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The Gain-Spread-Excitation Theorem

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The Gain-Spread-Excitation Theorem

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1.0 INTRODUCTION

The possible utility of storage rings as electron sources for efficient high power free electron lasers has been under study for a number of years. Studies of the storage ring behavior of uniform one dimensional wigglers of the sort employed in the original Stanford free electron laser showed them to be unsuitable for this purpose.^{1,2,3} The limitation arises from the fact that, while electrons passing through the wiggler on the average deliver energy to the laser field, individual electrons may gain or lose energy. As a result, emerging electrons have a larger energy spread than entering electrons. The growth of energy spread has to be damped by the synchrotron radiation, a circumstance which limits the laser power output to be a small fraction of the synchrotron radiation and hence limiting the overall efficiency.

Attempts were made to remedy this situation by introducing axial variations in the wiggler structure, but these proved to be unsuccessful. An understanding of this lack of success was provided by the gain-spread theorem discovered by John Madey⁴ and proved with increasing generality⁵⁻⁸ over the years. This theorem showed induced energy spread to be an intrinsic part of any one dimensional FEL system operating close to a linear regime and hence subject to the efficiency limitation noted above.

The gain expanded free electron laser was proposed as a solution to the above described problems. In its originally proposed form it consisted of a standard transverse FEL wiggler modified so that the wiggler field acquired a transverse gradient which included a non-alternating component. Electrons of different energies were to traverse the wiggler in different transverse positions, with the transverse gradients arranged so as to provide equal transit velocities for all of the electrons. Since the FEL amplification process is based upon a velocity resonance, it was expected that gain could, in this way, be made energy independent. Since the transverse gradient wigglers are two dimensional structures, the proofs of the gain-spread theorem did not apply to them, and the insensitivity of gain to electron energy was thought to make efficient storage ring operation possible.⁹ Subsequent analysis of the uniform transverse gradient wiggler in the form originally proposed showed that while energy independent gain could be achieved, the amplification process acted as a driver for transverse betatron oscillations.¹⁰ Numerical analysis suggested that this would result in limitations for storage ring operation similar to those of one dimensional wigglers. There followed an extended series of investigations of various two dimensional configurations, which, while incorporating the principle of gain expansion, attempted to avoid transverse excitation (i.e., the excitation of betatron oscillations). The systems were complex and

the analyses ultimately numerical, often containing plausible but not well controlled approximations. Some of them produced quite encouraging results, but there was no plausible pattern of behavior with respect to parameter variation, and no physical understanding of what purported to be successful designs was provided by the numerical analyses.

In a recent report,¹¹ Rosenbluth and Wong obtained exact analytic results for one of the configurations that had been studied numerically. These results included exact gain formulas, Manley Rowe relations, and an extension of the gain-spread theorem to include transverse excitations. These results proved to be of considerable assistance in refining the numerical analysis that preceded them.

Previous unreported analyses as well as the above stated result suggested that the gain-spread-excitation theorem for two dimensional systems was of comparable generality to the one dimensional gain-spread theorem. In sections 2, 3, and 4, we shall show this to be the case and will see that it (1) implies the impossibility of designing a two-dimensional wiggler which produces linear gain simultaneously with vanishing lowest order spread and excitation, and (2) a wiggler which produces vanishing lowest order excitation is governed by the gain-spread theorem. In section 5 we

shall discuss the implications of these results for the quasi-linear operation of storage ring FELs and conclude that they constitute a possibly insuperable obstacle to the avoidance of the limitations found in refs. 1, 2, and 3.

2.0 THEORETICAL FRAMEWORK AND STATEMENT OF THE GAIN-SPREAD-EXCITATION THEOREM

As in ref. 6 we begin with a Hamiltonian of general form

$$K(E, t, p, x; z) = K_0(E, t, p, x; z) + K_1(E, t, p, x; z) + \dots \quad (2.1)$$

where z is taken to be the independent variable, and Hamilton's Equations take the form

$$\begin{aligned} \frac{dE}{dz} &= \frac{\partial K}{\partial t} & \frac{dt}{dz} &= -\frac{\partial K}{\partial E} \\ \frac{dp}{dz} &= -\frac{\partial K}{\partial x} & \frac{dx}{dz} &= \frac{\partial K}{\partial p} \end{aligned} \quad (2.2)$$

The time-independent Hamiltonian K_0 describes the motion of electrons in the wiggler in the absence of radiation, and z is the axial coordinate of the wiggler. While we have magnetic fields primarily in mind, electrostatic field may also be present. The treatment will be sufficiently general that K_0 could include the static storage ring fields as well as the wiggler fields and z can represent any conveniently chosen variable (e.g., the azimuthal angle about the center of the storage ring) that describes the progress of the electrons around the ring.

Radiation is taken into account by adding time-dependent terms K_n to the Hamiltonian of order ϵ^n , where ϵ is some small

parameter. The time averages of the K_n are assumed to vanish, and some additional conditions on the time dependence will be specified later.

We designate a one parameter family of orbits (the parameter is E) of K_0 as nominal orbits, represented by $x_0(z,E)$, $p_0(z,E)$, and $t_0(z,E)$ with $t_0(0,E)$ equal to zero. If one were considering the entire storage ring, we would specify these orbits to repeat themselves on performing a complete cycle around the ring, so that they would be periodic in z with period corresponding to the complete cycle. Alternatively, if the wiggler structure were periodic, one might choose them to have the period of the structure. In an ideal gain expanded design one arranges to have t_0 independent of E . Since, however, we wish to be general, we do not specify any additional conditions.

