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SIMPLE EXAMPLES OF NON LINEAR TRUSS BEHAVIOR

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Technical Report

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Nonlinear material models are defined to represent elastic/perfectly-plastic and elastic/buckling behavior. A simple three-bar truss is used to demonstrate that under a monotonically increasing prescribed displacement the truss may exhibit reverse stressing or non-uniqueness, and that when two different control displacements are applied the principal of superposition does not hold.

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SIMPLE EXAMPLES OF NONLINEAR TRUSS BEHAVIOR

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1. Introduction. The theory of linear elasticity is a very "nice" mathematical theory with many convenient features such as superposition, uniqueness, and the equivalence of proportional loading to proportional stressing. Theories involving nonlinear material behavior do not necessarily have these nice features, and it is instructive to consider very simple examples which dramatically illustrate this fact.

In 1951 Drucker [1] considered a simple 3-bar truss essentially similar to the one shown in Fig. 1 which is subjected to a single monotonically increasing vertical load corresponding to the load Q. In Drucker's example each bar is made of an elastic/perfectly-plastic material with identical elastic properties but different yield strengths. For a suitable choice of yield strengths, bar 1 will first yield in compression but will then unload and will eventually yield in tension when the truss fails. The present author [2, 3] has made frequent use of trusses similar to the one in Fig. 1.

The present note shows how this truss can be used to discuss two different models of non-linear material behavior in relation to the above-mentioned features of superposition, uniqueness, and proportional stressing.

The two models to be treated are the familiar elastic/perfectly-plastic (E/PP) one illustrated by the dashed curve in Fig. 2, and an idealized elastic buckling (E/B) model indicated by the solid curve. Since we are primarily concerned with compressive stresses, we define the bar shortening and the negative bar force by

$$s_i^* = -L_i \varepsilon_i \quad , \qquad C_i^* = -A_i \sigma_i \qquad (1)$$

The buckling model places no limit on negative values of C_i^* (it would be a trivial extension to construct a combined model wnich yielded in tension and buckled in compression) but an equally important difference in the two models occurs on unloading. At point B, for example, the E/PP model unloads along BCD whereas the E/B model retraces the loading path BAOG.

The next section lists the defining equations for the truss in Fig. 1 according to the two models, and the concluding section defines three specific examples chosen to illustrate reverse stressing under a monotonic load, non-uniqueness, and superposition, in that order. A summary of results are given in Section 3, and some details of the solutions are included in the Appendix.

2. Equations. The cross bar AB is assumed rigid and the three vertical bars all have the same modulus E, area A, and minimum moment of inertia I. We find it convenient to use asterisks to represent physical quantities and define dimensionless variables as indicated below. For both models we define the stiffness of bar 2 by

$$k = AE/H \tag{2}$$

where H is the length of the bar. For the E/B model we use the simple Euler buckling formula for a pinned bar and denote the shortening of bar 2 at the onset of buckling by

$$\bar{s} = \frac{C_{cr}}{k} = \frac{\pi^2 EI}{H^2} \cdot \frac{H}{AE} = \frac{\pi^2 I}{AH} \tag{3}$$

We then define dimensionless displacements v and w, loads P and Q, shortenings s_i, and compressive forces C_i by

$$\begin{array}{lll} v = v^*/\bar{s} & w = w^*/\bar{s} & s_i = s_i^*/\bar{s} \\ P = P^*/k\bar{s} & Q = Q^*/k\bar{s} & C_i = C_i^*/k\bar{s} \end{array} \tag{4}$$

For the E/PP model, let Y be the yield force in bar 2 and define the above variables by the similar formulas

$$\begin{array}{lll} v = kv^*/Y & w = kw^*/Y & s_i = ks_i^*/Y \\ P = P^*/Y & Q = Q^*/Y & C_i = C_i^*/Y \end{array} \tag{5}$$

We denote the length of bar i by $\alpha_i H$ where $\alpha_2 = 1$ and we will choose α_1 and α_3 in each example. In view of the above definitions, the shortening at which bar i buckles is then given by

$$\bar{s}_i = 1/\alpha_i \tag{6}$$

So that the two models will have the same scale in Fig. 2, we choose the yield stress in bar i to be inversely proportional to the square of its length, hence

$$Y_i = 1/\alpha_i^2 \tag{7}$$

We assume that all displacements are small so that the only non-linearity is the material behavior, and the three bars can be taken to remain vertical. The kinematics are given by

$$\Delta s_i = (4\Delta w - 3\Delta v,\ \Delta v,\ 2\Delta v - \Delta w) \tag{8}$$

where we find it convenient to use the incremental form for all equations. The statics are obtained from moment equilibrium about the two loads:

$$\Delta P = -3\Delta C_1 + \Delta C_2 + 2\Delta C_3 \tag{9a}$$

$$\Delta Q = 4\Delta C_1 - \Delta C_3 \tag{9b}$$

Finally, the constitutive equation for each bar in the E/PP model may be written

$$\text{IF } (C_i = \pm 1/\alpha_i^2 \text{ AND } C_i \Delta s_i \geq 0)$$

$$\text{THEN } \Delta C_i = 0 \quad \text{ELSE} \quad \Delta C_i = \Delta s_i/\alpha_i \tag{10}$$

For the E/B model the equation is similar but significantly different:

$$\text{IF } s_i > 1/\alpha_i \text{ THEN } \Delta C_i = 0$$

$$\text{ELSE} \quad \Delta C_i = \Delta s_i/\alpha_i \tag{11}$$

In each case the ELSE clause represents elastic behavior and the IF clause represents the idealized inelastic behavior. When all three bars are elastic, the solution for either model may be written

$$\Delta P = \left(\frac{9}{\alpha_1} + 1 + \frac{4}{\alpha_3}\right)\Delta v - \left(\frac{12}{\alpha_1} + \frac{2}{\alpha_3}\right)\Delta w \tag{12a}$$

$$\Delta Q = -\left(\frac{12}{\alpha_1} + \frac{2}{\alpha_3}\right)\Delta v + \left(\frac{16}{\alpha_1} + \frac{1}{\alpha_3}\right)\Delta w \tag{12b}$$

Equations (12) could be solved for Δv and Δw. However, in each of our examples we will apply only one load at a time so that the zero load equation is trivially solved to relate Δv and Δw and the other equation relates load and displacement. In all cases, we will regard one of the displacements as the control variable.

3. Examples. As an example to illustrate unstressing we take $\alpha_i = (2, 1, 4)$, let Q be the only load, and increase the control displacement w under Q from 0 to 3. For the E/PP model this example is essentially similar to Drucker's [1].

