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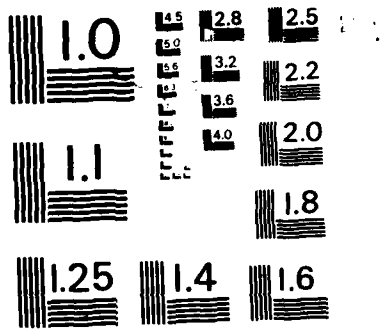

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STATE ESTIMATION AND CONTROL OF CONDITIONALLY LINEAR SYSTEMS*

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STATE ESTIMATION AND CONTROL OF CONDITIONALLY LINEAR SYSTEMS*

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Abstract. The filtering problem for a partially observable stochastic system, with linear in observable states dynamics and non-Gaussian initial conditions is studied here. It is shown that the conditional expected value of the unobservable states, given the past observations, can be expressed in terms of a finite dimensional set of statistics. This result, which generalizes the conditionally Gaussian filter is used to derive a separation principle for a linear-quadratic control problem.

Key Words. Optimal filtering, stochastic control, non-Gaussian stochastic systems.

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Introduction

Stochastic, partially observable systems, with linear-in-observable state dynamics are termed conditionally linear systems here. It is well known that the solution of a state estimation problem for a conditionally linear system with Gaussian distribution of the initial state is given in terms of two sets of sufficient statistics, satisfying stochastic differential equations [4].

Solved here is the state estimation problem which generalizes the above result for the case of an arbitrary a priori distribution. The method applied in this study is based on the derivation of an explicit formula for the conditional characteristic function of the state, given the past and present observations. This approach seems to impose less restrictive conditions on the system structure than the methods based on the derivation of the conditional distribution function. The latter can be found in [1] where the filter is derived for a linear system with a priori distribution having a well-defined density function.

It is shown here that the conditional characteristic function of the present and past states, given the present and past observations, is parametrically determined by a finite number of sufficient statistics. This result leads to the derivation of a filter, in the form of a finite set of stochastic differential equations which extends the result of [1] in a similar manner as a conditionally Gaussian filter generalizes a Kalman filter.

Also discussed here and illustrated by the examples, is the suitability of the filter structure for the study of stochastic control and parameter estimation.

1. Problem Formulation and the Main Result

Given the following system of stochastic differential equations

$$(1.1) \qquad dx_t = \left(f_0(t,y) + f_1(t,y)x_t\right) dt + g_0(t,y)dw_t + q_0(t,y)dv_t,$$

$$(1.2) \qquad dy_t = \left(h_0(t,y) + h_1(t,y)x_t\right)dt + dv_t, \qquad 0 \leq t \leq T,$$

where $f_0, f_1, g_0, q_0, h_0, h_1$ are the nonanticipative functionals of y (i.e., $\mathcal{Y}_t$ measurable with $\mathcal{Y}_t = \sigma - \text{alg}\ \{y_s,\ 0 \leq s \leq t\}$), and w_t, v_t are independent Wiener processes.

The objective is to find $\hat{x}_t = E(x_t/\mathcal{Y}_t)$, assuming that x_t, y_t satisfy (1.1) and (1.2), and that the conditional distribution of the initial states $F(a) = P(x_0 \leq a|y_0)$ is given.

The organization of this section starts with Lemma 1, whereby it is shown that the conditional characteristic function of $(x_{t0}, x_{t1}, \ldots, x_{tn})|\mathcal{Y}_t$, for an arbitrary decomposition $0 \leq t_0 < t_1 < \ldots < t_n \leq t \leq T$, of the interval $[0,T]$ is of a particular form. Results from the theory of conditionally Gaussian processes are used here.

Next, Lemma 2, the explicit formula for the characteristic function of $x_t|\mathcal{Y}_t$ is derived, and finally, in Lemma 3, all the results are organized to yield the recursive, finite-dimensional set of filter equations.

The assumptions used in the proof of Lemma 1 and 2 are listed below:

Let C_T denote the space of continuous functions $\eta = \{\eta_t,\ 0 \leq t \leq T\}$. It is assumed that for each $\eta \in C_t$

$$(1.3) \qquad \int_0^T \Big(\sum_{k=0}^{1} \left(|f_k(t,\eta)|+|h_k(t,\eta)|\right)+|g_0(t,\eta)|^2+|q_0(t,\eta)|^2\Big)\, dt < \infty$$

The above assumption assures the existence of the (Ito) integrals in (1.1) and (1.2) [3]. In order to use the results for conditionally Gaussian processes it is also assumed that [4]:

(1.4) for all $\eta \in C_t$, $t \in [0,T]$, $|f_1(t,\eta)| + |h_1(t,\eta)| \leq \text{const}$,

and

(1.5) $$\int_0^T \mathbb{E}\left(|f_0(t,y)|^4 + |g_0(t,y)|^4 + |q_0(t,y)|^4\right)dt < \infty, \quad \mathbb{E}\left(|x_0|^4\right) < \infty$$

Lemma 1.

