \cdot \phi(\hat{Y}, \hat{s})^{\circ X_s^{-1}} \right. \\
 & \quad \left. - \frac{1}{r_Y^\beta} \cdot \Gamma(Y, s) \cdot \phi(Y, s)^{\circ X_s^{-1}} \right\} JX_s^{-1} d\lambda_{\partial B_s} \Big| \\
 & + \left| \int_{C_{2,1\delta}^\partial(Y, s)} \left\{ \frac{1}{r_Y^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \right\}^{\circ X_{ss}} \cdot \phi(\hat{Y}, \hat{s})^{\circ X_s^{-1}} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| \\
 & + \left| \int_{C_{2,1\delta}^\partial(Y, s)} \frac{1}{r_Y^\beta} \cdot \Gamma(Y, s) \cdot \phi(Y, s)^{\circ X_s^{-1}} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| \\
 & \leq \sum_{j=1}^8 I_j^\beta(Y, s; \hat{Y}, \hat{s}),
 \end{aligned} \tag{20}$$

in which the $I_j^\beta(Y, s; \hat{Y}, \hat{s}) = I_j^\beta$, $j = 1, \dots, 8$, are given by

$$\begin{aligned}
 I_1^\beta := & \left| \int_{\partial B_s \cap C_d^\partial(Y, s)} \left\{ \frac{1}{r_Y^\beta \circ X_{ss}} - \frac{1}{r_Y^\beta} \right\} \cdot \Gamma(\hat{Y}, \hat{s})^{\circ X_{ss}} \right. \\
 & \left. \cdot \phi(\hat{Y}, \hat{s})^{\circ X_s^{-1}} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|,
 \end{aligned} \tag{21}$$

$$I_2^\beta := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y^\beta} \cdot \{ \Gamma(\hat{Y}, \hat{s}) \circ \chi_{s\hat{s}}^{-1} \Gamma(Y,s) \} \right. \\ \left. \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (22)$$

$$I_3^\beta := \left| \int_{\partial B_s \cap C_d^\partial(Y,s)} \frac{1}{r_Y^\beta} \cdot \Gamma(Y,s) \cdot \{ \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1} \} \right. \\ \left. \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (23)$$

$$I_4^\beta := \left| \int_{C_d^\partial(Y,s) \cap C_{2,\delta}^\partial(Y,s)} \left\{ \frac{1}{r_Y^\beta \circ \chi_{s\hat{s}}} - \frac{1}{r_Y^\beta} \right\} \cdot \Gamma(\hat{Y}, \hat{s}) \circ \chi_{s\hat{s}} \right. \\ \left. \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (24)$$

$$I_5^\beta := \left| \int_{C_d^\partial(Y,s) \cap C_{2,\delta}^\partial(Y,s)} \frac{1}{r_Y^\beta} \{ \Gamma(\hat{Y}, \hat{s}) \circ \chi_{s\hat{s}}^{-1} \Gamma(Y,s) \} \right. \\ \left. \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (25)$$

$$I_6^\beta := \left| \int_{C_d^\partial(Y,s) \cap C_{2,\delta}^\partial(Y,s)} \frac{1}{r_Y^\beta} \cdot \Gamma(Y,s) \right. \\ \left. \cdot \{ \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} - \phi(Y,s) \circ \chi_s^{-1} \} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (26)$$

$$I_7^\beta := \left| \int_{C_{2,1}^\partial(Y,s)} \left\{ \frac{1}{r_Y^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \right\} \circ \chi_{s\hat{s}} \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|, \quad (27)$$

and

$$I_8^\beta := \left| \int_{C_{2,1}^\partial(Y,s)} \frac{1}{r_Y^\beta} \cdot \Gamma(Y,s) \cdot \phi(Y,s) \circ \chi_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \right|. \quad (28)$$

Let M_ϕ^K , M_Γ , and M_J^K denote positive numbers such that

$$|\phi| \leq M_\phi^K \quad \text{on} \quad \partial R \times \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\},$$

$$|\Gamma| \leq M_\Gamma,$$

and

$$|J\chi_\xi^{-1}(Z)| \leq M_J^K \quad \text{for} \quad (Z, \xi) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}.$$

We shall first prove (i)', then verify (ii)' and (iii)' together, proceeding in each case from inequality (20). Of course, we maintain the notation which has already been established.

(i)' Here, it is sufficient to show that $w_{3\beta}(\phi)|_{\partial B}$ is continuous at (Y,s) , for each $\beta \in (0,2)$. In turn, choosing $\epsilon > 0$ and writing

$$\delta_0 := \min \left\{ \frac{1}{2\epsilon} d, |s-t_1|, |s-t_2| \right\},$$

it suffices to produce numbers $\{\delta_j\}_{j=1}^8 \subset (0, \delta_0)$, perhaps depending upon ϵ , β , and (Y,s) , such that

$$\begin{aligned}
 I_j^\beta(Y, s; \hat{Y}, \hat{s}) &< \epsilon \quad \text{whenever} \quad (\hat{Y}, \hat{s}) \in \partial B \\
 \text{with} \quad 0 < \delta &:= |(\hat{Y}, \hat{s}) - (Y, s)|_4 < \delta_j, \\
 \text{for} \quad j &= 1, \dots, 8 \quad \text{and} \quad \beta \in (0, 2).
 \end{aligned} \tag{29}$$

(Note that $0 < \delta < \delta_0$ implies that inequality (20) is valid.)

We proceed to establish that this can be done. Let us assume at the outset that

$$0 < \delta < \delta'_0 := \min \left\{ \frac{d}{4_1}, |s-t_1|, |s-t_2|, \Delta_1 \right\} \quad (\leq \delta_0), \tag{30}$$

wherein Δ_1 is the positive number corresponding to K which figures in hypothesis (i). In particular, it is then certainly true that inequality (20) is valid. Fix $\beta \in (0, 2)$.

I_1^β : If $Z \in \partial B_s \cap C_{2, \delta}^\partial(Y, s)'$, then $r_Y(Z) \geq 2\delta$. Since

$$\begin{aligned}
 |r_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - r_Y(Z)| &\leq |\hat{Y} - Y|_3 + |\chi_{s\hat{s}}(Z) - Z|_3 \\
 &\leq |\hat{Y} - Y|_3 + c^* |\hat{s} - s| \\
 &\leq \delta \quad \text{for} \quad Z \in \partial B_s,
 \end{aligned} \tag{31}$$

the usual argument yields the inequalities

$$\frac{1}{2} \leq \frac{r_{\hat{Y}}(\chi_{s\hat{s}}(Z))}{r_Y(Z)} \leq \frac{3}{2} \quad \text{for each} \quad Z \in \partial B_s \cap C_{2, \delta}^\partial(Y, s)'. \tag{32}$$

Now, choose $Z \in \partial B_s \cap C_{2, \delta}^\partial(Y, s)'$, and suppose for the moment that the numbers $r_{\hat{Y}}(\chi_{s\hat{s}}(Z))$ and $r_Y(Z)$ are unequal; (32) implies that each of these numbers is positive, whence the mean-value theorem shows that there is some $\xi = \xi(Z; Y, s; \hat{Y}, \hat{s})$ lying between them for

which

$$\frac{1}{r_Y^\beta(\chi_{ss}(Z))} - \frac{1}{r_Y^\beta(Z)} = -\frac{\beta}{\xi^{\beta+1}} \cdot (r_Y^\beta(\chi_{ss}(Z)) - r_Y^\beta(Z)). \quad (33)$$

If $\xi > r_Y(\chi_{ss}(Z))$, then (32) gives

$$\frac{1}{\xi} = \frac{1}{r_Y(Z)} \cdot \frac{r_Y(Z)}{\xi} < \frac{1}{r_Y(Z)} \cdot \frac{r_Y(Z)}{r_Y(\chi_{ss}(Z))} < \frac{2}{r_Y(Z)},$$

while if $\xi > r_Y(Z)$, we have simply

$$\frac{1}{\xi} = \frac{1}{r_Y(Z)} \cdot \frac{r_Y(Z)}{\xi} < \frac{1}{r_Y(Z)}.$$

Thus, from (31) and (33),

$$\begin{aligned} \left| \frac{1}{r_Y^\beta(\chi_{ss}(Z))} - \frac{1}{r_Y^\beta(Z)} \right| &\leq \frac{\beta \cdot 2^{\beta+1}}{r_Y^{\beta+1}(Z)} \cdot |r_Y^\beta(\chi_{ss}(Z)) - r_Y^\beta(Z)| \\ &\leq \frac{1\beta \cdot 2^{\beta+1}}{r_Y^{\beta+1}(Z)} \cdot \delta \end{aligned} \quad (34)$$

for each $Z \in \partial B_s \cap C_{2\delta}^\partial(Y, s)'$

(since (34) is certainly true even when $r_Y(\chi_{ss}(Y)) = r_Y(Z)$). If $Z \in \partial B_s \cap C_d^\partial(Y, s)'$, then $r_Y(Z) \geq d$, so (21) gives, with (34),

$$\begin{aligned} I_1^\beta &\leq M_\Gamma \cdot M_\phi^K \cdot M_J^K \cdot \frac{1\beta \cdot 2^{\beta+1}}{d^{\beta+1}} \cdot \delta \cdot \int_{\partial B_s \cap C_{2\delta}^\partial(Y, s)'} d\lambda_{\partial B_s} \\ &< M_\Gamma \cdot M_\phi^K \cdot M_J^K \cdot \frac{1\beta \cdot 2^{\beta+1}}{d^{\beta+1}} \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta) \cdot \delta. \end{aligned} \quad (35)$$

Obviously, (35) allows us to choose $\delta_1 = \delta_1(\varepsilon, \beta, s, K) \in (0, \delta_0')$ such that (29) is true for $j = 1$. Strictly speaking, the second

inequality in (35) is not needed here, but will be of use later.

I₂^B: Using (i), in view of (30) we have

$$|\Gamma_{(\hat{Y}, \hat{s})} \circ \chi_{ss}^{-1}(Z) - \Gamma_{(Y, s)}(Z)| \leq \kappa_1 \cdot \delta^{\alpha_1} + \frac{\kappa_1'}{r_Y(Z)} \cdot \delta$$

for each $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{ss}^{-1}(\hat{Y})\}'$,

(36)

$\kappa_1 > 0$, $\kappa_1' > 0$, and $\alpha_1 \in (0, 1]$ depending on only K and M .

Then, from (22),

$$I_2^B \leq M_\phi^K \cdot M_J^K \cdot \frac{1}{d^B} \left\{ \kappa_1 \delta^{\alpha_1} + \frac{\kappa_1'}{d} \cdot \delta \right\} \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta),$$
(37)

whence it is easy to see that there exists $\delta_2 = \delta_2(\epsilon, \beta, s, K) \in (0, \delta_0')$

for which (29) holds with $j = 2$.

I₃^B: ϕ is continuous on $\partial R \times R^4$, hence, in particular, uniformly continuous on the compact set $\partial R \times \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$. Thus, there exists a number $\delta_3 = \delta_3(\epsilon, \beta, s, K) \in (0, \delta_0')$ such that

$$|\phi_{(\hat{Y}, \hat{s})}(P) - \phi_{(Y, s)}(P)| < \epsilon \cdot d^B \cdot \{M_\Gamma \cdot M_J^K \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta)\}^{-1}$$

for $\delta < \delta_3$, and $P \in \partial R$,

and so

$$|\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(Z) - \phi_{(Y, s)} \circ \chi_s^{-1}(Z)| < \epsilon \cdot d^B \cdot \{M_\Gamma \cdot M_J^K \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta)\}^{-1},$$

for $\delta < \delta_3$, and $Z \in \partial B_s$.

(38)

But now inequality (29) with $j = 3$ results directly from (23)

and (38), as one can easily verify.

I₄^β: Clearly, we can apply (34) in the estimation of this term, obtaining, from (24),

$$\begin{aligned} I_4^\beta &\leq M_T \cdot M_\phi^K \cdot M_J^K \cdot 1\beta \cdot 2^{\beta+1} \cdot \delta \int_{C_d^\partial(Y,s) \cap C_{21\delta}^\partial(Y,s)} \frac{1}{r_Y^{\beta+1}} d\lambda_{\partial B_s} \\ &\leq 2^{\beta+\frac{5}{2}} \pi \cdot 1\beta \cdot M_T \cdot M_\phi^K \cdot M_J^K \cdot \delta \int_{21\delta}^d \frac{1}{\zeta^\beta} d\zeta, \end{aligned} \quad (39)$$

while

$$\delta \cdot \int_{21\delta}^d \frac{1}{\zeta^\beta} d\zeta = \begin{cases} \delta \cdot \frac{1}{1-\beta} \cdot \zeta^{1-\beta} \Big|_{21\delta}^d < \frac{d^{1-\beta}}{1-\beta} \cdot \delta, & \text{if } \beta \in (0,1), \\ \delta \cdot \ln \frac{d}{21\delta} \leq M_0 \delta^{\alpha'}, & \text{if } \beta \in (1,2), \\ \delta \cdot \frac{1}{1-\beta} \cdot \zeta^{1-\beta} \Big|_{21\delta}^d < \frac{(21)^{1-\beta}}{\beta-1} \cdot \delta^{2-\beta}, & \text{if } \beta \in (1,2), \end{cases} \quad (40)$$

wherein $\alpha' \in (0,1)$, and M_0 is such that

$$\zeta^{1-\alpha'} \cdot \ln \frac{d}{21\zeta} \leq M_0 \quad \text{for} \quad 0 < \zeta < \delta'_0. \quad (41)$$

Upon combining (39) and (40), it becomes clear that there exists some $\delta_4 = \delta_4(\varepsilon, \beta, s, K) \in (0, \delta'_0)$ such that (29) is true when $j = 4$.

I₅^β: Immediately from (25) and (36),

$$\begin{aligned} I_5^\beta &\leq M_J^K \cdot M_\phi^K \cdot \int_{C_d^\partial(Y,s) \cap C_{21\delta}^\partial(Y,s)} \left\{ \frac{\kappa_1}{r_Y^\beta} \cdot \delta^{\alpha_1} + \frac{\kappa'_1}{r_Y^{\beta+1}} \cdot \delta \right\} d\lambda_{\partial B_s} \\ &\leq 2^{3/2} \pi \cdot M_J^K \cdot M_\phi^K \cdot \left\{ \kappa_1 \cdot \delta^{\alpha_1} \cdot \int_{21\delta}^d \frac{1}{\zeta^{\beta-1}} d\zeta + \kappa'_1 \cdot \delta \cdot \int_{21\delta}^d \frac{1}{\zeta^\beta} d\zeta \right\} \end{aligned}$$

$$< 2^{3/2} \pi \cdot M_J^K \cdot M_\phi^K \cdot \left\{ \frac{\kappa_1 \cdot d^{2-\beta}}{2-\beta} \cdot \delta^{\alpha_1 + \kappa_1' \cdot \delta} \cdot \int_{2\delta}^d \frac{1}{\zeta^\beta} d\zeta \right\}. \quad (42)$$

Now, (40) and (42) show that the desired result can be achieved for I_5^β .