The results are shown in Table 1 and Figs. 3 and 4; details are given in the Appendix. The "status" column in Table 1 shows for each bar if it is elastic (E), yielding in tension (T) or compression (C), or buckling (B). A "stage" is the time spent with no bar changing status, stage 1L is the limit of stage 1 as bar 1 reaches yield and changes from E to C, etc.

As shown in Table 1 as w is increased first bar 3 and then bar 1 yield in compression. However, as is clear from Fig. 1, a mechanism motion with bar 2 rigid would require bar 3 to lengthen. Therefore, in stage 3 bar 1 elastically unloads through zero and eventually yields in tension as the yield-point load is reached. The dashed curves in Figs. 3 and 4 respectively, show the load-displacement history and the shortening history of bar 3.

Stages 1 and 2 do not involve unstressing or tensile yield so that they are the same for the E/B model. However, in stage 3 as bar 3 starts to reclaim its buckling deformation, there is no change in its force. Since bar 1 is now buckling C_1 also remains constant. Therefore, equilibrium shows that C_2 and the load are also constant. Thus, as w increases bar AB rotates about the unchanged location of the top of bar 2. However, this situation lasts only until s_3 is reduced to $\bar{s}_3$ at which point it resumes elastic behavior in stage 4. Since tensile yielding is not considered, stage 4 continues until bar 2 reaches buckling. The buckling collapse mechanism of the truss thus involves a rotation about the top of bar 1. The results are shown by the solid curves in Figs. 3 and 4. Notice the finite horizontal portion of the load-displacement curve in stage 3, followed by further increase of load in stage 4.

As a second example to illustrate non-uniqueness,* we consider a truss with a_i = (4, 1, 2) where P is the only load. The control variable v is to be increased from 0 to 1. For the E/PP model the all-elastic stage 1, shown in the top line of Table 2 ends when bars 1 and 3 both reach compressive yield at the same instant. Therefore, in stage 2, $\Delta C_1 = \Delta C_3 = 0$ and since there is no load Q Eq. (9b) becomes an identity. Therefore, the only information about Δw comes from (8) and the inequalities in (10) for bars 1 and 3:

*Other simple examples of non-uniqueness are discussed in [4] and [5].

$$\Delta s_1 = 4\Delta w - 3\Delta v \geq 0 \qquad \Delta s_3 = 2\Delta v - \Delta w \geq 0 \qquad (13)$$

These must apply for any infinitesimal increment in stage 2, which leads to

$$3/4 \leq dw/dv \leq 2 \qquad (14)$$

as shown in the last column for stage 2. During this stage the load P must increase with v and the bar forces are all unique, but w may take any value permitted by (14). However, we note that once w has been established for any particular v, the restrictions (14) apply from that point. Therefore, although not unique as v is increased, any solution reached is stable if v is held constant at any time.

Figure 5 shows the strain-path trajectory of bars 1 and 3. It is unique along OA in stage 1, but during stage 2 it may follow any path with positive slope in the domain ABC. In particular, at stage 2L it may have reached any point on the line BC. However, if the solution at v = 3/4 is observed to be at point D, say, then the possible solutions when v = 1 are restricted to the segment EF.

For the E/B model the equations and unique part of the results are exactly the same, so that we have not repeated them in Table 2. However, the inequalities apply to the total shortening rather than instantaneous increments, so that (13) and (14) must be replaced by

$$s_1 \geq \bar{s}_1 = 1/4 \qquad s_2 \geq \bar{s}_2 = 1/2$$

$$(3/4)v + 1/16 \leq w \leq 2v - 1/2 \qquad (15)$$

Not only is the solution for w not unique, it is only neutrally stable and could change from one value to another with no change in v. Thus, at stage 2L it could be any point on BC regardless of its earlier values at $v = 3/4$.

The final example discusses superposition. We return to the first truss where $u_i = (2, 1, 4)$ and control both v and w, increasing them from zero to $w = 1/4$, $v = 1$. For the E/PP model the final loads depend upon the order in which the displacements are increased. Suppose that they are applied in the order v, w; i.e., v is increased to 1 with w held at zero, then v is held at 1 while w is increased to 1/4. The light solid curve ABCDE shows the history of load Q; P would have a similar curve. Complete results may be found in the Appendix. The final state when $v = 1$, $w = 1/4$ is shown in line 1 of Table 3.

On the other hand, if w is first increased to 1/4 and then v is increased to 1, the history of Q is given by the heavy solid curve AFGHIJ in Fig. 6. Not only is the history quite different, but the final load values (points E and J) do not even agree in sign. Line 2 of Table 3 lists all final values.

If the principle of superposition were valid, we could find two components corresponding to v-only and w-only displacements and then add them. Lines 3 - 5 in Table 3 show the final results, and we see that line 5 is quite different than either lines 1 or 2. In terms of Fig. 6, we could add the w-only solution AFG to the v-only solution ABCD by translating the first curve so that A is at D. The resulting curve ABCDNM has its new part shown light dashed.

The order of superposition, of course, does not matter. If v-only is added to w-only, the curve AFGKLM has exactly the same terminal point M.

For the E/B model the shortenings, bar forces, and loads are all unique functions of the instantaneous displacements, so that curves ABCD and AFGHD in Fig. 7 both end at the same state D. The complete solution at the final point $v = 1$, $w = 1/4$ is shown in line 6 of Table 3. Observe that the loads are very much different from those required by the E/PP model, essentially because the latter had bar 1 with substantial yielding in tension.

However, as shown by lines 7, 8, 9 in Table 3 and by curves ABCL or AFKL in Fig. 7, the results of superposition still do not agree with the actual solution.

APPENDIX

We present here some of the intermediate steps of the examples in Section 3. During a stage in which all bars are elastic, it follows from Eqs. (8) and the "ELSE" part of (10) or (11) that

$$\Delta C_i = \left(\frac{4\Delta w - 3\Delta v}{\alpha_1},\ \Delta v,\ \frac{2\Delta v - \Delta w}{\alpha_3}\right) \tag{A1}$$

Further, if any bar is inelastic, the corresponding component of ΔC_i is replaced by zero. Therefore, we can immediately write an explicit expression for ΔC_i in terms of Δw and Δv for any stage of loading.

Example 1. In this example

$$\alpha_i = (2,\ 1,\ 4) \qquad Y_i = (4,\ 16,\ 1)/16 \qquad \ddot{S}_i = (2,\ 4,\ 1)/4 \tag{A2}$$

so that Eq. (A1) can be written

$$\Delta C_i = (8,\ 0,\ -1)\Delta w/4 + (-3,\ 2,\ 1)\Delta v/2 \tag{A3}$$

The only non-zero load is Q, hence it follows from Eq. (9a)

$$\Delta P = -3\Delta C_1 + \Delta C_2 + 2\Delta C_3 = 0 \tag{A4}$$

Substitution of (A3) or its partially inelastic replacement in (A4) produces an equation which is easily solved for Δv in terms of the control variable Δw. Thus we begin with

Stage 1. EEE

$$\Delta P = (-24-2)\Delta w/4 + (9 + 2 + 2)\Delta v/2 = 0$$

$$\Delta v = \Delta w \tag{A5}$$

For the E/PP model this stage will end when bar 3 reaches compressive yield, i.e., when

$$C_3 = 0 + \Delta w/4 = 1/16 \qquad \Delta w = w = 1/4 \tag{A6}$$

Equations (A5), (A6), and (8) - (11) then determine the complete solution for stage 1L as given in line 1 of Table 1.

