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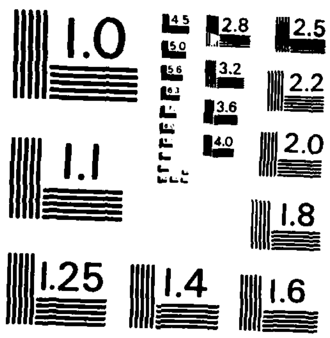

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ON DYNAMICAL FORMULATION OF A TETHERED SATELLITE SYSTEM WITH MASS TRANSPORT

TECHNICAL REPORT

AU-AFIT-EN-TR-84-1

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SATELLITE SYSTEM WITH MASS TRANSPORT

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RESEARCH REPORT

ON DYNAMICAL FORMULATION OF A TETHERED SATELLITE SYSTEM WITH MASS TRANSPORT

by

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May 25, 1984

PREFACE

This research report was prepared by Frank C. Liu, Professor of Mechanical Engineering, The University of Alabama in Huntsville during his leave at the Air Force Institute of Technology. The objective of this research is to investigate dynamic response of a tether connected satellite system due to mass transport along the tether. This problem may have useful application to NASA "Skyhook" program. The author is regretful that due to limited time and computer programming skills, and the large number of computer runs required, that numerical results are not available at the present time. They will be presented in a later report.

ACKNOWLEDGEMENTS

I am highly honored to have been invited for the 1983-1984 Distinguished Visiting Professorship at the Air Force Institute of Technology. I am grateful to the Institute for giving me a fine atmosphere to do this research. I wish to thank the Dean for Research and Professional Development, Dr. L. E. Wolaver, and the Chairman of the Department of Aeronautics and Astronautics Engineering, Dr. Peter J. Torvik, for helping to make my stay there enjoyable and fruitful. I would like to render grateful thanks to the department secretary Ms. Amy Whitehead for her painstaking effort in typing this report.

ABSTRACT

Two satellites connected by a long flexible tether along the earth radial direction comprise a stable equilibrium state. This research investigates the manner in which a third mass transporting from one satellite to the other disturbs the equilibrium state. A system of four equations of in-plane motion has been derived based on the assumptions that the tether remains straight between the masses and of constant length. A combination of computer subroutines DGEAR and ZANAYT is suggested for the approximate solution of the system of nonlinear differential equations with one constraint condition on the variables. Alternatively, a system of four independent differential equations are derived by eliminating the Lagrange multiplier.

TABLE OF CONTENTS

NOMENCLATURE

A_{ij}, $\bar{A}_{ij}$	coefficients of Equations (2.8) (4.2) and (4.4), defined in Apendix B
B_{ij}, $\bar{B}_{ij}$	same
C_{ij}, $\bar{C}_{ij}$	same
D_{ij}, $\bar{D}_{ij}$	same
D_n	defined by Equation (4.3)
$\bar{D}_n$	defined by Equation (4.5)
E_1, E_2, E_3	defined in Appendix A
E_{min}	min. energy of transfer defined by Equation (3.11)
f_{ni}	defined by Equation (4.3)
$\bar{f}_{ni}$	defined by Equation (4.5)
$g(y)$	constraint equation, Equation (4.2e)
G_1 to G_7	defined in Appendix A
$\underline{i}$	unit vector along orbital radius
$\underline{j}$	unit vector along orbital velocity
ℓ	length of tether
m	$= m_1 + m_2 + m_3$
M	$= m/m_1$
M_a, M_b, M_c	defined by Equation (1.5)
m_1	mass of outer satellite
m_2	mass of inner satellite
m_3	mass of transport mass
m_t	mass of tether
M_i	$= m_i/m_1$, $i = 2,3,t$
M_{mt}	$= M + M_t$
M_{23}	$= M_2 + M_3 = M - 1$

NOMENCLATURE

$\bar{M}$	$= \bar{M}_{23} + 1$
$\bar{M}_{23}$, $\bar{M}_3$	defined by Equation (1.3)
$p(t)$	driving force applied to m_3
$p^*(t)$	$= p(t)/m_3 \ell \omega_o^2$
p^*_D	defined by Equation (3.13)
$p_{min}^{(0)}$	min. magnitude of driving force to start transfer motion
q_i	generalized coordinate
Q_i	generalized force corresponding to q_i
Q^*_i	defined by Equations (2.8), (3.7) and (3.13)
Q^*_n	subscript designates the corrresponding equation, defined by (4.3)
Q_{ab}, Q_{ca}, Q_{cd}	defined by Equation (4.4)
$\bar{Q}_2$, $\bar{Q}_4$, $\bar{Q}_6$, $\bar{Q}_8$	defined by Equation (4.5)
$\underline{r}_i$	position vector of m_i from center of the Earth, $i = 1, 2, 3$
$\underline{r}_o$	position vector of center of mass of satellite system
R_o	$= r_o/\ell$
s	integration variable along tether
t	time variable
T_a to T_o	kinetic energy terms defined in Table 1.
T_m	kinetic energy of masses
T_t	kinetic energy of tether
V_a to V_i	potential energy terms defined in Table 2.
V_i	$= - \mu/r_i$
$\underline{v}_i$	velocity vector of m_i, $i = 1, 2, 3$

NOMENCLATURE

V_m	potential energy of masses
$\underline{v}_o$	orbital velocity of center of mass
V_o	$= - \mu/r_o$
V_t	potential energy of tether
x, y	coordinates defined in section 4.4
y_1 to y_8	variable of equations of motion defined by Equation (4.1)
α_a, α_b, α_c	defined by Equation (2.5)
α_{23}, α_3	defined by Equation (1.3)
β	$= \theta - \gamma$, angle between $\underline{\rho}$ and $\underline{\rho}_2$
γ	angle formed by $\underline{\rho}$ and $\underline{r}_o$
δ	$=\rho_2/\ell$
ξ,η	coordinates defined in section 4.4
θ	angle between $\underline{\rho}_2$ and $\underline{r}_o$
λ	Lagrange multiplier
λ^*	$= \lambda/m\ell\omega_o^2$; also used in Section 4.3
μ	gravity constant of Earth
$\underline{\rho}_i$	position of vector of m_i, i = 1, 2, 3
$\underline{\rho}$	$= \underline{\rho}_3 - \underline{\rho}_2$
σ	$= \rho/\ell$
τ	$= \omega_o t$, non-dimensional independent time variable
ω_o	$= (\mu/r_o{}^3)^{1/2}$, angular orbital velocity

SYMBOLS:

$\_$	a letter underlined denotes vector
$\cdot$	dot between vectors denotes dot product
$\dot{}$	dot on top of a letter denotes time derivative
$\ddot{}$	double dots on top of a letter denotes second time derivative
$'$	prime denotes derivative with respect to τ
$''$	double prime denotes second derivative with respect to τ
$*$	normalized quantity
Σ	summation of all variables in the equation

I. INTRODUCTION

Since Colombo [1] developed the concept of connecting a heavy mass to a satellite by a very long flexible tether in 1974, the subject has stimulated the science and engineering community. A recent article by Ivan Bekey [2] of the NASA Office of Space Flight gives detailed descriptions of many scientific applications of this idea. Two contractors, Martin Marietta (Denver), Aerospace and Ball Aerospace, have been given responsibility for the design of the "skyhook" project which is scheduled for first flight by NASA in 1987.

Preliminary analyses, feasibility studies, and design of a Tethered Satellite System (TSS) were conducted at Marshall Space Flight Center by Rupp and Lane [3], and Baker, et al [4]. Numerous papers have been published in the last decade dealing with the dynamics of deployment and retrieving the mass from a space shuttle. Various dynamical models have been developed by many investigators. These models may include one or more of the following actions:

(a) tether mass,

(b) three-dimensional motion,

(c) longitudinal vibration of the tether,

(d) transverse vibration of the tether,

(e) rotational motion of masses,

(f) offset distance of attachment to C.M. of masses, and

(g) eccentricity of TSS orbit.

Comparisons of the models can be found in the paper by Misra and Modi [5]. A formulation of a general dynamical model for TSS by the same authors is given in Reference 6.

Another aspect of the problem is optimal control of the tension in the tether for dynamic stability during deployment and retrieving of the mass. See, for example, a paper by Bainum and Kumar [7].

The objective of this research concerns with a problem which is quite different from that described above. The TSS aligned along earth radial direction is a stable equilibrium state. Consider the requirement that a third mass must be transported from one sub-satellite to the other along the tether. The mass transfer operation can be accomplished by free motion with sufficient initial velocity, or by applying a small thrust. It is desired to find the dynamic response of the TSS due to the motion of the transport mass.

The dynamical model to be treated here will include only one of the factors, (a), mentioned above. Hence, the TSS has three degrees-of-freedom There is no convenient method for reducing the dynamical model to three independent variables. Thus, one constraint condition must be induced between the four variables used for the formulation of equations of motion. This constraint condition is that the tether remains straight between masses and of constant length. Due to inclusion of the mass of the tether, the system of four second order differential equations will be very lengthy.

To the author's knowledge, there is no existing computer subroutine for the approximate solution of a system of differential equations with a constraint. A combined computer subroutine for the solution of such problems is suggested, and will be tested on an example with known solution. Further, a new system of four independent second order differential equations has been derived from the original system by eliminating Lagrange multiplier. This approach will be verified by direct comparison of solution of both systems.

II. ANALYSIS

1. FORMULATION OF KINETIC AND POTENTIAL ENERGIES OF THE SATELLITE SYSTEM

The following assumptions have been made for the formulation of the kinetic and potential energies of the three masses and of the tether connecting them.

(a) The motion of the center of mass of the system is undisturbed by the relative motions of the masses and the tether, i.e., the center of mass maintains uniform motion in a circular orbit.

(b) The tether is flexible and inextensible. It remains straight at all times between the transport mass and each of the others.

(c) The transport mass, m_3, is free to move along the tether, i.e., friction is neglected.

(d) There is no motion normal to the orbital plane.

(e) The main satellite m_1 and the sub-satellite m_2 are treated as point masses, i.e., the off-set distances from the attachment point to center of masses are neglected in the development of the equations of motion. However, satellite dimension will be partially accounted for through the distance Δ (See Fig. 1) in the initial and final radii.

(f) For the same reason, the rotational inertias of all masses are neglected.

(h) Mass m_1 is much greater than m_2 and both mass m_3 and the mass of the tether m_t are much smaller than m_2.

(i) The driving force applied on m_3 had negligible effect on the orbital motion of the system.