I_6^β : Arguing from the continuity of ϕ , as in the analysis of I_3^β , there exists a $\delta_6 = \delta_6(\varepsilon, \beta, s, K) \in (0, \delta_0')$ such that

$$|\phi(\hat{Y}, \hat{s}) \circ \chi_s^{-1}(Z) - \phi(Y, s) \circ \chi_s^{-1}(Z)| < \varepsilon \cdot \left\{ M_I \cdot M_J^K \cdot 2^{3/2} \pi \cdot \frac{d^{2-\beta}}{2-\beta} \right\}^{-1} \quad (43)$$

for $\delta < \delta_6$ and $Z \in \partial B_s$.

Since

$$\int_{C_d^\partial(Y, s) \cap C_{2\delta}^\partial(Y, s)} \frac{1}{r_Y^\beta} d\lambda_{\partial B_s} < 2^{3/2} \pi \cdot \frac{d^{2-\beta}}{2-\beta}, \quad (44)$$

it is evident that (29), with $j = 6$, follows from (26) and (43).

I_7^β : We begin here with the observation that

$$\chi_{s\hat{s}}(C_{2\delta}^\partial(Y, s)) \subset C_{4\delta}^\partial(\hat{Y}, \hat{s}). \quad (45)$$

Indeed, supposing that $Z \in C_{2\delta}^\partial(Y, s)$, we have $r_Y(Z) \leq \frac{9}{7} r_Y(\pi_Y(Z)) < \frac{9}{7} \cdot 2\delta$, and so

$$\begin{aligned} |\pi_{\hat{Y}}(\chi_{s\hat{s}}(Z)) - \hat{Y}|_3 &\leq |\chi_{s\hat{s}}(Z) - \hat{Y}|_3 \\ &\leq |Y - \hat{Y}|_3 + |\chi_{s\hat{s}}(Z) - Z|_3 + |Z - Y|_3 \\ &< \delta + \frac{9}{7} \cdot 2\delta = \frac{25}{7} \cdot \delta < 4\delta, \end{aligned}$$

whence (45) follows; note that $4_1\delta < d < \frac{7}{9} d_K$, by (18) and (30), so $C_{4_1\delta}^\partial(\hat{Y}, \hat{s})$ is defined, since $\hat{s} \in K$ (again, by (30)). Transforming in (27) to integration over a subset of $\partial B_{\hat{s}}$, keeping (15) and (16) in mind, and using (45), we can write

$$\begin{aligned} I_7^\beta &= \left| \int_{X_{\hat{s}}(C_{2_1\delta}^\partial(Y, s))} \frac{1}{r_{\hat{Y}}^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \cdot J \chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \right| \\ &\leq \left| \int_{C_{4_1\delta}^\partial(\hat{Y}, \hat{s})} \frac{1}{r_{\hat{Y}}^\beta} \cdot \Gamma(\hat{Y}, \hat{s}) \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \cdot J \chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \right| \\ &\leq M_\Gamma \cdot M_\phi^K \cdot M_J^K \cdot 2^{3/2} \pi \cdot \int_0^{4_1\delta} \frac{1}{\zeta^{\beta-1}} d\zeta \\ &= 2^{3/2} \pi \cdot M_\Gamma \cdot M_\phi^K \cdot M_J^K \cdot \frac{1}{2-\beta} \cdot (4_1)^{2-\beta} \cdot \delta^{2-\beta}. \end{aligned} \tag{46}$$

Evidently, the latter inequality allows us to find a $\delta_7 = \delta_7(\epsilon, \beta, s, K) \in (0, \delta'_0)$ for which (29) holds when $j = 7$.

I_8^β : Directly from (28),

$$I_8^\beta \leq 2^{3/2} \pi \cdot M_\Gamma \cdot M_\phi^K \cdot M_J^K \cdot \frac{1}{2-\beta} \cdot (2_1)^{2-\beta} \cdot \delta^{2-\beta}, \tag{47}$$

with which we can surely choose $\delta_8 = \delta_8(\epsilon, \beta, s, K) \in (0, \delta'_0)$ possessing the requisite properties.

The proof of (i)' is now complete.

(ii)' and (iii)' As noted, we shall present the proofs of these statements together, supposing hypothesis (ii) to be in effect and, if $\beta = 2$, that (iii) holds. Select $\beta \in (0, 2]$. It

is sufficient to show that $w_{3\beta}\{\phi\} | \cup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$ is Hölder continuous; in turn, for this it is sufficient to show that, for some $\Delta_0 > 0$, there exist $\ell_0 > 0$ and $\alpha_0 \in (0,1]$, perhaps depending upon some or all of ϕ , M , K , β , and Δ_0 , such that

$$|w_{3\beta}\{\phi\}(\hat{Y}, \hat{s}) - w_{3\beta}\{\phi\}(Y, s)| \leq \ell_0 \cdot \delta^{\alpha_0} \quad \text{for } 0 < \delta < \Delta_0 \quad (48)$$

(of course, we retain the convention that (Y, s) and $(\hat{Y}, \hat{s})$ are points of $\cup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$). In fact, we shall assume throughout that

$$0 < \delta < \Delta_0 := \min \left\{ \frac{d}{4}, \Delta_1, \Delta_2 \right\}, \quad (49)$$

Δ_1 and Δ_2 being the positive numbers corresponding to K which appear in hypotheses (i) and (ii), respectively. Concerning d , we may, and shall, suppose (in addition to (18))

$$d < \frac{1}{2} \Delta_3, \quad (50)$$

wherein Δ_3 is the positive number corresponding to K and occurring in hypothesis (iii). Now, with (49) and (18), it is clear that inequality (20) is valid, so that we need only show that there exist $\{\ell_j\}_{j=1}^8 \subset (0, \infty)$ and $\{\alpha'_j\}_{j=1}^8 \subset (0,1]$, depending upon at most ϕ , M , K , and β , such that

$$I_j^{\beta}(Y, s; \hat{Y}, \hat{s}) \leq \ell_j \cdot \delta^{\alpha'_j} \quad \text{whenever (49) holds, for } j = 1, \dots, 8. \quad (51)$$

Once this has been established, there can certainly be produced $\ell_0 > 0$ and $\alpha_0 \in (0,1]$ such that (48) holds. Consequently, the proof has been reduced to verifying that each term on the right in

(20) possesses an estimate of the required form, which we now do.

I₁^β: It is easy to check that (34) is valid here, even if $\beta = 2$, and so (35) is as well; the latter is an equality of the form of (51), for $j = 1$.

I₂^β: By (i) and (49), (36) is true. Then, (37) follows as before, even if $\beta = 2$, whence an estimate of the required form can be derived for I_2^β .

I₃^β: Here, we appeal to hypothesis (ii), which, in view of (49), allows us to assert that

$$|\phi_{(\hat{Y}, \hat{s})} \circ \chi_s^{-1}(Z) - \phi_{(Y, s)} \circ \chi_s^{-1}(Z)| \leq \kappa_2 \cdot \delta^{\alpha_2} \quad \text{for each } Z \in \partial B_s, \quad (52)$$

with $\kappa_2 > 0$ and $\alpha_2 \in (0, 1]$ depending upon ϕ and K at most.

Using this with (23),

$$\begin{aligned} I_3^\beta &\leq \frac{1}{d^\beta} \cdot M_I \cdot M_J^K \cdot \kappa_2 \cdot \delta^{\alpha_2} \cdot \int_{\partial B_s \cap C_d^\partial(Y, s)} d\lambda_{\partial B_s} \\ &\leq \frac{1}{d^\beta} \cdot M_I \cdot M_J^K \cdot \kappa_2 \cdot \max_{\zeta \in K} \lambda_{\partial B_\zeta}(\partial B_\zeta) \cdot \delta^{\alpha_2}, \end{aligned} \quad (53)$$

so I_3^β satisfies an inequality of the requisite form.

I₄^β: Suppose first that $\beta \in (0, 2)$: (39) rests upon (34), and so is valid in the present setting, giving, with (40), an estimate of the form required in (51).

Now assume that $\beta = 2$. In this case, by (iii), we know

that there exist $\kappa_3 > 0$, $\Delta_3 > 0$, and $\alpha_3 \in (0,1]$, depending on ϕ , K , and M alone, such that

$$|\phi_{(\tilde{Y}, \tilde{s})} \circ \chi_{\tilde{s}}^{-1}(\tilde{Z})| \leq \kappa_3 \cdot r_{\tilde{Y}}^{\alpha_3}(\tilde{Z}) \quad \text{for} \quad (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\} \quad (54)$$

and $\tilde{Z} \in \partial B_{\tilde{s}} \cap B_{\Delta_3}^3(\tilde{Y})$.

If $Z \in C_d^{\partial}(Y, s)$, (49) and (50) show that

$$\begin{aligned} |\chi_{s\hat{s}}(Z) - \hat{Y}|_3 &\leq |Y - \hat{Y}|_3 + |\chi_{s\hat{s}}(Z) - Z|_3 + |Z - Y|_3 \\ &\leq 1\delta + \frac{9}{7} \cdot |\Pi_Y(Z) - Y|_3 \\ &< \frac{1}{4} \delta + \frac{9}{7} \delta < 2\delta < \Delta_3. \end{aligned} \quad (55)$$

But (55) allows us to use (54) in conjunction with (32) to write

$$\begin{aligned} |\phi_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}}^{-1}(Z)| &= |\phi_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}}^{-1}(\chi_{s\hat{s}}(Z))| \\ &\leq \kappa_3 \cdot r_{\hat{Y}}^{\alpha_3}(\chi_{s\hat{s}}(Z)) \\ &\leq \kappa_3 \cdot (3/2)^{\alpha_3} \cdot r_Y^{\alpha_3}(Z) \end{aligned} \quad (56)$$

for each $Z \in C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'$.

Employing (24), (34), and (56), we produce

$$\begin{aligned} I_4^2 &\leq M_T \cdot M_J^K \cdot \kappa_3 \cdot (3/2)^{\alpha_3} \cdot 1\beta \cdot 2^{\beta+1} \cdot \delta \cdot \int_{C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'} \frac{1}{r_Y^{3-\alpha_3}} d\lambda_{\partial B_s} \\ &\leq 2^{\beta+\frac{3}{2}} \cdot (3/2)^{\alpha_3} \cdot \pi \cdot 1\beta \kappa_3 \cdot M_T \cdot M_J^K \cdot \delta \cdot \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_3}} d\zeta \end{aligned} \quad (57)$$

and observe that

$$\delta \cdot \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_3}} d\zeta = \begin{cases} \frac{\delta}{\alpha_3-1} \cdot \zeta^{\alpha_3-1} \Big|_{2\delta}^d < \frac{(2\delta)^{\alpha_3-1}}{1-\alpha_3} \cdot \delta^{\alpha_3}, \\ \delta \cdot \ln \frac{d}{2\delta} \leq M'_0 \cdot \delta^{\alpha'}, \end{cases} \quad \text{if } \alpha_3 \in (0,1), \quad (58)$$

(with $\alpha' \in (0,1)$ and M'_0 sufficiently large), from which the desired estimate for I_4^2 follows.

I₅^B: Suppose that $\beta \in (0,2)$: then the estimation (42) can be carried through from (25) and (36). Having (40) and (42), we can clearly derive an inequality of the form (51) with $j = 5$, in this case.

If $\beta = 2$, however, we should use (56) along with (25) and (36), obtaining

$$\begin{aligned} I_5^2 &\leq M_J^K \cdot \kappa_3 \cdot (3/2)^{\alpha_3} \cdot \int_{C_d^\partial(Y,s) \cap C_{2\delta}^\partial(Y,s)} \left\{ \frac{\kappa_1}{2-\alpha_3} \cdot \delta^{\alpha_1} + \frac{\kappa'_1}{3-\alpha_3} \cdot \delta \right\} d\lambda_{\partial B_s} \\ &\leq 2^{3/2} \pi \cdot (3/2)^{\alpha_3} \cdot \kappa_3 \cdot M_J^K \cdot \left\{ \frac{\kappa_1 \cdot d^{\alpha_3}}{\alpha_3} \cdot \delta^{\alpha_1 + \kappa'_1 \cdot \delta} \cdot \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_3}} d\zeta \right\}. \end{aligned} \quad (59)$$

Recalling (58), it is plain that an inequality of the form (51) follows for I_5^2 from (59).

I₆^B: Once again applying (52),

that there exist $\kappa_3 > 0$, $\Delta_3 > 0$, and $\alpha_3 \in (0,1]$, depending on ϕ , K , and M alone, such that

$$|\phi_{(\tilde{Y}, \tilde{s})} \circ \chi_{\tilde{s}}^{-1}(\tilde{Z})| \leq \kappa_3 \cdot r_{\tilde{Y}}^{\alpha_3}(\tilde{Z}) \quad \text{for} \quad (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\} \quad (54)$$

and $\tilde{Z} \in \partial B_{\tilde{s}} \cap B_{\Delta_3}^3(\tilde{Y})$.

If $Z \in C_d^{\partial}(Y, s)$, (49) and (50) show that

$$\begin{aligned} |\chi_{ss}(Z) - \hat{Y}|_3 &\leq |Y - \hat{Y}|_3 + |\chi_{ss}(Z) - Z|_3 + |Z - Y|_3 \\ &\leq 1\delta + \frac{9}{7} \cdot |\Pi_Y(Z) - Y|_3 \\ &< \frac{1}{4}d + \frac{9}{7}d < 2d < \Delta_3. \end{aligned} \quad (55)$$

But (55) allows us to use (54) in conjunction with (32) to write

$$\begin{aligned} |\phi_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}}^{-1}(Z)| &= |\phi_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}}^{-1}(\chi_{ss}(Z))| \\ &\leq \kappa_3 \cdot r_{\hat{Y}}^{\alpha_3}(\chi_{ss}(Z)) \\ &\leq \kappa_3 \cdot (3/2)^{\alpha_3} \cdot r_Y^{\alpha_3}(Z) \end{aligned} \quad (56)$$

for each $Z \in C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'$.