For stage 2, Eq. (A5) is replaced by

Stage 2. EEC

$$\Delta C_i = (2,\ 0,\ 0)\Delta w + (-3,\ 2,\ 0)\Delta v/2$$

$$\Delta P = -6\Delta w + (9 + 2)\Delta v/2 = 0 \qquad \Delta v = (12/11)\Delta w \tag{A7}$$

This stage ends when bar 1 yields in compression. Since $C_1 = 1/8$ at the end of stage 1 (see line 1 of Table 1), we have

$$C_1 = (1/8) + (4/11)\Delta w = 1/4 \qquad \Delta w = 11/32 \tag{A8}$$

Equations (A7), (A8), and (8) - (11) then give the complete increment solution during stage 2:

$$\Delta v = 3/8 \qquad \Delta s_i = (1/32)(8,\ 12,\ 13)$$

$$\Delta C_i = (1/32)(4,\ 12,\ 0) \qquad \Delta Q = 1/2 \tag{A9}$$

When these are added to the stage 1L values in line 1 of Table 1, we obtain the complete solution at stage 2L as shown in line 2.

At this point it might appear logical to assume

Stage 3X. CEC

$$\Delta C_i = (0,\ 1,\ 0)\Delta v$$

$$\Delta P = \Delta v = 0 \qquad \Delta s_i = (4,\ 0,\ -1)\Delta w \tag{A10}$$

However, since Δw is positive, we would be predicting $\Delta s_3 < 0$ which is not consistent with the assumption of yielding. Therefore, instead we write

Stage 3. CEE

$$\Delta C_i = (0,\ 0,\ -1)\Delta w/4 + (0,\ 2,\ 1)\Delta v/2$$

$$\Delta P = -\Delta w/2 + (2+2)\Delta v/2 = 0 \qquad \Delta v = \Delta w/4 \qquad \text{(A11)}$$

This stage ends when bar 3 yields in tension:

$$C_3 = 1/16 - \Delta w/8 = -1/16$$

$$\Delta w = 1 \qquad \Delta v = 1/4 \qquad \Delta s_i = (13,\ 1,\ -2)/4$$

$$\Delta C_i = (0,\ 2,\ -1)/8 \qquad \Delta Q = 1/8 \qquad \text{(A12)}$$

Addition of these values to those in line 2 of Table 1 produces the values in line 3.

In the final stage

Stage 4. CET

$$C_i = (0,\ \Delta v,\ 0)$$

$$\Delta P = \Delta v = \Delta C_i = 0 \qquad \Delta s_i = \Delta w(4,\ 0,\ -1) \qquad \text{(A13)}$$

which is the collapse mechanism about the end of bar 2. Since our program calls for w to increase to 3, we set $\Delta w = 45/32$ in Eqs. (A13) and add to line 3 to get line 4 in Table 1.

For the E/B model, Eqs. (A5) through (A9) are still applicable hence the solution is exactly the same through stage 2L.

However, since $s_3 = 21/32$ is beyond the buckling value $\bar{s}_3 = 1/4$, Eq. (A10) is a valid description with bar 3 unstressing, but remaining buckled, until

$$s_3 = 21/32 - \Delta w = 1/4 \qquad \Delta w = 13/32$$

$$\Delta v = \Delta P = \Delta Q = \Delta C_i = 0 \qquad \Delta s_i = (13/32)(4,\ 0,\ -1) \tag{A14}$$

Addition of these values to line 2 gives line 5 in Table 1.

Bar 3 now resumes elastic behavior, hence stage 4 is described by Eq. (A8). Since tensile yield is not considered, this stage continues until bar 2 buckles:

$$s_2 = 5/8 + \Delta w/4 = 1 \qquad \Delta w = 5/2 \qquad \Delta v = 5/8$$

$$\Delta C_i = 1/16(0,\ 6,\ -3) \qquad \Delta Q = 3/16$$

$$\Delta s_i = (1/8)(39,\ 3,\ -6) \tag{A15}$$

Addition of (A15) and line 5 of Table 1 produces line 6.

In the final stage

Stage 5. BBE

$$\Delta C_i = (0,\ 0,\ 2\Delta v - \Delta w)/4$$

$$\Delta P = \Delta v - \Delta w/2 = 0 \qquad \Delta v = \Delta w/2$$

$$\Delta Q = \Delta C_i = 0 \qquad \Delta s_i = (5,\ 1,\ 0)\Delta w/2 \tag{A16}$$

This is a mechanism motion of rotation about bar 3 and will continue until w reaches its final value of 3. Thus line 7 of Table 1 is obtained by setting $\Delta w = 1/2$ in (A16) and adding the result to line 6.

Example 2. In this example Eqs. (A2) - (A4) are replaced by

$$\alpha_i = (4,\ 1,\ 2) \qquad \gamma_i = (1,\ 16,\ 4)/16$$

$$\bar{s}_i = (1,\ 4,\ 2)/4 \tag{A17}$$

$$\Delta Q = 4\Delta C_1 - \Delta C_3 = 0 \tag{A18}$$

$$\Delta C_i = (2,\ 0,\ -1)\Delta w/2 + (-3,\ 4,\ 4)\Delta v/4 \tag{A19}$$

Therefore, we can find Δw in terms of the control variable Δv. When all bars are elastic, we use (A19) and obtain

Stage 1. EEE

$$\Delta Q = (8 + 1)\Delta w/2 + (-12 - 4)\Delta v/4 = 0$$

$$\Delta w = (8/9)\Delta v \tag{A20}$$

For the E/PP model this stage ends when bars 1 and 3 reach yield simultaneously at $v = \Delta v = 9/20$ which leads to the values in line 1 of Table 2.

In stage 2 bars 1 and 3 are both yielding hence

Stage 2. CEC

$$\Delta C_i = (0,\ 1,\ 0)\Delta v$$

$$\Delta Q = 0 \qquad \Delta P = \Delta v$$

$$\Delta s_i = \Delta w(4,\ 0,\ -1) + \Delta v(-3,\ 1,\ 2) \tag{A21}$$

The only information available for Δw is in the Inequalities (13). This stage ends when the remaining bar yields:

$$C_2 = 9/20 + \Delta v = 1 \qquad \Delta v = 11/20 \tag{A22}$$

With this value of Δv, the sum of lines 1 and 2 in Table 3 gives line 3.