While E is a constant of the motion in the absence of radiation, the presence of radiation will cause it to change. The aim of a gain expanded wiggler design is to arrange things so that a particle entering on a nominal orbit of energy E and changing its energy to E' on traversal of the wiggler emerges on the nominal orbit of energy E' . To assess the extent to which the objective is achieved, one considers deviations from the nominal orbit designated by

$$x_{\beta} = x - x_0, \quad p_{\beta} = p - p_0. \quad (2.3)$$

In the absence of radiation and for small amplitudes, a Courant-Snyder invariant for these variables can be defined which is z -independent and characterizes the extent to which the orbit deviates from the nominal orbit. We refer to this quantity as an action J because we shall construct a canonical transformation which reduces the Hamiltonian to angle action form. In the presence of radiation, both J and E became z -dependent. Designating these z -dependent corrections by E_n , J_n , where n refers to the order in ϵ , we shall show that

$$\langle E_2 \rangle = \frac{1}{2} \frac{\partial}{\partial E_0} \langle E_1^2 \rangle + \frac{1}{2} \frac{\partial}{\partial J_0} \langle E_1 J_1 \rangle \quad (2.4)$$

$$\langle J_2 \rangle = \frac{1}{2} \frac{\partial}{\partial J_0} \langle J_1^2 \rangle + \frac{1}{2} \frac{\partial}{\partial E_0} \langle E_1 J_1 \rangle. \quad (2.5)$$

We refer to these two equations as the gain-spread-excitation (GSE) relations. The indicated averages are taken over time and the action phase.

3.0 TRANSFORMATION TO ANGLE ACTION VARIABLES

Following the pattern of ref. 11, we shall perform a series of canonical transformations all based upon a generating function of the form $G(E, t', p', x; z)$ with

$$x' = \frac{\partial G}{\partial p'},$$

$$p = \frac{\partial G}{\partial x}$$

$$E' = \frac{\partial G}{\partial t'},$$

$$t = \frac{\partial G}{\partial E}$$

$$K' = K + \partial G / \partial z \quad .$$

The transformations to be employed affect the K_n , $n > 1$, only by virtue of the fact that the variables upon which they depend must be expressed in terms of the final canonical variables. We shall therefore keep track of these relations among variables but retain only K_0 in the discussion given below.

Recall that $x_0(z, E)$, $p_0(z, E)$, $t_0(z, E)$ are any conveniently chosen one parameter set of solutions to the equations

of motion generated by K_0 . We begin with

$$G_a(E, t_\beta, p_\beta, x, z) = p_\beta(x - x_0) + xp_0 + t_\beta E \quad (3.2)$$

which yields

$$p_\beta = p - p_0 \quad (3.3)$$

$$x_\beta = x - x_0 \quad (3.4)$$

$$t = t_\beta + x \frac{\partial p_0}{\partial E} - p_\beta \frac{\partial x_0}{\partial E} \quad (3.5)$$

$$E_\beta = E \quad (3.6)$$

$$K_{0a} = K_0(E, p_0 + p_\beta, x_0 + x_\beta, z) + x \frac{\partial p_0}{\partial z} - p_\beta \frac{\partial x_0}{\partial z} \quad (3.7)$$

In order to simplify (3.7), we make a small amplitude expansion of K_0 about the nominal orbit. Hence we write

$$\begin{aligned}
K_o(E, p_o + p_\beta, x_o + x_\beta, z) &= K_{oo} + x_\beta \frac{\partial K_{oo}}{\partial x_o} + p_\beta \frac{\partial K_{oo}}{\partial p_o} + \\
&+ \frac{1}{2} \frac{\partial^2 K_{oo}}{\partial x_o^2} x_\beta^2 + \frac{\partial^2 K_{oo}}{\partial x_o \partial p_o} x_\beta p_\beta + \\
&+ \frac{1}{2} \frac{\partial^2 K_{oo}}{\partial p_o^2} p_\beta^2 + \dots \approx \\
&\approx K_{oo} - x_\beta \frac{\partial p_o}{\partial z} + p_\beta \frac{\partial x_o}{\partial z} + \frac{1}{2} A x_\beta^2 + \\
&+ B x_\beta p_\beta + \frac{1}{2} C p_\beta^2
\end{aligned} \tag{3.8}$$

where $K_{oo} = K_o(E, p_o, x_o, z)$. Although we have omitted higher order terms in x_β , p_β in (3.8), we shall not do so in what follows. This is justified because the higher order terms may vanish or make smaller contributions than higher order terms which will appear subsequently.¹¹ For compactness we have designated the second derivatives by A , B , C . They are to be thought of as determined functions of E and z . We assume A to be non-vanishing and positive. To complete the first transformation, we combine (3.7) and (3.8) and obtain

$$K_{oa} = \frac{1}{2} A p_{\beta}^2 + B x_{\beta} p_{\beta} + \frac{1}{2} C x_{\beta}^2 + K_{oo} + x_o \frac{\partial p_o}{\partial z} \quad (3.9)$$

Next we choose

$$G_b = \hat{p}_{\beta} x_{\beta} + \hat{t}_{\beta} E - \int^E \left(x_o \frac{\partial p_o}{\partial E} - t_o \right) dE' \quad (3.10)$$

which yields

$$p_{\beta} = \hat{p}_{\beta}$$

$$\hat{x}_{\beta} = x_{\beta}$$

$$\hat{E}_{\beta} = E$$

$$t = \hat{t}_{\beta} + t_o + x_{\beta} \frac{\partial p_o}{\partial E} - p_{\beta} \frac{\partial x_o}{\partial E} \quad (3.11)$$