Let

$$\phi_t = \exp\left(i \sum_{k=0}^{n} z_k x_{tk}\right), \quad z = \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} \in R^n, \quad 0 \leq t_0 < t_1 < \ldots < t_n \leq t \leq T .$$

Then the conditional characteristic function of $(x_{t0}, x_{t1}, \ldots, x_{tn})|\mathcal{Y}_t$ is given by

(1.6) $$e_t(z) = \mathbb{E}\left(\phi_t | \mathcal{Y}_t\right) = \int_{-\infty}^{\infty} \exp\left(Q(t,a,z,y)\right) dF(a)$$

where $Q(t,a,z,y)$ is quadratic in the variables a and z.

Proof of Lemma 1

First notice that (1.1) solves as

(1.7) $$x_t = \Phi_t\left(x_0 + \int_0^t \Phi_s^{-1}\,(f_0 - q_0 h_0)ds + \int_0^t \Phi_s^{-1}\, q_0 dy_s + \int_0^t \Phi_s^{-1} g_0 dw_s\right)$$

where $\Phi_t = \exp\left(\int_0^t (f_1 - g_0 h_1)ds\right)$.

Rewrite (1.7) in the symbolic way as

(1.8) $$x_t = \psi_t(x_0, w, y).$$

Now, the following version of the Bayes formula will be used [2, p. 8]: Let $\phi_t(x_0,w,y)$ be a nonanticipative functional of its arguments with $\mathbb{E}(|\phi_t|) < \infty$ for all $t \in [0,T]$. Then

$$\mathbb{E}(\phi_t|\mathcal{Y}_t) = \int_{-\infty}^{\infty}\int_{C_T} \phi_t(a,\eta,y)\rho_t(a,\eta,y)d\mu_w(\eta)dF(a) \tag{1.9}$$

where μ_w is a Wiener measure in the measurable space of continuous functions η on $[0,T]$,

$$\rho_t(a,\eta,y) = \exp\Big(\int_0^t h_1\big(\psi_s(a,\eta,y) - \hat{x}_s(y)\big)d\nu_s - \frac{1}{2}\int_0^t h_1^2\big(\psi_s(a,\eta,y) - \hat{x}_s(y)\big)^2 ds\Big)$$

(1.10)

with $d\nu_s = dy_s - (h_0 + h_1\hat{x}_s)ds$, and $\psi_s(a,\eta,y)$ defined by (1.8). The random process ν_t can be represented by

$$\nu_t = \int_0^t \big(dy_s - \big(h_0(s,y) + h_1(s,y)\hat{x}_s(y)\big)ds\big) = v_t + \int_0^t h_1(s,y)\big(x_s - \hat{x}_s(y)\big)ds \; .$$

Now using the Ito formula we have

$$e^{iz\nu_t} = e^{iz\nu_s} + iz\int_s^t h_1(\tau,y)e^{iz\nu_\tau}\big(x_\tau - \hat{x}_\tau(y)\big)d\tau$$

$$+ iz\int_s^t e^{iz\nu_\tau}dv_\tau - \frac{z^2}{2}\int_s^t e^{iz\nu_\tau}d\tau \; .$$

Multiplying both sides of the above equation by $e^{iz\nu_s}$ and taking the conditional expectation $E(\cdot|Y_s)$ gives

$$E(e^{iz(\nu_t-\nu_s)}|Y_s) = 1 - \frac{z^2}{2}\int_s^t E(e^{iz(\nu_\tau-\nu_s)}|Y_s)\,d\tau .$$

Solving the last equation yields

$$E(e^{iz(\nu_t-\nu_s)}|Y_s) = e^{-\frac{z^2}{2}(t-s)} , \tag{1.11}$$

which shows that (ν_t, Y_t) is a Wiener process.

Now rewrite $\rho_t(a,\eta,y)$ in a more convenient form. To this end introduce the following notation:

$$A_1(t,y) = h_1\left(\Phi_t\left(\int_0^t \Phi_s^{-1}(f_0-q_0h_0)ds + \int_0^t \Phi_s^{-1}\, q_0 dy_s\right) - \hat{x}_t\right) ,$$

$$A_2(t,y) = h_1\Phi_t ,$$

$$A_3(t,y) = \Phi_t^{-1}g_0 ,$$

$$C_1(t,y) = \int_0^t A_1(s,y)d\nu_s - \frac{1}{2}\int_0^t A_1^2(s,y)ds ,$$

$$C_2(t,y) = \int_0^t A_2(s,y)d\nu_s - \int_0^t A_1(s,y)A_2(s,y)ds ,$$

$$C_3(t,y) = \left(\int_0^t A_2^2(s,y)ds\right)^{1/2} ,$$

$$C_4(t,y,w) = \int_0^t A_2(s,y)\int_0^s A_3(\tau,y)dw_\tau d\nu_s - \int_0^t A_1(s,y)A_2(s,y)\int_0^s A_3(\tau,y)dw_\tau ds ,$$

$$C_5(t,y,w) = -\int_0^t A_2^2(s,y)\int_0^s A_3(s,y)dw_\tau ds \ .$$

Note also that $C_4(t,y,w)$ and $C_5(t,y,w)$ can be rewritten with the use of the Ito formula by:

$$C_4(t,y,w) = \int_0^t A_4(t,s,y)dw_s \ ,$$

$$C_5(t,y,w) = \int_0^t A_5(t,s,y)dw_s \ ,$$

where

$$A_4(t,s,y) = \Big(\int_0^t A_2(s,y)d\nu_s - \int_0^s A_2(\tau,y)d\nu_\tau\Big)$$

$$- \Big(\int_0^t A_1(s,y)A_2(s,y)ds - \int_0^s A_1(\tau,y)A_2(\tau,y)d\tau\Big)A_3(s,y)$$

$$A_5(t,s,y) = \Big(\int_0^s A_2^2(\tau,y)d\tau - \int_0^t A_2^2(s,y)ds\Big)A_3(s,y) \ .$$