Example 3. In these final examples the truss properties are again given by (A2) and the fully elastic force-increment solution by (A3). Both v and w are controlled, and we will find it convenient to express the kinematic and static equations (8) and (9) in integrated form:

$$s_i = (4,\ 0,\ -1)w + (-3,\ 1,\ 2)v \tag{A23}$$

$$Q = 4C_1 - C_3$$

$$P = -3C_1 + C_2 + 2C_3 \tag{A24}$$

Thus, given the control variables, (A23) gives an explicit expression for the shortenings. Using (A2) or its non-elastic replacement, we find the force increments, but we can wait and use the integrated forces to find the loads directly from (A24).

In example 3A we first increase v to 1 and then increase w to 1/4; in 3B we reach the same final values by increasing first w and then v.

The computations fall into a simple pattern and are conveniently presented in tabular form in Table 4. We consider first example 3A for an E/PP material. The zeroes in the unnumbered top line emphasize that we start from the zero state and obtain the solution by finite increments.

In stage 1 all bars are elastic and v is the active control variable. We denote its increment from zero by Δ in line 1, column 1, and express the bar-force increments in terms of Δ in column 3. In this incremental stage we do not need the items in columns 2 or 4. Stage 1 will terminate when bar 3 reaches its yield value in compression. The resulting equation for the increment Δ is written in column 5, and its solution is entered in column 6.

The solution at stage 1L is then written in line 2. Values for the displacements in column 1 and bar forces in column 3 are obtained by putting the value $\Delta = 1/8$ in line 1 and adding the result to the zero values at the beginning of stage 1. Although we could treat the shortenings s_i and loads Q and P in the same incremental fashion, it is more convenient to use (A23) and (A24) and complete columns 2 and 4 directly from the w, v, and C_i values in line 2.

Line 3 treats the next increment in the same way. In column 3, ΔC_1 and ΔC_2 still have their elastic values from (A3), but $\Delta C_3 = 0$ since it is already yielding. Stage 2 terminates when bar 1 yields in tension, which leads to the equation in column 5 and the solution $\Delta = 1/24$ in column 6.

For line 4 we set $\Delta = 1/24$ in line 3 and add the result to line 2 to obtain values of w, v, and C_i. Again, the values of s_i, Q, and P are obtained directly from (A23) and (A24).

Lines 5 and 6 contain the solution for stages 3 and 3L figured in the same way. No bar changes state before v reaches its final value, hence the equation in column 5 is simply v = 1. Line 6 is also recorded as line 3 in Table 3 as the v-component for "superposition".

In stage 4 control switches to w. Since the elastic solution for w increasing produces a positive ΔC_1 and negative ΔC_3, both plastic bars now unload and all three bars are elastic in line 7. Now w reaches its final value with no bar changing state so the equation in column 5 is w = 1/4. Line 8 gives the final values for example 3A; the same values are recorded in line 1 of Table 3.

Example 3B for the E/PP material is treated in exactly the same way. The details are displayed in lines 9-18 of Table 4, and the final results from line 18 are recorded as line 2 of Table 3. Also, line 12 of Table 4 is repeated as line 4 of Table 3 for the w-component for "superposition".

For the E/B model and example 3A, stages 1, 1L, and 2 are the same as for the E/PP model. However, since the E/B model does not have any constraint on allowable tensile loads, the terminating condition is

$$v = 1/8 + \Delta = 1 \qquad \Delta = 7/8 \tag{A25}$$

Substitution of this value in line 3 of Table 4 and addition to line 2 gives the stage 2L solution in line 19 for the E/B model. This line is also written as line 7 of Table 3.

Since bar 3 has undergone a finite amount of buckling, it will remain in the buckled state in stage 3 as shown in line 20 of Table 4. In fact, it is still buckled when w reaches its final value of 1/4. The final values at stage 3L are listed in line 21 of Table 4 and line 6 of Table 3. From column 2 and the last Eq. (A2) we verify that bar 3 is still well within the buckled state, bar 1 is in tension, and bar 2 is on the verge of buckling.

In example 3B the two models are exactly the same through stage 2L. In stage 3 bar 1 will remain in the buckled state until its shortening reduces to $\bar{s}_1$. To determine buckling or unbuckling criteria in this example it is necessary to have the Δs_1 expressions available as given for example in line 22 column 2. In this case the stage ends when bar 1 unbuckles at $\delta = 1/6$.

Line 23 shows the solution at stage 3L. The shortening in column 2 is, of course, the same whether it is computed by adding the increment in line 22 to the value at stage 2L in line 12, or directly from Eq. (A23).

The rest of the table is completed in the same fashion. As noted, lines 27 and 21 are identical for this model.

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Line	Model	Stage	Status	w	v	Q	$16s_i$			$16C_i$		
1	E/PP	1L	EEE	1/4	1/4	7/16	4	4	4	2	4	1
2		2L	EEC	19/32	5/8	15/16	8	10	10.5	4	10	1
3		3L	CEE	51/32	7/8	17/16	60	14	2.5	4	14	-1
4		4L	CET	3	7/8	17/16	150	14	-20	4	14	-1
5	E/B	3L	BEB	1	5/8	15/16	34	10	4	4	10	1
6		4L	BEE	5/2	1	9/8	112	16	-8	4	16	-2
7		5L	BBE	3	5/4	9/8	132	20	-8	4	16	-2

TABLE 1

UNSTRESSING

Control: w (P = 0) α_i = (2,1,4)

Line	Model	Stage	Status	v	P	C_i			(w)
1	E/PP	1L	EEE	9/20	61/80	1/16	9/20	1/4	$w = 2/5$
2		2	CEC	Δv	Δv	0	Δv	0	$3/4 \le dw/dv \le 2$
3		2L	CCC	1	21/16	1/16	1	1/4	$13/16 \le w \le 3/2$

TABLE 2

NONUNIQUENESS

Control: v (Q = 0) α_i = (4,1,2)

Line	Model	History	w	v	Q	P	C_i		
1	E/PP	v then w	1/4	1	1	1/4	1/4	1	0
2		w then v	1/4	1	-17/16	15/8	-1/4	1	1/16
3		w = 0	0	1	-17/16	15/8	-1/4	1	1/16
4		v = 0	1/4	0	17/16	-7/8	1/4	0	-1/16
5		Superposition	1/4	1	0	1	0	1	0
6	E/B	Actual	1/4	1	-65/16	33/8	-1	1	1/16
7		w = 0	0	1	-97/16	45/8	-3/2	1	1/16
8		v = 0	1/4	0	17/16	-7/8	1/4	0	-1/16
9		Superposition	1/4	1	-5	19/8	-5/4	1	0

