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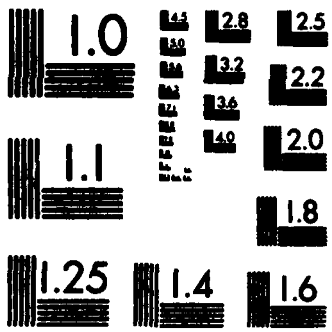

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A SPLINE-BASED APPROXIMATION METHOD FOR INVERSE PROBLEMS FOR A HYPERBOLIC SYSTEM INCLUDING UNKNOWN BOUNDARY PARAMETERS

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Summary

~~We discuss~~ a method for the estimation of unknown parameters (variable as well as constant) occurring in a hyperbolic system, in the context of a seismic application. ~~We~~ present both theoretical results and some numerical (test) examples.

Introduction

We have developed a numerical algorithm, and a corresponding convergence theory to solve a one dimensional "seismic" inverse problem. The response in certain classes of seismic experiments can be modeled by the following hyperbolic partial differential equation with associated boundary and initial conditions:

$$q_1(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(q_2(x)\frac{\partial u}{\partial x}\right) \qquad t > 0,\ x \in [0,1]$$

$$\frac{\partial u}{\partial x}(t,0) + k_1 u(t,0) = s(t;\tilde{k})$$

$$\frac{\partial u}{\partial t}(t,1) + k_2\frac{\partial u}{\partial x}(t,1) = 0 \tag{1}$$

$$u(0,x) = \phi(x)$$

$$\frac{\partial u}{\partial t}(0,x) = \psi(x).$$

Here x represents depth below the surface of the earth, u represents displacement, $q_1(x)$ is the mass density of the medium (in the most general case an unknown), and $q_2(x)$ is an unknown elastic modulus. The boundary condition at the surface $(x=0)$ is an elastic boundary condition involving the unknown (negative constant) k_1, and an unknown source term $s(t;\tilde{k})$. For our treatment of the problem it is not necessary to assume that s is an impulse. In the numerical examples presented below, it has been assumed that $s(t;\tilde{k})$ has a known form, and only the unknown $\tilde{k}$ (a constant vector in $\mathbb{R}^k$) is to be identified, but this also is not essential. We show for example, that $q_2(x)$ (similar remarks are valid for s) can be identified as a function without a priori knowledge of its shape. The ideas in this case are similar to those in [2] where problems with coefficients which are unknown functions of both space and time are discussed for parabolic equations. At the bottom boundary $(x=1)$, an absorbing boundary condition is imposed, involving an unknown (positive constant) k_2. The purpose of this condition is to limit the interval of computation without producing artificial reflections; it allows down-going waves to pass through the boundary undisturbed, while annihilating up-going waves.

We assume we have data observations, $\hat{y}_i$, corresponding to $u(t_i,0)$, a solution of (1) evaluated at the surface. The inverse, or identification problem consists of minimizing a least-squares function $J(q) \equiv \sum_{i=1}^{n}|\hat{y}_i - u(t_i,0;q)|^2$ over an appropriately chosen constraint set Q. Here $(t,x) \to u(t,x;q)$ is the solution of (1) corresponding to $q(x) \equiv (q_1(x), q_2(x), k_1, k_2, \tilde{k})$. We follow the general approach developed in [3] and [1]; we first reformulate the identification problem in an abstract setting, then define a sequence of approximate finite dimensional identification problems, the solution of which generate parameter estimates which converge to a solution of the original identification problem.