1.1 Coordinates and Definition of Variables

Figure 1 illustrates a tether connected satellite system. The left side figure shows a disturbance from the original undisturbed configuration (right). A rotating coordinate system is chosen, with origin at

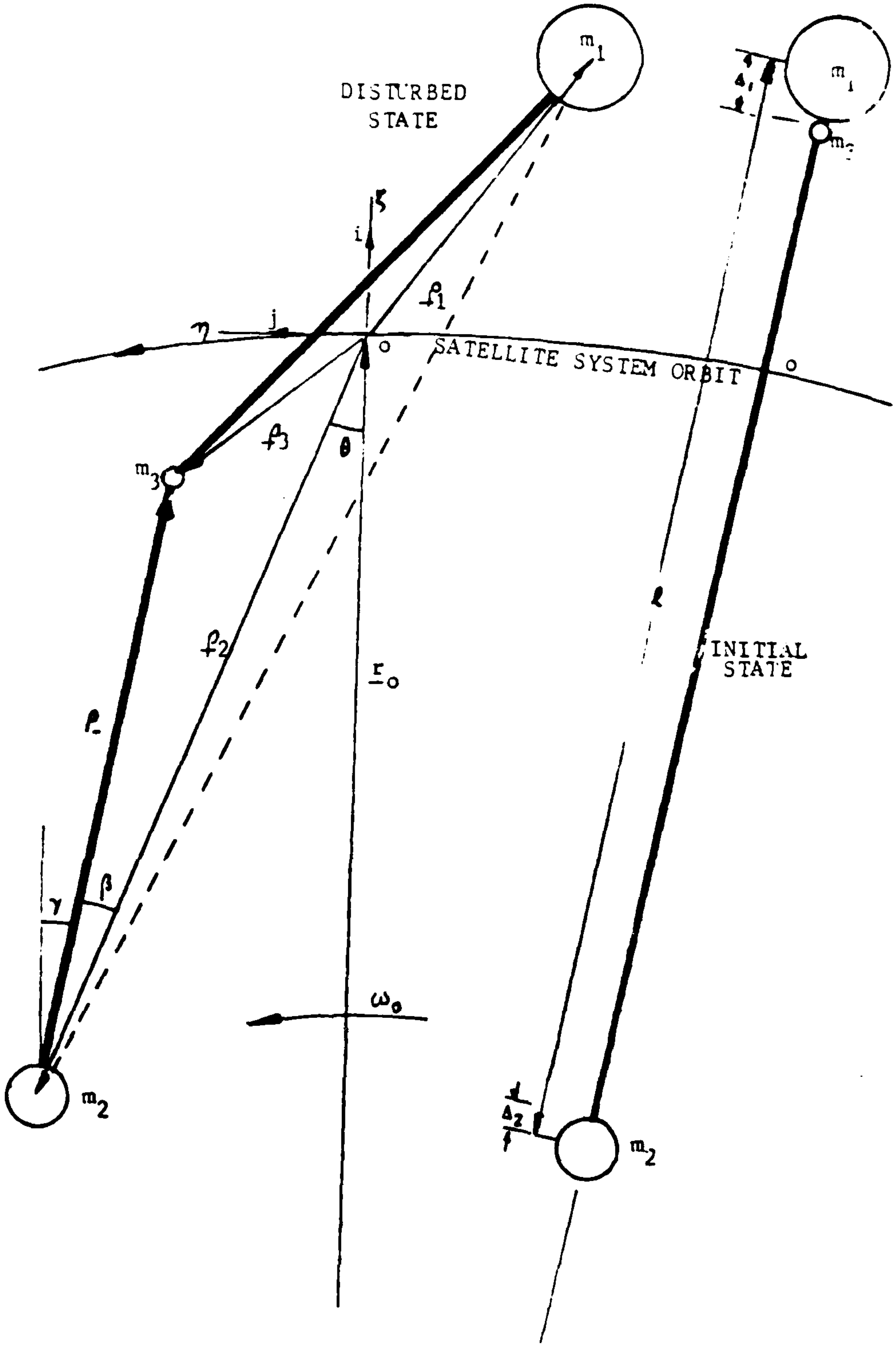


FIGURE 1 SATELLITE SYSTEM CONFIGURATIONS

the center of mass of the system, ξ-axis along the radial direction, and η-axis in the direction of the orbital velocity. Let $\underline{\rho}_i$ be the position vectors of m_i ($i = 1, 2, 3$) relative to the rotating coordinate system. Denote by $\underline{\rho}$ the position vector of m_3 relative to m_2 and by θ and γ the angles of $\underline{\rho}_2$ and $\underline{\rho}$ with respect to ξ-axis.

The tethered satellite has three degrees-of-freedom. Two sets of variables, (ρ_2, θ) and (ρ,γ) are chosen for formulating equations of motion of the system. Hence, the four variables must satisfy the assumed constraint condition that the length of tether is constant. Now, set the vectors

$$\underline{\rho}_2 = \rho_2 \, (-\cos\theta \underline{i} + \sin\theta \underline{j}) \tag{1.1a}$$

$$\underline{\rho} = \rho \, (\cos\gamma \underline{i} - \sin\gamma \underline{j}) = \underline{\rho}_3 - \underline{\rho}_2 \tag{1.1b}$$

and express $\underline{\rho}_1$ and $\underline{\rho}_3$ in terms of $\underline{\rho}_2$ and $\underline{\rho}$.

It follows from assumption (a) that

$$m_1\underline{\rho}_1 + m_2 \underline{\rho}_2 + m_3\underline{\rho}_3 + m_t\rho \, (\underline{\rho}_2 + \underline{\rho}_3)/2\ell + m_t(\ell - \rho)(\underline{\rho}_1 + \underline{\rho}_3)/2\ell = 0$$

This equation gives

$$- \underline{\rho}_1 = \{[M_{23} + ½M_t(1 + \sigma) \, \underline{\rho}_2 + (M_3 + ½M_t) \, \rho\} \, [1 + ½M_t(1 - \sigma)]^{-1} \tag{1.2}$$

where

$$m = m_1 + m_2 + m_3, \qquad M_i = m_i/m_1 \quad (i = 1,2,t)$$

$$M = m/m_1, \qquad \bar{M}_{23} = \bar{M}_2 + \bar{M}_3 = M - 1 \qquad \sigma = \rho/\ell$$

After eliminating second and higher order terms of M_t, one may write

$$\underline{\rho}_1 = - (\bar{M}_{23}\underline{\rho}_2 + \bar{M}_3 \, \underline{\rho}) \tag{1.3}$$

where

$$\bar{M}_{23} = M_{23}(1 + \alpha_{23}) \qquad \bar{M}_3 = M_3(1 + \alpha_3)$$

$$\alpha_{23} = M_t(M\sigma + 1 - M_{23})/2M_{23} \qquad \alpha_3 = M_t(M_3\sigma + 1 - M_3)/2M_3$$

Note that $\bar{M}_{23}$ and $\bar{M}_3$ are variables through inclusion of $\rho/\ell = \sigma$ in the expressions for α_{23} and α_3. Due to assumption (h), the quantities which involve M_t as much smaller than unity.

1.2 Formulation of the Kinetic Energy of the Masses

The velocity vector of mass m_i is

$$v_i = v_o + \dot{\underline{\rho}}_i + \underline{\omega}_o \times \underline{\rho}_i \qquad i = 1, 2, 3 \tag{1.4}$$

Substituting Equation (1.1) and (1.2), results in

$$\underline{v}_1 = \bar{M}_{23}\,[\dot{\rho}_2\cos\theta - \rho_2(\dot{\theta} - \omega_o)\sin\theta] - [\bar{M}_3\,\dot{\rho}\cos\gamma - \rho(\dot{\gamma} - \omega_o)\sin\gamma]\,\underline{i}$$

$$+ \{r_o\omega_o - \bar{M}_{23}[\dot{\rho}_2\sin\theta + \rho_2\,(\dot{\theta} - \omega_o)\cos\theta] + \bar{M}_3[\dot{\rho}\sin\gamma + \rho(\dot{\gamma} - \omega_o)\cos\gamma]\}\,\underline{j}$$

$$\underline{v}_2 = [\rho_2(\dot{\theta} - \omega_o)\sin\theta - \dot{\rho}_2\cos\theta]\underline{i} + [r_o\omega_o + \dot{\rho}_2\sin\theta + \rho_2(\dot{\theta} - \omega_o)\cos\theta]\underline{j}$$

$$\underline{v}_3 = \underline{v}_2 + [\dot{\rho}\cos\gamma - \rho(\dot{\gamma} - \omega_o)\sin\gamma]\underline{i} - [\dot{\rho}\sin\gamma + \rho(\dot{\gamma} - \omega_o)\cos\gamma]\underline{j}$$

It is helpful to present in tabulated form as in Table 1 the individual terms which constitute the kinetic energy.

More mass parameters are defined in the following for the formulation of the kinetic energy terms in Table 1.

$$\text{for } T_a:\quad M_a = (m_1\bar{M}_{23}^2 + m_2 + m_3)/m = M_{23}(1 + 2\alpha_{23}M_{23}/M) \tag{1.5a}$$

$$\text{for } T_b:\quad M_b = (m_1\bar{M}_3^2 + m_3)/m_3 = 1 + M_3 + 2\alpha_3 M_3 \tag{1.5b}$$

$$\text{for } T_c:\quad M_c = (m_1\bar{M}_{23}^2 M_3 + m_3)/m = M_3\,[1 + (\alpha_3 + \alpha_{23})M_{23}/M] \tag{1.5c}$$

It is helpful to list derivatives of the above mass parameters which will be used in formulation of Lagrange's equations.

$$\frac{\partial M_a}{\partial \rho} = M_t M_{23}\,/\ell \qquad\qquad \dot{M}_a = M_t M_{23}\dot{\sigma} \tag{1.5d}$$

$$\frac{\partial M_b}{\partial \rho} = M_t M_3/\ell \qquad\qquad \dot{M}_b = M_t M_3\dot{\sigma} \tag{1.5e}$$

$$\frac{\partial M_c}{\partial \rho} = M_t(1 - 1/2M)M_3/\ell \qquad\qquad \dot{M}_c = M_t(1 - 1/2M)M_3\dot{\sigma} \tag{1.5f}$$

From Table 1 the various velocity terms can be formed by summing up the products of the elements in the corresponding column and that in the first column. For example.

Table 1 Formation of Kinetic Energy

Terms in K.E.	v_1	v_2^2	v_3^2	$2\underline{v}_1\cdot\underline{v}_2$	$2\underline{v}_2\cdot\underline{v}_3$	$\sum_{i=1}^{3} m_i v_i^2$	$6T_t/m_3$
$T_a = \dot{\rho}_2^2 + \rho_2^2(\dot{\theta} - \omega_o)^2$	$\bar{M}_{23}^2$	1	1	$-2\bar{M}_{23}$	2	mM_a	$1/2E_1$
$T_b = \dot{\rho}^2 + \rho^2(\dot{\gamma} - \omega_o)^2$	$\bar{M}_3^2$	0	1	$-2\bar{M}_3$	0	m_aM_b	$1/2G_1$
$T_c = 2[\dot{\rho}_2\dot{\rho} + \rho_2\rho(\dot{\theta}-\omega_o)(\dot{\gamma}-\omega_o)]\cos(\theta-\gamma)$	$-\bar{M}_3\bar{M}_{23}$	0	-1	$\bar{M}_3+\bar{M}_{23}$	-1	$-mM_c$	$1/2G_3$
$T_d = 2[\dot{\rho}_2\dot{\rho}(\dot{\gamma}-\omega_o)+\rho_2\dot{\rho}(\theta-\omega_o)]\sin(\theta-\gamma)$	$-\bar{M}_3\bar{M}_{23}$	0	-1	$\bar{M}_3+\bar{M}_{23}$	-1	$-mM_c$	$1/2G_3$
$T_e = 2r_o\omega_o[\dot{\rho}_2\sin\theta + \rho_2(\dot{\theta}-\omega_o)\cos\theta]$	$-\bar{M}_{23}$	1	1	$1-\bar{M}_{23}$	2	$-m_{23}\alpha_{23}$	$\frac{3}{2}(1-M_{23}+M\sigma)$
$T_f = 2r_o\omega_o[\dot{\rho}\sin\gamma + \rho(\dot{\gamma}-\omega\ )\cos\gamma]$	$\bar{M}_3$	0	-1	$\bar{M}_3-1$	-1	$m_3\alpha_3$	$-\frac{3}{2}(1-M_3+M_3\sigma)$
$T_o = r_o^2\omega_o^2$	1	1	1	2	2	m	m_t

$$v_1^2 = \bar{M}_{23}^2 T_a + \bar{M}_3^2 T_b - \bar{M}_3\bar{M}_{23}(T_c + T_d) - \bar{M}_{23}T_e + \bar{M}_3 T_f + T_o$$

The total kinetic energy of the three masses is

$$2T_m = mM_a T_a + m_3 M_b T_b - mM_c(T_c + T_d) - m_{23}\alpha_{23}T_e + m_3\alpha_3 T_f + mT_o \quad (1.6)$$

In Table 1 the second column from the right is $2T_m$ and the last column gives all terms in the kinetic energy of the tether which will be formulated later.