Using the equations of motion, we see that

$$\begin{aligned}
\frac{\partial}{\partial z} \left(x_o \frac{\partial p_o}{\partial E'} - t_o \right) &= \frac{\partial x_o}{\partial x} \frac{\partial p_o}{\partial E'} + \frac{\partial}{\partial E'} \left(x_o \frac{\partial p_o}{\partial z} \right) - \frac{\partial x_o}{\partial E'} \frac{\partial p_o}{\partial z} - \frac{\partial t_o}{\partial z} \\
&= \frac{\partial}{\partial E'} \left(x_o \frac{\partial p_o}{\partial z} \right) + \frac{\partial K_{oo}}{\partial p_o} \frac{\partial p_o}{\partial E'} + \frac{\partial K_{oo}}{\partial x_o} \frac{\partial x_o}{\partial E'} + \\
&\quad + \left(\frac{\partial K_{oo}}{\partial E'} \right)_{p_o, x_o} \\
&= \frac{\partial}{\partial E'} \left(K_{oo} + x_o \frac{\partial p_o}{\partial z} \right) . \tag{3.12}
\end{aligned}$$

Hence

$$\frac{\partial G}{\partial z} = - \left(K_{oo} + x_o \frac{\partial p_o}{\partial z} \right) \tag{3.13}$$

and completing the second transformation we find

$$K_{ob} = \frac{1}{2} A p_\beta^2 + B p_\beta x_\beta + \frac{1}{2} C x_\beta^2 . \tag{3.14}$$

This Hamiltonian can be reduced to the harmonic oscillator form, $k_\beta(p^2 + Q^2)/2$, by means of a linear transformation. Thus we write

$$G_c = \frac{1}{2} P^2 f_1 + \frac{1}{2} x_\beta^2 f_2 + x_\beta P f_3 + \tau E \tag{3.15}$$

which yields

$$Q = Pf_1 + x_\beta f_3 \quad (3.16)$$

$$p_\beta = x_\beta f_2 + Pf_3 \quad (3.17)$$

$$\hat{t}_\beta = \tau + \frac{1}{2} P^2 \frac{\partial f_1}{\partial E} + \frac{1}{2} x_\beta^2 \frac{\partial f_2}{\partial E} + x_\beta P \frac{\partial f_3}{\partial E} \quad (3.18)$$

$$E_\tau = E \quad .$$

We determine the functions f_i by requiring that (3.16) and (3.17) transform a solution $Q = \cos k_\beta z$ of the harmonic oscillator Hamiltonian to a solution of the equations of motion of K_{ob} . To this end we make use of the easily verified fact that a K_{ob} solution may be expressed in the form

$$x_\beta = \alpha \sqrt{A\beta} \cos \psi \quad (3.19)$$

where

$$\psi = \int^z \frac{dz'}{\beta} \quad (3.20)$$

and β is a solution of

$$\frac{1}{2} \beta \beta^{*'} - \frac{1}{4} \beta^{*2} + AC \beta^2 = 1 \quad (3.21)$$

with

$$\beta^* = \beta' + \frac{A'}{A} \beta - 2B \beta \quad . \quad (3.22)$$

In the above expression α is an arbitrary constant to be determined along with f 's, and the prime in (3.21) and (3.22) indicates differentiation with respect to z . Note that solutions of (3.21) can never vanish and may be taken positive. Using Hamilton's equations to determine the related P and p_β and applying (3.16) and (3.17) we find $\alpha = 1$ and

$$f_1 = -\tan\phi \quad (3.23)$$

$$f_2 = \left(\frac{1}{2} \beta^* - \tan\phi \right) / A\beta \quad (3.24)$$

$$f_3 = \sec\phi / \sqrt{AB} \quad (3.25)$$

$$\phi \equiv \psi - k_\beta z \quad . \quad (3.26)$$

Combining these with (3.16) and (3.17) we find

$$x_\beta = \sqrt{AB} (Q \cos\phi + P \sin\phi) \quad (3.27)$$

$$p_{\beta} = \frac{1}{\sqrt{AB}} \left[Q \left(\frac{1}{2} \beta^* \cos \phi - \sin \phi \right) + P \left(\frac{1}{2} \beta^* \sin \phi + \cos \phi \right) \right]. \quad (3.28)$$

It is, of course, not necessary to know the origin of the f 's to carry out the transformation. Using only (3.15), (3.16), (3.17), (3.20) and (3.22) through (3.26) we find

$$\begin{aligned} K_{oc} &= K_{ob} + \frac{\partial G}{\partial z} \\ &= \frac{1}{2} k_{\beta} (P^2 + Q^2) \\ &\quad + \frac{1}{2\beta} (Q \cos \phi + P \sin \phi)^2 \left(\frac{1}{2} \beta \beta^{*'} - \frac{1}{4} \beta^{*2} + A C \beta^2 - 1 \right). \end{aligned}$$

Now applying (3.21) we find

$$K_{oc} = \frac{1}{2} k_{\beta} (P^2 + Q^2). \quad (3.29)$$

Since K_{oc} is z -independent

$$J \equiv \frac{1}{2} (P^2 + Q^2) \quad (3.30)$$

is a constant of the motion. Inverting (3.27) and (3.28) and substituting, we find

$$J = \frac{1}{2} \left[A\beta p_{\beta}^2 + \frac{1}{A\beta} \left(1 + \frac{\beta^{*2}}{4} \right) x_{\beta}^2 - x_{\beta} p_{\beta} \beta^* \right] \quad (3.31)$$

which we recognize as the Courant-Snyder invariant. Evidently, for a given J , all associated values of p_{β} , x_{β} must lie on an ellipse of area $2\pi J$. The orientation and aspect ratio, of course, depend upon z . It should be noted that a whole set of Courant-Snyder invariants exists because we have only required that β satisfy (3.21). For a study of storage ring operation one would normally choose β to have the same periodicity properties as the nominal orbits. If such periodic solutions do not exist, then the nominal orbits are unstable and unsuitable for storage ring operation. One would, of course, require of a design that stable nominal orbits exist. For a periodic linear structure one would have the option, as for the nominal orbits, of choosing a β function of the same periodicity. The parameter k_{β} is also completely arbitrary and has no effect on (3.21) nor on the relation between J and (x_{β}, p_{β}) . Again there is a natural choice when dealing with a periodic situation, and one would choose k_{β} so that ϕ has the same periodicity as β . In that case k_{β} represents the frequency (wave number, really) of betatron oscillations, and, as noted in ref. 10 has an important bearing on the amplification process. None of the above has any bearing on what we are about to prove, however, as the theorem holds for every choice of β .