Convergence

Motivated by the fact that our differential equation can be written as a system using the variables (u,u_t), we define a Hilbert space $X(q) \equiv V(q) \times L^2(q)$ where $V(q)$ is $H^1(0,1)$ with inner product $\langle v,w\rangle_{V(q)} \equiv \int_0^1 q_2 DvDw dx - q_2(0)k_1 v(0)w(0)$, and $L^2(q)$ is $H^0(0,1)$ with inner product $\langle v,w\rangle_{0,q} \equiv \int_0^1 q_1 vw dx$. The $X(q)$ inner product is then given by $\langle x,y\rangle_q \equiv \langle x_1,y_1\rangle_{V(q)} + \langle x_2,y_2\rangle_{0,q}$ where $x = (x_1,x_2)^T$, $y = (y_1,y_2)^T$. After a straightforward transformation to a system with homogeneous boundary conditions, system (1) can be rewritten in $X(q)$ as

$$\dot{z}(t) = A(q)z(t) + G(t;q)$$
$$z(0) = z_0(q) \tag{2}$$

where $z(t) \in X(q)$ is identified with $\begin{pmatrix} u(t,\cdot) \\ u_t(t,\cdot)\end{pmatrix}$, the boundary conditions are incorporated into the domain of the operator $A(q)$ by defining $V_B(q) = \{v \in V(q) \cap H^2(0,1) \mid Dv(0) + k_1 v(0) = 0\}$ and $\operatorname{dom} A(q) = \left\{\begin{pmatrix} u \\ v\end{pmatrix} \in V_B(q) \times H^1(0,1) \mid v(1) + k_2 Du(1) = 0\right\}$, and $A(q)$ is the unbounded linear operator defined by

$$A(q) = \begin{pmatrix} 0 & I \\ (1/q_1)D(q_2D) & 0 \end{pmatrix}.$$

With $X(q)$ and $A(q)$ so chosen, for each q $A(q)$ is a dissipative operator in $X(q)$ and it can be shown that in fact $A(q)$ is the generator of a C_0-semigroup $T(t;q)$ on $X(q)$. Standard semigroup theory can then be used to show that equation (2) has a unique mild solution:

$$z(t;q) = T(t;q)z_0(q) + \int_0^t T(t-s;q)G(s;q)ds \qquad (3)$$

and the identification problem can be restated as

(ID) Minimize $J(q) \equiv \sum_{i=1}^{n} \left|\hat{y}_i - z_1(t_i)|_{x=0}\right|^2$ over $q \in Q$

subject to $z(\cdot;q)$ satisfying (3), where z_1 represents the first component of z.

Before formulating the approximate identification problems, we first define finite dimensional subspaces $X^N(q)$. Let $S^3(\Delta^N)$ be the subspace of C^2 cubic splines (as in [6], pp. 78-81) corresponding to the partition $\Delta^N = \{x_i\}_{i=0}^N$, $x_i = i/N$, and then define $X^N(q)$ to be that subspace of $S^3(\Delta^N) \times S^3(\Delta^N)$ which satisfies the boundary conditions corresponding to q, i.e., $X^N(q) \subset \operatorname{dom} A(q)$. The space $X^N(q)$ can be expressed as the span of a set of $2N+3$ basis elements, which are straightforward modifications of the standard spline basis elements of $S^3(\Delta^N) \times S^3(\Delta^N)$ (see [5] or [4, p. 38] for details). As a result of these modifications, the new basis elements, and thus the subspaces, depend on the unknown parameter q. It is clear then, that as we iterate on q, these spaces will change.

One assumption we make about the constraint set Q is that each component is uniformly bounded above and below, implying that as q ranges over Q, the $X(q)$ norms will be uniformly equivalent, and hence the spaces $X(q)$ will be equal as sets. With this in mind, let $P_{\bar{q}}^N(q)$: $X(\bar{q}) \to X^N(q)$ be the orthogonal projection of $X(\bar{q})$ onto $X^N(q)$ in the $X(\bar{q})$ norm (for a precise statement of this, one should introduce the canonical isomorphism which associates elements of $X(\bar{q})$ with those in the equivalent space $X(q)$, but to shorten this presentation, we will omit such notation); whenever q and $\bar{q}$ are the same the projection will be written as $P^N(q)$. Define $A^N(q) = P^N(q)A(q)P^N(q)$ and define the approximate system in $X^N(q)$ as

$$\begin{aligned} \dot{z}^N(t) &= A^N(q)z^N(t) + P^N(q)G(t;q) \\ z^N(0) &= P^N(q)z_0(q). \end{aligned} \qquad (4)$$