TABLE 3

SUPERPOSITION

Controls: v,w a_i = (2,1,4)

Line	Stage	Status	(1) w	(1) v	(2) ai	(3) Ci	(4) Q	(4) P	(5) Terminate	(6) Δ
Ex 3A (E/PP)			0	0	0	0	0	0		
1	1	EEE	0	Δ		(-3,2,1)Δ/2			$C_3 = 0 + \Delta/2 = 1/16$	1/8
2	1L		0	1/8	(-3,1,2)/8	(-3,2,1)/16	-13/16	13/16		
3	2	EEC	0	Δ		(-3,2,0)Δ/2			$C_1 = -3/16 - 3\Delta/2 = -1/4$	1/24
4	2L		0	1/6	(-3,1,2)/6	(-12,16,3)/48	-17/16	25/24		
5	3	TEC	0	Δ		(0,2,0)Δ/2			$v = 1/6 + \Delta = 1$	5/6
6	3L		0	1	(-3,1,2)	(-4,16,1)/16	-17/16	15/8		
7	4	EEE	Δ	0		(8,0,-1)/4			$w = 0 + \Delta = 1/4$	1/4
8	4L		1/4	1	(-8,4,7)/4	(1,4,0)/4	1	1/4		
Ex 3B (E/PP)			0	0	0	0	0	0		
9	1	EEE	Δ	0		(8,0,-1)Δ/4			$C_1 = 0 + 2\Delta = 1/4$	1/8
10	1L		1/8	0	(4,0,-1)/8	(8,0,-1)/32	33/32	-13/16		
11	2	CEE	Δ	0		(0,0,-1)Δ/4			$w = 1/8 + \Delta = 1/4$	1/8
12	2L		1/4	0	(4,0,-1)/4	(4,0,-1)/16	17/16	-7/8		
13	3	EEE	0	Δ		(-3,2,1)Δ/2			$C_3 = -1/16 + \Delta/2 = 1/16$	1/4
14	3L		1/4	1/4	(1,1,1)/4	(-2,4,1)/16	-9/16	3/4		
15	4	EEC	0	Δ		(-3,2,0)Δ/2			$C_1 = -1/8 - 3\Delta/2 = -1/4$	1/12
16	4L		1/4	1/3	(0,4,5)/12	(-12,16,3)/48	-17/16	29/24		
17	5	TEC	0	Δ		(0,2,0)Δ/2			$v = 1/3 + \Delta = 1$	2/3
18	5L		1/4	1	(-8,4,7)/4	(-4,16,1)/16	-17/16	15/8		
Ex 3A (E/B)										
19	2L		0	1	(-3,1,2)	(-24,16,1)/16	-97/16	45/8		
20	3	EEB	Δ	0		(2,0,0)Δ			$w = 0 + \Delta = 1/4$	1/4
21	3L		1/4	1	(-8,4,7)/4	(-16,16,1)/16	-65/16	33/8		
Ex 3B (E/B)										
22	3	BEE	0	Δ	(-3,1,2)Δ	(0,2,1)Δ/2			$a_1 = 1-3\Delta = 1/2$	1/6
23	3L		1/4	1/6	(6,2,1)/12	(12,8,1)/48	47/48	-13/24		
24	4	EEE	0	Δ	(-3,1,2)Δ	(-3,2,1)Δ/2			$a_3 = 1/12 + 2\Delta = 1/4$	1/12
25	4L		1/4	1/4	(1,1,1)/4	(6,12,3)/48	7/16	0		
26	5	EEB	0	Δ	(-3,1,2)Δ	(-3,2,0)Δ/2			$v = 1/4 + \Delta = 1$	3/4
27	5L		1/4	1	(-8,4,7)/4	(-16,16,1)/16	-65/16	33/8		

TABLE 4

SUPERPOSITION DETAILS

Controls: v,w $\qquad a_i = (2,1,4)$

LIST OF FIGURES

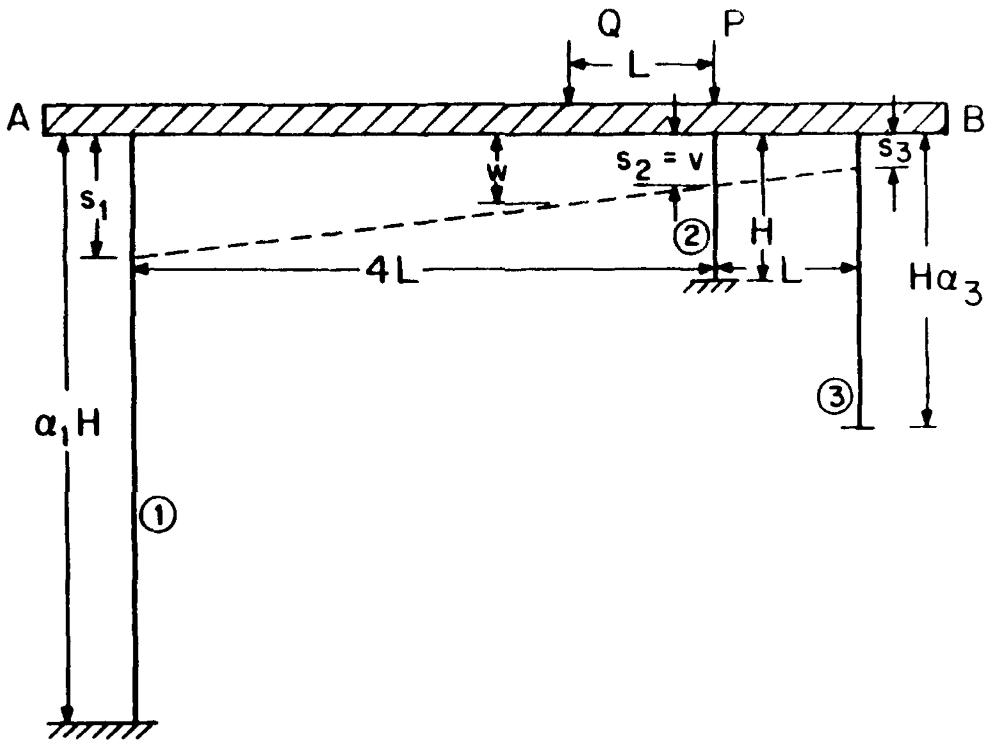
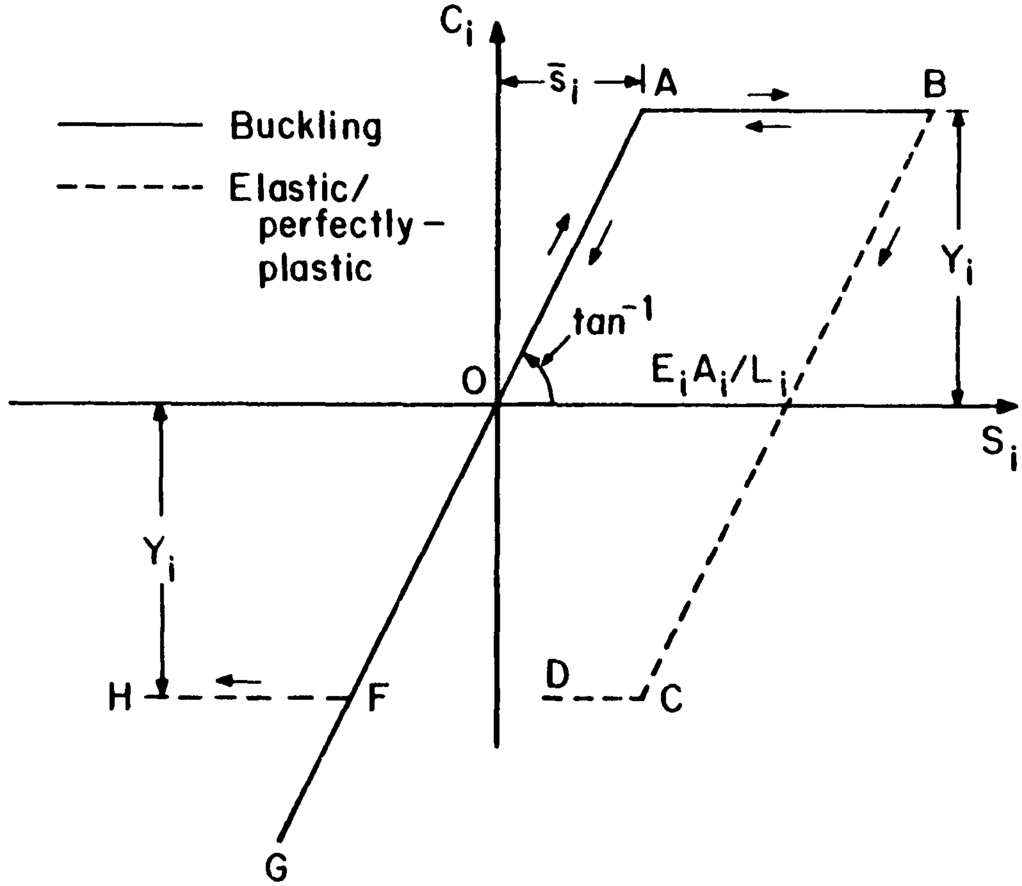