Now, using the above notation we have from (1.8) and (1.10)

$$\rho_t(a,w,y) = \exp\Big(C_1 + a\big(C_2 + C_5\big) + C_4 - \frac{a^2}{2}C_3^2 - \frac{1}{2}\int_0^t A_2^2\Big(\int_0^s A_3 dw_\tau\Big)^2 ds\Big)$$

$$= \exp\Big(C_1 + aC_2 - \frac{a^2}{2}C_3^2 + \int_0^t \big(aA_5 + A_4\big)dw_s - \frac{1}{2}\int_0^t A_2^2\Big(\int_0^s A_3 dw_\tau\Big)^2 ds\Big) \ .$$

(1.12)

The arguments in (1.12) were omitted for brevity.

From (1.8) it follows that

$$x_t = \psi_t(x_0, w, y) = \Phi_t\left(x_0 + A_6(t,y) + \int_0^t A_3(s,y)\,dw_s\right)$$

where

$$A_6(t,y) = \int_0^t \Phi_s^{-1}\,(f_0 - q_0 h_0)\,ds + \int_0^t \Phi_2^{-1}\, q_0\,dy_s\ .$$

Combining (1.12) and the above

$$\exp(Q(t,a,z,y)) = \int_{C_T} \phi_t(a,\eta,y)\rho_t(a,\eta,y)\,d\mu_w(\eta)$$

$$= \exp\Big(C_1 + aC_2 - \frac{a^2}{2}C_3^2 + a\Big(\sum_{k=1}^{n} \Phi_{tk} i z_k\Big)$$

$$+ \sum_{k=1}^{n} \Phi_{tk} A_6(t_k, y) i z_k\Big) \int_{C_T} \exp\Big(\int_0^t (aA_5 + A_4)\,d\eta_s$$

$$+ \sum_{k=1}^{n} i z_k \Phi_{tk} \int_0^{t_k} A_3\,d\eta_s - \frac{1}{2}\int_0^t A_2^2\Big(\int_0^s A_3\,d\eta_\tau\Big)^2 ds\Big)\,d\mu_w(\eta).$$

(1.13)

In order to evaluate the integral in (1.13) the following results will be used:

(i) Since the above integral represents a conditional expected value of its integrand, under the condition that y_s, $s \in [0,t]$ and $x_0 = a$ are given, the resulting distributions are of conditionally Gaussian type [4]. Note that this fact does <u>not</u> depend on the F(a).

(ii) With all the variables in (1.13) being conditionally Gaussian we can use a convenient theorem:

Theorem [4, pp. 12-13]

Let w_t, $t \in [0,T]$ be a Wiener process and let $R(t)$, $G(t)$, and $H(t) \geq 0$ be such that

$$\int_0^T (|R(t)| + G(t)^2 + H(t))dt < \infty .$$

Then for all $t \in [0,T]$

$$(1.14) \qquad \mathbb{E}\Big(\exp\Big(\int_0^t R(s)G(s)dw_s - \int_0^t H(s)\Big(\int_0^s G(\tau)dw_\tau\Big)^2 ds\Big)$$

$$= \exp\Big(\frac{1}{2} D(t) + \frac{1}{2}\int_0^t G(s)^2\Gamma(s)ds\Big)$$

where $\quad d\Gamma(s) = (2H(s) - \Gamma(s)^2 G(s)^2)ds \; , \; \Gamma(t) = 0 \; ,$

and $D(t)$ is the covariance of $\int_0^t R(s)d\xi_s$, where

$$d\xi_s = G(s)^2\Gamma(s)\xi_s ds + G(s)dw_s \; , \; \xi_0 = 0 \; .$$

Comparing the last integral in (1.13) with the equation given by (1.14), we note that the corresponding $R(t)$ is a linear function of a and z. Now (1.9), (1.13), (1.14), and the definition of $D(t)$ conclude the proof of Lemma 1.

From Lemma 1 it follows in particular that for $z \in \mathbb{R}$, the characteristic function of $x_t | \mathcal{Y}_t$ is given by

$$(1.15)\qquad e_t(z) = C(t,y)\int_{-\infty}^{\infty} \exp\Big(a^2F_1(t,y) + aF_2(t,y) + izaF_3(t,y) + izF_4(t,y) + z^2F_5(t,y)\Big)dF(a) ,$$

where F_1, F_2, F_3, F_4, F_5 do not depend on $F(a)$. Normalizing $e_t(z)$ (i.e., requiring that $e_t(0) = 1$) yields

$$(1.16)\qquad e_t(z) = \exp\big(izF_4 + z^2F_5\big)\frac{\int_{-\infty}^{\infty} \exp\big(a^2F_1 + a(F_2 + izF_3)\big)dF(a)}{\int_{-\infty}^{\infty} \exp\big(a^2F_1 + aF_2\big)dF(a)}$$