The operator $A^N(q)$ inherits the dissipativity of $A(q)$, and is also the generator of a C_0-semigroup, $T^N(t;q)$ on $X(q)$. Moreover, we can establish the existence and uniqueness of mild solutions to (4) and write them as

$$z^N(t;q) = T^N(t;q)P^N(q)z_0(q) + \int_0^t T^N(t-s;q)P^N(q)G(s;q)ds. \qquad (5)$$

We now pose the approximate identification problem as

(ID^N) Minimize $J^N(q) \equiv \sum_{i=1}^{n} \left|\hat{y}_i - z_1^N(t_i)|_{x=0}\right|^2$ over

$q \in Q$ subject to $z^N(\cdot;q)$ satisfying (5).

It is important to note that since $X^N(q)$ is a finite dimensional space, (4) is in fact an initial value problem for a system of ordinary differential equations. Furthermore, due to the nature of the spline basis functions, this system possesses desirable numerical properties; for example, the matrix representation for $A^N(q)$ is sparse and banded. We will discuss below a spline representation for $q_2(x)$, which makes solving (4) even more tractable.

The identification problems (ID^N) can be solved using IMSL routines (a Levenberg-Marquardt algorithm for the optimization, Gear's method to solve the differential equations) with a sequence of parameter estimates $\{q^N\}$ thus generated. One then would like to verify that this sequence, or some subsequence thereof, converges to a solution, q^*, of the original problem, (ID).

We use the following version of the Trotter-Kato Theorem.

<u>Theorem</u>: Let $(B,|\cdot|)$ and $(B^N,|\cdot|_N)$, $N = 1,2,\ldots$, be Banach spaces and let $\pi^N : B \to B^N$ be bounded linear operators. Further assume that $T(t)$ and $T^N(t)$ are C_0-semigroups on B and B^N with infinitesimal generators $\tilde{A}$ and $\tilde{A}^N$, respectively. If

(i) $\lim_{N\to\infty} |\pi^N x|_N = |x|$ for all $x \in B$,

(ii) there exist constants M, ω independent of N such that $|T^N(t)|_N \le Me^{\omega t}$, for $t \ge 0$,

(iii) there exists a set $\mathcal{D} \subset B$, $\mathcal{D} \subset \operatorname{dom}(\tilde{A})$, with $\overline{(\lambda_0 - \tilde{A})\mathcal{D}} = B$ for some $\lambda_0 > 0$, such that for all $x \in \mathcal{D}$ we have

$$|\tilde{A}^N\pi^N x - \pi^N\tilde{A}x|_N \to 0 \quad \text{as } N \to \infty$$

then $|T^N(t)\pi^N x - \pi^N T(t)x|_N \to 0$ as $N \to \infty$, for all $x \in B$, uniformly in t on compact intervals in $[0,\infty)$.

We apply this theorem by identifying $(B,|\cdot|)$ with $(X(\bar{q}),|\cdot|_{\bar{q}})$, $(B^N,|\cdot|_N)$ with $(X(q^N),|\cdot|_{q^N})$, $\tilde{A}$ with $A(\bar{q})$, and $\tilde{A}^N$ with $A^N(q^N)$, and we assume $\{q^N\}$ is an arbitrary sequence such that $q^N \to \bar{q}$ in an appropriate sense.

Any version of the Trotter-Kato Theorem would involve some convergence statement about the generators $A^N(q^N)$ and $A(\bar{q})$. As mentioned earlier, the spaces $X^N(q^N)$ and the domains of the approximate operators $A^N(q^N)$ change with q^N. Moreover, elements of $\operatorname{dom} A^N(q^N)$ satisfy the boundary conditions corresponding to q^N, while elements of $\mathrm{dom}\,A(\bar{q})$ satisfy the boundary conditions corresponding to $\bar{q}$, and in general there is no inclusion relation between these sets. This necessitates the use of an operator $\pi^N : X(\bar{q}) \to X(q^N)$ which maps elements of $\mathrm{dom}\,A(\bar{q})$ into those of $\mathrm{dom}\,A(q^N)$ so that it will be possible to compare these elements.