1.3 Formulation of the Potential Energy of the Masses

Denote $V_i = -\mu/r_i$, where r_i is the distance from center of the earth to m_i and μ is the gravitational constant. The potential energy of masses is then

$$V_m = m_1V_1 + m_2V_2 + m_3V_3 \quad (1.7)$$

Let

$$\frac{1}{r_i} = [(\underline{r}_o + \underline{\rho}_i)\cdot(\underline{r}_o + \underline{\rho}_i)]^{-1/2} \quad (1.8)$$

Making use of Equations (1.1), (1.2) and the relationships,

$$\underline{r}_o\cdot\underline{\rho}_2 = -r_o\,\rho_2\cos\theta, \qquad \underline{r}_o\cdot\underline{\rho} = r_o\,\rho\cos\gamma$$

$$r_i^2 = r_o^2\,[1 + (2\underline{r}_o\cdot\underline{\rho}_i + \rho_i^2)/r_o^2]$$

and taking the binomial expansion,

$$\frac{1}{r_i} \cong \frac{1}{r_o}[1 - (2\underline{r}_o\cdot\underline{\rho}_i + \rho_i^2)/2r_o^2 + 3(\underline{r}_o\cdot\underline{\rho}_i/r_o^2)/2]$$

one obtains the following expressions

$$\frac{1}{r_1} = \frac{1}{r_o}\{1 + (\bar{M}_3\rho\cos\gamma - \bar{M}_{23}\,\rho_2\cos\theta)/r_o + \bar{M}_{23}^2\rho_2^2\,(3\cos^2\theta - 1)/2r_o^2$$

$$+ \bar{M}_3^2\rho^2(3\cos^2\gamma - 1)/2r_o^2 - \bar{M}_3\bar{M}_{23}\rho_2\rho[\cos\theta\cos\gamma + \cos(\theta+\gamma)]/r_o^2\} \quad (1.8a)$$

$$\frac{1}{r_2} = \frac{1}{r_o}\{1 + \frac{1}{r_o}\rho_2\cos\theta + \rho_2^2(3\cos^2\theta - 1)/2r_o^2\} \quad (1.8b)$$

$$\frac{1}{r_3} = \frac{1}{r_o}\{1 + \frac{1}{r_o}(\rho_2\cos\theta - \rho\cos\gamma) + \rho_2^2(3\cos^2\theta - 1)/2r_o^2$$

$$+ \rho^2(3\cos^2\gamma - 1)/2r_o^2 - \rho_2\rho[\cos\theta\cos\gamma + \cos(\theta + \gamma)]/r_o^2\} \qquad (1.8c)$$

Individual terms constituting potential energy are tabulated in Table 2. Thus, the potential energy of the three masses can be written in the form

$$V_m = mV_o - m_{23}\alpha_{23}V_a + m_3\alpha_3V_b + mM_a(3V_c - V_d)$$
$$+ m_3M_b(3V_e - V_f) - mM_c(V_g + V_h) \qquad (1.10)$$

where the V's in Equation (1.10) are defined in Table 2. The last column of Table 2 gives terms in the potential energy of the tether which will be formulated later.

1.4 Formulation of Kinetic Energy of the Tether

It is assumed that the tether remains straight between the masses as shown in Figure 2. Thus, velocity of points P_1 and P_2 can be written in the forms,

$$\underline{v}_{p_1} = [s_1\underline{v}_1 + (\ell - \rho - s_1)\underline{v}_3]/(\ell - \rho) \qquad 0 < s_1 < \ell - \rho \qquad (1.11a)$$

$$\underline{v}_{p_2} = [s_2\underline{v}_3 + (\rho - s_2)\underline{v}_2]/\rho \qquad 0 < s_2 < \rho \qquad (1.11b)$$

The kinetic energy of the tether can be obtained by integration

$$2T_t = (m_t/\ell)\int_0^{\ell-\rho}(\underline{v}_{p_1}\cdot\underline{v}_{p_1})ds_1 + \int_0^{\rho}(\underline{v}_{p_2}\cdot\underline{v}_{p_2})ds_2$$

$$= (m_t/3\ell)[(\ell - \rho)v_1^2 + \rho v_2^2 + \ell v_3^2 + (\ell-\rho)\underline{v}_1\cdot\underline{v}_3 + \rho\underline{v}_2\cdot\underline{v}_3] \qquad (1.12)$$

Making use of Table 1, Equation (1.12) yields

$$2T_t = (m_t/6)\,[E_1T_a + G_1T_b + G_3(T_c + T_d) + 3(1 - M_{23} + M\sigma)T_e$$
$$- 3(1 - M_3 + M_3\sigma)T_f] + m_tT_o \qquad (1.13)$$

Table 2 Formation of Potential Energy

Terms in P.E.	V_1	V_2	V_3	$V_m = m_1 V_1$ $m_2 V_2 + m_3 V_3$	V_+/m_+
$V_o = -\mu/r_o$	1	1	1	m	1
$V_a = (-\mu/r_o^2)\rho_2\cos\theta$	$-\bar{M}_{234}$	1	1	$-m_{23}\alpha_{23}$	$1/2(1-M_{23}+M\sigma)$
$V_b = (-\mu/r_o^2)\rho\cos\gamma$	$\bar{M}_3$	0	-1	$m_3\alpha_3$	$-1/2(1-M_3+M_3\sigma)$
$V_c = (-\mu/2r_o^3)\rho_2^2\cos^2\theta$	$3\bar{M}_{23}^2$	3	3	$3mM_a$	$1/2E_1$
$V_d = (-\mu/2r_o^3)\rho_2^2$	$-\bar{M}_{23}^2$	-1	-1	$-mM_a$	$-\frac{1}{3}(1+2\sigma)$
$V_e = (-\mu/2r_o^3)\rho^2\cos^2\gamma$	$3\bar{M}_3^2$	0	3	$3m_3M_b$	$1/2G_1$
$V_f = (-\mu/2r_o^3)\rho^2$	$-\bar{M}_3^2$	0	-1	$-m_3M_b$	$-\frac{1}{3}$
$V_g = (-\mu/r_o^3)\rho_2\rho\cos\theta\cos\gamma$	$-\bar{M}_3\bar{M}_{23}$	0	-1	$-mM_c$	0
$V_h = (-\mu/r_o^3)\rho_2\rho\cos(\theta+\gamma)$	$-\bar{M}_3\bar{M}_{23}$	0	-1	$-mM_c$	0
$V_i = (-\mu 6r_o^3)\rho_2\rho\cos(\theta-\gamma)$	0	0	0	0	E_2

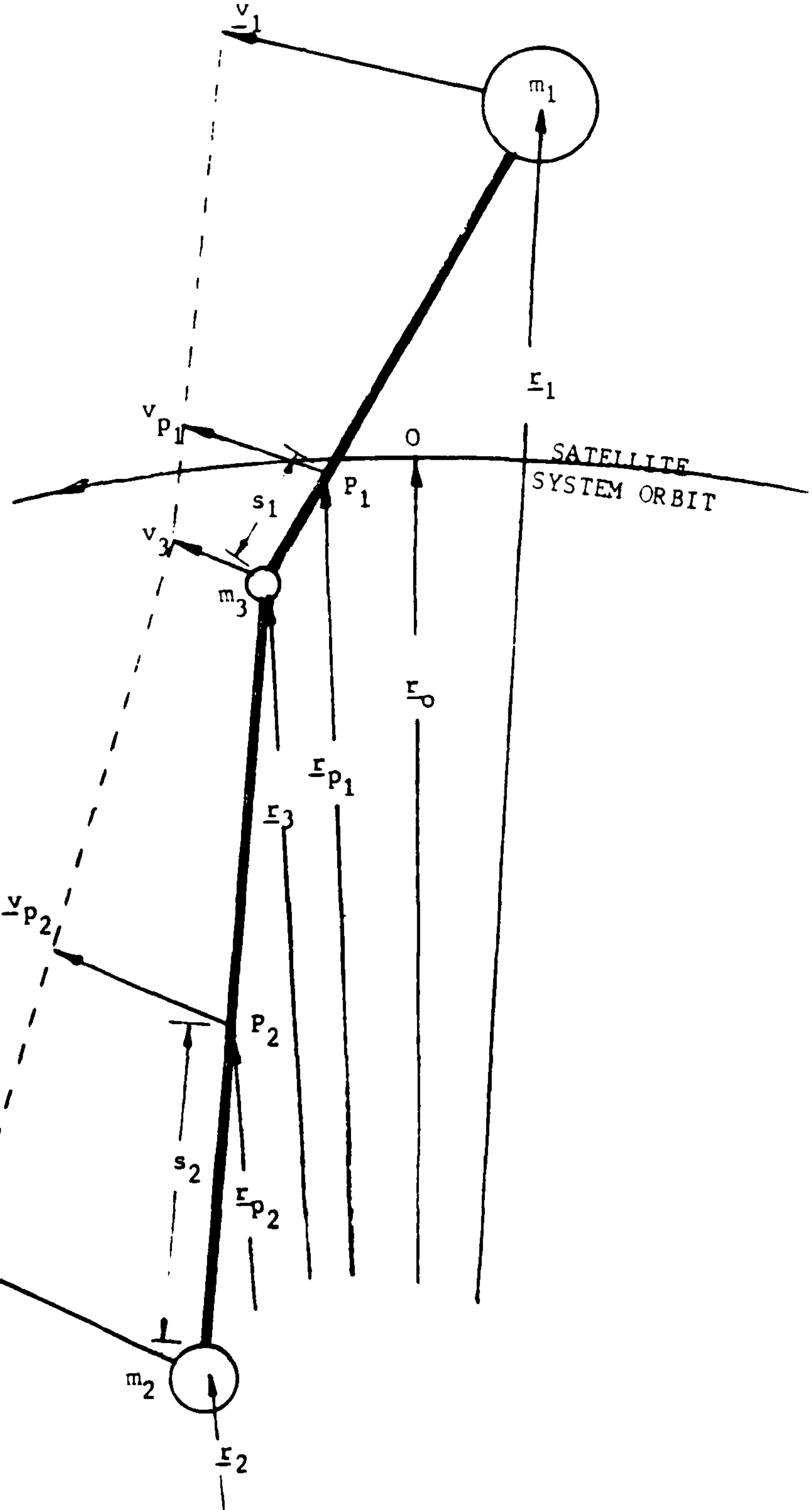


FIGURE 2 POSITION AND VELOCITY VECTORS

where the coefficients are defined in Appendix A. Note that the bar on top of M, M_3, and M_{23} in Equation (1.13) and in the expressions for its coefficients have been removed to retain only first order terms of M_t. This is justified through assumption (h).