Employing (24), (34), and (56), we produce

$$\begin{aligned} I_4^2 &\leq M_T \cdot M_J^K \cdot \kappa_3 \cdot (3/2)^{\alpha_3} \cdot 1\beta \cdot 2^{\beta+1} \cdot \delta \cdot \int_{C_d^{\partial}(Y, s) \cap C_{2\delta}^{\partial}(Y, s)'} \frac{1}{r_Y^{3-\alpha_3}} d\lambda_{\partial B_s} \\ &\leq 2^{\beta+\frac{3}{2}} \cdot (3/2)^{\alpha_3} \cdot \pi \cdot 1\beta \kappa_3 \cdot M_T \cdot M_J^K \cdot \delta \cdot \int_{2\delta}^d \frac{1}{\zeta^{2-\alpha_3}} d\zeta \end{aligned} \quad (57)$$

and observe that

$$\begin{aligned}
 I_6^\beta &\leq M_\Gamma \cdot M_J^K \cdot \kappa_2 \cdot \delta^{\alpha_2} \cdot \int_{C_d^\partial(Y,s) \cap C_{21\delta}^\partial(Y,s)} \frac{1}{r_Y^\beta} d\lambda_{\partial B_s} \\
 &\leq 2^{3/2} \pi \cdot M_\Gamma \cdot M_J^K \cdot \kappa_2 \cdot \delta^{\alpha_2} \cdot \int_{21\delta}^d \frac{1}{\zeta^{\beta-1}} d\zeta,
 \end{aligned} \tag{60}$$

while

$$\delta^{\alpha_2} \cdot \int_{21\delta}^d \frac{1}{\zeta^{\beta-1}} d\zeta = \begin{cases} \frac{\delta^{\alpha_2}}{2-\beta} \cdot \zeta^{2-\beta} \Big|_{21\delta}^d < \frac{d^{2-\beta}}{2-\beta} \cdot \delta^{\alpha_2}, & \text{if } \beta \in (0,2), \\ \delta^{\alpha_2} \cdot \ln \frac{d}{21\delta} \leq M_1 \delta^{\hat{\alpha}_2}, & \text{if } \beta = 2, \end{cases} \tag{61}$$

having chosen $\hat{\alpha}_2 \in (0, \alpha_2)$ and M_1 sufficiently large. With (60) and (61), whatever the value of $\beta \in (0,2]$, we have secured an inequality of the form (51) with $j = 6$.

I_7^β : The inclusion (45) is certainly true here, and we still have $41\delta < d < \frac{7}{9} d_K$. Thus, if $\beta \in (0,2)$, we can proceed just as before to arrive at (46). On the other hand, if $\beta = 2$, we can reason initially as in (46) and then use (54) with $(\tilde{Y}, \tilde{s}) = (\hat{Y}, \hat{s})$, noting that, for $Z \in C_{41\delta}^\partial(\hat{Y}, \hat{s})$,

$$r_{\hat{Y}}(Z) \leq \frac{9}{7} r_{\hat{Y}}(\Pi_{\hat{Y}}(Z)) < \frac{9}{7} \cdot 41\delta < \frac{9}{7} d < \frac{9}{14} \Delta_3 < \Delta_3,$$

by (50):

$$I_7^\beta \leq \left| \int_{C_{41\delta}^\partial(\hat{Y}, \hat{s})} \frac{1}{r_{\hat{Y}}^2} \cdot \Gamma(\hat{Y}, \hat{s}) \cdot \phi(\hat{Y}, \hat{s}) \circ \chi_{\hat{s}}^{-1} \cdot J\chi_{\hat{s}}^{-1} d\lambda_{\partial B_{\hat{s}}} \right|$$

$$\begin{aligned}
 &\leq M_T \cdot M_J^K \cdot \kappa_3 \cdot \int_{C_{4, \delta}^{\partial}(\hat{Y}, \hat{s})} \frac{1}{r_Y^{2-\alpha_3}} d\lambda_{\partial B_{\hat{s}}} \\
 &\leq 2^{3/2} \pi \cdot \kappa_3 \cdot M_T \cdot M_J^K \cdot \int_0^{4, \delta} \frac{1}{\zeta^{1-\alpha_3}} d\zeta \\
 &= 2^{3/2} \pi \cdot \kappa_3 \cdot M_T \cdot M_J^K \cdot \frac{1}{\alpha_3} \cdot (4, \delta)^{\alpha_3} \cdot \delta^{\alpha_3}.
 \end{aligned} \tag{62}$$

Consequently, whether $\beta \in (0, 2)$ or $\beta = 2$, we can produce an estimate as in (51) for $j = 7$.

I_8^β : If $\beta \in (0, 2)$, one can check that (47) holds. However, if $\beta = 2$ we must use (54) with $(\tilde{Y}, \tilde{s}) = (Y, s)$, after pointing out that

$$r_Y(Z) \leq \frac{9}{7} r_Y(\Pi_Y(Z)) < \frac{9}{7} \cdot 2, \delta < 3, \delta < d < \Delta_3$$

whenever $Z \in C_{2, \delta}^{\partial}(Y, s)$. Thus, from (28),

$$\begin{aligned}
 I_8^2 &\leq M_T \cdot M_J^K \cdot \kappa_3 \cdot \int_{C_{2, \delta}^{\partial}(Y, s)} \frac{1}{r_Y^{2-\alpha_3}} d\lambda_{\partial B_s} \\
 &\leq 2^{3/2} \pi \cdot M_T \cdot M_J^K \cdot \kappa_3 \cdot \int_0^{2, \delta} \frac{1}{\zeta^{1-\alpha_3}} d\zeta \\
 &= 2^{3/2} \pi \cdot M_T \cdot M_J^K \cdot \kappa_3 \cdot \frac{1}{\alpha_3} \cdot (2, \delta)^{\alpha_3} \cdot \delta^{\alpha_3}.
 \end{aligned} \tag{63}$$

In (47) or (63), as the case may be, we have an inequality of the required form (51) for I_8^β .

This concludes the proofs of statements (ii)' and (iii)'.

During the remainder of the proof, we suppose that, for each compact $\tilde{K} \subset \mathbb{R}$, $\chi_{,4}(P, \cdot)|_{\tilde{K}}$ is Hölder continuous, uniformly for $P \in \partial R$.

(iv)' As usual, we suppose that (Y, s) and $(\hat{Y}, \hat{s})$ are points of ∂B , and continue to write $\delta := |(\hat{Y}, \hat{s}) - (Y, s)|_4$. We observe first that, just as in (IV.24.31),

$$\begin{aligned} |\text{grad } r_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z)) - \text{grad } r_Y(Z)|_3 &= \left| \frac{\chi_{\hat{s}\hat{s}}(Z) - \hat{Y}}{r_{\hat{Y}}(\chi_{\hat{s}\hat{s}}(Z))} - \frac{Z - Y}{r_Y(Z)} \right|_3 \\ &\leq \frac{2}{r_Y(Z)} \{ |\hat{Y} - Y|_3 + |\chi_{\hat{s}\hat{s}}(Z) - Z|_3 \} \\ &\leq \frac{2\delta}{r_Y(Z)} \end{aligned} \quad (64)$$

whenever $Z \in \partial B_s \cap \{Y\}' \cap \{\chi_{\hat{s}\hat{s}}(\hat{Y})\}'$.

Next, recalling the definition of $W: \partial B \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, given by (IV.14.28), and the bounds $|V^c|_3 \leq c^*/c$, $|\chi_{,4}^c|_3 \leq c^*/c$, and using (15), it is clear that there exist c_1^* and c_2^* , depending on the ratio c^*/c alone, such that

$$\begin{aligned} |W_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}\hat{s}}(Z) - W_{(Y, s)}(Z)|_3 &\leq c_1^* |V_{(\hat{Y}, \hat{s})}^c \circ \chi_{\hat{s}\hat{s}}(Z) - V_{(Y, s)}^c(Z)|_3 \\ &\quad + c_2^* |[\chi_{,4}^c]_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}\hat{s}}^{-1}(Z) - [\chi_{,4}^c]_{(Y, s)} \circ \chi_s^{-1}(Z)|_3, \end{aligned} \quad (65)$$

for each $Z \in \partial B_s$.

For $m, n \in \mathbb{N} \cup \{0\}$, $\Gamma^{mn}(Z, t; X) := \Gamma_{(X, t)}^{mn}(Z)$ is given by (IV.14.29) for $(Z, t) \in \partial B$ and $X \in \mathbb{R}^3 \cap \{Z\}'$; according to (IV.14.57), the denominator appearing in the definition of Γ^{mn} is bounded below

by a positive number depending on only n and c^*/c . From this fact, (65), and the bounds cited for $|V^c|_3$ and $|X_{,4}^c|_3$, one can easily check that there exist positive numbers c_1^{mn} , c_2^{mn} , and c_3^{mn} which depend upon m , n , and c^*/c alone, such that

$$\begin{aligned} |\Gamma_{(\hat{Y}, \hat{s})}^{mn} \circ X_{ss}^{-1}(Z) - \Gamma_{(Y, s)}^{mn}(Z)| &\leq c_1^{mn} |[X_{,4}^c]_{(\hat{Y}, \hat{s})} \circ X_s^{-1}(Z) - [X_{,4}^c]_{(Y, s)} \circ X_s^{-1}(Z)|_3 \\ &\quad + c_2^{mn} |V_{(\hat{Y}, \hat{s})}^c \circ X_{ss}^{-1}(Z) - V_{(Y, s)}^c(Z)|_3 \\ &\quad + c_3^{mn} |\text{grad } r_{\hat{Y}}(X_{ss}^{-1}(Z)) - \text{grad } r_Y(Z)|_3 \end{aligned} \quad (66)$$

for each $Z \in \partial B_s \cap \{Y\} \cap \{X_{ss}^{-1}(\hat{Y})\}'$.

To prove that (i) is fulfilled when Γ is replaced by any one of the functions given in (iv)', it is sufficient to consider the compact interval $K = [t_1, t_2]$. Fix any $\Delta_1 > 0$, and suppose now that (Y, s) and $(\hat{Y}, \hat{s})$ lie on $\bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$ with $\delta \leq \Delta_1$. We set

$$t_0 := \frac{1}{c-c^*} \{ \Delta_1 + \max_{\zeta \in K} \text{diam } B_\zeta \}$$

and

$$\hat{K} := [t_1 - t_0, t_2].$$

It is a simple matter to show that, for each $Z \in \partial B_s$, we have $\tau(X_s^{-1}(Z); \hat{Y}, \hat{s}) \leq t_0$ and $\tau(X_s^{-1}(Z); Y, s) \leq t_0$ (cf., (IV.24.24 and 25)), so $\hat{s} - \tau(X_s^{-1}(Z); \hat{Y}, \hat{s})$ and $s - \tau(X_s^{-1}(Z); Y, s)$ are in $\hat{K}$. Now, we know that there exist $\hat{A} > 0$ and $\hat{a} \in (0, 1]$, depending upon M , K , and Δ_1 alone, for which

$$|x_{,4}^c(P, s_2) - x_{,4}^c(P, s_1)|_3 \leq \hat{A} \cdot |s_2 - s_1|^{\hat{\alpha}} \quad (67)$$

for $s_1, s_2 \in \hat{K}$ and $P \in \partial R$.

Then, reasoning as in (IV.24.26),

$$|[x_{,4}^c]_{(\hat{Y}, \hat{s})} \circ x_s^{-1}(Z) - [x_{,4}^c]_{(Y, s)} \circ x_s^{-1}(Z)|_3 \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1+c^2\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}} \quad (68)$$

for each $Z \in \partial B_s$,

and, proceeding as in the derivation of (IV.24.29),

$$|v_{(\hat{Y}, \hat{s})}^c \circ x_{s\hat{s}}(Z) - v_{(Y, s)}^c(Z)|_3 \leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot \{1+c^2\}^{\hat{\alpha}/2} \cdot \delta^{\hat{\alpha}} \quad (69)$$

for each $Z \in \partial B_s$.

From (66), with (64), (68), and (69), it is now evident that

$$|\Gamma_{(\hat{Y}, \hat{s})}^{mn} \circ x_{s\hat{s}}(Z) - \Gamma_{(Y, s)}^{mn}(Z)| \leq k_1^{mn} \cdot \delta^{\hat{\alpha}} + k_2^{mn} \cdot \frac{1}{r_Y(Z)} \cdot \delta \quad (70)$$

for each $Z \in \partial B_s \cap \{Y\}' \cap \{x_{s\hat{s}}(\hat{Y})\}'$,

wherein the positive k_1^{mn} and k_2^{mn} depend on at most m , n , Δ_1 , M , and K .

In view of the boundedness of each Γ^{mn} and $\text{grad } r_Y$, it is surely obvious that the validity of (iv)' follows from (64) and (70).

Henceforth in the proof, we suppose that $\mu: \partial B \rightarrow K$ is (at least) continuous.

(v)' Here, we shall in fact prove the following stronger statement:

(v)" When ϕ is equal to either $\Lambda_{41}^1\{\mu\}$ or $\bar{\Lambda}_3^1\{\mu\}$, for each compact $\bar{K} \subset \mathbb{R}$ and positive $\bar{\Delta}_3$ there exist $\bar{\kappa}_3 > 0$ and $\bar{\alpha}_3 \in (0,1]$ such that if $(Y,s) \in \bigcup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}$ and $X \in B_{\bar{\Delta}_3}^3(Y)$, then

$$|\phi_{(X,s)} \circ \chi_s^{-1}(Z)| \leq \bar{\kappa}_3 \cdot r_X^{\bar{\alpha}_3}(Z) \quad \text{for each } Z \in \partial B_s. \quad (71)$$

In order to prove (v)", it suffices to consider the compact interval $K = [t_1, t_2]$. Select $\Delta_3 > 0$ and $(Y,s) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$, then $X \in B_{\Delta_3}^3(Y)$. Since

$$\begin{aligned} \tau(\chi_s^{-1}(Z); X, s) &= |\tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Z, s)| \\ &\leq \frac{1}{c-c^*} |Z-X|_3 \\ &\leq \frac{1}{c-c^*} \{|Z-Y|_3 + |Y-X|_3\} \\ &< \frac{1}{c-c^*} \cdot \left\{ \max_{\zeta \in K} \text{diam } B_\zeta + \Delta_3 \right\}, \quad \text{for } Z \in \partial B_s, \end{aligned}$$

we see that if we set

$$t'_0 := \max_{\zeta \in K} \text{diam } B_\zeta + \Delta_3$$

and

$$K_0 := [t_1 - t'_0, t_2],$$

then $s - \tau(\chi_s^{-1}(Z); X, s) \in K_0$ for each $Z \in \partial B_s$. Let $M_u^{K_0}$ and $M_j^{K_0}$ be positive numbers such that

$$\text{and } \left. \begin{array}{l} |u^0| \leq M_\mu^{K_0} \\ jx \leq M_j^{K_0} \end{array} \right\} \text{ on } \partial R \times K_0.$$

From (IV.14.36 and 48), respectively, it is clear that

$$\begin{aligned} \Lambda_{4i}^1\{u\}_{(X,s)} \circ x_s^{-1}(Z) &\leq \frac{1}{\{1-(c^*/c)^2\}^3} \cdot M_\mu^{K_0} \cdot M_j^{K_0} \cdot \{1+2(c^*/c)^2\} \\ &\cdot |v_{(X,s)}^c(Z) - [x_{,4}^c]_{(X,s)} \circ x_s^{-1}(Z)|_3, \end{aligned} \quad (72)$$

and

$$\begin{aligned} \bar{\Lambda}_3^1\{u\}_{(X,s)} \circ x_s^{-1}(Z) &\leq \frac{1}{\{1-(c^*/c)^2\}^3} \cdot M_\mu^{K_0} \cdot M_j^{K_0} \cdot (c^*/c) \\ &\cdot |v_{(X,s)}^c(Z) - [x_{,4}^c]_{(X,s)} \circ x_s^{-1}(Z)|_3, \end{aligned} \quad (73)$$

for each $Z \in \partial B_s$.