Once the Trotter-Kato Theorem has been used to show the convergence of the semigroups, it can be shown that also the (mild) solutions, $z^N(t;q^N)$ of (4) converge in $X(q^N)$ to the (mild) solution $z(t;\bar{q})$ of (2) (again, a precise statement of this convergence would require the use of the canonical isomorphism) whenever $q^N \to \bar{q}$ in an appropriate sense. With this result and the following theorem (from [4] or [5]) it can be shown that q^* is a solution to the inverse problem.

Theorem: Assume Q is compact in the $C \times H^1 \times \mathbb{R}^{2+k}$ topology. If $q \to z_0(q)$, $q \to P^N(q)z$, $q \to T^N(t;q)z$, $z \in X(q)$ are continuous in this same Q-topology, with the latter uniformly in $t \in [0,T]$, then

(i) there exists for each N a solution $\hat{q}^N$ of (ID^N) and the sequence $\{\hat{q}^N\}$ possesses a convergent subsequence $\hat{q}^{N_k} \to \hat{q}$.

(ii) If we further assume that, for any sequence $\{q^j\}$ in Q with $q^j \to \bar{q}$, we have $z^j(t;q^j) \to z(t;\bar{q})$ as $j \to \infty$ uniformly in $t \in [0,T]$, then $\hat{q}$ is a solution of (ID).

The proofs and details of all the results stated above can be found in [5] and are variations of the general framework developed in [3].

Estimation of Functional Coefficients

If we make further smoothness assumptions on the variable coefficients to be estimated, and stronger compactness assumptions on Q, we can search for an approximation to each of these coefficients as a finite linear combination of cubic splines, reducing the infinite dimensional optimization problem to a finite dimensional one. In our numerical examples, we assume for computational ease that $q_1(x) \equiv 1$, and search for $q_2(x)$ in a function space. For notational convenience, we will write Q as $Q_1 \times Q_2 \times Q_3$, so that if $q = (q_1(x), q_2(x), k_1, k_2, \tilde{k}) \in Q$, then $q_1 \in Q_1$, $q_2 \in Q_2$, and $(k_1, k_2, \tilde{k}) \in Q_3$. If we let $I^M(q_2(\cdot))$ denote the interpolate of q_2 in the space of cubic splines $S^3(\Delta^M)$, then following arguments similar to those in [2], we can conclude that $I^M(Q_2)$ has the following representation:

$$I^M(Q_2) = \left\{ q_2^M(x) : [0,1] \to \mathbb{R} \mid q_2^M(x) = \sum_{i=1}^{M+3} c_i B_i^M(x), c_i \in \mathfrak{C}_i \right\}$$

where $\{B_i^M(x)\}$ are the basis functions for $S^3(\Delta^M)$ and each $\mathfrak{C}_i$ is a compact subset of $\mathbb{R}$. Searching for (an approximate) q_2 in $I^M(Q_2)$ is then equivalent to searching for $(c_1, c_2, \ldots, c_{M+3})$ in $\mathfrak{C}_1 \times \mathfrak{C}_2 \times \ldots \times \mathfrak{C}_{M+3} \subset \mathbb{R}^{M+3}$.

Define $Q^M \equiv \{(I^M(q_2), k_1, k_2, \tilde{k}) \mid q_2 \in Q_2, (k_1, k_2, \tilde{k}) \in Q_3\}$. We can fix M and iteratively solve (ID^N) over Q^M to obtain a sequence $\{q^N(M)\}_N$ with $q^N(M) \equiv (q_2^N(M), k_1^N, k_2^N, \tilde{k}^N)$ and $q_2^N(M) \in I^M(Q_2)$. The convergence theorems discussed in the previous section guarantee that this sequence contains a subsequence (relabeled for convenience) such that $q^N(M) \to \hat{q}(M)$, where $\hat{q}(M)$ satisfies $J(\hat{q}(M)) \le J(q)$ for any $q \in Q^M$. That this sequence $\{\hat{q}(M)\}_M$ is in fact a set of good approximations to a solution of (ID) can be argued as follows. Under the proper compactness assumptions about Q, Q^M will also be compact, ensuring the existence of a subsequence (relabeled, if necessary) of $\hat{q}(M)$ such that $\hat{q}(M) \to q^*$ in Q. Under further (although not too restrictive) assumptions, among them that the solution $z(t;q)$ is continuous in q (which can be proved using the Trotter-Kato Theorem), one can prove that q^* is a solution to (ID). This proof involves the compactness of Q and standard spline error estimates such as those found in [7].