For our final transformation we convert (3.30) to angle

(θ) action (J) form in the usual way. Thus

$$G_d = \hat{\tau} E + J \left[\sin^{-1} \frac{Q}{\sqrt{2J}} + \frac{Q}{\sqrt{2J}} \left(1 - Q^2/2J \right)^{1/2} \right] \quad (3.32)$$

which yields

$$\hat{E} = E, \quad \hat{\tau} = \tau$$

$$\theta = \sin^{-1} Q/\sqrt{2J} \quad \text{or} \quad Q = \sqrt{2J} \sin \theta \quad (3.33)$$

$$P = \sqrt{2J - Q^2} = \sqrt{2J} \cos \theta \quad (3.34)$$

$$K_{od} = k_{\beta}^J \quad . \quad (3.35)$$

Our final set of variables (E, τ, J, θ) are related to the original set (E, t, p, x) as follows:

$$x = \sqrt{2J/\alpha\beta} \sin(\phi + \theta) + x_0 \quad (3.36)$$

$$p = \sqrt{2J/\alpha\beta} \left(\cos(\phi + \theta) + \frac{1}{2} \beta^* \sin(\phi + \theta) \right) + p_0 \quad (3.37)$$

$$\begin{aligned}
t = \tau + t_0 - J \frac{\partial \phi}{\partial E} + \frac{1}{2} J A \beta \sin^2(\phi + \theta) \frac{\partial}{\partial E} \frac{\beta^*}{A \beta} \\
+ \frac{1}{2} J A \beta \sin 2(\phi + \theta) \frac{\partial}{\partial E} \frac{1}{A \beta} + \sqrt{2 J A \beta} \sin(\phi + \theta) \frac{\partial p_0}{\partial E} \\
- \sqrt{2 J / A \beta} \left(\cos(\phi + \theta) + \frac{1}{2} \beta^* \sin(\phi + \theta) \right) \frac{\partial x_0}{\partial E} . \quad (3.38)
\end{aligned}$$

Our final Hamiltonian is simply

$$H = k_\beta J + \sum_{n=1} K_n(E, t, p, x; z) \quad (3.39)$$

where (3.36), (3.37), and (3.38) are to be used to reexpress t , p , x in terms of τ , J , and θ . Note that despite the singularity in G_c where $\cos \phi$ vanishes, the connections between the old and new variables and between K and H are non-singular. As an overall algebraic check we have verified that the Lagrange bracket relations are satisfied.

4.0 PROOF OF THE GAIN-SPREAD-EXCITATION THEOREM

In this section it will be convenient to choose k_β to be zero. As a consequence the zero order (in ϵ) values of the canonical variables $(E_0, \tau_0, J_0, \theta_0)$ are all independent of z and will be taken to be the initial values. In addition we shall be interested only in first order corrections and time averaged second order corrections to these quantities. Recalling that the time average of K_2 vanishes, we easily see that only K_1 can contribute and hence we retain only this term in the sum over n . Finally, we note that as a consequence of (3.36), (3.37), and (3.38) K_1 is a periodic function of θ and hence may be represented by a Fourier series.

We could proceed by assuming only a steady state stationary property for the time dependence as in ref. 6, but in the interest of simplicity we specialize to a single frequency (taken positive) and write

$$K_1 = \epsilon e^{i\omega\tau} \sum_{m=-\infty}^{\infty} H_m(E, J, z) e^{im\theta} + \text{complex conjugate} \quad . \quad (4.1)$$

Then from Hamilton's equations

$$\frac{dJ}{dz} = -i\epsilon e^{i\omega\tau} \sum_m H_m e^{im\theta} + c.c. \quad (4.2)$$

$$\frac{dE}{dz} = i\epsilon\omega e^{i\omega\tau} \sum_m H_m e^{im\theta} + c.c. \quad (4.3)$$

$$\frac{d\theta}{dz} = \epsilon e^{i\omega\tau} \sum_m \frac{\partial H_m}{\partial J} e^{im\theta} + c.c. \quad (4.4)$$

$$\frac{d\tau}{dz} = -\epsilon e^{i\omega\tau} \sum_m \frac{\partial H_m}{\partial E} e^{im\theta} + c.c. \quad (4.5)$$

We define

$$I_m(E_o, J_o, z) = \int_0^z H_m(E_o, J_o, z') dz' \quad (4.6)$$

Then for the first-order quantities we find

$$J_1 = -i\epsilon e^{i\omega\tau_o} \sum_m I_m e^{im\theta_o} + c.c. \quad (4.7)$$

$$E_1 = i\epsilon\omega e^{i\omega\tau_o} \sum_m I_m e^{im\theta_o} + c.c. \quad (4.8)$$

$$\theta_1 = \epsilon e^{i\omega\tau_o} \sum_m \frac{\partial I_m}{\partial J_o} e^{im\theta_o} + c.c. \quad (4.9)$$

$$\tau_1 = -\epsilon e^{i\omega\tau_o} \sum_m \frac{\partial I_m}{\partial E_o} e^{im\theta_o} + c.c. \quad (4.10)$$

Note that the time average of all first-order quantities vanishes.