1.5 Formulation of the Potential Energy of the Tether

The potential energy of the tether may be written in the form

$$V_t = (\mu m_t/\ell)\left\{\int_0^{\ell-\rho}(1/r_{p_1})ds_1 + \int_0^{\rho}(1/r_{p_2})ds_2\right\} \tag{1.14}$$

where r_{p_1} and r_{p_2} are respectively distances of P_1 and P_2 from the center of the earth, as shown in Figure 2. The position vector of points P_1 and P_2 on the tether may be expressed in the forms

$$\underline{r}_{p_1} = [s_1\underline{r}_1 + (\ell - \rho s_1)\underline{r}_3]/(\ell-\rho) \tag{1.14a}$$

$$\underline{r}_{p_2} = [s_2\underline{r}_3 + (\rho - s_2)\underline{r}_2]/\rho \tag{1.14b}$$

$$1/r_{p_i} = (\underline{r}_{p_i}\cdot\underline{r}_{p_i})^{-1/2} \quad i = 1, 2 \tag{1.14c}$$

Using the following relationships,

$$r_i^2 = r_o^2 + 2\underline{r}_o\cdot\underline{\rho}_i + \rho_i^2$$

$$\underline{r}_i\cdot\underline{r}_j = r_o^2 + \underline{r}_o\cdot\underline{\rho}_i + \underline{r}_o\cdot\underline{\rho}_j + \underline{\rho}_i\cdot\underline{\rho}_j$$

one obtains

$$\frac{1}{r_{p_1}} = \frac{1}{r_o}\{1 - (\underline{r}_o\cdot\underline{\rho}_1)[s_1^2 + s_1(\ell - \rho - s_1)] - (\underline{r}_o\cdot\underline{\rho}_3)\times$$

$$[(\ell - \rho - s_1)^2 + s_1(\ell - \rho - s_1)] + 3(\underline{r}_o\cdot\underline{\rho}_1)^2[s_1^2 + s_1(\ell - \rho - s_1)/2r_o^2$$

$$+ 3(\underline{r}_o\cdot\underline{\rho}_3)^2[(\ell - \rho - s_1)^2 + s_1(\ell - \rho - s_1)]/2r_o^2$$

$$- 1/2[s_1^2\rho_1^2 + (\ell - \rho - s_1)^2\rho_3^2 + s_1(\ell - \rho - s_1)\underline{\rho}_1\cdot\underline{\rho}_3]\}/r_o^2(\ell - \rho)^2$$

$$\frac{1}{r_{p_2}} = \frac{1}{r_o}\{1 - (\underline{r}_o\cdot\underline{\rho}_3)[s_2^2 + s_2(\rho - s_2)] - (\underline{r}_o\cdot\underline{\rho}_2)\,[(\rho - s_2)^2 + s_2(\rho - s_2)]$$

$$+ 3(\underline{r}_o\cdot\underline{\rho}_3)^2\,[s_2^2 + s_2(\rho - s_2)]/2r_o^2 + 3(\underline{r}_o\cdot\underline{\rho}_2)^2[(\rho - s_2)^2 + s_2(\rho - s_2)]/2r_o^2$$

$$- 1/2\,[s_2^2\,\rho_3^2 + (\rho - s_2)^2\rho_2^2 + s_2(\rho - s_2)\underline{\rho}_2\cdot\underline{\rho}_3]\;/r_o^2\rho^2\}$$

The integrals lead to

$$\int_0^{\ell-\rho}\frac{1}{r_{p_1}}ds_1 + \int_o^{\rho}\frac{1}{r_{p_2}}ds_2 = \frac{1}{r_o}\{1 - \frac{1}{6r_o^2}\,[\ell\rho_3^2 + \rho\rho_2^2 + \rho(\underline{\rho}_2\cdot\underline{\rho}_3) + (\ell - \rho)(\underline{\rho}_1\cdot\underline{\rho}_3)]$$

$$- \frac{1}{2r_o^2}\,[\ell(\underline{r}_o\cdot\underline{\rho}_3) + \rho(\underline{r}_o\cdot\underline{\rho}_2) + (\ell - \rho)(\underline{r}_o\cdot\underline{\rho}_1)\,]$$

$$+ \frac{1}{2r_o^4}\,[\ell(\underline{r}_o\cdot\underline{\rho}_3)^2 + \rho(\underline{r}_o\cdot\underline{\rho}_2)^2 + (\ell - \rho)(\underline{r}_o\cdot\underline{\rho}_1)^2]$$

$$+ \frac{1}{r_o^2}\,(\underline{r}_o\cdot\underline{\rho}_3)[\,(\underline{r}_o\cdot\underline{\rho}_2) + (\ell - \rho)(\underline{r}_o\cdot\underline{\rho}_1)]\}$$

Substitution of the following into the above equation

$$\rho_1^2 = M_{23}^2\,\rho_2^2 + M_3^2\,\rho^2 - 2M_3M_{23}\rho_2\rho\cos(\theta - \gamma)$$

$$\rho_3^2 = \rho_2^2 + \rho^2 - 2\rho_2\rho\cos(\theta - \gamma)$$

$$\underline{r}_o\cdot\underline{\rho}_1 = r_o(M_{23}\rho_2\cos\theta - M_3\,\rho\cos\gamma)$$

$$\underline{r}_o\cdot\underline{\rho}_2 = - r_o\rho_2\cos\theta$$

$$\underline{r}_o\cdot\underline{\rho}_3 = r_o(\rho\cos\gamma - \rho_2\cos\theta)$$

results in the terms given in the last column of Table 2. Thus, the potential energy of the tether is

$$V_t = \frac{1}{6}m_t\,\{3(1 - M_{23} + M\sigma)V_a - 3(1 - M_3 + M_3\sigma)V_b + 3E_1V_c - 2(1 + 2\sigma)V_d$$

$$+ 3G_1V_e - 2V_f + E_2V_i\} + m_tV_o \qquad (1.15)$$

where the V's are defined in Table 2.

2. LAGRANGE'S EQUATIONS OF THE SYSTEM

It is straightforward to apply Lagrange's method to obtain equations of motion in terms of the chosen variables (ρ_2, θ) and (ρ, γ). The constraint that the length of the tether must be a constant may be satisfied by introducing a Lagrange multiplier into the formulation.

2.1 Equation of Constraint

From Figure 1, it can be seen that

$$| \underline{\rho}_1 - \underline{\rho}_3 | + | \underline{\rho} | = \ell \qquad (2.1)$$

Making use of Equation (1.3), Equation (2.1) becomes

$$[(1 + \bar{M}_{23}) \underline{\rho}_2 + (1 + \bar{M}_3)\underline{\rho}].[(1 + \bar{M}_{23})\underline{\rho}_2 + (1 + \bar{M}_3)\underline{\rho}] = (\ell - \rho)^2 \qquad (2.2)$$

The above equation yields

$$\bar{M}^2 \rho_2^2 + (1 + \bar{M}_3)^2 \rho^2 - 2\bar{M}(1 + \bar{M}_3)\rho_2\rho\cos(\theta - \gamma) - (\ell - \rho)^2 = 0 \qquad (2.3)$$

Direct differentiation of Equation (2.3) leads to

$$a_{\rho_2} \dot{\rho}_2 + a_\rho \dot{\rho} + a_\theta \dot{\theta} + a_\gamma \dot{\gamma} = 0 \qquad (2.4)$$

where, after second and higher order terms of M_t have been eliminated,

$$a_{\rho_2} = [M\rho_2 - (1 + M_3)\rho\cos(\theta - \gamma) + M_t(\alpha_a\rho_2 - \alpha_c\rho)]/\ell \qquad (2.5a)$$

$$a_\rho = \{[(1 + M_3)^2 - 1]\rho/M + \ell/M - (1 + M_3)\rho_2\cos(\theta - \gamma)$$

$$+ (M_t/2\ell)[M\rho_2^2 + (1 + M_3)(M_3/M)\rho^2 - (1 + 2M_3)\rho\rho_2$$

$$+ 2\ell(\alpha_b\rho - \alpha_c\rho_2)]\} /\ell \qquad (2.5b)$$

$$a_\theta = (1 + M_3 + M_t\alpha_c)\rho_2\sigma\sin(\theta - \gamma) \qquad (2.5c)$$

$$a_\gamma = -(1 + M_3 + M_t\alpha_c)\rho_2\sigma\sin(\theta - \gamma) \qquad (2.5d)$$

In the above expressions the following notations have been used

$$\alpha_a = M\sigma + 1 - M_{23}$$

$$\alpha_b = (M_3\sigma + 1 - M_3)(1 + M_3)/M$$

$$\alpha_c = (1/2 + M_3)\sigma + (1 - M_3M_{23})/M$$

Note that Equation (2.4) has been divided by a factor $2M\ell$ to obtain Equation (2.5) which will give the Lagrange multiplier used in the equations of motion a dimension of force.

2.2 Lagrange's Equations of Motion

Let the kinetic energy and potential energy of the satellite system respectively be

$$T = T_m + T_t , \qquad V = V_m + V_t$$

where the terms on the right hand side are given by Equations (1.6), (1.7), (1.12) and (1.15). Four equations of motion of the satellite system are obtained from Lagrange equation,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = \lambda a_i + Q_i \tag{2.6}$$

$$q_i = \rho_2, \rho, \theta, \text{ and } \gamma$$

where a_i are given by Equation (2.5) and Q_i are generalized forces due to force applied on m_3. Lagrange multiplier λ is introduced as a result of the constraint condition given by Equation (2.4).

2.2.1 Elimination of Nonlinear Terms

Note that the variable ρ which represents the distance of m_3 from m_2 varies from ℓ to zero during the transfer and is the only variable which cannot be treated as a first order small quantity. Usually the variable ρ_2 is replaced by its initial equilibrium length plus a displacement variable; but since the system equations of motion cannot be linearized due to ρ, ρ_2 will also be kept as a finite variable. The derivatives, ρ_2 and ρ, as well as θ, γ and their derivative $\dot{\theta}$ and $\dot{\gamma}$ are considered as first order small quantities. All the second and higher order quantities will be neglected from the equations of motion.