Numerical Examples

In the examples to be presented below, the "data" has been generated using an independent finite difference scheme, where known "true" values of the parameters were preassigned. We begin each example with an initial guess q^0 and a choice of N and solve (ID^N) to obtain q^N. We then use this q^N as the initial guess for the next value of N. All examples were produced either on an IBM VM/370 or a CDC 6600.

Example 1: For this example we parameterized q_2 as $q_2(x) = 3/2 + (1/\pi)\tan^{-1}[q_{21}(x - q_{22})]$, where q_{21} and q_{22} are to be estimated. We used $s(t;\tilde{k}) = 0$, and initial conditions $\phi(x) = e^x$, $\psi(x) = -3e^x$. Data points were chosen at x = 0 and fifteen equally spaced time values in (0,1]. We obtained the following results:

	q_{21}^N	q_{22}^N	k_1^N	k_2^N	$J^N(q^N)$
N = 4	5.873	0.503	-0.995	3.005	0.15×10^{-3}
N = 8	5.929	0.497	-1.001	3.001	0.12×10^{-4}
True Values	6.0	0.5	-1.0	3.0	
Start Up (q^0)	5.0	1.0	-2.0	2.0	

Example 2: We added random noise to the data in this example, at a level of about 3%. We searched for q_2 as a constant, we used $s(t;\tilde{k}) = \tilde{k}_1(1-e^{-5t})e^{\tilde{k}_2 t}$, and used zero initial conditions. Data points were chosen at x = 0 and fifteen equally spaced time values in (0,2]. We obtained these results:

	q_2^N	$\bar{k}_1^N$	$\bar{k}_2^N$	k_1^N	k_2^N	$J^N(q^N)$
N = 4	2.887	2.076	-0.996	-2.110	1.009	0.12×10^{-3}
N = 8	2.947	2.032	-1.013	-2.038	1.027	0.17×10^{-3}
True Values	3.0	2.0	-1.0	-2.0	1.0	
Start Up (q^0)	2.0	1.5	-0.5	-1.0	2.0	

Example 3: In this example, we searched for $q_2(x)$ in the space of cubic splines; the true q_2 used was $q_2^*(x) = 1.5 + \tanh[6(x - \frac{1}{2})]$. We used $s(t;\tilde{k}) = 0$, and $\phi(x) = e^x$, $\psi(x) = -3e^x$. We did not search for the boundary parameters in this example; the true values, $k_1^* = -1.0$, $k_2^* = 3.0$ were used. The data points were chosen at seven equally spaced spatial values in [0,1] and three equally spaced time values in (0,1]. Our initial guess for $q_2(x)$ was the constant function $q_2^0(x) = 6.0$. With N = 4 (for the state approximation) and M = 1 (coefficient approximation) we obtained an estimate, q_2^4, for our functional coefficient such that $|q_2^4 - q_2^*|_0 = 0.099$, and $J^4(q^4) = 0.48\times10^{-2}$. We have several spatial observations in this example rather than only the one at the surface; this is more representative of problems that arise in treating data from "bore-hole" type of seismic experiments, in which receivers are located at various points down a well.

Acknowledgments

This work is the outcome of collaboration with Dr. H. T. Banks; some of the findings reported here constitute a part of the results in the author's Ph.D. Dissertation [5]. This research was supported in part by NSF grant MCS-8205335, by AFOSR contract 81-0198 and by ARO contract ARO-DAAG-29-79-C-0161.

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