We may suppose that estimate (67) holds here whenever s_1 and s_2 lie in K_0 , and $P \in \partial R$, with $\hat{A} > 0$ and $\hat{\alpha} \in (0,1]$ depending upon at most M , K , and Δ_3 . From the definition (IV.14.11), for each $Z \in \partial B_s$ there is some $\hat{t}^1(Z;X,s) \in [s-\tau(x_s^{-1}(Z);X,s), s]^\dagger$ with which we can therefore write

$$\begin{aligned} &|v_{(X,s)}^1(Z) - [x_{,4}^1]_{(X,s)} \circ x_s^{-1}(Z)| \\ &= |x_{,4}^1(x_s^{-1}(Z), \hat{t}^1(Z;X,s)) - x_{,4}^1(x_s^{-1}(Z), s-\tau(x_s^{-1}(Z);X,s))| \\ &\leq \hat{A} \cdot |\hat{t}^1(Z;X,s) - \{s-\tau(x_s^{-1}(Z);X,s)\}|^{\hat{\alpha}} \end{aligned}$$

[†]Note that this interval might be degenerate to $\{s\}$.

$$\begin{aligned}
 &\leq \hat{A} \cdot \tau^{\hat{\alpha}}(\chi_s^{-1}(Z); X, s) \\
 &= \hat{A} \cdot |\tau(\chi_s^{-1}(Z); X, s) - \tau(\chi_s^{-1}(Z); Z, s)|^{\hat{\alpha}} \\
 &\leq \frac{\hat{A}}{(c-c^*)^{\hat{\alpha}}} \cdot r_X^{\hat{\alpha}}(Z). \tag{74}
 \end{aligned}$$

Statement (v)" now follows by using (74) with each of (72) and (73).

(vi)' The local Hölder continuity of the functions on ∂B which are given by (4) and (5) for each $(Y, s) \in \partial B$ shall be an immediate consequence of (iii)', (iv)', and (v)', once we have shown that, with the stated conditions on μ and $\hat{J}\chi$, hypothesis (ii) is satisfied when either $\phi = \Lambda_{41}^1\{\mu\}$ or $\phi = \tilde{\Lambda}_3^1\{\mu\}$. Now, in [IV.24], under somewhat stronger hypotheses on M and (R, χ) , we have shown that (ii) holds with $\phi = \Lambda_{11}^1\{\mu\}$; one can easily check that the sort of reasoning carried through there also serves here to demonstrate that (ii) holds when ϕ is replaced by either of the functions presently under consideration, even under the weakened conditions which we have now available for M and (R, χ) . We shall allow these brief remarks to suffice for the proof of (vi)'.

(vii)' Since we know now that $\chi_{,44} \in C(\partial R \times R)$ (by [IV.10]) it is clear from the definitions (IV.14.38-41, 50, and 51) that $\Lambda_{11}^2\{\mu\}$, $\Lambda_{21j}^2\{\mu\}$, $\Lambda_3^2\{\mu\}$, $\Lambda_{41}^2\{\mu\}$, $\tilde{\Lambda}_{11}^2\{\mu\}$, and $\tilde{\Lambda}_2^2\{\mu\}$ are in $C(\partial R \times R^4)$. Thus, with (iv)', the continuity of each function defined on ∂B by using (6), ..., (10), and (11) follows from (i)'. To verify that these six functions are locally Hölder continuous on ∂B when μ is locally Hölder continuous on ∂B and $\chi_{,44}(P, \cdot) \in \tilde{K}$ and

$\hat{J}X(P, \cdot) | \bar{K}$ are Hölder continuous, uniformly for $P \in \partial R$, for each compact $\bar{K} \subset R$, we can apply (ii)' (and (iv)'), provided we first show that (ii) is true when ϕ is replaced therein by $\Lambda_{11}^2\{\mu\}, \dots, \bar{\Lambda}_{11}^2\{\mu\}$, or $\bar{\Lambda}_2^2\{\mu\}$. The latter is easily accomplished by constructing an argument modeled on the proof in [IV.24] that [IV.24.iii] holds when $\phi = \Lambda_{11}^1\{\mu\}$; cf., the proof of (vi)', *supra*. We omit the details.

(viii)' Here, it is given that $D_4^0 \mu \in C(\partial R \times R)$ and, by [IV.10], $D_4 \hat{J}X \in C(\partial R \times R)$, so (IV.14.43, 44, and 53) show that $\Lambda_1^3\{\mu\}$, $\Lambda_{21}^3\{\mu\}$, and $\bar{\Lambda}_1^3\{\mu\}$ are in $C(\partial R \times R^4)$. The continuity of each of the three functions defined on ∂B via (12), (13), and (14) is therefore an obvious implication of (i)'. The local Hölder continuity of these functions, under the additional hypotheses cited for $D_4^0 \mu$ and $D_4 \hat{J}X$, shall result from (ii)' and (iv)' after we have shown that (ii) holds with $\phi = \Lambda_1^3\{\mu\}$, $\Lambda_{21}^3\{\mu\}$, or $\bar{\Lambda}_1^3\{\mu\}$. For this, once again we shall simply note that the requisite reasoning follows by obvious modifications of the proof that [IV.24.iii] holds when $\phi = \Lambda_{11}^1\{\mu\}$. $\square$.

Using the preceding Theorem [IV.31] in conjunction with Lemma [IV.21], we can identify conditions under which a function of the form $w_{3\beta}\{\phi\}$ is continuous on R^4 , for $\beta \in (0, 2)$ or $\beta = 2$.

[IV.32] T H E O R E M. Let M be a motion in $M(1)$ such that $\{B_t^0\}_{t \in R}$ is locally uniformly Lyapunov. Let (R, X) be a reference pair for M with the properties listed in [I.3.25], and ϕ and Γ

be as in [IV.15], viz., $(P, X, t) \mapsto \phi_{(X, t)}(P)$ is in $C(\partial R \times R^4)$, while $(Z, X, t) \mapsto \Gamma_{(X, t)}(Z)$ is continuous and bounded on $\{(Z, X, t) \mid (X, t) \in R^4, Z \in \partial B_t \cap \{X\}\}$ into R . Suppose further that

- (i) whenever $\tilde{K}$ is compact in R , there exist positive numbers $\tilde{\kappa}_1$, $\tilde{\kappa}'_1$, and $\tilde{\delta}_1$, and a number $\tilde{\alpha}_1 \in (0, 1]$ such that

$$\begin{aligned} & |\Gamma_{(\hat{Y}, \hat{s})} \circ \chi_{\hat{s}\hat{s}}(Z) - \Gamma_{(Y, s)}(Z)| \\ & \leq \tilde{\kappa}_1 \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4^{\tilde{\alpha}_1} + \frac{\tilde{\kappa}'_1}{r_Y(Z)} \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4 \\ & \text{for } (Y, s) \text{ and } (\hat{Y}, \hat{s}) \in \bigcup_{\zeta \in \tilde{K}} \{\partial B_\zeta \times \{\zeta\}\} \quad (1) \\ & \text{with } |(\hat{Y}, \hat{s}) - (Y, s)|_4 \leq \tilde{\delta}_1, \\ & \text{and } Z \in \partial B_s \cap \{Y\} \cap \{\chi_{\hat{s}\hat{s}}(\hat{Y})\}. \end{aligned}$$

Then

- (i)' for $0 < B < 2$, $w_{3B}\{\phi\}$ is continuous on R^4 .

If it is also known that

- (ii) whenever $\tilde{K} \subset R$ is compact, there exist positive numbers $\tilde{\kappa}_2$, $\tilde{\kappa}_3$, $\tilde{\delta}_2$, and $\tilde{\delta}_3$, and numbers $\tilde{\alpha}_2, \tilde{\alpha}_3 \in (0, 1]$ such that

$$|\phi_{(\hat{Y}, \hat{s})}(P) - \phi_{(Y, s)}(P)| \leq \tilde{\kappa}_2 \cdot |(\hat{Y}, \hat{s}) - (Y, s)|_4^{\tilde{\alpha}_2}$$

for (Y, s) and $(\hat{Y}, \hat{s}) \in \cup_{\zeta \in \tilde{K}} \{\partial B_\zeta \times \{\zeta\}\}$ (2)

with $|(\hat{Y}, \hat{s}) - (Y, s)|_4 \leq \tilde{\Delta}_2$, and $P \in \partial R$,

and

$$|\phi_{(X, s)} \circ \chi_s^{-1}(Z)| \leq \tilde{\kappa}_3 \cdot r_X^{\tilde{\alpha}_3}(Z) \quad \text{for each } Z \in \partial B_s \cap B_{\tilde{\Delta}_3}^3(Y)$$

whenever $(Y, s) \in \cup_{\zeta \in \tilde{K}} \{\partial B_\zeta \times \{\zeta\}\}$ (3)

and $X \in L_\nu(Y, s) \cap B_{\tilde{\Delta}_3}^3(Y)$,

then

(ii)' $W_{32}\{\phi\}$ is continuous on $\mathbb{R}^4$.

Next, suppose that $\mu \in C(\partial B)$. We consider the application of (i)' and (ii)' to those functions defined by certain of the terms appearing in (IV.14.32, 37, 42, 45, 49, and 52): for each compact $\tilde{K} \subset \mathbb{R}$, let $\chi_{,4}(P, \cdot)|_{\tilde{K}}$ be Hölder continuous, uniformly for $P \in \partial R$. Then

(iii)' hypothesis (i) is fulfilled by taking $\Gamma_{(X, t)}$ to be $\Gamma_{(X, t)}^{01}$, $\Gamma_{(X, t)}^{12}$, $\Gamma_{(X, t)}^{13}$, $\Gamma_{(X, t)}^{23}$, $\Gamma_{(X, t)}^{02} r_{X, i}$, $\Gamma_{(X, t)}^{13} r_{X, i}$, or $\Gamma_{(X, t)}^{03} r_{X, i} r_{X, j}$ on $\partial B_t \cap \{X\}$ for each $(X, t) \in \mathbb{R}^4$.

(iv)' If it is known that μ is locally Hölder continuous

on ∂B , and, for each compact $\bar{K} \subset \mathbb{R}$, $\hat{J}_X(P, \cdot)|_{\bar{K}}$ is Hölder continuous, uniformly for $P \in \partial R$, then hypothesis (ii) is satisfied when ϕ is replaced by either $\Lambda_{41}^1\{u\}$ or $\bar{\Lambda}_3^1\{u\}$. Consequently, the functions given on $\mathbb{R}^4$ by (IV.31.4 and 5) for each $(Y, s) \in \mathbb{R}^4$ are continuous in that case.

(v)' Assume now that $M \in \mathcal{M}(1;0)$, and the reference pair (R, X) also possesses the properties listed in [IV.10]. Then the functions on $\mathbb{R}^4$ given by (IV.31.6-11) for each $(Y, s) \in \mathbb{R}^4$ are continuous. Moreover, if $D_4^0 u \in C(\partial R \times \mathbb{R})$, then the functions given on $\mathbb{R}^4$ by (IV.31.12-14) for each $(Y, s) \in \mathbb{R}^4$ are continuous.

(vi)' If $M \in \mathcal{M}(2)$, then there exists a reference pair (R, X) for M so that each condition on M or (R, X) used in the preceding assertions is fulfilled.

P R O O F. (i)' and (ii)' We shall present the proofs of these statements together, supposing, of course, that (ii) is in force whenever we are considering $w_{32}\{\phi\}$. We first point out that $w_{3\beta}\{\phi\}|_{\partial B} \in C(\partial B)$ for each $\beta \in (0, 2]$ ($w_{32}\{\phi\}|_{\partial B}$ is even locally Hölder continuous on ∂B); these results follow from [IV.31], specifically, conclusions (i)' (if $0 < \beta < 2$) and (iii)' (if $\beta = 2$) of that theorem, the satisfaction of the requisite conditions being easily checked from those imposed here. Having already

remarked that each $w_{3\beta}\{\phi\}$ is continuous at each point of $B^{\alpha\cup\sigma}$ (cf., [IV.16]), we see that if we can succeed in proving that, for each $\beta \in (0,2]$, for each $(\tilde{Y}, \tilde{s}) \in \partial B$,

$$\lim_{\substack{X \rightarrow \tilde{Y} \\ X \in L_{\tilde{Y}}(\tilde{Y}, \tilde{s})}} w_{3\beta}\{\phi\}(X, \tilde{s}) = w_{3\beta}\{\phi\}(\tilde{Y}, \tilde{s}), \quad (4)$$

∂B -locally uniformly in $(\tilde{Y}, \tilde{s})$, then it shall follow directly from Lemma [IV.21] that $w_{3\beta}\{\phi\} \in C(R^4)$ for $\beta \in (0,2]$, i.e., that (i)' and (ii)' are true.