Then from the general properties of the characteristic function, it fo ws that

$$\frac{1}{i}\frac{de_t(z)}{dz}\Bigg|_{z=0} = \hat{x}_t ,$$

$$\left(\frac{1}{i}\right)^2\frac{d^2e_t(z)}{dz^2}\Bigg|_{z=0} = P_t + \hat{x}_t^2 ,$$

where $P_t = E((x_t - \hat{x}_t)^2|Y_t)$ i.e., the conditional variance of $x_t|Y_t$. From the above and (1.16)

$$(1.17)\qquad \hat{x}_t = F_3I_t(1) + F_4 ,$$

$$(1.18)\qquad P_t = -2F_5 + F_3^2\,\big(I_t(2) - I_t^2(1)\big) ,$$

where

$$(1.19)\qquad I_t(n) = \frac{\int_{-\infty}^{\infty} a^n\exp\big(a^2F_1 + aF_2\big)dF(a)}{\int_{-\infty}^{\infty} \exp\big(a^2F_1 + aF_2\big)dF(a)} ,\ n = 1,2 .$$

The following Lemma defines F_i $i = 1,2,3,4,5$ in (1.16).

Lemma 2

The characteristic function of $x_t | Y_t$ is given by

$$(1.20)\qquad e_t(z) = \exp\left(-\frac{1}{2} z^2 \bar{P}_t(0)\right) \frac{\int_{-\infty}^{\infty} \exp\left(a^2 F_1 + aF_2 + iz\bar{x}_t(a,0)\right) dF(a)}{\int_{-\infty}^{\infty} \exp\left(a^2 F_1 + aF_2\right) dF(a)} ,$$

where $\bar{x}_t(a,0)$, $\bar{P}_t(0)$ are given as the solutions to the following set of differential equations with $\sigma = 0$:

$$(1.21)\qquad d\bar{x}_t(a,\sigma) = \left(f_0 + f_1 \bar{x}_t(a,\sigma)\right)dt + \left(q_0 + \bar{P}_t(\sigma)h_1\right)\left(dy_t - \left(h_0 + h_1\bar{x}_t(a,\sigma)\right)dt\right)$$

$$\bar{x}_0(a,\sigma) = a ,$$

$$(1.22)\qquad d\bar{P}_t(\sigma) = \left(2f_1\bar{P}_t(\sigma) + g_0^2 + q_0^2 - \left(q_0 + \bar{P}_t(\sigma)h_1\right)^2\right)dt , \quad \bar{P}_0(\sigma) = \sigma^2 ,$$

and

$$(1.23)\qquad F_1 = -\frac{1}{2} \int_0^t h_1^2 \phi_s^2 \, ds ,$$

$$(1.24)\qquad F_2 = \int_0^t \phi_s h_1 \left(d\nu_s + h_1 \phi_s I_s(1) ds\right) ,$$

$$(1.25)\qquad \phi_t = \exp\left(\int_0^t \left(f_1 - h_1\left(q_0 + \bar{P}_s(0)h_1\right)\right)ds\right) .$$

Proof of Lemma 2

Since the F_i do not depend on $F(a)$ (see Lemma 1) take

$$(1.26)\qquad dF(a) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(a-m)^2}{2\sigma^2}\right)da , \quad a, \ \sigma > 0 .$$

In this case the resulting conditionally Gaussian distribution allows for explicit $e_t(z)$ calculation [4]. Accordingly,

$$(1.27)\qquad e_t(z) = \exp\left(iz\bar{x}_t(m,\sigma) - \frac{1}{2} z^2\bar{P}_t(\sigma)\right) ,$$

where $\bar{x}_t(m,\sigma)$ and $\bar{P}_t(\sigma)$ satisfy (1.21) and (1.22) respectively. With F(a) given by (1.26) it follows from (1.16) that

$$(1.28)\qquad e_t(z) = \exp\left(iz\left(F_4 + \hat{\sigma}^2\left(F_2 + \frac{m}{\sigma^2}\right)F_3\right) + z^2\left(F_5 - \frac{1}{2}\hat{\sigma}^2 F_3^2\right)\right) ,$$

where $\qquad \hat{\sigma}^{-2} = \sigma^{-2} - 2F_1$.

Comparing (1.27) and (1.28), we have

$$(1.29)\qquad \bar{x}_t(m,\sigma) = F_4 + \hat{\sigma}^2\left(F_2 + \frac{m}{\sigma^2}\right)F_3 ,$$

and

$$(1.30)\qquad \bar{P}_t(\sigma) = \hat{\sigma}^2 F_3^2 - 2F_5 .$$

Letting now $\sigma \to 0$ in (1.29) and (1.30), it follows that

$$(1.31)\qquad F_4 + mF_3 = \bar{x}_t(m,0) ,$$

and

$$(1.32)\qquad F_5 = -\frac{1}{2}\bar{P}_t(0) .$$

The above allows $e_t(z)$ to be of the form of (1.20) with F_1 and F_2 yet to be defined. Using now (1.17) and (1.18) and explicitly calculating $I_t(n)$, $n = 1,2$,

$$\Delta_t\left(\sigma^{-2} - 2F_1\right) = F_3^2 \tag{1.33}$$

and

$$\Delta_t\left(\frac{m}{\sigma^2} + F_2\right) = F_3\left(\hat{x}_t - F_4\right) , \tag{1.34}$$

with $\qquad \Delta_t = P_t - \bar{P}_t(0)$.