2.2.2 Normalization of Variables

All the mass quantities, m, m_2, m_3, and m_t are normalized in terms of the outer satellite m_1 and are denoted by their capital letters. All the length variables are normalized in terms of the length of the tether, ℓ, as follows:

$$\rho_2/\ell = \delta \text{ , } \rho/\ell = \sigma \text{, and } r_o/\ell = R_o \tag{2.6}$$

The time derivatives of variables are normalized by the orbital period. Setting

$$(\mu/r_o^3)^{1/2} = \omega_o \text{ and } \tau = \omega_o t$$

one may change the time derivatives

$$\dot{q}_i = \omega_o q_i' \qquad \ddot{q}_i = \omega_o^2 q_i'' \qquad (q_i = \rho_2, \rho, \theta. \text{ and } \gamma) \tag{2.7}$$

where the prime denotes differentiation with respect to τ.

To perform the normalization process, the four equations obtained by the Lagrange's method for the variables ρ_2, ρ, θ, and γ are divided respectively by $m_1\ell\omega_o^2$, $m_3\ell\omega_o^2$, $m_1\ell\omega_o^2\rho_2$, and $m_3\ell\omega_o^2\rho$. Thus, all the variables and their coefficients in the equations of motion are dimensionless.

2.2.3 Change Variable

It is more meaningful to use $\beta = \theta - \gamma$ to replace γ as a dependent variable, while the angle γ is used only for the convenience of forming the energies. As shown in Figure 1, β represents the angle between the vectors ρ_2 and ρ. The initial and final values of β are always zero.

2.2.4 Remark on the Differentiation of Enery Terms

It is important to note that all the coefficients of the energy terms given in Tables 1 and 2 vary with σ. Therefore, in formulating Lagrange equations differentiations must be carried out on energy terms as well as their coefficients. However, in the coefficient of the energy terms for the tether all variable mass parameters $\bar{M}$, $\bar{M}_{23}$, and $\bar{M}_3$ are replaced by their constant counterpart M, M_{23}, and M_3 respectively, since all tether energies already have a first order coefficient M_t. See assumption (b).

2.3 Equations of Motion

Four equations of motion obtained from Lagrange's equations are presented in the forms

$$C_{11}\delta'' + C_{12}\sigma'' - [A_{11}\delta + A_{13}\sigma + A_{16}\theta' + A_{18}\beta'] = A_{19}\lambda^* + Q^*_{\rho_2} \quad (2.8a)$$

$$C_{21}\delta'' + C_{22}\sigma'' - [A_{21}\delta + A_{23}\sigma + A_{26}\theta' + A_{28}\beta'] = A_{29}\lambda^* + Q^*_{\rho} \quad (2.8b)$$

$$D_{11}\theta'' + D_{12}\beta'' - [B_{12}\delta' + B_{14}\sigma' + B_{15}\theta + B_{17}\beta] = B_{19}\lambda^* + Q^*_{\theta} \quad (2.8c)$$

$$D_{21}\theta'' + D_{22}\beta'' - [B_{22}\delta' + B_{24}\sigma' + B_{25}\theta + B_{27}\beta] = B_{29}\lambda^* + Q^*_{\beta} \quad (2.8d)$$

where

$$\lambda^* = \lambda/m\ell\omega_o^2, \qquad Q^*_{\rho_2} = Q_{\rho_2}/m\ell\omega_o^2, \qquad Q^*_{\theta} = Q_{\theta}/m\ell\omega_o^2\rho_2,$$

$$Q^*_{\rho} = Q_{\rho}/m_3\ell\omega_o^2, \text{ and } Q^*_{\beta} = Q_{\gamma}/m_3\ell\omega_o^2\rho$$

Next, rewriting the equation of constraint given by Equations (2.3) and (2.4) in terms of the non-dimensional variables gives

$$\bar{M}^2\delta^2 + (1 + \bar{M}_3)^2\sigma^2 - 2\bar{M}(1 + \bar{M}_3)\sigma\delta\cos\beta - (1 - \sigma)^2 = 0 \quad (2.8e)$$

and

$$[M\delta - (1 + M_3)\sigma\cos\beta + M_t(\alpha_a\delta - \alpha_c\sigma)]\dot{\delta} + \{[(1 + M_3)^2 - 1]\sigma + 1$$

$$- M(1 + M_3)\delta\cos\beta + \tfrac{1}{2}M_t[M^2\delta^2 + (1 + M_3)M_3\sigma^2 - M(1 + 2M_3)\sigma\delta$$

$$+ 2M(\alpha_b\sigma - \alpha_c\delta)]\ \dot{\sigma}/M + \dot{\beta}\ M(1 + M_3 + M_t\alpha_c)\sigma\delta\sin\beta = 0 \qquad (2.8f)$$

The solution of the four equations of motion must satisfy the constraint equation given by either (2.8e) or (2.8f).

Note that at the initial equilibrium state, $\beta = \beta'' = \delta' = \sigma' = 0$, and Equations (2.8c) and (2.8d) both reduce to a single equation.

$$\theta'' + 3\theta = 0 \qquad (2.9)$$

This equation represents oscillations of a dumbbell in a circular orbit[8].

2.3.1 Equations of Motion With $M_t = 0$

The system of equations of motion can be simplifed considerably if M_t is set to zero. Equations (2.8) reduces to

$$M_{23}\delta'' - M_3\sigma'' + 2(M_{23}\delta - M_3\sigma)\theta' + 2M_3\sigma\beta' - 3M_{23}\delta + 3M_3\sigma$$
$$= \lambda^*[M\delta - (1 + M_3)\sigma\cos\beta] + Q^*_{\rho_2} \qquad (2.10a)$$

$$(1 + M_3)\sigma'' - M\delta'' + 2\,[(1 + M_3)\sigma - M\delta]\theta' - 2(1 + M_3)\sigma\beta' + 3(1 + M_3)\sigma + 3M\delta$$
$$= \lambda^*\ \{[(1 + M_3)^2 - 1]\sigma + 1 - M(1 + M_3)\delta\cos\beta\}\ M/M_3 + Q^*_\rho \qquad (2.10b)$$

$$(M_{23}\delta - M_3\sigma)(\theta'' + 3\theta) + M_3\sigma\beta'' - 2M_{23}\delta' + 2M_3\sigma'$$
$$= \lambda^*(1 + M_3)\sigma\sin\beta + Q^*_\theta \qquad (2.10c)$$

$$[(1 + M_3)\sigma - M\delta](\theta'' + 3\theta - 3\beta) - (1 + M_3)\sigma\beta'' + 2M\delta' - 2(1 + M_3)\sigma'$$
$$= -\lambda^*\ [(1 + M_3)M/M_3]\ \delta\sin\beta + Q^*_\beta \qquad (2.10d)$$

It is important to find out numerically whether neglecting the contribution of the tether mass leads to any significant differences.

3. GENERALIZED APPLIED FORCES AND INITIAL CONDITIONS

Two classes of mass transfers are considered in this paper. One class of transfer is that m_3 is given an initial velocity with sufficient magnitude to cross the orbit. In this case all the generalized forces in Equation (2.8) are zero. The other class of mass transfer is that m_3 is driven by a continuous thrust with sufficient initial magnitude to get motion of m_3 started. The free motion of the transfer mass is treated first.

3.1 Free Motion of M_3

Motion in this type of transfer depends entirely on initial conditions. Two cases are considered, one is that m_3 departs from the outer satellite, m_1, and the other is that m_3 starts from the inner satellite, m_2.

3.1.1 Departure From m_1

To derive the necessarily initial value of δ_0, disregard the off-set distance from center of m_1 at starting point, Δ_1. From definition of center of mass of the system,

$$1/2\, m_t \ell + (m_1 + m_3)\ell = (m + m_t)\, \rho_2(0)$$

It gives

$$\delta_0 = (1 + M_3 + 1/2\, M_t)/M_{mt}, \qquad M_{mt} = M + M_t \tag{3.1a}$$

Now, let

$$\sigma_0 = 1 - (\Delta_1/\ell) \tag{3.1b}$$

The minimum initial velocity of m_3 to cross the orbit is determined by work and energy principle. It follows

$$1/2\, \dot{\delta}_0^2 = \int_{r_0}^{r_1} (r\omega_0^2 - \mu/r^2)dr = 1/2(r_1^2 - r_0^2)\omega_0^2 + \mu/r_1 - \mu/r_0$$

$$= 1/2\, r_0^2\, [(1 - \rho_{20}/r_0)^2 - 1]\omega_0^2 + [(1 + \rho_{20}/r_0)^{-1} - 1]\mu/r_0$$

$$= \frac{3}{2} (\ell - \rho_{20})^2 \omega_o^2 \quad (3.2)$$

Substituting Equation (3.1a) and changing to dimensionless form, it gives

$$\sigma_o' \leq - \sqrt{3} [1 - (1 + M_3 + \frac{1}{2} M_t)/M_{mt}] = - \sqrt{3} (M_2 + \frac{1}{2} M_t)/M_{mt} \quad (3.3)$$

The negative sign shows that the velocity is in the direction of decreasing σ as m_3 moves toward m_2.

3.1.2 Departure From m_2

The initial position of the three masses gives

$$1/2\ m_t \ell + m_1 \ell = (m + m_t)\rho_2(0)$$

Hence,

$$\delta_o = (1 + \frac{1}{2} M_t)/M_{mt} \qquad \sigma_o = \Delta_2/\ell \quad (3.4)$$

Similarly, applying the work and energy principle,

$$\frac{1}{2} \dot{\rho}_o^2 = \frac{1}{2} (r_2^2 - r_o^2)\omega_o^2 + \mu/r_2 - \mu/r_o$$

$$= 1/2\ r_o^2 \omega_o^2 [(1 - \rho_{20}/r_o)^2 - 1] + [(1 - \rho_{20}/r_o)^{-1} - 1]\mu/r_o$$

$$= \frac{3}{2} \rho_{20}^2 \omega_o^2 \quad (3.5)$$

Thus, the minimum initial velocity is

$$\sigma_o' \geq \sqrt{3} (1 + \frac{1}{2} M_t)/M_{mt} \quad (3.6)$$

3.2 Forced Motion and Generalized Forces

3.2.1 Forced Motion Departure From m_1

Consider that the driving force p(t) on m_3 is directed toward m_2, i.e., the force vector is aligned along the negative direction of vector, $\underline{\rho}$. Therefore, the virtual work is

$$\delta W = \underline{p}(t).\delta\ \underline{\rho} = -\ p(t)\delta\rho$$

Here, the sumbol "δ" denotes variation and it should not be confused with the variable "δ".