The quadratic averages which appear in the CSE relations may now be

computed. Denoting averages over τ_0 and θ_0 by $\langle \rangle$, we find

$$\langle J_1^2 \rangle = 2 \epsilon^2 \sum m^2 |I_m|^2 \quad (4.11)$$

$$\langle E_1^2 \rangle = 2 \omega^2 \epsilon^2 \sum |I_m|^2 \quad (4.12)$$

$$\langle E_1 J_1 \rangle = -2 \omega \epsilon^2 \sum m |I_m|^2 \quad (4.13)$$

Because general relations have proved to be useful in checking both numerical and analytic treatments of specific realizations, we note that (4.7) and (4.8) imply

$$\frac{\partial J_1}{\partial \tau_0} = - \frac{\partial E_1}{\partial \theta_0} \quad (4.14)$$

Proceeding to second order we have

$$\frac{dJ_2}{dz} = -i\epsilon e^{i\omega\tau_0} \sum m e^{im\theta_0} \left(J_1 \frac{\partial}{\partial J_0} + E_1 \frac{\partial}{\partial E_0} + i\omega\tau_1 + im\theta_1 \right) H_{m0} + c.c. \quad (4.15)$$

$$\frac{dE_2}{dz} = i\epsilon\omega e^{i\omega\tau_0} \sum e^{im\theta_0} \left(J_1 \frac{\partial}{\partial J_0} + E_1 \frac{\partial}{\partial E_0} + i\omega\tau_1 + im\theta_1 \right) H_{m0} + c.c. \quad (4.16)$$

Averaging, we find

$$\left\langle \frac{dJ_2}{dz} \right\rangle = \epsilon^2 \int m^2 R_m - \omega m S_m \quad (4.17)$$

$$\left\langle \frac{dE_2}{dz} \right\rangle = \epsilon^2 \int \omega^2 S_m - \omega m R_m \quad (4.18)$$

where

$$R_m = \frac{\partial H_{mo}}{\partial J_o} I_m^* + \frac{\partial H_{mo}^*}{\partial J_o} I_m + H_{mo} \frac{\partial I_m^*}{\partial J_o} + H_{mo}^* \frac{\partial I_m}{\partial J_o} \quad (4.19)$$

$$S_m = \frac{\partial H_{mo}}{\partial E_o} I_m^* + \frac{\partial H_{mo}^*}{\partial E_o} I_m + H_{mo} \frac{\partial I_m^*}{\partial E_o} + H_{mo}^* \frac{\partial I_m}{\partial E_o} \quad (4.20)$$

Next observe

$$\begin{aligned} & \int_0^z (\partial H_{mo} I_m^* + H_{mo}^* \partial I_m) dz_1 \\ &= \int_0^z dz_1 \partial H_{mo}(z_1) \int_0^{z_1} H_{mo}^*(z_2) dz_2 + \int_0^z dz_2 H_{mo}^*(z_2) \int_0^{z_2} \partial H_{mo}(z_1) dz_1 \\ &= \int_0^z dz_1 \partial H_{mo}(z_1) \cdot \int_0^{z_1} + \int_{z_1}^z H_{mo}^*(z_2) dz_2 \\ &= \partial I_m I_m^* \quad (4.20) \end{aligned}$$

Hence

$$\int R_m dz = \frac{\partial}{\partial J_0} |I_m|^2 \quad (4.21)$$

$$\int S_m dz = \frac{\partial}{\partial E_0} |I_m|^2 \quad . \quad (4.22)$$

Combining these with (4.17), (4.18), and (4.11) through (4.13) we have the GSE relations

$$\langle J_2 \rangle = \frac{1}{2} \frac{\partial}{\partial J_0} \langle J_1^2 \rangle + \frac{1}{2} \frac{\partial}{\partial E_0} \langle J_1 E_1 \rangle \quad (4.23)$$

$$\langle E_2 \rangle = \frac{1}{2} \frac{\partial}{\partial E_0} \langle E_1^2 \rangle + \frac{1}{2} \frac{\partial}{\partial J_0} \langle J_1 E_1 \rangle \quad . \quad (4.24)$$

Since the case of small excitation has been of particular interest, we examine the $J_0 \rightarrow 0$ limit of these relations. It is clear from the relations (3.36)-(3.38) that we have

$$I_m = L_m(E_0, J_0, z) J_0^{\frac{|m|}{2}} \quad (4.25)$$

where $L_{m0} \equiv L_m(E_0, 0, z)$ is finite or zero. Hence in the limit of small J_0 we have

$$\langle J_1^2 \rangle = 2\epsilon^2 J_0 (|L_{10}|^2 + |L_{-10}|^2) \quad (4.26)$$

$$\langle E_1^2 \rangle = 2\epsilon^2 \omega^2 |L_{00}|^2 \quad (4.27)$$

$$\langle E_1 J_1 \rangle = 2\epsilon^2 J_0 (|L_{-10}|^2 - |L_{10}|^2) \quad (4.28)$$

which yields at $J_0 = 0$

$$\langle J_2 \rangle = \epsilon^2 (|L_{10}|^2 + |L_{-10}|^2) \quad (4.29)$$

$$\langle E_2 \rangle = \epsilon^2 \omega^2 \frac{\partial |L_{00}|^2}{\partial E_0} + \epsilon^2 \omega (|L_{-10}|^2 - |L_{10}|^2) \quad (4.30)$$

It is therefore clear that if $\langle J_2 \rangle$ vanishes, that we have the usual gain-spread relation since the $\frac{\partial}{\partial J_0} \langle E_1 J_1 \rangle$ term vanishes. Furthermore, if $\langle E_1^2 \rangle$ also vanishes, then, as has been noted previously,⁵ so does $\langle E_2 \rangle$, and there can be no gain.

Some comment on the time dependence assumed in Eq. (4.1) is appropriate. We could easily have replaced $e^{i\omega\tau} H_m$ by

$\sum_k e^{i\omega_k\tau} H_{mk}$. The proof is essentially the same; expressions like $\omega^2 \sum_m |I_m|^2$ in (4.12) became $\sum_{m,k} \omega_k^2 |I_{mk}|^2$, etc., (4.23) and (4.24) are unchanged and (4.29) and (4.30) are replaced by

$$\langle J_2 \rangle = \epsilon^2 \sum_k (|L_{1k0}|^2 + |L_{-1k0}|^2) \quad (4.31)$$

$$\langle E_2 \rangle = \epsilon^2 \sum_k \left[\omega_k^2 \frac{\partial |L_{0ko}|^2}{\partial E_0} + \omega_k (|L_{1ko}|^2 - |L_{-1ko}|^2) \right] \quad (4.32)$$

Of course, the time averages must be carried out over time intervals which cause the frequency cross terms to vanish, and if the frequencies are incommensurate the GSE relations may contain errors of order $1/\omega T$ where T is the averaging time.