Figure 1



Buckling
Elastic/ perfectly- plastic
C_i
s̄_i
A
B
Y_i
tan^-1
E_iA_i/L_i
O
S_i
Y_i
H
F
D
C
G

Figure 2

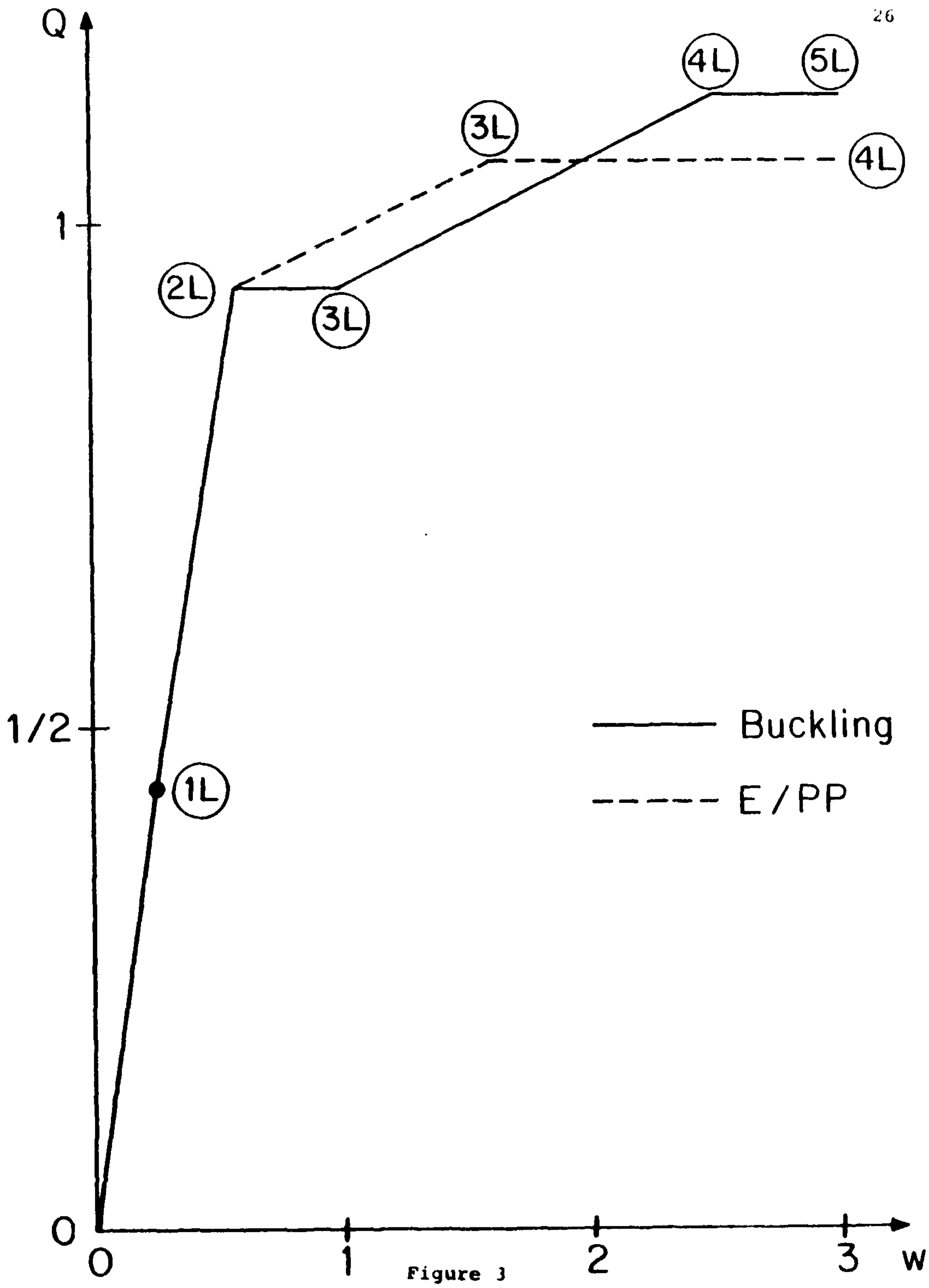


Figure 3

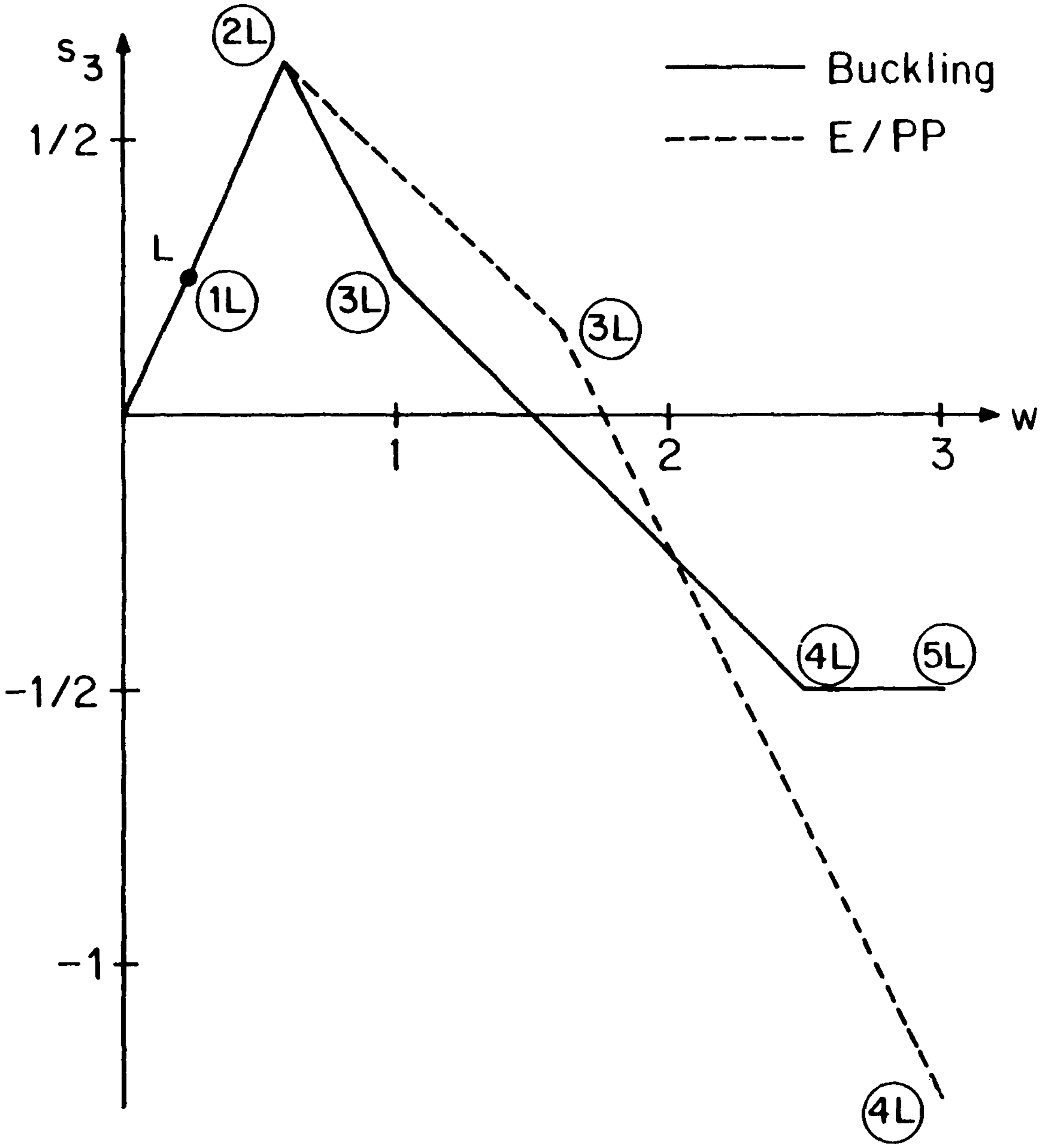


Figure 4

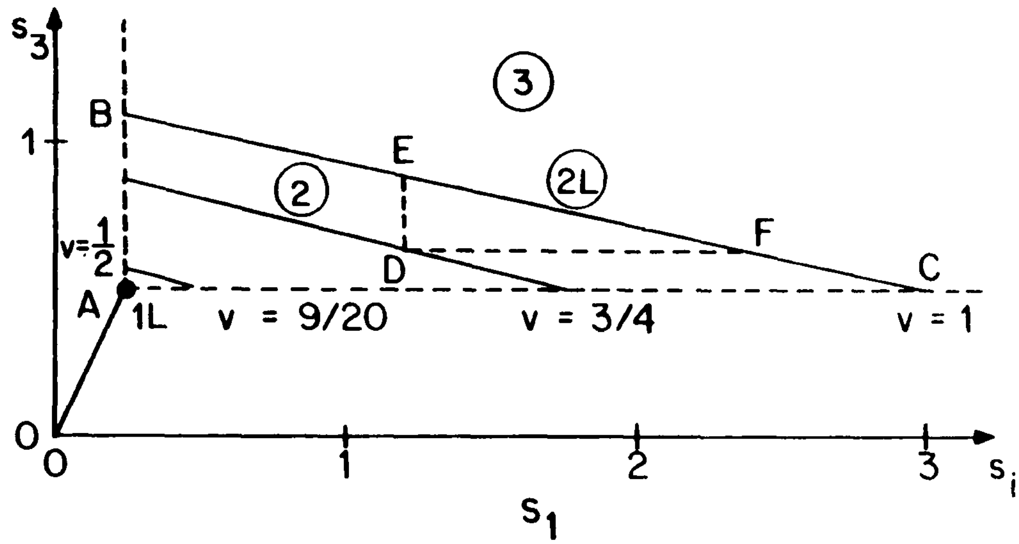