Fix $\beta \in (0,2]$ and choose a compact set $K \subset R$. We shall show that (4) holds uniformly for $(\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$, which shall imply (i)' and (ii)', as just remarked. The family $\{B_{\zeta}^0\}_{\zeta \in K}$ is uniformly Lyapunov, for which we can find a set of uniform Lyapunov constants, (a_K, α_K, d_K) . Moreover, we may, and shall, suppose here that the positive d_K is chosen so small that there exists a $\gamma_K \in (0,1)$, depending only on a_K , α_K , and d_K , i.e., only on M and K , such that

$$\gamma_K < \frac{r_X(Z)}{r_X(\pi_{\tilde{Y}}(Z))} < \frac{1}{\gamma_K} \quad \text{whenever} \quad (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}, \quad (5)$$

$$X \in L_{\tilde{Y}}(\tilde{Y}, \tilde{s}), \quad \text{and} \quad Z \in \partial B_{\tilde{s}} \cap B_{d_K}^3(\tilde{Y}) \cap \{X\}'.$$

Suppose that $\delta \in (0, (7/9)d_K)$. Let $s \in K$, $Y \in \partial B_s$, $X \in L_{\tilde{Y}}(Y, s) \cap B_{d_K}^3(Y)$, and write

$$\begin{aligned}
 |w_{3\beta}\{\phi\}(X,s) - w_{3\beta}\{\phi\}(Y,s)| &= \left| \int_{\partial B_s} \frac{1}{r_X^\beta} \cdot \Gamma(X,s) \cdot \phi(X,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right. \\
 &\quad \left. - \int_{\partial B_s} \frac{1}{r_Y^\beta} \cdot \Gamma(Y,s) \cdot \phi(Y,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right| \\
 &\leq \sum_{j=1}^3 I_j^\beta(\delta; Y, s, X),
 \end{aligned} \tag{6}$$

wherein the $I_j^\beta(\delta; Y, s, X) = I_j^\beta$, $j = 1, 2$, and 3 , are given by

$$I_1^\beta := \left| \int_{\partial B_s \cap B_\delta^3(Y)} \frac{1}{r_X^\beta} \cdot \Gamma(X,s) \cdot \phi(X,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|, \tag{7}$$

$$I_2^\beta := \left| \int_{\partial B_s \cap B_\delta^3(Y)} \frac{1}{r_Y^\beta} \cdot \Gamma(Y,s) \cdot \phi(Y,s) \circ X_s^{-1} \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|, \tag{8}$$

and

$$\begin{aligned}
 I_3^\beta &:= \left| \int_{\partial B_s \cap B_\delta^3(Y)} \left\{ \frac{1}{r_X^\beta} \cdot \Gamma(X,s) \cdot \phi(X,s) \circ X_s^{-1} - \frac{1}{r_Y^\beta} \cdot \Gamma(Y,s) \cdot \phi(Y,s) \circ X_s^{-1} \right\} \right. \\
 &\quad \left. \cdot JX_s^{-1} d\lambda_{\partial B_s} \right|.
 \end{aligned} \tag{9}$$

Now, let $\epsilon > 0$. To show that (4) holds uniformly for $(\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}$, it suffices to produce positive numbers depending on at most ϵ , β , ϕ , M , and K , $\delta_j = \delta_j(\epsilon, \beta, \phi, M, K)$, for $j = 0, 1, 2$, and 3 , such that $\delta_0 < \frac{7}{9} d_K$, $\delta_j \leq d_K$ for $j = 1, 2$, and 3 , and (maintaining the notation introduced)

$$I_j^\beta(\delta_0; Y, s, X) < \epsilon \quad \text{for} \quad |X - Y|_3 < \delta_j, \quad \text{for} \quad j = 1, 2, \text{ and } 3. \tag{10}$$

We shall show that this can be done. Let M_T , M_ϕ^K , and M_J^K be such that

$$|r| \leq M_T,$$

$$|\phi(\tilde{X}, \tilde{s})(P)| \leq M_\phi^K \quad \text{for } P \in \partial R, \quad \tilde{s} \in K, \quad \text{and} \quad \text{dist}(\tilde{X}, \partial B_{\tilde{s}}) \leq d_K,$$

and

$$|J\chi_\xi^{-1}(Z)| \leq M_J^K \quad \text{for } (Z, \xi) \in \bigcup_{\zeta \in K} \{\partial B_\zeta \times \{\zeta\}\}.$$

I₁^β: If $Z \in C_\delta^\partial(Y, s)$, then

$$r_Y(Z) \leq \frac{9}{7} r_Y(\pi_Y(Z)) < \frac{9}{7} \delta < d_K,$$

so we can apply (5) to write

$$\frac{1}{r_X(Z)} < \frac{1}{\gamma_K} \cdot \frac{1}{r_X(\pi_Y(Z))} \leq \frac{1}{\gamma_K} \cdot \frac{1}{r_Y(\pi_Y(Z))} \quad \text{for } Z \in C_\delta^\partial(Y, s) \cap \{Y\}', \quad (11)$$

since it is obvious that $r_X(\pi_Y(Z)) \geq r_Y(\pi_Y(Z))$. Thus, if we first suppose $\beta \in (0, 2)$, (7) and the inclusion $\partial B_s \cap B_\delta^3(Y) \subset C_\delta^\partial(Y, s)$ give

$$\begin{aligned} I_1^\beta &\leq M_T \cdot M_\phi^K \cdot M_J^K \cdot \frac{1}{\gamma_K} \int_{C_\delta^\partial(Y, s)} \frac{1}{r_Y^\beta \circ \pi_Y} d\lambda_{\partial B_s} \\ &\leq 2^{3/2} \pi \cdot \gamma_K^{-\beta} \cdot M_T \cdot M_\phi^K \cdot M_J^K \cdot \int_0^\delta \frac{1}{\zeta^{\beta-1}} d\zeta \\ &= 2^{3/2} \pi \cdot \gamma_K^{-\beta} M_T \cdot M_\phi^K \cdot M_J^K \cdot \frac{1}{2-\beta} \cdot \delta^{2-\beta}. \end{aligned} \quad (12)$$

Next, suppose that $\beta = 2$: we may here invoke (11), which says that

there exist $\kappa_3 > 0$, $\Delta_3 > 0$ and $\alpha_3 \in (0,1]$, depending on at most ϕ , M , and K , for which

$$|\phi_{(X,s)} \circ \chi_s^{-1}(Z)| \leq \kappa_3 \cdot r_X^{\alpha_3}(Z) \quad \text{for each } Z \in \partial B_s \cap B_{\Delta_3}^3(Y), \quad (13)$$

provided we also suppose, as we shall, that $|X-Y|_3 < \Delta_3$. Assuming that $\delta < \frac{7}{9} \Delta_3$, so that $Z \in \partial B_s \cap B_{\Delta_3}^3(Y)$ whenever $Z \in C_\delta^\partial(Y,s)$,

(7) gives in this case

$$\begin{aligned} I_1^2 &\leq M_T \cdot M_J^K \cdot \kappa_3 \cdot \int_{C_\delta^\partial(Y,s)} \frac{1}{r_X^{2-\alpha_3}} d\lambda_{\partial B_s} \\ &\leq M_T \cdot M_J^K \cdot \kappa_3 \cdot \gamma_K^{2-\alpha_3} \int_{C_\delta^\partial(Y,s)} \frac{1}{r_Y^{2-\alpha_3} \pi_Y} d\lambda_{\partial B_s} \\ &\leq 2^{3/2} \pi \cdot \kappa_3 \cdot \gamma_K^{2-\alpha_3} \cdot M_T \cdot M_J^K \cdot \frac{1}{\alpha_3} \cdot \delta^{\alpha_3}. \end{aligned} \quad (14)$$

In view of (12) and (14), it is now obvious that we can find

$\delta_0 = \delta_0(\epsilon, \beta, \phi, M, K) \in (0, (7/9)d_K)$, and less than $\frac{7}{9} \Delta_3$ if $\beta = 2$, such that

$$I_1^\beta(\delta_0; Y, s, X) < \epsilon \quad \text{for } |X-Y|_3 < \delta_1, \quad (15)$$

wherein $\delta_1 := d_K$ if $\beta \in (0,2)$ and $\delta_1 := \min\{d_K, \Delta_3\}$ if $\beta = 2$.

I_2^β : Since we can take $X = Y$ in the analysis of I_1^β , from (15) we have, with $\delta_2 := \delta_1$,

$$I_2^\beta(\delta_0; Y, s) < \epsilon \quad \text{for } |X-Y|_3 < \delta_2. \quad (16)$$

I_3^β : Now take $\delta = \delta_0$ in (9). On the set

$$\{(\tilde{Y}, \tilde{s}, \tilde{X}, Z) \mid (\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}, \tilde{X} \in B_{\frac{1}{2}\delta_0}^3(\tilde{Y})^-, \\ Z \in \partial B_{\tilde{s}} \cap B_{\delta_0}^3(\tilde{Y})'\} \quad (17)$$

in $\mathbb{R}^{10}$, let us define the function f according to

$$f(\tilde{Y}, \tilde{s}, \tilde{X}, Z) := \frac{1}{r_{\tilde{X}}^{\beta}(Z)} \cdot \Gamma(\tilde{X}, \tilde{s})(Z) \cdot \phi(\tilde{X}, \tilde{s})^{\circ \chi_{\tilde{s}}^{-1}}(Z). \quad (18)$$

By the properties of Γ and ϕ , f is certainly continuous.

Moreover, the set (17) is compact: to see this, first note that

its boundedness follows from the obvious fact that $\bigcup_{\zeta \in K} \partial B_{\zeta}$ is bounded in $\mathbb{R}^3$. Next, let $((Y_n, s_n, X_n, Z_n))_{n=1}^{\infty}$ be a sequence in the set which converges to (Y_0, s_0, X_0, Z_0) . Since $\bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\} = \chi^*(\partial R \times K)$ is closed, while $(Y_n, s_n) \rightarrow (Y_0, s_0)$ and $(Z_n, s_n) \rightarrow (Z_0, s_0)$, we find that (Y_0, s_0) and (Z_0, s_0) lie in $\bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$. Finally, the inequalities $|X_n - Y_n|_3 \leq \frac{1}{2} \delta_0$, $|Z_n - Y_n|_3 \geq \delta_0$, holding for each $n \in \mathbb{N}$, lead to $|X_0 - Y_0|_3 \leq \frac{1}{2} \delta_0$, $|Z_0 - Y_0|_3 \geq \delta_0$. Clearly, then, (Y_0, s_0, X_0, Z_0) is in the set (17), whence the latter is indeed compact. Thus, f is uniformly continuous, so there is a $\delta_3 = \delta_3(\epsilon, \beta, \phi, M, K) \in (0, \min\{d_K, 1/2 \delta_0\})$ such that, in particular,

$$\left| \frac{1}{r_{\tilde{X}}^{\beta}(Z)} \cdot \Gamma(\tilde{X}, \tilde{s})(Z) \cdot \phi(\tilde{X}, \tilde{s})^{\circ \chi_{\tilde{s}}^{-1}}(Z) - \frac{1}{r_{\tilde{Y}}^{\beta}(Z)} \cdot \Gamma(\tilde{Y}, \tilde{s})(Z) \cdot \phi(\tilde{Y}, \tilde{s})^{\circ \chi_{\tilde{s}}^{-1}}(Z) \right| \\ = |f(\tilde{Y}, \tilde{s}, \tilde{X}, Z) - f(\tilde{Y}, \tilde{s}, \tilde{Y}, Z)| \\ < \epsilon \cdot (M_J^K \cdot \max_{\zeta \in K} \lambda_{\partial B_{\zeta}}(\partial B_{\zeta}))^{-1} \quad (19)$$

for $(\tilde{Y}, \tilde{s}) \in \bigcup_{\zeta \in K} \{\partial B_{\zeta} \times \{\zeta\}\}$, $Z \in \partial B_{\tilde{s}} \cap B_{\delta_0}^3(\tilde{Y})'$,

and $\tilde{X} \in \mathbb{R}^3$ with $|\tilde{X} - \tilde{Y}|_3 < \delta_3$.

Directly from (9) and (19), it is clear that

$$I_3^B(\delta_0; Y, s, X) < \varepsilon \quad \text{for} \quad |X-Y|_3 < \varepsilon_3. \quad (20)$$

Having obtained, in (15), (16), and (20), inequalities as required in (10), the proofs of (i)' and (ii)' are complete, by the previously outlined reasoning.

(iii)' This is just conclusion (iv)' of [IV.31]; the proof is the same.

(iv)' For the proof of statement (3) of hypothesis (ii) when ϕ is either $\Lambda_{41}^1\{\mu\}$ or $\tilde{\Lambda}_3^1\{\mu\}$, one should consult the proof of assertion (v)" in [IV.31], where a stronger result has been prepared. Now suppose that μ is locally Hölder continuous on ∂B , and, for each compact $\bar{K} \subset \mathbb{R}$, $\hat{J}_X(P, \cdot)|_{\bar{K}}$ is Hölder continuous, uniformly for $P \in \partial R$: to see that (2) of hypothesis (ii) holds when ϕ is either $\Lambda_{41}^1\{\mu\}$ or $\tilde{\Lambda}_3^1\{\mu\}$, we refer to the remarks made in the proof of conclusion (vi)' of [IV.31]. Now the continuity of the functions constructed on $\mathbb{R}^4$ by using (IV.31.4 and 5) for each $(Y, s) \in \mathbb{R}^4$ is a direct consequence of these observations, (ii)', and (iii)'.

(v)' With statement (iii)' in hand, each assertion made here follows as a simple application of (i)'.

(vi)' The reasoning required for the verification of this fact is set forth in the proof of [IV.22]; cf., also, [IV.23.a]. $\square$.

In particular, Theorem [IV.32] implies the following fact:

[IV.33] C O R O L L A R Y. Let M be a motion in $\mathcal{M}(1)$ such that $\{B_t^0\}_{t \in \mathbb{R}}$ is locally uniformly Lyapunov. Suppose that there exists a reference pair (R, X) for M which possesses the properties of [I.3.25] and is also such that, for each compact $K \subset \mathbb{R}$, $X_4(P, \cdot)|_K$ is Hölder continuous, uniformly for $P \in \partial R$; each of these conditions is satisfied if $M \in \mathcal{M}(2)$. Further, let $\mu \in C(\partial B)$. Then $V\{\mu\} \in C(\mathbb{R}^4)$, i.e., $V\{\mu\}$ possesses a continuous extension to all of $\mathbb{R}^4$.

P R O O F. According to Proposition [IV.9], we have $V\{\mu\} \in C(B^0 \cup \Omega^0)$. Now, using (IV.3.5-7), (IV.14.20 and 22), and definitions (IV.14.28 and 29), one can check that

$$\begin{aligned} V\{\mu\}(X, t) &= \frac{1}{4\pi} \int_{\partial R} \kappa(\cdot; X, t) \cdot [\mu \cdot \hat{J}X]_{(X, t)} d\lambda_{\partial R} \\ &= \frac{1}{4\pi} \int_{\partial B_t} \kappa(X_t^{-1}(\cdot); X, t) \cdot [\mu \cdot \hat{J}X]_{(X, t)} \circ X_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t} \\ &= \frac{1}{4\pi} \int_{\partial B_t} \{r_X^0[X]_{(X, t)} \cdot \{1 + r_{X, \ell}^0[X]_{(X, t)} \cdot [X_{, 4}^{\ell c}]_{(X, t)}\}^{-1} \circ X_t^{-1} \\ &\quad \cdot [\mu \cdot \hat{J}X]_{(X, t)} \circ X_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t} \\ &= \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \frac{1}{\{w_{(X, t)}^k \cdot r_{X, k} + \sqrt{(r_{X, \ell} \cdot v_{(X, t)}^{\ell c})^2 + (1 - |v_{(X, t)}^c|^2)}\}} \\ &\quad \cdot \frac{1 - |v_{(X, t)}^c|^2}{1 - v_{(X, t)}^c \cdot [X_{, 4}^c]_{(X, t)} \circ X_t^{-1}} \cdot [\mu \cdot \hat{J}X]_{(X, t)} \circ X_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t} \end{aligned}$$

$$= \frac{1}{4\pi} \int_{\partial B_t} \frac{1}{r_X} \cdot \Gamma_{(X,t)}^{01} \cdot \Lambda\{\mu\}_{(X,t)} \cdot \alpha_X^{-1} \cdot J_X^{-1} d\lambda_{\partial B_t} \quad (1)$$

for each $(X,t) \in B^0 \cup \Omega^0$,

wherein

$$\Lambda\{\mu\}_{(X,t)} := \frac{1 - |v_{(X,t)}^c \alpha_X|_3^2}{1 - v_{(X,t)}^{kc} \alpha_X \cdot [X, 4]_{(X,t)}^c} \cdot [\mu \cdot J_X]_{(X,t)} \quad \text{on } \partial R. \quad (2)$$

In (1), $V\{\mu\}$ is displayed as a function of the form $W_{31}\{\Lambda\{\mu\}\}$; as in Definition [IV.30.i], we can extend $V\{\mu\}$ to $\mathbb{R}^4$ by asserting that equality (1) is the definition of $V\{\mu\}(X,t)$ for $(X,t) \in \partial B$. By conclusion (iii)' of [IV.32], hypothesis [IV.32.i] is fulfilled when $\Gamma_{(X,t)}$ is replaced by $\Gamma_{(X,t)}^{01}$ on $\partial B_t \cap \{X\}'$ for each $(X,t) \in \mathbb{R}^4$, because of the Hölder-type property imposed on $X, 4$. Consequently, once we have taken into account the other conditions which have been required here of M and μ , it is easy to see that (i)' of [IV.32] implies the continuity of $V\{\mu\}$ on $\mathbb{R}^4$. $\square$.