For application to the next section we note that if k_β is not set equal to zero, equations (4.11), (4.12), (4.13), (4.23), and (4.24) are unchanged. The expression for I_m [Eq. (4.6)] acquires a formal dependence upon k_β , viz.

$$I_m(E_0, J_0, z) = \int_0^z dz' H_m(E_0, J_0, z'; k_\beta) \exp imk_\beta z' \quad (4.6')$$

but from (3.36), (3.37), and (3.38) one sees that H contains

$\phi = \psi - k_\beta z$ only in the combination $\theta + \phi$ from which we conclude

$$H_m(E_0, J_0, z'; k_\beta) = H_m(E_0, J_0, z') \exp - imk_\beta z' \quad (4.33)$$

and therefore that I_m is in fact also independent of k_β .

5.0 IMPLICATIONS FOR STORAGE RING OPERATION

In this section we present a discussion of the implications of the GSE relations for quasi-linear storage ring operation of the FEL.

In the absence of the laser fields, the combined storage ring and wiggler can be regarded as a particular example of a storage ring with a somewhat unusual system of static fields. The usual concepts and principles of storage ring design apply. Thus z will be specified to represent a cyclic variable such that (x, z) and $(x, z+Z)$ represent the same physical point. Hence the Hamiltonian will be periodic in z with period Z . The system will be designed so that stable nominal orbits of period Z exist and are uniquely defined for each energy lying within some specified acceptance range. The natural and standard choice for β is then the (unique) periodic solution of (3.21), which, on account of the stability assumption, must exist. The natural and standard specification of k_β is

$$k_\beta = \frac{1}{Z} \int_0^Z dz' / \beta(z') \quad (5.1)$$

which causes ϕ to be periodic as well. The conventional "betatron number" ν is defined by $\nu = k_\beta Z / 2\pi$. Because we require the

nominal orbits to be unique, ν cannot be an integer, and to avoid stability problems one designs the ring so that ν is quasi-irrational, that is to say, not expressible as the ratio of small integers. As a result, a given electron with a specified J and E tends to uniformly sweep through all action phases θ in successive transits around the ring. Accordingly, in the absence of laser radiation the electron distribution function will be θ independent. We have neglected the effect of synchrotron radiation and the RF cavity in the above but it is clear that the conclusion will continue to hold when they are taken into account.

We can formalize the above by means of the Liouville Equation, which for the zero'th order Hamiltonian takes the form

$$\frac{\partial f_0}{\partial z} + k_\beta \frac{\partial f_0}{\partial \theta} = 0 \quad (5.2)$$

The general solution to (5.2) is

$$f_0 = F(\theta - k_\beta z, \tau, J, E) \quad (5.3)$$

where F is an arbitrary function of its arguments except for the periodicity conditions which follow from

$$f_0(\theta, \tau, J, E, z) = f_0(\theta + 2\pi, \tau, J, E, z) \quad (5.4)$$

$$f_0(\theta, \tau - T(E), J, E, z + Z) = f_0(\theta, \tau, J, E, z) \quad (5.5)$$

where $T(E)$ is the one cycle transit time of the nominal orbits and is given by

$$T = \int_0^Z t_0 dz \quad . \quad (5.6)$$

Equations (5.4) and (5.5) are consequences of the fact that the arguments on the left and right hand side represent the same physical space time point [see (3.36), (3.37), and (3.38)].

We note that the space time structure of the micropulses is contained in the dependence of F on τ . On account of the RF cavity and synchrotron radiation the distribution function at fixed z will settle down to a periodic function of τ with period T_c determined by the cavity frequency. We simulate this effect here by neglecting the energy dependence of $T(E)$, assuming $T(E) = T_c$, and requiring that F be periodic in τ with period T . Hence the steady state F must have both the 2π periodicity in θ and the Z periodicity in z . Thus F will be invariant to any shift $\Delta\theta = 2\pi(r + vs)$ where r and s are integers. If v can be expressed in the form m/n where m and n are relatively prime integers, then a periodicity in θ of $2\pi/n$ is implied; while if v is irrational, F must be θ independent. We have, however,

assumed that n is not small, and noting that the synchrotron oscillation-synchrotron radiation processes will cause some additional smearing not included in the above argument, we conclude that f_0 must be effectively θ independent.

We now wish to take account of the presence of the laser field. We recall that the CSE relations involve an average over τ and θ . In order to apply them to a discussion of storage ring operation we need to establish the appropriateness of such averages. We first note that the τ variation due to the micropulse structure is slow compared to that due to the optical frequency. The τ averages are really averages over optical phase and are not significantly affected by the micropulse structure. We consider the distribution function to consist of a zero'th, first, and second order part in ϵ , and we consider their effect on first and second order energy transfer and transverse excitation processes which take place as the electrons pass through the wiggler. The first order part, f_1 , contains $\exp i\omega\tau$ factors. Due, however, to the rapid energy dependence in $\exp i\omega T$, and the effects of J and E changes induced by synchrotron radiation and the cavity on cycle times for individual electrons, there can be no phase correlation between the $\omega\tau$ dependence of f_1 and that of the laser field present when the pulse is passing through the wiggler. Thus second order effects which could in principle be produced by

the action of K_1 on the entering f_1 contribute nothing on the average. Time stationary terms do appear in f_2 , which may also be θ dependent. However, since f_2 is already second order, its effect on changes in E , J must be of higher order (the average effect is fourth order). One might ask whether the laser could induce a cumulative buildup of θ dependent f_2 of the same order as f_0 . We consider this to be precluded by the quasi-irrational character of ν , which has the same θ smearing effect upon f_2 as it has upon f_0 .