Thus, the generalized forces in Equation (2.8) are

$$Q^*_\rho = -\ p(t)/m_3\ell\omega_o^2 = -\ p^*(t) \tag{3.7}$$

$$Q^*_{\rho 2} = Q^*_\theta = Q^*_\beta = 0$$

The initital magnitude of p(t) must be sufficnent to overcome the difference of centripetal and gravity forces. This gives

$$p_{min}(0) = m_3(r_1\omega_o^2 - \mu/r_1^2) = m_3\ell\omega_o^2(1 - \delta_o)$$

Substitution of Equation (3.1a) leads to

$$p^*_{min}(0) = (M_2 + \tfrac{1}{2}\ M_t)/M_{mt} \tag{3.8}$$

3.2.2 Forced Motion Departure From m_2

Again, the driving force is considerd to be in the direction toward the end point, m_1. The virtual work is

$$\delta W = \underline{p}(t).\delta\underline{s}_1 = p(t)\delta\ |\ \underline{\rho}_1 - \underline{\rho}_3\ | \tag{3.9}$$

Making use of the constraint condition given by Equation (2.1), yields

$$\delta W = p(t)\delta(\ell - \rho) = -\ p(t)\delta\rho$$

Results indentical to Equation (3.7) are obtained. The minimum magnitude of p(t) to get the motion of m_3 started is

$$p_{min}(0) = m_3(\mu/r_2^2 - r_2\omega_o^2)$$

$$= m_3\ [(1 - \rho_{20}/r_o)^{-2} - (1 - \rho_{20}/r_o)]\mu/r_o^2$$

$$= 3m_3\omega_o^2\rho_{20}$$

Thus,

$$p^*_{min}(0) = 3(1 + \frac{1}{2} M_t)/M_{mt} \tag{3.10}$$

3.2.3 The Minimum Energy Expense For Transfer Operation

The first phase of transfer motion is to drive m_3 across the system orbit and the second phase is a braking action which reduces velocity of m_3 to zero when it reaches the end. If the velocity of m_3 at the instant it passes the orbit is zero, the required energy has minimum magnitude which is the sum of the kinetic energy given by Equation (3.2) and (3.5). That is

$$E_{min} = \frac{3}{2} m_3 (\ell \omega_o)^2 [(M_2 + \frac{1}{2} M_t)^2 + (1 + \frac{1}{2} M_t)^2]/M_{mt}^2 \tag{3.11}$$

3.2.4 Generalized Force Due To Air Drag

Since the transfer operation is accomplished in less than one orbit, air drag has no significant influence on the motion of the system. However, it can easily be included. Consider that the drag force acts on the inner satellite alone since it is much closer to earth atmosphere if a very long tether is used. Let the drag force

$$F_D = cdAv_2^2 \tag{3.12}$$

where

A = effective area of m_2

d = air density at altitude r_2

c = drag coefficient

$v_2^2 = T_a + T_e + T_o$ (see Table 1)

Then the virtual work is

$$\delta W = - F_D \rho_2 \delta\theta$$

and results in

$$Q^*_\theta = - p^*_D[R_o + 2\,(1 - \theta')] \tag{3.13}$$

where $p^*_D = cdA_o/m$. Note that the air is treated non-rotating.

3.2.5 Summary of Initital Conditions and Generalized Forces

To summarize the initital conditions of the variables and the generalized forces of all cases treated, Table 3 is presented.

TABLE 3 Summary of Initial Conditions and Generalized Forces

	FREE TRANSFER		POWERED TRANSFER	
	Depart from m_1	Depart from m_2	Depart from m_1	Depart from m_2
δ'_o, θ_o, θ'_o, β_o, β'_o	0	0	0	0
δ_o	$(1+M_3+ \tfrac{1}{2}M_t)/M_{mt}$	$(1 + \tfrac{1}{2}M_t)/M_{mt}$	$(1+M_3 + \tfrac{1}{2}M_t)/M_{mt}$	$(1 + \tfrac{1}{2}M_t)/M_{mt}$
σ_o	$1 - \Delta_1/\ell$	Δ_2/ℓ	$1 - \Delta_1/\ell$	Δ_2/ℓ
$(\sigma'_o)_{min}$	$- 3(M_2+\tfrac{1}{2}M_t)/M_{mt}$	$3(1 + \tfrac{1}{2}M_t)/M_{mt}$	*none*	*none*
$\lambda^*(0)$	0	0	0	0
$p^*_{min}(0)$	none	none	none	none
$Q^*(t)$	0	0	0	0
$Q^*(t)$	0	0	$-p^*(t)$	$-p^*(t)$
$Q^*_\theta(t)$	$-p_D[R_o+2\delta(1-\theta')]$	$-p_D[R_o+2\delta(1-\theta')]$	$-p_D[R_o+2\delta(1-\theta')]$	$-p_D[R_o+2\delta(1-\theta')]$
$Q^*_\beta(t)$	0	0	0	0

3.3 Reversed Position of the Satellite System

Formulation has been based on a configuration that the main satellite m_1 is moving outside the system orbit. If the two tether connected satellites interchange their positions, referred to as the reversed system, equations of motion can be applied without changes. One simply refers m_1 be the mass of the outside satellite. The two systems have different values for the mass ratios; the reversed system has $M_2 >> 1$ and $M_3 << 1$ while the regular system has $M_2 << 1$ and $M_3 << M_2$.

3.4 Initial Value of λ*

To show that $\lambda^*(0) = 0$, one may use equations of motion for zero tether mass given by Equation (2.10). At an initital equilibrium state: $\theta''_o = \theta_o = 0$, $\beta'' = \beta = 0$, $\delta' = \sigma' = 0$, the third and fourth equations are satisfied for any value of $\lambda^*(0) = 0$. Now, setting the rest of initial values, $\theta'_o = \beta'_o = 0$, $\sigma_o = 1$ and $\delta_o = (1 + M_3)/M$ and $\lambda^*(0) = 0$ in the first and second equations and solving for δ''_o and σ''_o, yields

$$\delta''_o = 3(1 + M_3)/M \qquad \text{and } \sigma''_o = 3$$

This result satisfies the vector equation,

$$\ddot{\underline{\rho}}_1(0) = \ddot{\underline{\rho}}_2(0) + \ddot{\underline{\rho}}\ (0)$$

$$= \ell\omega_o^2[- 3(1 + M_3)/M + 3]\underline{i} = 3(M_2/M)\ell\omega_o^2\underline{i}$$

It has been proved that the assumption $\lambda^*(0) = 0$ is correct. The inclusion of m_t will change the magnitude of δ''_o and σ''_o but not the value $\lambda^*(0)$.

4. NUMERICAL METHODS OF SOLUTION

An approximate solution of Equation (2.8), the equations of motion, cannot be found directly by any existing computer integration subroutine due to the constraint condition on the variables. The author has been unsuccessful in seeking a special computer program for solution of a system of differential equations with constraints. A program which combines an integration subroutine and a subroutine for the determination of zeros of analytical functions has been suggested.

4.1 Integration by Combined Subroutines

The commonly used subroutine "DGEAR" is a differential equation solver which finds approximations to the solution of a system of first order ordinary differential equations of the form $y_N' = f_N(\tau, y)$ with initial conditions. The basic methods used for the solution are of implicit linear multistep type. The user may use either the implicit Adams methods (up to order twelve), or the backward differentiation formula methods (up to order five), also called Gear's stiff methods. See Appendix C.

4.1.1 Transformation of Equations of Motion to First Order System

To convert the system of equations given by Equations (2.8) into a first order system, the variables are redefined as follows:

$$y_1 = \delta \qquad y_2 = y_1' = \delta' \tag{4.1a}$$

$$y_3 = \sigma \qquad y_4 = y_3' = \sigma' \tag{4.1b}$$

$$y_5 = \theta \qquad y_6 = y_5' = \theta' \tag{4.1c}$$

$$y_7 = \beta \qquad y_8 = y_7' = \beta' \tag{4.1d}$$

Thus, Equation (2.8) can be rewritten in the form

$$C_{11}y_2' + C_{12}y_4' = A_{11}y_1 + A_{13}y_3 + A_{16}y_6 + A_{18}y_8 + A_{19}\lambda^* + Q^*_{\rho 2} \quad (4.2a)$$

$$C_{21}y_2' + C_{22}y_4' = A_{21}y_1 + A_{23}y_3 + A_{26}y_6 + A_{28}y_8 + A_{29}\lambda^* + Q^*_{\rho} \quad (4.2b)$$

$$D_{11}y_6' + D_{12}y_8' = B_{12}y_2 + B_{14}y_4 + B_{15}y_5 + B_{17}y_7 + B_{19}\lambda^* + Q^*_{\theta} \quad (4.2c)$$

$$D_{21}y_6' + D_{22}y_8' = B_{22}y_2 + B_{24}y_4 + B_{25}y_5 + B_{27}y_7 + B_{29}\lambda^* + Q^*_{\beta} \quad (4.2d)$$

where all the coefficients are defined in Appendix B. The constraint equation becomes

$$g(y) = \bar{M}^2 y_1^2 + (1 + \bar{M}_3)^2 y_3^2 - 2\bar{M}(1 + \bar{M}_3)y_1 y_3 \cos y_7 - (1 - y_3)^2 = 0 \quad (4.2e)$$

Solving for y_2', y_4' from Equations (4.2a) and (4.2b) and y_6', y_8' from (4.2c) and (4.2d), results in

$$y_n' = \frac{1}{D_n}\left[\sum_{i=1}^{8} f_{ni}y_i + f_{n9}\lambda^* + Q_n^*\right] \qquad n = 2, 4, 6, 8 \quad (4.3a)$$

and four more equations from Equation (4.1),

$$y_n' = y_{n+1} \qquad n = 1, 3, 5, 7 \quad (4.3b)$$

where the coefficients are defined as follows:

$$D_2 = D_4 = C_{11}C_{22} - C_{12}C_{21}, \quad D_6 = D_8 = D_{11}D_{22} - D_{12}D_{21} \quad (4.3c)$$

$$f_{2i} = C_{22}A_{1i} - C_{12}A_{2i} \qquad i = 1, 3, 6, 8, 9 \quad (4.3d)$$

$$f_{4i} = C_{11}A_{2i} - C_{21}A_{1i}$$

$$f_{2i} = f_{4i} = 0 \qquad i = 2, 4, 5, 7 \quad (4.3e)$$

$$f_{6i} = D_{22}B_{1i} - D_{12}B_{2i} \qquad i = 2, 4, 5, 7, 9 \quad (4.3f)$$

$$f_{8i} = D_{11}B_{2i} - D_{21}B_{1i}$$

$$f_{6i} = f_{8i} = 0 \qquad i = 1, 3, 6, 8 \quad (4.3g)$$

$$Q_2^* = C_{22}Q^*_{\rho 2} - C_{12}Q^*_{\rho} \qquad Q_4^* = C_{11}Q^*_{\rho} - C_{21}Q^*_{\rho 2} \quad (4.3h)$$

$$Q_6^* = D_{22}Q^*_{\theta} - D_{12}Q^*_{\beta} \qquad Q_8^* = D_{11}Q^*_{\gamma} - D_{21}Q^*_{\beta}$$

4.1.2 Integration Procedure and Subroutine-ZANLYT

With the initial conditions $y_n(0)$ for n = 1 through 8, Q*(0) and $\lambda^*(0)$ given in Table 3, $y_n(\tau_1)$ can be obtained by using integration subroutine-DGEAR provided that $\lambda^*(\tau_1)$ is known. Let

$$\lambda^*_{j+1}(\tau_1) = \lambda^*_j(\tau_1) + \Delta\lambda^*_j(\tau_1) \qquad j = 1,2..,k \tag{a}$$

and rewrite Equation (4.3) in the form

$$y'_{nj}(\tau_1) = f_n(\tau_1, y_{nj}(\tau_1), \lambda^*_j(\tau_1)) \tag{b}$$

Now, the above equation is integrated for j = 1,2,:..", k by using DGEAR to obtain $y_{n1}(\tau_1),\ldots, y_{nk}(\tau_1)$. Substituting these in the constraint equation, Equation (4.2e), yields

$$g_j(y_{nj}(\tau_1)) = R_j \qquad j = 1, 2,.., k \tag{c}$$

where R_j denotes the residual of fucntion $g(y_n(\tau_1))$ from zero.