The formulae for F_1 and F_2 will be obtained by differentiating (1.33) and (1.34). However, before this is done recall from the theory of nonlinear filtering [3] that in general for x_t, y_t given as a solution to (1.1) and (1.2) $\hat{x}_t$, P_t satisfy

$$d\hat{x}_t = \left(f_0 + f_1\hat{x}_t\right)dt + \left(q_0 + P_th_1\right)d\nu_t \ , \ \hat{x}_0 = \int_{-\infty}^{\infty} a\,dF(a) \ , \tag{1.35}$$

$$dP_t = \left(2f_1P_t + g_0^2 + q_0^2 - \left(q_0 + P_th_1\right)^2\right)dt + h_1R_td\nu_t \ , \ P_0 = \int_{-\infty}^{\infty}\left(a-\hat{x}_0\right)^2dF(a) \ ,$$

(1.36)

where $\qquad R_t = E\left(\left(x_t - \hat{x}_t\right)^3 \middle| Y_t\right)$.

Remark. Direct application of Eqs. (1.35) and (1.36) meets the difficulty of infinite coupling between the subsequent moments.

From Eqs. (1.22), (1.35), and (1.36), and the fact that for conditionally Gaussian processes $R_t = 0$,

(1.37) $$d\Delta_t = \Delta_t\big(2f_1 - h_1\big(2q_0 + h_1\big(P_t + \bar{P}_t(0)\big)\big)\big)dt \ , \ \Delta_0 = \sigma^2$$

Now from (1.33) and (1.34) (upon differentiation) and using (1.35), (1.37) it follows that

(1.38) $$dF_1 = -\frac{1}{2} h_1^2 F_3^2 \, dt \ , \ F_1(0) = 0 \ ,$$

and

(1.39) $$dF_2 = F_3 h_1 \big(d\nu_t + h_1 F_3 I_t(1) dt\big) \ , \ F_2(0) = 0 \ .$$

To define F_3, notice that Eq. (1.21) solves as

$$\bar{x}_t(a,\sigma) = \phi_t\Big(a + \int_0^t \phi_s^{-1}\big(f_0 - h_0\big(q_0 + \bar{P}_s(\sigma)h_1\big)\big)ds + \int_0^t \phi_s^{-1}\big(q_0 + \bar{P}_s(\sigma)h_1\big)dy_s\Big) \ ,$$

(1.40)

where

(1.41) $$\phi_t = \exp\Big(\int_0^t \big(f_1 - h_1\big(q_0 + \bar{P}_s(\sigma)h_1\big)\big)ds$$

Comparing the above with (1.31) shows that $F_3 = \phi_t$ for $\sigma = 0$, which ends the proof of Lemma 2.

Lemma 3 below merely organizes all the results into the filter equations and the final form of the conditional characteristic function.

Lemma 3

Given the system (1.1) and (1.2) together with the a priori distribution $F(a) = P(x_0 < a|y_0)$. The following are the filter equations (i.e., formulae of the recursive type, which calculate $\hat{x}_t = E(x_t|\mathcal{Y}_t)$).

(1.42) $$d\hat{x}_t = (f_0 + f_1\hat{x}_t)dt + (q_0 + P_t h_1)d\nu_t \ , \ \hat{x}_0 = \int_{-\infty}^{\infty} a dF(a) \ ,$$

(1.43) $$d\nu_t = dy_t - (h_0 + h_1\hat{x}_t)dt \ ,$$

(1.44) $$P_t = \bar{P}_t + \phi_t^2(I_t(2) - I_t^2(1)) \ ,$$

(1.45) $$d\bar{P}_t = (2f_1\bar{P}_t + g_0^2 + q_0^2 - (q_0 + \bar{P}_t h_1)^2)dt \ , \ \bar{P}_0 = 0 \ ,$$

(1.46) $$I_t(n) = \frac{\int_{-\infty}^{\infty} a^n \exp(a^2F_1 + aF_2)dF(a)}{\int_{-\infty}^{\infty} \exp(a^2F_1 + aF_2)dF(a)} \ , \ n = 1,2 \ ,$$

(1.47) $$dF_1 = -\frac{1}{2} h_1^2\phi_t^2 dt \ , \ F_1(0) = 0 \ ,$$

(1.48) $$dF_2 = \phi_t h_1(d\nu_t + \phi_t h_1 I_t(1)dt) \ , \ F_2(0) = 0 \ ,$$

(1.49) $$d\phi_t = (f_1 - h_1(q_0 + \bar{P}_t h_1))\phi_t \ dt \ , \ \phi_0 = 1 \ .$$

The characteristic function of $x_t|\mathcal{Y}_t$ is given by:

$$e_t(z) = \exp\left(iz\left(\hat{x}_t - \phi_t I_t(1)\right) - \frac{1}{2} z^2 \bar{P}_t\right) \frac{\int_{-\infty}^{\infty} \exp\left(a^2F_1 + a\left(F_2 + iz\phi_t\right)\right)dF(a)}{\int_{-\infty}^{\infty} \exp\left(a^2F_1 + aF_2\right)dF(a)} .$$