Having accumulated sufficient information concerning the various auxiliary functions arising out of the computation of the partial derivatives of a kinematic single layer potential $V\{\mu\}$ associated with an appropriate motion M and density μ , as in [IV.15 and 17], we are finally prepared to supply the most elementary properties of those partial derivatives in B^0 and in Ω^0 , under a reasonable set of sufficient conditions on M and μ . The facts to be presented here prove to be invaluable in the

reformulation of the scattering problem, undertaken in Chapter [I.6].

[IV.34] D E F I N I T I O N S. Let M be a motion in $M(1;0)$.

Suppose that

- (i) $\{B_{\zeta}^0\}_{\zeta \in \mathbb{R}}$ is locally uniformly Lyapunov,
- (ii) there exists a reference pair (R, χ) for M which possesses the properties set forth in [IV.10] and is also such that $\chi_{,4}$ and $\tilde{J}\chi$ are locally Hölder continuous on $\partial R \times \mathbb{R}$,

and

- (iii) $\mu: \partial B \rightarrow K$ is locally Hölder continuous and such that $D_4 \overset{\circ}{\mu} \in C(\partial R \times \mathbb{R})$.

Then, prompted by the results (IV.17.1 and 2), in view of Definition [IV.20], Proposition [IV.27], Definition [IV.30], and conclusion (v)' of Theorem [IV.31], it is clear that we may define $V_i^*(\mu): \partial B \rightarrow K$ for $i = 1, 2, 3$, and 4 according to

$$\begin{aligned} V_i^*(\mu)(Y, s) := & \omega_1^*(\Lambda_{11}^1\{\mu\})(Y, s) - \omega_{2i}^*(\Lambda_2^1\{\mu\})(Y, s) - \omega_{2j}^*(\Lambda_{3ij}^1\{\mu\})(Y, s) \\ & - \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Y^2} \cdot r_Y^{13}(Y, s) \cdot \Lambda_{4i}^1\{\mu\}(Y, s) \circ x_s^{-1} \cdot J\chi_s^{-1} d\lambda_{\partial B_s} \\ & + V\{\mu\}_i^2(Y, s) + V\{\mu\}_i^3(Y, s), \end{aligned} \quad (1)$$

for $i = 1, 2$, and 3,

and

$$\begin{aligned}
 V_4^*\{\mu\}(Y,s) &:= -c\omega_1^*\{\tilde{\lambda}_1^1\{\mu\}\}(Y,s) + c\omega_{2j}^*\{\tilde{\lambda}_{2j}^1\{\mu\}\}(Y,s) \\
 &\quad - \frac{c}{4\pi} \int_{\partial B_s} \frac{1}{r_Y^2} \cdot r_{Y^3}^{13}(Y,s) \cdot \tilde{\lambda}_3^1\{\mu\}(Y,s) \circ x_s^{-1} \cdot Jx_s^{-1} d\lambda_{\partial B_s} \\
 &\quad + V\{\mu\}_4^2(Y,s) + V\{\mu\}_4^3(Y,s),
 \end{aligned} \tag{2}$$

for each $(Y,s) \in \partial B$.

Here, of course, $V\{\mu\}_1^2(Y,s)$, $V\{\mu\}_1^3(Y,s)$ ($i = 1, 2, 3$), $V\{\mu\}_4^2(Y,s)$, and $V\{\mu\}_4^3(Y,s)$ denote the values at $(Y,s) \in \partial B$ of the expressions appearing in (IV.14.37, 42, 49, and 52), respectively, each being a sum of values of functions of the form $\omega_{31}\{\phi\}$. ■.

[IV.35] T H E O R E M. Let M be a motion in $\mathbb{M}(1;0)$. Suppose that

- (i) $\{B_\zeta^0\}_{\zeta \in \mathbb{R}}$ is strongly locally uniformly Lyapunov, i.e., whenever $\bar{K} \subset \mathbb{R}$ is compact, $\nu \mid \cup_{\zeta \in \bar{K}} \{\partial B_\zeta \times \{\zeta\}\}$ is Hölder continuous,
- (ii) there exists a reference pair (R,x) for M possessing the properties recounted in [IV.10] and such that $x_{,4}$ and $\hat{J}x$ are locally Hölder continuous on $\partial R \times \mathbb{R}$,

and

- (iii) $\mu: \partial B \rightarrow \mathbb{K}$ is locally Hölder continuous and such that $D_4^0 \mu$ is in $C(\partial R \times \mathbb{R})$.

Then

(i)' $V^0\{\mu\} \in C^1(\Omega^{\sigma-})$ and $V^I\{\mu\} \in C^1(B)$, with

$$\left. \begin{aligned} V^{[I]}_{\{\mu\},i} &= [\mp] \frac{1}{2\{1-(v^c)^2\}} \mu v^i + V^*_i\{\mu\}, \\ \text{for } i &= 1, 2, \text{ and } 3, \\ \text{and} \\ V^{[I]}_{\{\mu\},4} &= [\mp] \frac{1}{2\{1-(v^c)^2\}} \mu \mu + V^*_4\{\mu\} \end{aligned} \right\} \text{ on } \partial B, \quad (1)$$

wherein, for each $(Y,s) \in \partial B$, $V^0\{\mu\}_j(Y,s)$ $[V^I\{\mu\}_j(Y,s)]$, $j = 1, 2, 3$, and 4 , denotes the value of the continuous extension of $V^0\{\mu\}_j$ $[V^I\{\mu\}_j]$ from Ω^σ to $\Omega^{\sigma-}$ [from B^0 to B].

(ii)' If $M \in M(2)$, hypotheses (i) and (ii) are fulfilled.

(iii)' If it is also known that, for each compact $\bar{K} \subset \mathbb{R}$, $x_{,44}(P,\cdot)|_{\bar{K}}$, $D_4 \hat{J}x(P,\cdot)|_{\bar{K}}$, and $D_4^0 \hat{u}(P,\cdot)|_{\bar{K}}$ are Hölder continuous, uniformly for $P \in \partial R$, then $V^*_i\{\mu\}$ is locally Hölder continuous on ∂B for each $i \in \{1,2,3,4\}$.

P R O O F. (i)' Upon observing that the hypotheses of both [IV.33] and [IV.17] are satisfied here, we can immediately assert that $V\{\mu\} \in C^1(B^0 \cup \Omega^\sigma) \cap C(\mathbb{R}^4)$, so $V^0\{\mu\} \in C^1(\Omega^\sigma) \cap C(\Omega^{\sigma-})$ and $V^I\{\mu\} \in C^1(B^0) \cap C(B)$. Consequently, it must be shown that, for $j = 1, 2, 3$, and 4 , $V^0\{\mu\}_j$ $[V^I\{\mu\}_j]$ can be extended continuously to $\Omega^{\sigma-}$

[to B], and that the continuous extensions are given on ∂B by the expressions displayed in (1) and (2).

Suppose first that $i \in \{1, 2, 3\}$. $V\{\mu\}_i$ is given in $B^0 \cap \Omega^\sigma$ by (IV.17.1); we shall examine the behavior in $\Omega^{\sigma-}$ and in B of the functions given by each term in this expression. First, it is clear from conclusion (iv)' of [IV.32] that the function

$$(X, t) \mapsto \int_{\partial B_t} \frac{1}{2} \cdot r_X^{13} \cdot \Lambda_{4i}^1\{\mu\}(X, t) \circ X_t^{-1} \cdot JX_t^{-1} d\lambda_{\partial B_t}$$

is continuous on $\mathbb{R}^4$, while (v)' of that same theorem shows that $(X, t) \mapsto V\{\mu\}_i^2(X, t)$ and $(X, t) \mapsto V\{\mu\}_i^3(X, t)$ are also continuous on $\mathbb{R}^4$ (cf., (IV.14.37 and 42)). Next, to study $\omega_1^0\{\Lambda_{1i}^1\{\mu\}\}$ in Ω^σ [$\omega_1^I\{\Lambda_{1i}^1\{\mu\}\}$ in B^0], we shall use Lemma [IV.21]: we have already pointed out that $\omega_1^0\{\Lambda_{1i}^1\{\mu\}\}$ is continuous in Ω^σ [$\omega_1^I\{\Lambda_{1i}^1\{\mu\}\}$ is continuous in B^0], while Theorem [IV.22] says that, for each $(Y, s) \in \partial B$, locally uniformly in (Y, s) ,

$$\begin{aligned} \lim_{\substack{X \rightarrow Y \\ X \in L_V^+(Y, s) \\ [X \in L_V^-(Y, s)]}} \omega_1^{[I]} \{\Lambda_{1i}^1\{\mu\}\}(X, s) &= [^+] \frac{1}{2(1 - |X_{,4}^c(X_s^{-1}(Y), s)|_3^2)} \\ &\quad \cdot \{ (1 - |X_{,4}^c(X_s^{-1}(Y), s)|_3^2)^{\frac{1}{2}} v^i(Y, s) \\ &\quad + v^c(Y, s) \cdot X_{,4}^{ic}(X_s^{-1}(Y), s) \} \mu(Y, s) \\ &\quad + \omega_1^* \{\Lambda_{1i}^1\{\mu\}\}(Y, s), \end{aligned} \quad (3)$$

having accounted for (IV.22.4 and 5). Appealing to [IV.24], it is certainly true that $\omega_1^* \{\Lambda_{1i}^1\{\mu\}\}$ is locally Hölder continuous on ∂B , whence it follows that the function on ∂B which is given by

the right-hand side of (3), for each $(Y, s) \in \partial B$, is continuous. Thus, Lemma [IV.21] implies that $w_{1i}^0\{\Lambda_{1i}^1\{\mu\}\}$ $[w_{1i}^I\{\Lambda_{1i}^1\{\mu\}\}]$ can be extended continuously to Ω^{σ^-} [to B]. For the two remaining terms on the right in (IV.17.1), we reason similarly: we have noted that $w_{2i}^0\{\Lambda_2^1\{\mu\}\}$ and $w_{2j}^0\{\Lambda_{3ij}^1\{\mu\}\}$ are in $C(\Omega^{\sigma})$ $[w_{2i}^I\{\Lambda_2^1\{\mu\}]$ and $w_{2j}^I\{\Lambda_{3ij}^1\{\mu\}]$ are in $C(B^0)]$, while Theorem [IV.28] clearly shows that, for each $(Y, s) \in \partial B$, locally uniformly in (Y, s) ,

$$\begin{aligned} & \lim_{X \rightarrow Y} w_{2i}^{[I]} \{\Lambda_2^1\{\mu\}\}(X, s) \\ & \quad X \in L_V^+(Y, s) \\ & \quad [X \in L_V^-(Y, s)] \\ & = \frac{+}{[-]} \frac{v^c(Y, s) \{v^c(Y, s) v^i(Y, s) - x_{,4}^i(x_s^{-1}(Y), s)\}}{2\{1 - |x_{,4}^c(x_s^{-1}(Y), s)|^2\} \cdot \{1 - (v^c(Y, s))^2\}} \\ & \quad \cdot \{1 - |x_{,4}^c(x_s^{-1}(Y), s)|^2\} \cdot v(Y, s) + w_{2i}^* \{\Lambda_2^1\{\mu\}\}(Y, s), \end{aligned} \quad (4)$$

and

$$\begin{aligned} & \lim_{X \rightarrow Y} w_{2j}^{[I]} \{\Lambda_{3ij}^1\{\mu\}\}(X, s) \\ & \quad X \in L_V^+(Y, s) \\ & \quad [X \in L_V^-(Y, s)] \\ & = \frac{+}{[-]} \frac{v^c(Y, s) \{v^c(Y, s) v^j(Y, s) - x_{,4}^j(x_s^{-1}(Y), s)\}}{2\{1 - |x_{,4}^c(x_s^{-1}(Y), s)|^2\} \cdot \{1 - (v^c(Y, s))^2\}} \\ & \quad \cdot x_{,4}^j(x_s^{-1}(Y), s) \cdot x_{,4}^i(x_s^{-1}(Y), s) \cdot v(Y, s) + w_{2j}^* \{\Lambda_{3ij}^1\{\mu\}\}(Y, s), \end{aligned} \quad (5)$$

having used (IV.28.6 and 7). From Theorem [IV.29] we can conclude that $w_{2i}^* \{\Lambda_2^1\{\mu\}\}$ and $w_{2j}^* \{\Lambda_{3ij}^1\{\mu\}\}$ are locally Hölder continuous on

∂B , so that the functions on ∂B which are given by the respective expressions appearing on the right in (4) and (5), for each $(Y, s) \in \partial B$, are continuous. Once again utilizing Lemma [IV.21], it obviously follows that $w_{2i}^0\{\Lambda_2^1\{\mu\}\}$ and $w_{2j}^0\{\Lambda_{3ij}^1\{\mu\}\}$ possess continuous extensions to $\Omega^{\sigma-}$ [$w_{2i}^I\{\Lambda_2^1\{\mu\}\}$ and $w_{2j}^I\{\Lambda_{3ij}^1\{\mu\}\}$ possess continuous extensions to B]. It is now evident that $v^0\{\mu\}_{,1}$ can be extended continuously to $\Omega^{\sigma-}$ [$v^I\{\mu\}_{,1}$ can be extended continuously to B], whence $v^0\{\mu\}_{,1} \in C^1(\Omega^{\sigma-})$ [$v^I\{\mu\}_{,1} \in C^1(B)$]. Moreover, denoting its continuous extension again by $v^0\{\mu\}_{,1}$ [$v^I\{\mu\}_{,1}$], and using (IV.17.1), (IV.34.1), (3), (4), and (5), we must have

$$\begin{aligned} & v^{[I]}_{\{\mu\}_{,1}}(Y, s) \\ = & [+]\frac{1}{2\{1-|x_{,4}^c(x_s^{-1}(Y), s)|_3^2\}} \cdot \{1-|x_{,4}^c(x_s^{-1}(Y), s)|_3^2\} v^1(Y, s) \\ & + v^c(Y, s) \cdot x_{,4}^{1c}(x_s^{-1}(Y), s) \cdot \mu(Y, s) \\ & - \left\{ \begin{array}{l} + \\ [-] \end{array} \frac{v^c(Y, s) \cdot \{v^c(Y, s) v^j(Y, s) - x_{,4}^j(x_s^{-1}(Y), s)\}}{2\{1-|x_{,4}^c(x_s^{-1}(Y), s)|_3^2\} \cdot \{1-(v^c(Y, s))^2\}} \right\} \\ & \cdot \{1-|x_{,4}^c(x_s^{-1}(Y), s)|_3^2\} \delta_{1j} + x_{,4}^j(x_s^{-1}(Y), s) \cdot x_{,4}^{1c}(x_s^{-1}(Y), s) \} \mu(Y, s) \\ & + w_{1i}^*\{\Lambda_{1i}^1\{\mu\}\}(Y, s) - w_{2i}^*\{\Lambda_2^1\{\mu\}\}(Y, s) - w_{2j}^*\{\Lambda_{3ij}^1\{\mu\}\}(Y, s) \\ & - \frac{1}{4\pi} \int_{\partial B_s} \frac{1}{r_Y} \cdot r_Y^{13}(Y, s) \cdot \Lambda_{4i}^1\{\mu\}(Y, s) \circ x_s^{-1} \cdot J x_s^{-1} d\lambda_{\partial B_s} \\ & + v\{\mu\}_1^2(Y, s) + v\{\mu\}_1^3(Y, s) \end{aligned}$$

$$= [\bar{+}] \frac{1}{2\{1-(v^c(Y,s))^2\}} \cdot \mu(Y,s) v^1(Y,s) + v_1^*(\mu)(Y,s)$$

for each $(Y,s) \in \partial B$,

with which (1) is proven.