We now proceed to a discussion of a phase and θ averaged distribution function $F(J, E, z)$. (We ignore the micropulse structure here.) Designating the wiggler entrance and exit by $z = 0$ and L , respectively, noting that wiggler induced changes in F are second order and hence small, and carrying out a standard Fokker-Planck type development, we find that

$$\begin{aligned} \Delta F_L &\equiv F(J, E, L) - F(J, E, 0) \\ &= - \frac{\partial}{\partial E} (\langle E_2 \rangle F) - \frac{\partial}{\partial J} (\langle J_2 \rangle F) \\ &\quad + \frac{1}{2} \frac{\partial^2}{\partial E^2} (\langle E_1^2 \rangle F) + \frac{\partial^2}{\partial E \partial J} (\langle E_1 J_1 \rangle F) + \frac{1}{2} \frac{\partial^2}{\partial J^2} (\langle J_1^2 \rangle F) . \end{aligned} \quad (5.7)$$

We are not going to write out a complete Fokker-Planck equation for the entire storage ring and are instead going to aim for an upper limit estimate of laser efficiency based upon entropy considerations. To this end we compute the laser induced entropy change. Let

$$S \equiv - \int F \ln F \, dJdE \quad (5.8)$$

then

$$\Delta S_L = - \int \Delta F \ln F \, dJdE - \int \Delta F dJdE \quad . \quad (5.9)$$

Conservation of probability requires $\int \Delta F$ to vanish. To obtain this result formally from (5.7), we make use of the fact that F must vanish when either J or E are infinite to obtain

$$\begin{aligned} \int \Delta F_L \, dJdE &= \int \left[F \left(\langle E_2 \rangle - \frac{1}{2} \frac{\partial \langle E_1^2 \rangle}{\partial E} - \frac{1}{2} \frac{\partial \langle E_1 J_1 \rangle}{\partial J} \right) - \right. \\ &\quad \left. - \frac{1}{2} \langle E_1^2 \rangle \frac{\partial F}{\partial E} - \frac{1}{2} \langle E_1 J_1 \rangle \frac{\partial F}{\partial J} \right] dJ \Big|_{E=0} + \\ &+ \int \left[F \left(\langle J_2 \rangle - \frac{1}{2} \frac{\partial \langle J_1^2 \rangle}{\partial J} - \frac{1}{2} \frac{\partial \langle E_1 J_1 \rangle}{\partial E} \right) - \right. \\ &\quad \left. - \frac{1}{2} \langle J_1^2 \rangle \frac{\partial F}{\partial J} - \frac{1}{2} \langle E_1 J_1 \rangle \frac{\partial F}{\partial E} \right] dE \Big|_{J=0} \quad . \end{aligned}$$

Applying the GSE relations (4.23) and (4.24) and the small J formulas (4.26) and (4.28) we have

$$\int \Delta F_L dJ dE = -\frac{1}{2} \int \left(\langle E_1^2 \rangle \frac{\partial F}{\partial E} + \langle E_1 J_1 \rangle \frac{\partial F}{\partial J} \right) dJ \Big|_{E=0} = 0 . \quad (5.10)$$

The last line follows from the fact that the storage ring aperture will preclude non-zero values for $\frac{\partial F}{\partial E}$, $\frac{\partial F}{\partial J}$ at $E = 0$, as well as from the fact that all laser induced processes will become ineffective at $E = 0$. Combining (5.7) (5.9) and (5.10), we have

$$\Delta S_L = \int \left[\frac{\partial}{\partial E} (\langle E_2 \rangle F) + \frac{\partial}{\partial J} (\langle J_2 \rangle F) - \frac{1}{2} \frac{\partial^2}{\partial E^2} (\langle E_1^2 \rangle F) - \frac{\partial^2}{\partial E \partial J} (\langle E_1 J_1 \rangle F) - \frac{1}{2} \frac{\partial^2}{\partial J^2} (\langle J_1^2 \rangle F) \right] \ln F dJ dE . \quad (5.11)$$

Integrating by parts and applying the arguments used in connection with (5.10) to eliminate boundary terms, we obtain

$$\begin{aligned}
\Delta S_L = \int & \left[- \left(\langle E_2 \rangle - \frac{1}{2} \frac{\partial}{\partial E} \langle E_1^2 \rangle - \frac{1}{2} \frac{\partial}{\partial J} \langle E_1 J_1 \rangle \right) \frac{\partial F}{\partial E} + \right. \\
& + \frac{1}{2} \langle E_1^2 \rangle \frac{1}{F} \left(\frac{\partial F}{\partial E} \right)^2 + \frac{1}{2} \langle E_1 J_1 \rangle \frac{1}{F} \frac{\partial E}{\partial J} \frac{\partial F}{\partial E} \\
& - \left(\langle J_2 \rangle - \frac{1}{2} \frac{\partial}{\partial J} \langle J_1^2 \rangle - \frac{1}{2} \frac{\partial}{\partial E} \langle E_1 J_1 \rangle \right) \frac{\partial F}{\partial J} \\
& \left. + \frac{1}{2} \langle J_1^2 \rangle \frac{1}{F} \left(\frac{\partial F}{\partial J} \right)^2 + \frac{1}{2} \langle E_1 J_1 \rangle \frac{1}{F} \frac{\partial F}{\partial J} \frac{\partial F}{\partial E} \right] dJ dE .
\end{aligned}$$

Another application of the GSE relations provides us with the useful form

$$\Delta S_L = \frac{1}{2} \int \left[\langle E_1^2 \rangle \left(\frac{\partial F}{\partial E} \right)^2 + 2 \langle E_1 J_1 \rangle \frac{\partial F}{\partial E} \frac{\partial F}{\partial J} + \langle J_1^2 \rangle \left(\frac{\partial F}{\partial J} \right)^2 \right] \frac{dJ dE}{F} . \quad (5.12)$$

Next we apply (4.11), (4.12), and (4.13) to obtain

$$\Delta S_L = \epsilon^2 \sum_m \int |I_m|^2 \left(\omega \frac{\partial F}{\partial E} - m \frac{\partial F}{\partial J} \right)^2 \frac{dJ dE}{F} \quad (5.13)$$

from which we conclude $\Delta S_L \geq 0$ as one would expect.