(1.50)

2. Control and Special Cases

Two special cases of Eqs. (1.1) and (1.2) result in significant simplification of the filter equations. The first case occurs when $g_0(t,y) = 0$, $0 \le t \le T$. From (1.7) it follows then that x_t is of the form

$$x_t = A_t(y)x_0 + B_t(y) .$$

Using the above equation in (1.2) we have the following estimation problem: Let x_0 be a random variable with distribution $F(a) = P(x_0 \le a|y_0)$. Assume that the observation process y_t, $0 \le t \le T$, admits a differential

$$dy_t = \left(h_0(t,y) + h_1(t,y)x_0\right)dt + dv_t ,$$

where the notation stays the same as in (1.2) and h_0, h_1 satisfy (1.3) and (1.4).

From Lemma 2 it follows now that the conditional characteristic function of x_0 given $\mathbf{Y}_t$ is of the form

$$e_t(z) = \frac{\int_{-\infty}^{\infty} \exp\left(a^2F_1 + aF_2 + iza\right)dF(a)}{\int_{-\infty}^{\infty} \exp\left(a^2F_1 + aF_2\right)dF(a)} \tag{2.1}$$

The above results from the fact that $dx_0 = 0$ replaces Eq. (1.1) implying $\bar{P}_t(0) = 0$ and $\bar{x}_t(a,0) = a$, as defined by Eqs. (1.21) and (1.22). Now

$$\left.\frac{de_t(z)}{dz}\right|_{z=0} = i\hat{x}_t ,$$

where $\hat{x}_t = E(x_0 | Y_t)$, combined with the general filter equations (1.42) + (1.50) yields

$$\hat{x}_t = \frac{\int_{-\infty}^{\infty} a \exp\left(-\frac{1}{2} a^2 \int_0^t h_1^2 ds + a \int_0^t h_1 \left(dy_s - h_0 ds\right)\right) dF(a)}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} a^2 \int_0^t h_1^2 ds + a \int_0^t h_1 \left(dy_s - h_0 ds\right)\right) dF(a)} . \tag{2.2}$$

In particular if $dF(a) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp(-\frac{1}{2}(a - m_0)^2) da$, (2.2) results in

$$\hat{x}_t = \frac{m_0 + \sigma_0^2 \int_0^t h_1 \left(dy_s - h_0 ds\right)}{1 + \sigma_0^2 \int_0^t h_1^2 ds} \tag{2.3}$$

The above agrees with the result presented in [4, pp. 22-24].

The second special case for which the filter takes a simple form follows if $h_1(t,y) = 0$ (i.e., the state is not observable directly), $0 < t < T$. Now the filter equations (1.42) + (1.50) reduce to

$$\begin{aligned} d\hat{x}_t &= \left(f_0 + f_1 \hat{x}_t\right) dt + q_0 \left(dy_t - h_0 dt\right) , \\ \hat{x}_0 &= \int_{-\infty}^{\infty} a dF(a) , \\ dP_t &= \left(2 f_1 P_t + g_0^2\right) dt , \\ P_0 &= \int_{-\infty}^{\infty} \left(a - \hat{x}_0\right)^2 dF(a) . \end{aligned} \tag{2.4}$$

In order to discuss a control problem using the results obtained here, assume that all the coefficients in (1.1) and (1.2), except $f_0(t,y)$ which is denoted here by $u_t(y)$, are functions of time only. If $u_t(y)$ satisfies the assumption (1.5) we say that $u = \{u_t(y),\ 0 \le t \le T\}$ is an admissible control and write $u \in \mathbf{U}$.

Let x_t^u, $\hat{x}_t^u$ denote the solutions to (1.1) nad (1.42) respectively, for some $u \in \mathbf{U}$, and let x_t^o, $\hat{x}_t^o$ correspond to $u_t \equiv 0$, $0 \le t \le T$.

Define $e_t^u = x_t^u - \hat{x}_t^u$, $e_0^u = x_0 - \hat{x}_0 = e_0^o$, $e_t^o = x_t^o - \hat{x}_t^o$. Substracting Eq. (1.42) from (1.1) we have

$$(2.5)\qquad de_t^u = f_1e_t^u dt + g_0 dw_t + q_0 dv_t - \left(q_0 + P_t(e^u)h_1\right)\left(h_1e_t^u + dv_t\right) ,$$

where $P_t = P_t(e^u)$ shows that P_t depends only on e_s^u, $0 \le s \le t$ which is seen from Eq. (1.48) rewritten as:

$$(2.6)\qquad dF_2 = \phi_t h_1\left(e_t^u h_1 dt + dv_t + \phi_t h_1 I_t(1)\right)dt .$$

From (2.5) it follows that with probability one the values of e_t^u and e_t^o coincide for all $u \in \mathbf{U}$. Now, since

$$(2.7)\qquad d\nu_t^u = dy_t^u - \left(h_0 + h_1\hat{x}_t^u\right)dt = h_1e_t^u dt + dv_t = h_1e_t^o dt + dv_t = d\nu_t^o ,$$