Subroutine "ZANLYT" is a program for finding zeroes of an analytical function. It is expected that ZANLYT will determine $\lambda^*(\tau_1)$ such that as to make the residual approximately equal to zero.

Finally, $\lambda^*(\tau_1)$ is used to integrate Equation (4.3) one more time to obtain $y_n(\tau_1)$. This procedure completes one integration step and is ready to move forward to τ_2 and repeat program loop.

There are questions concerning the magnitude of k, the first guess of $\lambda^*_1(\tau_1)$, and magnitude of $\Delta\lambda^*_j(\tau_1)$ to be resolved. Note that λ^* does not appear in Equation (4.5) explicitly, and nor there exists an equation governing λ^*. More detailed study of ZANLYT is required to answer these questions.

4.2 Elimination of the Lagrange Multiplier

Two possible approaches for the elimination of the Lagrange multiplier have been attemped. One is to solve the constraint equation for one of the variables in terms of the other three and hence reduce the system to three independent variables. Unfortunately, the resulting expressions are so complex as to make an analytical formulation of the equations of motion unpractical. The other approach is to eliminate λ^* directly from any two equations of the system of four equations; and hence three independent equations without λ^* can be formed. A fourth second order differential equation is obtained by differentiation of the constraint equation twice. Expressing the equation so obtained in a similar form as Equation (4.2), one has

$$\bar{C}_{11}y_2' + \bar{C}_{12}y_4' = 0 \tag{4.4a}$$

$$\bar{C}_{21}y_2' + \bar{C}_{22}y_4' = \Sigma\ \bar{A}_{2i}y_i + Q_{ab} \tag{4.4b}$$

$$\bar{D}_{11}y_6' + \bar{D}_{12}y_8' = \Sigma\ \bar{B}_{1i}y_i + Q_{ca} \tag{4.4c}$$

$$\bar{D}_{21}y_6' + \bar{D}_{22}y_8' = \Sigma\ \bar{B}_{2i}y_i + Q_{cd} \tag{4.4d}$$

where

$$Q_{ab} = A_{29}Q^*_{\rho_2} - A_{19}Q^*_{\rho}\ , \qquad Q_{ca} = A_{19}Q^*_{\theta} - B_{19}Q^*_{\rho_2}, \qquad Q_{cd} = My_1Q^*_{\beta} + M_3y_3Q^*_{\theta}$$

and the coefficients are defined in Appendix B. Note that the above equations are formed with the use of:

$$(4.4a) = \frac{d^2}{d\tau^2}\ (4.2e)$$

$$(4.4b) = A_{29}(4.2a) - A_{19}(4.2b)$$

$$(4.4c) = A_{19}(4.2c) - B_{19}(4.2a)$$

$$(4.4d) = My_1(4.2c) + M_3y_3(4.2d)$$

Since second and higher order terms were neglected from the original equations of motion, they are not included in Equation (4.4a).

Now, Equation (4.4) can readily be reduced to the first order system,

$$y'_n = y_{n+1} \qquad n = 1, 3, 5, 7 \tag{4.5a}$$

$$y'_n = \frac{1}{\bar{D}_n}(\Sigma\ \bar{f}_{ni}y_i + \bar{Q}_n) \qquad n = 2, 4, 6, 8 \tag{4.5b}$$

where

$$\bar{D}_2 = \bar{D}_4 = \bar{C}_{11}\bar{C}_{22} - \bar{C}_{12}\bar{C}_{21}, \qquad \bar{D}_6 = \bar{D}_8 = \bar{D}_{11}\bar{D}_{22} - \bar{D}_{12}\bar{D}_{21}$$

$$\bar{f}_{2i} = -\bar{C}_{12}\bar{A}_{2i}, \qquad \bar{f}_{4i} = \bar{C}_{11}\bar{A}_{2i} \qquad i = 1, 3, 6, 8, 9$$

$$\bar{f}_{2i} = \bar{f}_{4i} = 0 \qquad i = 2, 4, 5, 7$$

$$\bar{f}_{6i} = \bar{D}_{22}\bar{B}_{1i} - \bar{D}_{12}\bar{B}_{2i}, \ \bar{f}_{8i} = \bar{D}_{11}\bar{B}_{2i} - \bar{D}_{21}\bar{B}_{1i} \qquad i = 2, 4, 5, 7, 9$$

$$\bar{f}_{6i} = \bar{f}_{8i} = 0 \qquad i = 6, 8$$

$$\bar{Q}_2 = -\bar{C}_{12}Q_{ab}, \qquad \bar{Q}_4 = \bar{C}_{11}Q_{ab}$$

$$\bar{Q}_6 = \bar{D}_{22}Q_{ca} - \bar{D}_{12}Q_{cd}, \ \bar{Q}_8 = \bar{D}_{11}Q_{ca} - \bar{D}_{21}Q_{ca}$$

The above system is referred to as the derived system while Equation (4.3) is called the original system.

4.3 An Example for Illustration

An example which has known solution is given here for illustration and also may be used for programming verification. A pendulum of mass m and length L with small initial angle θ_0 and zero initial velocity has the linear solution $\theta = \theta_0 \cos\tau = \sqrt{g/L}t$. If rectangular coordinates $x = L\sin\theta$ and $y = L\cos\theta$ are used for the Lagrange formulation, one obtains the equations of motion

$$m\ddot{x} = \lambda x \tag{4.6a}$$

$$m\ddot{y} - mg = \lambda y \tag{4.6b}$$

with a constraint equation,

$$x^2 + y^2 = L^2 \quad \text{or} \quad x\dot{x} + y\dot{y} = 0 \tag{4.6c}$$

Note that Equations (4.6a) and (4.6b) can also be obtained from conditions of equilibrium as shown in Figure 3(a). Setting $\xi = x/L$, $\eta = y/L$, $\tau = \sqrt{g/L}\,t$ and $\lambda^* = \lambda/(mg/L)$, the above equation become

$$\xi'' = \lambda^*\xi \tag{4.7a}$$

$$\eta'' = \lambda^*\eta + 1 \tag{4.7b}$$

$$\xi^2 + \eta^2 = 1 \quad \text{or} \quad \xi\xi' + \eta\eta' = 0 \tag{4.7c}$$

where prime denotes differentiation with respect to τ.

To form the derived system, one equation is obtained by eliminating λ^* from Equations (4.7a) and (4.7b) and a second equation by differentiating (4.7c). Thus,

$$\eta\xi'' - \xi\eta'' = -\xi \tag{4.8a}$$

$$\xi\xi'' - \eta\eta'' = -(\xi'^2 + \eta'^2) \tag{4.8b}$$

If a linearized pendulum is treated, the term on the righthand side of Equation (4.8b) can be neglected. Solving for ξ'' and η'' from the above equations results in

$$\xi'' = -\xi\eta - \xi(\xi'^2 + \eta'^2) \tag{4.9a}$$

$$\eta'' = \xi^2 - \eta(\xi'^2 + \eta'^2) \tag{4.9b}$$

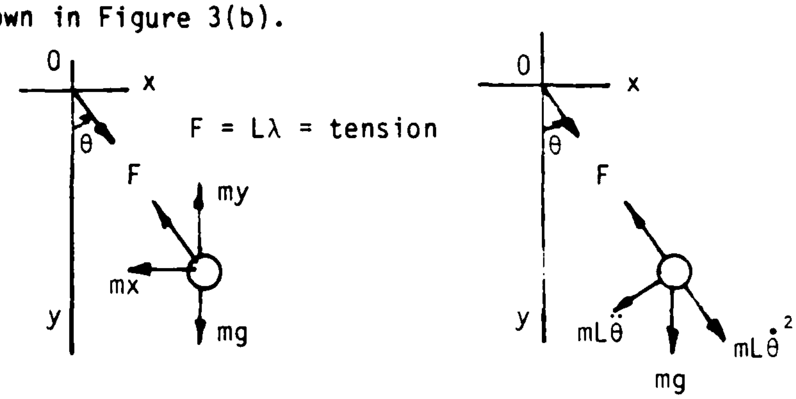
It can be shown by direct substitution that the following

$$\xi = \sin\theta \cong \theta_0\cos\tau$$

$$\eta = \cos\theta \cong 1 - 1/2(\theta_0\cos\tau)^2$$

$$\lambda^* = -\cos\theta + (\theta_0\sin\tau)^2 \cong -[1 - 1/2(\theta_0\cos\tau)^2 + (\theta_0\sin\tau)^2]$$

is an approximation to the original system, Equation (4.7) and the derived system, Equation (4.9). Note that λ^* is obtained from condition of equilibrium as shown in Figure 3(b).



(a) Forces in xy-coordinates (b) Forces in r θ-coordinates

Figure 3 Equilibrium of Forces

This example serves the following objectives:

(1) If a computer program is desinged to solve the original system, the example can be used as a test case to verify the program.

(2) This example illustrates that the proposed procedure for elimination of the Lagrange multiplier is a valid approach for a simple pendulum.

4.4 Numerical Verification

To verify that the program can find approximate solutions of a system differential equations with a constraint or that the derived system can truly represent the physical system, one should carry out numerical computation of the example given and compare with the known solution. The first order systems of differential equations are respectively:

(1) the original system

$$y_1' = y_2 \qquad y_2' = \lambda^* y_1$$

$$y_3' = y_4 \qquad y_4' = \lambda^* y_3 + 1$$

$$g(y) = y_1^2 + y_3^2 - 1 = 0 \tag{4.10}$$

(2) the derived system

$$y_1' = y_2 \qquad y_2' = - y_1 y_3 - (y_2^2 + y_4^2) y_1$$

$$y_3' = y_4 \qquad y_4' = y_1^2 - (y_2^2 + y_4^2) y_3 \tag{4.11}$$

(3) the physical system

$$\theta'' + \sin\theta = 0 \tag{4.12}$$

with initial conditions: $\theta(0) = \theta_0$ and $\theta'(0) = 0$,

for small oscillations, Equation (4.12) is replaced by an approximately nonlinear differential equation.

$$\theta'' + \theta' - c\theta^3 = 0 \ (c = 1/6) \tag{4.13}$$

which has known approximate solution,

$$\theta = \theta_0 \cos\omega\tau + \frac{1}{32} c\theta_0^2 (\cos\omega\tau - \cos 3\omega\tau) + O(c^2) \tag{4.14}$$

where

$$\omega = 1 - \frac{c}{8}\theta_0^2 \qquad \theta_0 << 1$$

Equation (4.14) is sufficient to represent the exact solution of a simple pendulum for small oscillations.

Now, from solutions of Equations (4.10) and (4.11) compute

$$\theta(\tau) = \tan^{-1}[y_1(\tau)/y_3(\tau)]$$

and compare them with Equation (4.14).

4.5 Parametric Study

The satellite system has three mass parameters, M_2, M_3, and M_t, and one length ratio R_0 and their numerical ranges may be given as follows:

Mass of sub-satellite /mass of main-satellite: 1/20 to 1/5

Transfer mass/mass of sub-satellite: 1/20 to 1/5

Mass of tether $\equiv$ transfer mass

Radius of satellite orbit/length of tether: 25 to 100

One additional parameter is the magnitude of initital velocity of the transfer mass for free transfer or the magnitude of the driving force for powered transfer.