Figure 5

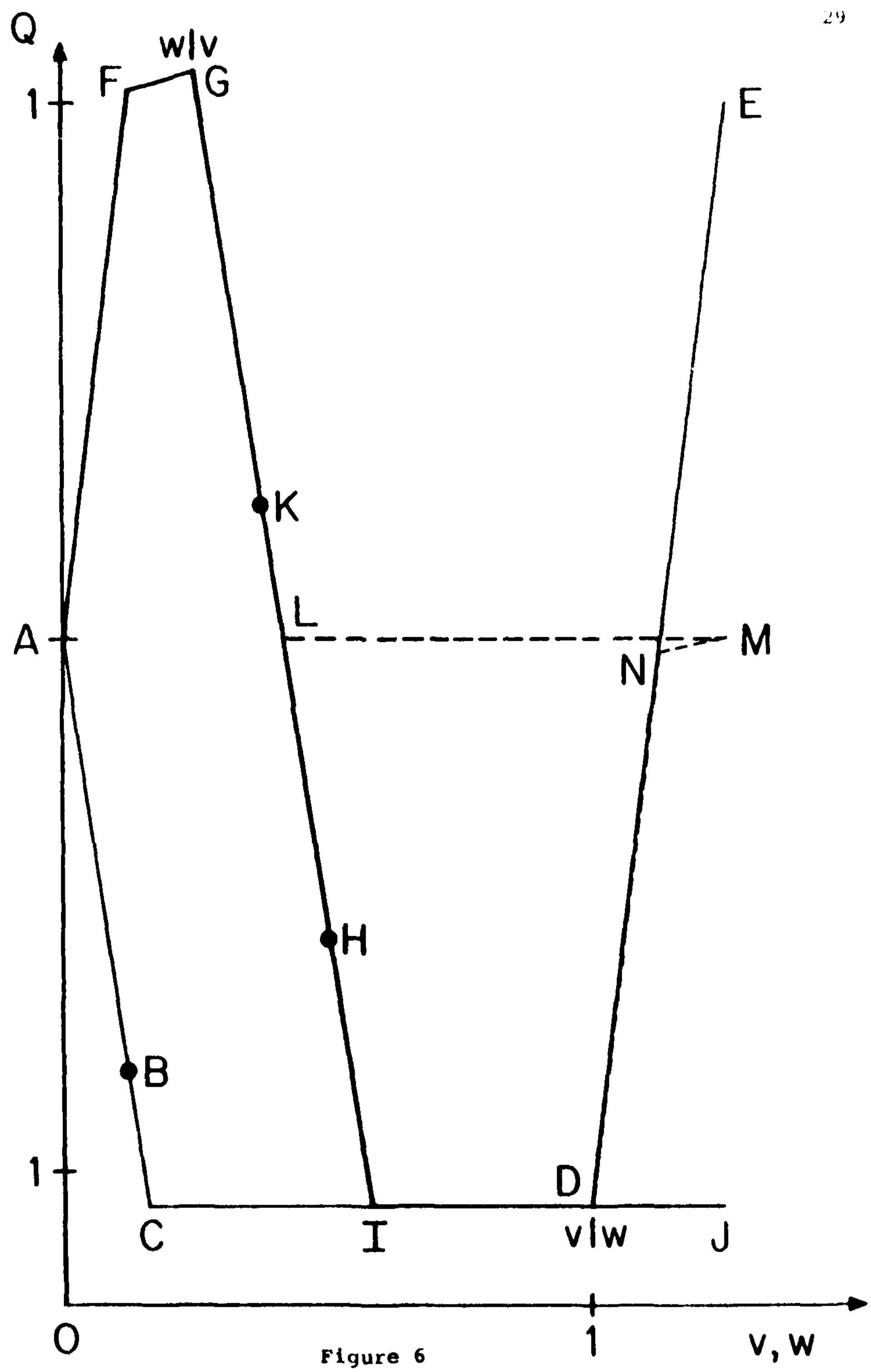


Figure 6

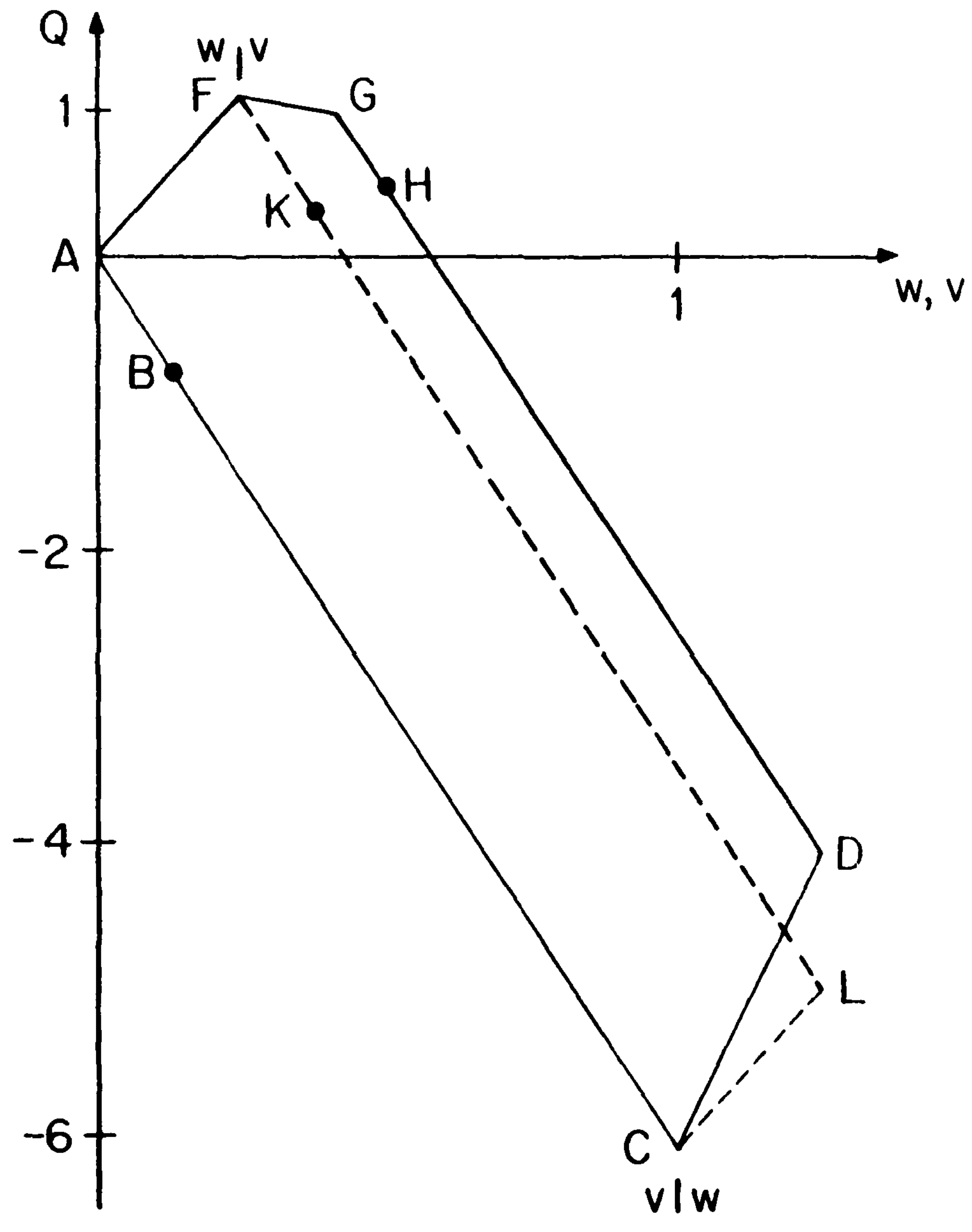


Figure 7