ν_t^u and ν_t^o coincide with probability one. From (2.7) it follows that Eq. (1.42) can be rewritten as

$$(2.8)\qquad d\hat{x}_t^u = \left(f_1\hat{x}_t^u + u_t\right)dt + \left(q_0 + h_1P_t\left(e_t^o\right)\right)d\nu_t^o ,\quad \hat{x}_0^u = \hat{x}_0 = \int_{-\infty}^{\infty} a dF(a)$$

Now let $\tilde{u}_t = F_t(\hat{\tilde{x}}^u)$, where F_t is a nonanticipative functional of $\hat{\tilde{x}}_s^u$, $0 \le s \le t$, and satisfies

$$(2.9) \qquad E\left(\int_0^T |F_t|^4 dt\right) < \infty .$$

From (2.8) it follows that $\tilde{u}_t$ is $\sigma\text{-alg}\{\nu_s^o, 0 \le s \le t\}$ measurable. Now, let $\bar{u}_t$ be any admissible control and let $y_t^{\bar{u}}$ be an observation process associated with $\bar{u}_t$. From (2.7),

$$Y_t^{\bar{u}} = \sigma\text{-alg}\{y_s^{\bar{u}}, 0 \le s \le t\} \supseteq \sigma\text{-alg}\{\nu_s^{\bar{u}}, 0 \le s \le t\} = \sigma\text{-alg}\{\nu_s^o, 0 \le s \le t\} .$$

(2.10)

The above shows that $\tilde{u}_t$ is $Y_t^{\bar{u}}$ measurable. This fact combined with (2.9) states that $\tilde{u}_t \in U$, and that we can expect the separation of the stochastic control of $\tilde{u}_t$ type and the filtering problem. As an illustration of the statement, consider the following control problem.

<u>Linear-Quadratic Control Problem with Non-Gaussian Initial Distributions</u>

The partially observable controlled process (x_t,y_t), $0 \le t \le T$, is given by the stochastic equations

$$(2.11) \qquad dx_t = (f_1(t)x_t + u_t)dt + g_0(t)dw_t,$$

$$dy_t = h_1(t)x_t dt + dv_t , \quad y_0 = 0 .$$

The independent Wiener processes w_t and v_t entering into (2.11) do not depend on the random variable x_0 (the initial state). x_0 is assumed to have distribution function $F(a) = P(x_0 \le a)$ with $\int_{-\infty}^{\infty} a^4 dF(a) < \infty$ (finite fourth order moment).

The $\mathcal{Y}_t = \sigma\text{-alg}\{y_s, 0 \le s \le t\}$ measurable, stochastic process u_t is called a control at time t and is assumed to satisfy

$$E\left(\int_0^T |u_t|^4 dt\right) < \infty .$$

For $u = \{u_t, 0 \le t \le T\}$, satisfying the above we write $u \in \mathcal{U}$, where $\mathcal{U}$ is the class of admissible controls. It is also assumed that f_1, g_0, h_1 satisfy the deterministic version of (1.3) ÷ (1.5).

Consider now the performance functional

$$J(u) = E\left(x_T^2 h_T + \int_0^T \left(x_t^2 H(t) + u_t^2 R(t)\right)dt\right) \tag{2.12}$$

where $h_T \ge 0$, $H(t) \ge 0$, $0 \le R^{-1}(t) \le \text{const.}$, $0 \le t \le T$. The admissible control $\hat{u} \in \mathcal{U}$ is called optimal if

$$J(\hat{u}) = \inf_{u \in \mathcal{U}} J(u) .$$

<u>Lemma</u>

The optimal control for the process (2.11) and the performance index (2.12) exists and is defined by

$$\hat{u}_t = -R^{-1}(t)Q(t)\hat{x}_t , \quad 0 \le t \le T \tag{2.13}$$

where $Q(t) \ge 0$ satisfies the Riccati equation

$$-\frac{dQ(t)}{dt} = 2f_1(t)Q(t) + H(t) - Q^2(t)R^{-1}(t) , \quad Q(T) = h_T , \tag{2.14}$$

and $\hat{x}_t$ is defined by

$$(2.15) \quad d\hat{x}_t = \left(f_1(t) - R^{-1}(t)Q(t)\right)\hat{x}_t dt + P_t(\nu)h_1(t)d\nu_t$$

$$\hat{x}_0 = \int_{-\infty}^{\infty} a dF(a) \ ,$$

$$d\nu_t = dy_t - h_1(t)\hat{x}_t dt$$

$$P_t = \bar{P}(t) + \phi(t)^2\left(I_t(2) - I_t^2(1)\right)$$

$$\frac{d\bar{P}(t)}{dt} = 2f_1(t)\bar{P}(t) + g_0^2(t) - h_1^2(t)\bar{P}^2(t) \ , \ \bar{P}(0) = 0$$

$$\phi(t) = \exp\left(\int_0^t \left(f_1(s) - h_1^2(s)\bar{P}(s)\right)ds\right.$$

$$F_1(t) = -\frac{1}{2}\int_0^t h_1^2(s)\phi^2(s)ds$$

$$F_2(t,\nu) = \int_0^t \phi(t)h_1(t)\left(d\nu_t + \phi(t)h_1(t)I_t(1)dt\right)$$

$$I_t(n) = \frac{\int_{-\infty}^{\infty} a^n \exp\left(a^2F_1(t) + aF_2(t,\nu)\right)dF(a)}{\int_{-\infty}^{\infty} \exp\left(a^2F_1(t) + aF_2(t,\nu)\right)dF(a)} \ , \ n = 1,2 \ .$$

Remark. The structure of the optimal control law is identical with the optimal controller for LQG problem.