Turning next to $v^0_{\{\mu\},4} [v^I_{\{\mu\},4}]$, we can use conclusions (iv)' and (v)' of Theorem [IV.32], Lemma [IV.21], and Theorems [IV.22, 24, 28, and 29] to show that $v^0_{\{\mu\},4}$ possesses a continuous extensions to $\Omega^{\sigma-} [v^I_{\{\mu\},4}$ possesses a continuous extension to B], hence that $v^0_{\{\mu\},4} \in C^1(\Omega^{\sigma-}) [v^I_{\{\mu\},4} \in C^1(B)]$; the required argument is so similar to that already presented for $v^0_{\{\mu\},1} [v^I_{\{\mu\},1}]$ for $i = 1, 2$, and 3 that we shall omit it. To demonstrate that (2) is correct, we employ (IV.17.2), (IV.34.2), (IV.22.4 and 6) and (IV.28.4 and 8) to write, retaining the same notation for the continuous extension of $v^0_{\{\mu\},4} [v^I_{\{\mu\},4}]$,

$$\begin{aligned} & v^{0,I}_{\{\mu\},4}(Y,s) \\ &= -c \left\{ [\bar{+}] \frac{1}{2\{1-|x^c_{,4}(x_s^{-1}(Y),s)|^2_3\}} \cdot v^c(Y,s) \cdot \mu(Y,s) \right\} \\ &+ c \left\{ [\bar{-}] \frac{v^c(Y,s) \{v^c(Y,s) v^j(Y,s) - x^{jc}_{,4}(x_s^{-1}(Y),s)\}}{2\{1-|x^c_{,4}(x_s^{-1}(Y),s)|^2_3\} \cdot \{1-(v^c(Y,s))^2\}} \cdot x^{jc}_{,4}(x_s^{-1}(Y),s) \cdot \mu(Y,s) \right. \\ &- c w^*_1 \{\tilde{\lambda}^1_1\} (Y,s) + c w^*_{2j} \{\tilde{\lambda}^1_{2j}\} (Y,s) \\ &- \frac{c}{4\pi} \int_{\partial B_s} \frac{1}{r_Y} \cdot \Gamma^{13}_{(Y,s)} \cdot \tilde{\lambda}^1_3(Y,s) \circ x_s^{-1} \cdot J x_s^{-1} d\lambda_{\partial B_s} \\ &+ v_{,4}^2(Y,s) + v_{,4}^3(Y,s) \end{aligned}$$

$$= \begin{bmatrix} + \\ - \end{bmatrix} \frac{1}{2\{1-(v^c(Y,s))^2\}} \cdot v(Y,s) \cdot \mu(Y,s) + v_4^*(\mu)(Y,s)$$

for each $(Y,s) \in \partial B$.

This gives (2), completing the proof of (i)'.

(ii)' The reasoning required to verify this statement is contained in the proof of Theorem [IV.22]; cf., also, [IV.23.a].

(iii)' In view of the definitions (IV.34.1 and 2), and noting (IV.14.37, 42, 49, and 52), it is easy to check that this assertion is a consequence of Theorems [IV.24 and 29] and conclusions (vi)'- (viii)' of Theorem [IV.31]. $\square$.

IV.A. APPENDIX

EVALUATION OF CERTAIN AUXILIARY INTEGRALS

In the proof of Lemma [IV.26], we encounter certain quite elementary integrals which must be explicitly evaluated; in this appendix, we give the major steps in the requisite evaluations. The computations, while entirely trivial, are somewhat tedious in various places. Since the results are so important to the main exposition, we have undertaken to display their derivations for inspection.

(i) Suppose that $\Delta > 0$, $\alpha > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$.

Set

$$I_1 := \int_0^{\Delta} \frac{\rho^2}{\{\alpha^2 + (\gamma\rho + \beta)^2\}^{3/2}} d\rho = \frac{1}{\alpha^3} \int_0^{\Delta} \frac{\rho^2}{\left\{1 + \left(\frac{1}{\alpha}(\gamma\rho + \beta)\right)^2\right\}^{3/2}} d\rho.$$

Employing the transformation given by

$$x(\rho) := \frac{1}{\alpha} (\gamma\rho + \beta),$$

and writing

$$x_1 := x(0) = \frac{\beta}{\alpha}, \quad x_2 := x(\Delta) = \frac{\gamma\Delta + \beta}{\alpha},$$

we find that

$$\begin{aligned}
 I_1 &= \frac{1}{\alpha^2 \gamma^3} \int_{x_1}^{x_2} \frac{\{\alpha x - \beta\}^2}{\{1+x^2\}^{3/2}} dx \\
 &= \frac{1}{\gamma^3} \int_{x_1}^{x_2} \frac{x^2}{\{1+x^2\}^{3/2}} dx - \frac{2\beta}{\alpha \gamma^3} \int_{x_1}^{x_2} \frac{x}{\{1+x^2\}^{3/2}} dx \\
 &\quad + \frac{\beta^2}{\alpha^2 \gamma^3} \int_{x_1}^{x_2} \frac{1}{\{1+x^2\}^{3/2}} dx.
 \end{aligned}$$

Now,

$$\begin{aligned}
 \int_{x_1}^{x_2} \frac{x}{\{1+x^2\}^{3/2}} dx &= \frac{1}{\{1+x_1^2\}^{1/2}} - \frac{1}{\{1+x_2^2\}^{1/2}}, \\
 \int_{x_1}^{x_2} \frac{1}{\{1+x^2\}^{3/2}} dx &= \sin(\tan^{-1} x_2) - \sin(\tan^{-1} x_1) \\
 &= \frac{x_2}{\{1+x_2^2\}^{1/2}} - \frac{x_1}{\{1+x_1^2\}^{1/2}},
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{x_1}^{x_2} \frac{x^2}{\{1+x^2\}^{3/2}} dx &= -\frac{x}{\{1+x^2\}^{1/2}} \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{1}{\{1+x^2\}^{1/2}} dx \\
 &= -\frac{x}{\{1+x^2\}^{1/2}} \Big|_{x_1}^{x_2} + \ln |x + \sec(\tan^{-1} x)| \Big|_{x_1}^{x_2}
 \end{aligned}$$

$$= \frac{x_1}{\{1+x_1^2\}^{1/2}} - \frac{x_2}{\{1+x_2^2\}^{1/2}} + \ln \left\{ \frac{x_2 + \{1+x_2^2\}^{1/2}}{x_1 + \{1+x_1^2\}^{1/2}} \right\}.$$

Collecting up these results and subsequently inserting the values of x_1 and x_2 yields

$$\begin{aligned} I_1 &= \frac{1}{\gamma^3} \left\{ \frac{x_1}{\{1+x_1^2\}^{1/2}} - \frac{x_2}{\{1+x_2^2\}^{1/2}} + \ln \frac{x_2 + \{1+x_2^2\}^{1/2}}{x_1 + \{1+x_1^2\}^{1/2}} \right. \\ &\quad \left. - \frac{2\beta}{\alpha} \left\{ \frac{1}{\{1+x_1^2\}^{1/2}} - \frac{1}{\{1+x_2^2\}^{1/2}} \right\} + \frac{\beta^2}{\alpha^2} \left\{ \frac{x_2}{\{1+x_2^2\}^{1/2}} - \frac{x_1}{\{1+x_1^2\}^{1/2}} \right\} \right\} \\ &= \frac{1}{\gamma^3} \left\{ \ln \frac{(\gamma\Delta + \beta) + \sqrt{\alpha^2 + (\gamma\Delta + \beta)^2}}{\beta + \sqrt{\alpha^2 + \beta^2}} + \frac{((\beta^2/\alpha^2) - 1)(\gamma\Delta + \beta) + 2\beta}{\sqrt{\alpha^2 + (\gamma\Delta + \beta)^2}} \right. \\ &\quad \left. - \frac{\beta}{\alpha^2} \sqrt{\alpha^2 + \beta^2} \right\}. \end{aligned} \quad (1)$$

(ii) Suppose that $\Xi_0 \in (0, 1)$ and $\Theta \in (0, \pi/2) \cup (\pi/2, \pi)$.

Writing

$$\hat{\mu}_0^2 := \frac{\Xi_0^2 \sin^2 \Theta}{1 - \Xi_0^2 \cos^2 \Theta}, \quad \text{where } \hat{\mu}_0 > 0, \quad (2)$$

we wish to evaluate

$$I_2 := \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \omega}{(1 - \hat{\mu}_0^2 \cos^2 \omega)(1 - \Xi_0^2 \sin^2 \Theta \cos^2 \omega)} d\omega.$$

We begin by introducing the transformation given by

$$\zeta(\omega) := \tan \omega/2, \quad -\pi/2 \leq \omega \leq \pi/2, \quad (3)$$

with which we obtain

$$I_2 = \int_{-1}^1 \frac{\left\{ \frac{2\zeta}{1+\zeta^2} \right\}^2}{\left\{ 1-\hat{u}_0^2 \left(\frac{1-\zeta^2}{1+\zeta^2} \right)^2 \right\} \left\{ 1-\hat{\Xi}_0^2 \sin^2 \Theta \left(\frac{1-\zeta^2}{1+\zeta^2} \right)^2 \right\}} \cdot \frac{2}{(1+\zeta^2)} d\zeta \quad (4)$$

$$= 16 \int_0^1 \frac{(1+\zeta^2)\zeta^2}{(a_1+a_2\zeta^2)(a_2+a_1\zeta^2)(b_1+b_2\zeta^2)(b_2+b_1\zeta^2)} d\zeta,$$

having written

$$a_1 := 1-\hat{u}_0 = \frac{1}{(1-\hat{\Xi}_0^2 \cos^2 \Theta)^{1/2}} \{ (1-\hat{\Xi}_0^2 \cos^2 \Theta)^{1/2} - \hat{\Xi}_0 \sin \Theta \}, \quad (5)$$

$$a_2 := 1+\hat{u}_0 = \frac{1}{(1-\hat{\Xi}_0^2 \cos^2 \Theta)^{1/2}} \{ (1-\hat{\Xi}_0^2 \cos^2 \Theta)^{1/2} + \hat{\Xi}_0 \sin \Theta \}, \quad (6)$$

$$b_1 := 1-\hat{\Xi}_0 \sin \Theta, \quad (7)$$

and

$$b_2 := 1+\hat{\Xi}_0 \sin \Theta.^\dagger \quad (8)$$

Next, we perform a partial fraction expansion of the integrand appearing in (4): with

$$A := - \frac{a_1 a_2}{a_1 + a_2} \cdot \frac{1}{(a_1 b_1 - a_2 b_2)(a_1 b_2 - a_2 b_1)} = - \frac{1-\hat{\Xi}_0^2}{8\hat{\Xi}_0^4 \sin^2 \Theta \cos^2 \Theta} \quad (9)$$

and

[†] Obviously, b_1 and $b_2 > 0$; since $\hat{u}_0 \in (0,1)$, also a_1 and $a_2 > 0$.

$$B := \frac{b_1 b_2}{b_1 + b_2} \cdot \frac{1}{(a_1 b_1 - a_2 b_2)(a_1 b_2 - a_2 b_1)} = \frac{(1 - \epsilon_0^2 \cos^2 \Theta)(1 - \epsilon_0^2 \sin^2 \Theta)}{8 \epsilon_0^4 \sin^2 \Theta \cos^2 \Theta}, \quad (10)$$

it follows that

$$\begin{aligned} I_2 &= 16 \left\{ A \int_0^1 \frac{d\zeta}{a_1 + a_2 \zeta^2} + A \int_0^1 \frac{d\zeta}{a_2 + a_1 \zeta^2} + B \int_0^1 \frac{d\zeta}{b_1 + b_2 \zeta^2} + B \int_0^1 \frac{d\zeta}{b_2 + b_1 \zeta^2} \right\} \\ &= \frac{16A}{\sqrt{(a_1 a_2)}} \{ \tan^{-1} \sqrt{(a_1/a_2)} + \tan^{-1} \sqrt{(a_2/a_1)} \} \\ &\quad + \frac{16B}{\sqrt{(b_1 b_2)}} \{ \tan^{-1} \sqrt{(b_1/b_2)} + \tan^{-1} \sqrt{(b_2/b_1)} \}; \end{aligned}$$

since, as it is easy to show,

$$\tan^{-1} \zeta + \tan^{-1} 1/\zeta = \pi/2 \quad \text{for each } \zeta > 0,$$

we can write further

$$\begin{aligned} I_2 &= 8\pi \left\{ \frac{A}{\sqrt{(a_1 a_2)}} + \frac{B}{\sqrt{(b_1 b_2)}} \right\} \\ &= 8\pi \left\{ \frac{A}{\sqrt{(1 - \epsilon_0^2)}} + \frac{B}{\sqrt{(1 - \epsilon_0^2 \sin^2 \Theta)}} \right\} \\ &= \frac{\pi}{\epsilon_0^4 \sin^2 \Theta \cos^2 \Theta} \left\{ - \frac{1 - \epsilon_0^2}{(1 - \epsilon_0^2 \cos^2 \Theta)^{1/2} (1 - \epsilon_0^2)^{1/2}} \right. \\ &\quad \left. + \frac{(1 - \epsilon_0^2 \cos^2 \Theta)(1 - \epsilon_0^2 \sin^2 \Theta)}{(1 - \epsilon_0^2 \sin^2 \Theta)^{1/2}} \right\} \end{aligned}$$

$$= \frac{\pi}{\Xi_0^4 \sin^2 \Theta \cos^2 \Theta} \{ (1-\Xi_0^2 \sin^2 \Theta)^{1/2} (1-\Xi_0^2 \cos^2 \Theta)^{1/2} - (1-\Xi_0^2)^{1/2} (1-\Xi_0^2 \cos^2 \Theta)^{1/2} \}. \quad (11)$$