Four cases are to be treated: namely free or powered transfer for both inward and outward transfers. This number is doubled if both regular and reversed satellite systems are treated. If three values (high, medium, and low) are taken for each parameter, this gives $8(3^4) = 648$ computer runs.

4.6 Computer Print-out and Computer Time Control

The amplitude of $\theta(\tau)$ and $\beta(\tau)$ are essential to the study of stability of the satellite system during mass transfer operations: $\sigma(\tau)$ and $\sigma'(\tau)$ give respectively the poisiton and velocity of the transport mass. These four variables are required to be ploted versus non-dimensional time τ.

The integration variable for the first order differential equations is the non-dimensional time τ, $\tau = 2\pi$ constitutes one orbital revolution. One may stop the computation when one of the following is reached:

(1) $\tau_f \geqq$ expected value for transfer operation,

(2) $|\sigma'(\tau)| \leqq \varepsilon$, where ε is some small positive value, and

(3) $\sigma(\tau) \leqq \Delta_2/\ell$ when departure from m_1 or

$\sigma(\tau) \leqq 1 - \Delta_1/\ell$, when departure from m_2.

IV. CONCLUSIONS AND RECOMMENDATIONS

Conclusions

A dynamical formulation of the equations of motion of a TSS with mass transport along the tether has been presented. Four second order differential equations are obtained directly by the Lagrange method with one constraint condition on the variables. A system of four independent second order differential equations are derived from the first by eliminating the Lagrange multiplier among the four equations. The fourth equation is obtained by differentiating the constraint equation twice. It is desired to obtain computer solutions of both systems to verify the validity of the derived system.

There are four cases for mass tranfer which are departures from the inner and outer sub-satellites and free and forced motions for each case. Initial conditions for all cases are presented in Table 3. To cover whole ranges of various combinations of parameters, a minimum of 648 computer runs are required, and many additional runs may be needed when critical state of equilibrium arises. Thus, systematic studies and records keeping for such a large volume data becomes a major problem.

The system of equations of motion can be reduced to very simple forms if the tether mass is disregarded. It is important to find out how significant is the contribution of tether mass. No conclusions can be drawn on how each parameter affects the stability of the TSS until numerical investigations are completed.

Recommendations:

The items (a) through (g) given in the Introduction may be included in a dynamical model one at a time so that contribution by each can be evaluated. If two or more items are treated simultaneously, analytical formulation of equations of motion becomes unpractical.

A paper[9] which will appear in the Journal of Applied Mechanics suggests that a dynamical system with constraints may be formulated without using a Lagrange multiplier. This new approach may provide another means of verification.

APPENDIX A DEFINITION OF NOTATION USED IN TABLES 1 AND 2

Notations related to kinetic energy of the tether in Table 1 are results of Equation (1.12) and related to potential energy of the tether in Table 2 are obtained from Equation (1.14). Definition of these notations are presented as follows:

$$E_1 = 2\{1 - M_{23}(1 - M_{23}) + [2 + M_{23}(1 - M_{23})]\sigma\}$$

$$E_2 = M_3 + M_{23} - 2M_3M_{23} - 2 - (1 + M_3 + M_{23} - 2M_3M_{23})\sigma$$

$$E_3 = 2\{2 - 3M_{23}(1 - M_{23}) + [4 - 3M_{23}(1 - M_{23})]\sigma\}$$

$$G_1 = 2[1 - M_3(1 - M_3) + M_3(1 - M_3)\sigma]$$

$$G_3 = 1 + M_3 + M_{23} - 2M_3M_{23}$$

$$G_4 = [2 + M_{23}(1 - M_{23})]\delta$$

$$G_5 = M_3(1 - M_3)\sigma$$

$$G_6 = [4 - 3M_{23}(1 - M_{23})]\delta$$

$$G_7 = 3[2 - 2M_3(1 - M_3) + 3M_3(1 - M_3)\sigma]$$

APPENDIX B DEFINITION OF COEFFICIENTS OF EQUATIONS OF MOTION

The coefficients used in the equations of motion, Equation (2.8), are defined as follows:

$$C_{11} = M_a + \tfrac{1}{6}M_tE_1 \qquad C_{12} = -M_c + \tfrac{1}{6}M_tE_2$$

$$C_{21} = -(MM_c - \tfrac{1}{6}M_tE_2)/M_3 \qquad C_{22} = (M_3M_b + \tfrac{1}{6}M_tG_1)/M_3$$

$$A_{11} = 3M_a + \tfrac{1}{6}M_t(E_1 + E_3) \qquad A_{13} = 3(M_c + \tfrac{1}{6}M_tE_2)$$

$$A_{16} = -2[M_a\delta - M_c\sigma + \tfrac{1}{6}M_t(E_1\delta - E_2\sigma)] \qquad A_{18} = -2(M_c + \tfrac{1}{6}M_tE_2)\sigma$$

$$A_{19} = M\delta - (1 + M_3)\sigma\cos\beta + M_t(\alpha_a\delta - \alpha_c\sigma)$$

$$A_{21} = -\{3MM_c - \tfrac{1}{6}M_t(9MM_{23} - 9(2M - 1)M_3 + 3MR_o + 3E_2 + G_3 + G_4 + G_6]\}/M_3$$

$$A_{23} = 3M_b + \tfrac{1}{6}M_t[18M_3 + (G_1 + G_5 - G_7)/M_3]$$

$$A_{26} = -2\{M_b - (MM_c/M_3)\delta + \tfrac{1}{6}M_t[6M_3\sigma^2 - 3(2 - M)M\sigma\delta + (3MM_{23}/M_3)\delta^2]\}$$

$$A_{28} = \{2M_b + \tfrac{1}{6}M_t[12M_3\sigma - 3(2M - 1)\delta - 3R_o + (2G_1 + 2G_5 + G_3)/M_3]\}\sigma$$

$$A_{29} = (M/M_3)\{[(1 + M_3)^2 - 1]\sigma + 1 - M(1 + M_3)\delta\cos\beta$$

$$+ 1/2M_t[M^2\delta^2 + (1 + M_3)M_3\sigma^2 - M(1 + 2M_3)\sigma\delta + 2(\alpha_b\sigma - \alpha_c\delta)]\}$$

$$D_{11} = M_a\delta - M_c\sigma + \tfrac{1}{6}M_t(E_1\delta + E_2\sigma) \qquad D_{12} = (M_c - \tfrac{1}{6}M_tE_2)\sigma$$

$$D_{21} = M_b\sigma - (MM_c/M_3)\delta + \tfrac{1}{6}M_t(G_1\sigma + E_2\delta)/M_3$$

$$D_{22} = -(M_b + \tfrac{1}{6}M_tG_1/M_3)\sigma \qquad B_{12} = 2(M_a + \tfrac{1}{6}M_tE_1)$$

$$B_{14} = -\{2M_c + \tfrac{1}{6}M_t[6M_{23}\delta - 3(2 - 1/M)M_3 - 3(M - 1)R_o + 2E_2 + G_3\sigma + 2G_4]\}$$

$$B_{15} = -3D_{11} \qquad B_{17} = 0 \qquad B_{19} = (1 + M_3 + M_t\alpha_c)\sigma\sin\beta$$

$$B_{22} = -2(MM_c - \tfrac{1}{6}M_tE_2)/M_3 \qquad B_{25} = -3D_{21}$$

$$B_{24} = 2M_b + \frac{1}{6}M_t\{[6M_3\sigma - 3(2M - 1)\delta + [3(M - M_3)R_o + 2G_1 + G_3\delta + 2G_5]/M_3\}$$

$$B_{27} = - 3 [M(M_c\delta - M_3M_b\sigma) - \frac{1}{6}M_t(G \sigma + E_2\delta)]/M_3$$

$$B_{29} = -(M/M_3)(1 + M_3 + M_t\alpha_c)\delta \sin\beta$$

Equation (4.4) is derived from Equation (4.2). It is straight forward to find coefficients of Equation (4.4), as follows:

$$\bar{C}_{11} = A_{19} \qquad \bar{C}_{22} = M_3 A_{29}$$

$$\bar{C}_{21} = A_{29}C_{11} - A_{19}C_{21} \qquad \bar{C}_{22} = A_{29}C_{12} - A_{19}C_{22}$$

$$\bar{A}_{21} = A_{29}A_{11} - A_{19}A_{21} \qquad \bar{A}_{23} = A_{29}A_{13} - A_{19}A_{23}$$

$$\bar{A}_{26} = A_{29}A_{16} - A_{19}A_{16} \qquad \bar{A}_{28} = A_{29}A_{18} - A_{19}A_{28}$$

$$\bar{D}_{11} = A_{19}D_{11} - B_{19}C_{11} \qquad \bar{D}_{12} = A_{19}D_{12} - B_{19}C_{12}$$

$$\bar{D}_{21} = My_1D_{11} + M_3y_3D_{21} \qquad \bar{D}_{22} = My_1D_{12} + M_3y_3D_{22}$$

$$\bar{B}_{2i} = -A_{19}A_{1i} \qquad (i = 1, 3, 5, 7)$$

$$\bar{B}_{2i} = B_{19}A_{1i} \qquad (i = 2, 4, 6, 8)$$

$$\bar{B}_{22} = My_1B_{12} + M_3y_3B_{22} \qquad \bar{B}_{24} = My_1B_{14} + M_3y_3B_{24}$$

$$\bar{B}_{25} = My_1B_{15} + M_3y_3B_{25} \qquad \bar{B}_{27} = My_1B_{17} + M_3y_3B_{27}$$

APPENDIX C COMPUTER SUBROUTINE

I. IMSL ROUTINE NAME: DGEAR

PURPOSE: Differential Equation Solver - Variable Order Adams Predictor Corrector Method or Gears Method

USAGE: Call DGEAR

Algorithm

DGEAR finds approximations to the solution of a system of first order ordinary differential equations of the form y' = f(x,y) with initial conditions. The basic methods used for the solution are of implicit linear multistep type. There are two classes of such methods availble to the user. The first is the implicit Adams methods (up to order twelve), and the second is the backward differentiation formular (BDF) methods (up to order five), also called Gear's stiff methods. In either case the implicitness of the basic formula required that an algebraic system of equations be solved at each step. A variety of corrector iteration methods is available for this.

DGEAR and the associated nuclei are adaptations of a package designed by A.C. Hindmarsh based on C.W. Gear's subroutine DIFSUB.

II. ISML ROUTINE NAME: ZANLYT

PURPOSE: Zeros of An Analytic Complex Function Using the Muller Method With Deflation

USAGE: Call ZANLYT

Algorithm

Muller's method with deflation is used.

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Two satellites connected by a long flexible tether along the earth radial direction comprise a stable equilibrium state. This research investigates the manner in which a third mass transporting from one satellite to the other disturb the equilibrium state. A system of four equations of in-plane motion has been derived based on the assumptions that the tether remains straight between the masses and of constant length. A combination of computer subroutines DGEAR and ZANAYT is suggested for the approximate solution of the system of nonlinear differential equations with one constraint condition on the variables. Alternately, a system of four independent differential equations are derived by eliminating the Lagrange multiplier.

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