Proof of the Lemma

First note that the assumptions made in the control problem statement assure well-defined filter for the system (2.11) and $u \in U$. Next rewrite the performance index as follows:

$$(2.16)\qquad J(u) = \mathbb{E}\Big(\mathbb{E}\big(x_T^2 h_T \,\big|\, \mathbf{Y}_T\big) + \int_0^T \mathbb{E}\big(x_t^2 H(t) + u_t^2 R(t) \,\big|\, \mathbf{Y}_t\big)dt\Big)$$

$$= \mathbb{E}\Big((\hat{x}_T^u)^2 h_T + \int_0^T \big((\hat{x}_t^u)^2 H(t) + u_t^2 R(t)\big)dt\Big)$$

$$+ \mathbb{E}\Big(h_T P_t^u + \int_0^T P_t^u H(t)dt\Big)$$

From (2.6) we conclude that P_t^u does not depend on the control u and coincides with the function P_t^o obtained from the filter equations for $u_t \equiv 0$, $0 \leq t \leq T$. The process $\hat{x}_t^u$ entering (2.15) satisfies equation (see 2.8)

$$(2.17)\qquad d\hat{x}_t^u = \big(f_1(t)\hat{x}_t^u + u_t\big)dt + h_1(t)P_t^o d\nu_t^o ,$$

where $\nu_t^o = \nu_t^u = \int_0^t (dy_s^u - h_1(s)\hat{x}_s^u ds)$, according to (2.7), and ν_t^o is a Wiener process (see (1.11)). Introduce now the function

$$(2.18)\qquad V(t,\xi) = \xi^2 Q(t) + \int_t^T Q(\tau)h_1^2(\tau)\big(P_\tau^o\big)^2 d\tau$$

where $0 \leq t \leq T$, $-\infty < \xi < \infty$, and Q(t) satisfies (2.14). It is easy to verify that $V(t,\xi)$ satisfies the following Bellman equation:

$$(2.19)\qquad \xi^2 H(t) + \xi f_1(t)\,\frac{\partial V(t,\xi)}{\partial \xi} + \frac{1}{2}\,\big(P_t^o\big)^2 h_1^2(t)\,\frac{\partial^2 V(t,\xi)}{\partial \xi^2} + \frac{\partial V(t,\xi)}{\partial t}$$

$$+ \min_\eta\Big(\eta^2 R(t) + \eta\,\frac{\partial V(t,\xi)}{\partial \xi}\Big) = 0 ,$$

and that $V(T,\xi) = \xi^2 h_T$.

Note that $\hat{\eta}$ which minimizes the above for positive definite R(t), is given by

$$(2.20)\qquad \hat{\eta} = -R^{-1}(t)Q(t)\xi .$$

Calculate now, with the use of the Ito formula,

$$V(T,\hat{x}^u_T) - V(0,\hat{x}_0) = \int_0^T \Big(\Big(\frac{\partial V(t,\xi)}{\partial t}\Big|_{\xi=\hat{x}^u_t} + \frac{1}{2} h_1^2(t)(P^o_t)^2 \frac{\partial^2 V(t,\xi)}{\partial \xi^2}\Big|_{\xi=\hat{x}^u_t}\Big)dt + \frac{\partial V(t,\xi)}{\partial \xi}\Big|_{\xi=\hat{x}^u_t} d\hat{x}^u_t \Big) .$$

Taking into account (2.8) and (2.19) we obtain

$$V(T,\hat{x}^u_t) - V(0,\hat{x}_0) \geq -\int_0^T ((\hat{x}^u_t)^2 H(t) + u_t^2 R(t))dt + 2\int_0^T \hat{x}^u_t Q(t) P^o_t h_1(t) d\nu^o_t .$$

After taking the expectation of both sides of the above inequality,

$$\text{(2.21)} \qquad V(0,\hat{x}_o) \leq E\Big((\hat{x}^u_T)^2 h_T + \int_o^T ((\hat{x}^u_t)^2 H(t) + u_t^2 R(t))dt\Big)$$

The equality in (2.21) holds, according to (2.20) only if

$$\text{(2.22)} \qquad \tilde{u}_t = -R^{-1}(t)Q(t)\,\hat{x}^{\tilde{u}}_t .$$

Comparing (2.22) with (2.16),

$$J(\tilde{u}) \leq J(u) \text{ for all } u \in U.$$

The admissibility of $\tilde{u}$ defined by (2.22) follows from (2.10) and the fact that

$$E\big(\sup_{0 \leq t \leq T} (\hat{x}^{\tilde{u}}_t)^4\big) < \infty .$$

The above can be proven in the same way as in the derivation of a conditionally Gaussian filter [Lemma 12.1; 4, pp. 18-19]. This ends the proof of the Lemma.

It seems to be possible, (following e.g. [2]) to show that the separation principle holds also for nonquadratic performance functionals.

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