(iii) Again with $\Xi_0 \in (0,1)$, $\Theta \in (0, \pi/2) \cup (\pi/2, \pi)$, and $\hat{u}_0$ as in (2), let us evaluate

$$I_3 := \int_{-\pi/2}^{\pi/2} \frac{\cos^2 \omega}{(1-\hat{u}_0^2 \cos^2 \omega)(1-\Xi_0^2 \sin^2 \Theta \cos^2 \omega)} d\omega.$$

Once again utilizing the transformation given by (3),

$$\begin{aligned} I_3 &= \int_{-1}^1 \frac{\left\{ \frac{1-\zeta^2}{1+\zeta^2} \right\}^2}{\left\{ 1-\hat{u}_0^2 \left(\frac{1-\zeta^2}{1+\zeta^2} \right)^2 \right\} \left\{ 1-\Xi_0^2 \sin^2 \Theta \left(\frac{1-\zeta^2}{1+\zeta^2} \right)^2 \right\}} \cdot \frac{2}{(1+\zeta^2)} d\zeta \\ &= 4 \int_0^1 \frac{(1+\zeta^2)(1-\zeta^2)^2}{(a_1+a_2\zeta^2)(a_2+a_1\zeta^2)(b_1+b_2\zeta^2)(b_2+b_1\zeta^2)} d\zeta. \end{aligned} \quad (12)$$

Working out the partial-fraction decomposition of the integrand in (12) and setting

$$A' := \frac{2}{(a_1b_1-a_2b_2)(a_1b_2-a_2b_1)} = \frac{1-\Xi_0^2 \cos^2 \Theta}{2\Xi_0^4 \sin^2 \Theta \cos^2 \Theta}, \quad (13)$$

we obtain, computing in a manner similar to that in (ii),

$$I_3 = 4A' \left\{ \int_0^1 \frac{d\zeta}{a_1+a_2\zeta^2} + \int_0^1 \frac{d\zeta}{a_2+a_1\zeta^2} - \int_0^1 \frac{d\zeta}{b_1+b_2\zeta^2} - \int_0^1 \frac{d\zeta}{b_2+b_1\zeta^2} \right\}$$

$$\begin{aligned}
 &= 4A' \left\{ \frac{1}{\sqrt{a_1 a_2}} (\tan^{-1} \sqrt{a_1/a_2} + \tan^{-1} \sqrt{a_2/a_1}) \right. \\
 &\quad \left. - \frac{1}{\sqrt{b_1 b_2}} (\tan^{-1} \sqrt{b_1/b_2} + \tan^{-1} \sqrt{b_2/b_1}) \right\} \\
 &= 2\pi A' \left\{ \frac{1}{\sqrt{a_1 a_2}} - \frac{1}{\sqrt{b_1 b_2}} \right\} \\
 &= 2\pi A' \left\{ \frac{1}{(1-\hat{\mu}_0^2)^{1/2}} - \frac{1}{(1-\hat{\mu}_0^2 \sin^2 \Theta)^{1/2}} \right\} \\
 &= \frac{\pi(1-\hat{\mu}_0^2 \cos^2 \Theta)}{\hat{\mu}_0^2 \sin^2 \Theta \cos^2 \Theta} \cdot \left\{ \frac{(1-\hat{\mu}_0^2 \cos^2 \Theta)^{1/2}}{(1-\hat{\mu}_0^2)^{1/2}} - \frac{1}{(1-\hat{\mu}_0^2 \sin^2 \Theta)^{1/2}} \right\}.
 \end{aligned} \tag{14}$$

(iv) As in (ii) and (iii), let $\hat{\mu}_0 \in (0,1)$, $\Theta \in (0, \pi/2) \cup (\pi/2, \pi)$, and $\hat{\mu}_0$ be given by (2). Define $\mu_1 > 0$, $\mu_2 > 0$, and μ_3 by

$$\mu_1^2 := 1 - \hat{\mu}_0^2 \sin^2 \Theta, \tag{15}$$

$$\mu_2^2 := 1 - \hat{\mu}_0^2 \cos^2 \Theta, \tag{16}$$

$$\mu_3 := \hat{\mu}_0^2 \sin \Theta \cos \Theta. \tag{17}$$

We shall evaluate

$$I_4 := \int_{-\pi/2}^{\pi/2} \frac{\sin \omega}{\{1 - \hat{\mu}_0^2 \cos^2 \omega\}^{3/2}} \cdot \ln \{ \mu_3 \sin \omega + \mu_1 \mu_2 (1 - \hat{\mu}_0^2 \cos^2 \omega)^{1/2} \} d\omega.$$

Observe first that

$$\mu_3 \sin \omega \pm \mu_1 \mu_2 (1 - \hat{\mu}_0^2 \cos^2 \omega)^{1/2} \geq 0 \quad \text{for each } \omega \in \mathbb{R}: \tag{18}$$

for,

$$\begin{aligned}
 & \mu_1^2 \mu_2^2 (1 - \mu_0^2 \cos^2 \omega) \\
 &= \mu_1^2 \mu_2^2 - \mu_1^2 \mu_2^2 \mu_0^2 \sin^2 \omega \cos^2 \omega \\
 &= 1 - \mu_0^2 + \mu_0^4 \sin^2 \omega \cos^2 \omega - \mu_0^2 \sin^2 \omega \cos^2 \omega + \mu_0^4 \sin^4 \omega \cos^2 \omega \\
 &= \{1 - \mu_0^2 - \mu_0^2 \sin^2 \omega \cos^2 \omega + \mu_0^4 \sin^2 \omega \cos^2 \omega\} \\
 &\quad + \{\mu_0^4 \sin^2 \omega \cos^2 \omega + \mu_0^4 \sin^4 \omega \cos^2 \omega - \mu_0^4 \sin^2 \omega \cos^2 \omega\} \\
 &= (1 - \mu_0^2)(1 - \mu_0^2 \sin^2 \omega \cos^2 \omega) \\
 &\quad + \mu_0^4 \sin^2 \omega \{\cos^2 \omega + \sin^2 \omega \cos^2 \omega - \cos^2 \omega\} \\
 &= (1 - \mu_0^2)(1 - \mu_0^2 \sin^2 \omega \cos^2 \omega) + \mu_0^4 \sin^2 \omega \cos^2 \omega \sin^2 \omega \\
 &= (1 - \mu_0^2)(1 - \mu_0^2 \sin^2 \omega \cos^2 \omega) + \mu_0^2 \sin^2 \omega,
 \end{aligned}$$

and so

$$\begin{aligned}
 & \mu_3 \sin \omega \pm \mu_1 \mu_2 (1 - \mu_0^2 \cos^2 \omega)^{1/2} \\
 &= \mu_3 \sin \omega \pm \sqrt{(1 - \mu_0^2)(1 - \mu_0^2 \sin^2 \omega \cos^2 \omega) + \mu_0^2 \sin^2 \omega};
 \end{aligned}$$

since the first term under the radical is positive, (18) must hold.

The evaluation of I_4 can be reduced to that of I_3 , by an integration by parts. To see this, we begin by pointing out that, with

$$F(\omega) := - \frac{\cos \omega}{(1 - \mu_0^2 \cos^2 \omega)^{1/2}}, \quad \omega \in \mathbb{R},$$

we have

$$F'(\omega) = \frac{\sin \omega}{(1-\hat{\mu}_0^2 \cos^2 \omega)^{3/2}}, \quad \omega \in \mathbb{R}.$$

Next, set

$$G(\omega) := \ln \{ \mu_3 \sin \omega + \mu_1 \mu_2 (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2} \}, \quad \omega \in \mathbb{R}.$$

Then

$$\begin{aligned} G'(\omega) &= \frac{\mu_3 \cos \omega + \mu_1 \mu_2 (1-\hat{\mu}_0^2 \cos^2 \omega)^{-1/2} \cdot \hat{\mu}_0^2 \cos \omega \sin \omega}{\mu_3 \sin \omega + \mu_1 \mu_2 (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2}} \\ &= \frac{\cos \omega}{(1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2}} \cdot \frac{\mu_3 (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2} + \mu_1 \mu_2 \hat{\mu}_0^2 \sin \omega}{\mu_3 \sin \omega + \mu_1 \mu_2 (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2}} \\ &\quad \cdot \frac{\mu_3 \sin \omega - \mu_1 \mu_2 (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2}}{\mu_3 \sin \omega - \mu_1 \mu_2 (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2}} \\ &= \frac{\cos \omega}{(1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2}} \{ \mu_3^2 \sin^2 \omega - \mu_1^2 \mu_2^2 (1-\hat{\mu}_0^2 \cos^2 \omega) \}^{-1} \\ &\quad \cdot \{ -\mu_1 \mu_2 \mu_3 (1-\hat{\mu}_0^2 \cos^2 \omega) + \mu_1 \mu_2 \mu_3 \hat{\mu}_0^2 \sin^2 \omega \\ &\quad + (\mu_3^2 - \mu_1^2 \mu_2^2) (1-\hat{\mu}_0^2 \cos^2 \omega)^{1/2} \sin \omega \}. \end{aligned}$$

Of course,

$$I_4 = \int_{-\pi/2}^{\pi/2} F'(\omega) \cdot G(\omega) d\omega = F(\omega) \cdot G(\omega) \Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} F(\omega) \cdot G'(\omega) d\omega;$$

thus, noting that $F(\pi/2) = F(-\pi/2) = 0$, and that any term in the product $F \cdot G'$ which is odd will vanish when integrated on

$[-\pi/2, \pi/2]$, we come to

$$\begin{aligned}
 I_4 &= \int_{-\pi/2}^{\pi/2} \frac{\cos^2 \omega}{(1-\hat{\mu}_0^2 \cos^2 \omega)} \cdot \frac{\{-\mu_1 \mu_2 \mu_3 (1-\hat{\mu}_0^2 \cos^2 \omega) + \mu_1 \mu_2 \mu_3 \hat{\mu}_0^2 \sin^2 \omega\}}{\{\mu_3^2 \sin^2 \omega - \mu_1^2 \mu_2^2 (1-\hat{\mu}_0^2 \cos^2 \omega)\}} d\omega \\
 &= -\frac{\mu_1 \mu_2 \mu_3 (\hat{\mu}_0^2 - 1)}{1-\hat{\mu}_0^2} \cdot \int_{-\pi/2}^{\pi/2} \frac{\cos^2 \omega}{(1-\hat{\mu}_0^2 \cos^2 \omega)(1-\hat{\mu}_0^2 \sin^2 \omega \cos^2 \omega)} d\omega \quad (19) \\
 &= \frac{\mu_1 \mu_2 \mu_3 (1-\hat{\mu}_0^2)}{1-\hat{\mu}_0^2} \cdot I_3,
 \end{aligned}$$

because

$$\begin{aligned}
 &\mu_3^2 \sin^2 \omega - \mu_1^2 \mu_2^2 (1-\hat{\mu}_0^2 \cos^2 \omega) \\
 &= \hat{\mu}_0^4 \sin^2 \omega \cos^2 \omega \sin^2 \omega \\
 &\quad - (1-\hat{\mu}_0^2 \sin^2 \omega)(1-\hat{\mu}_0^2 \cos^2 \omega - \hat{\mu}_0^2 \sin^2 \omega \cos^2 \omega) \\
 &= -(1-\hat{\mu}_0^2 \sin^2 \omega \cos^2 \omega) + \hat{\mu}_0^2 \cos^2 \omega + \hat{\mu}_0^2 \sin^2 \omega - \hat{\mu}_0^4 \sin^4 \omega \cos^2 \omega \\
 &\quad - \hat{\mu}_0^4 \sin^2 \omega \cos^2 \omega + \hat{\mu}_0^4 \sin^2 \omega \cos^2 \omega \sin^2 \omega \\
 &= -(1-\hat{\mu}_0^2 \sin^2 \omega \cos^2 \omega) + \hat{\mu}_0^2 - \hat{\mu}_0^4 \sin^4 \omega \cos^2 \omega - \hat{\mu}_0^4 \sin^2 \omega \cos^2 \omega \cos^2 \omega \\
 &= -(1-\hat{\mu}_0^2 \sin^2 \omega \cos^2 \omega) + \hat{\mu}_0^2 - \hat{\mu}_0^4 \sin^2 \omega \cos^2 \omega \\
 &= -(1-\hat{\mu}_0^2)(1-\hat{\mu}_0^2 \sin^2 \omega \cos^2 \omega).
 \end{aligned}$$

Continuing, since

$$1-\hat{\mu}_0^2 = \frac{1-\hat{\mu}_0^2}{1-\hat{\mu}_0^2 \cos^2 \omega},$$

and using the expression (14) for I_3 in (19), we obtain

$$I_4 = \frac{\mu_1 \mu_2 \mu_3}{1 - \varepsilon_0^2} \cdot \frac{1 - \varepsilon_0^2}{\mu_2^2} \cdot \frac{\pi \mu_2^2}{\mu_3^2} \cdot \left\{ \frac{\mu_2}{(1 - \varepsilon_0^2)^{1/2}} - \frac{1}{\mu_1} \right\}$$

$$= \pi \cdot \frac{\mu_1 \mu_2}{\mu_3} \left\{ \frac{\mu_2}{(1 - \varepsilon_0^2)^{1/2}} - \frac{1}{\mu_1} \right\}.$$
(20)

We shall conclude by summarizing the results of (ii) and (iv), for ease of reference: with $\varepsilon_0 \in (0, 1)$ and $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$, let $\hat{\mu}_0 > 0$, $\mu_1 > 0$, $\mu_2 > 0$, and μ_3 be defined by (2), (15), (16), and (17), respectively. Then, from (11) and (20),

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^2 \omega}{(1 - \hat{\mu}_0^2 \cos^2 \omega)(1 - \varepsilon_0^2 \sin^2 \theta \cos^2 \omega)} d\omega$$

$$= \frac{\pi}{\mu_3} \{ \mu_1 \mu_2^2 - (1 - \varepsilon_0^2)^{1/2} \mu_2 \},$$
(21)

and

$$\int_{-\pi/2}^{\pi/2} \frac{\sin \omega}{(1 - \hat{\mu}_0^2 \cos^2 \omega)^{3/2}} \cdot \ln \{ \mu_3 \sin \omega + \mu_1 \mu_2 (1 - \hat{\mu}_0^2 \cos^2 \omega)^{1/2} \} d\omega$$

$$= \pi \cdot \frac{\mu_1 \mu_2}{\mu_3} \left\{ \frac{\mu_2}{(1 - \varepsilon_0^2)^{1/2}} - \frac{1}{\mu_1} \right\}.$$
(22)

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