Now consider the laser energy generation per particle,

ΔE_L , given by

$$\Delta E_L = - \int \langle E_2 \rangle F dJdE \quad . \quad (5.14)$$

Applying (4.24) and integrating by parts we find

$$\Delta E_L = \frac{1}{2} \int \left(\langle E_1^2 \rangle \frac{\partial F}{\partial E} + \langle E_1 J_1 \rangle \frac{\partial F}{\partial J} \right) dJdE \quad . \quad (5.15)$$

Again applying (4.12) and (4.13) we have

$$\Delta E_L = \epsilon^2 \sum_m \int |I_m|^2 \omega \left(\omega \frac{\partial F}{\partial E} - m \frac{\partial F}{\partial J} \right) dJdE \quad . \quad (5.16)$$

From (5.13) we see that in order to have zero ΔS_L we must have

either $|I_m|^2$ or $\omega \frac{\partial F}{\partial E} - m \frac{\partial F}{\partial J}$ zero for all m . If this were the case, however, (5.16) tells us that we also have zero ΔE_L .

Hence we have rigorously shown that gain (or loss) requires entropy production. In order to have a steady state the total entropy production in a cycle must vanish. Since the only entropy sink available is the synchrotron radiation, we see that laser gain must be limited by the synchrotron radiation.

We now seek a lower bound on the entropy production associated with a specified ΔE_L . To simplify progress in this direction we restrict our attention to fully gain expanded systems, that is, systems for which the energy variation of $\langle E_1^2 \rangle$, $\langle E_1 J_1 \rangle$, and $\langle J_1^2 \rangle$ within the storage ring aperture can be

neglected. We have also had to assume that E and J are uncorrelated in F , that is

$$F(J,E) = F_1(J)F_2(E) \quad (5.17)$$

so that our lower bound will be rigorous only for uncorrelated distributions.

Applying (5.17) to (5.12) and neglecting the energy dependence of the various $\langle \rangle$ averages, we can carry out the energy integration to obtain

$$\Delta S_L = \frac{1}{2} \int dJ \left(\langle J_1^2 \rangle \frac{1}{F_1} \frac{dF_1}{dJ}^2 + \langle E_2^2 \rangle \frac{F_1}{\Delta E^2} \right) \quad (5.18)$$

where

$$\frac{1}{\Delta E^2} \equiv \int \frac{1}{F_2} \left(\frac{dF_2}{dE} \right)^2 dE \quad (5.19)$$

The aperture width is typically related to the RMS spread given by

$$\delta E^2 = \int (E - E_0)^2 F_2 dE \quad (5.20)$$

with

$$E_0 = \int E F_2 dE \quad . \quad (5.21)$$

A lower limit on ΔS_L may be specified in terms of these quantities by choosing F_2 so as to minimize (5.19) subject to specified values of δE^2 and E_0 . A simple calculation shows the proper form to be Gaussian, for which $\Delta E^2 = \delta E^2$. Hence

$$\frac{1}{\Delta E^2} > \frac{1}{\delta E^2} \quad . \quad (5.22)$$

Again making use of the gain expansion assumption we find

$$\begin{aligned} \Delta E_L &= -\frac{1}{2} \int F_1 \frac{d}{dJ} \langle E_1 J_1 \rangle dJ \\ &= \frac{1}{2} \int \langle E_1 J_1 \rangle \frac{dF_1}{dJ} dJ \end{aligned} \quad (5.23)$$

and applying (4.11), (4.12), and (4.13) we have

$$\Delta S_L > \Delta S_L \equiv \epsilon^2 \sum_m \int I_m^2 \left[m^2 \left(\frac{dF_1}{dJ} \right)^2 + \omega^2 \frac{F_1^2}{\delta E^2} \right] dJ / F_1 \quad (5.24)$$

$$\Delta E_L = -\epsilon^2 \int |I_m|^2 \omega m \frac{dF_1}{dJ} dJ \quad . \quad (5.25)$$

Combining (5.24) with (5.25) we obtain

$$\Delta \bar{S}_L = \epsilon^2 \sum_m \int I_m^2 \left(m \frac{dF_1}{dJ} + \frac{\omega F_1}{\delta E} \right)^2 dJ/F_1 + \frac{2\Delta E_L}{\delta E} \quad (5.26)$$

and hence that

$$\Delta S_L > \frac{2\Delta E_L}{\delta E} . \quad (5.27)$$

It is possible to design a system and determine an F for which the lower limit is actually achieved. The system must be designed so that a single $|I_m|^2$ dominates (with $m > 1$ in order to get gain). F_2 must be chosen to be Gaussian. The extra term then becomes negligible if

$$m \frac{dF_1}{dJ} + \frac{\omega F_1}{\delta E} = 0$$

or

$$F_1 = \frac{\omega}{m\delta E} \exp - \frac{\omega J}{m\delta E} . \quad (5.28)$$

We have, of course, no reason to believe that the solution to the Fokker-Planck equation would actually take the form which we have found for the minimizing F . The general form found for F , F_1 ,

and F_2 are all quite plausible but we see no reason to assume that $\langle J \rangle = \frac{m\delta E}{\omega}$, as implied by (5.28), will emerge.

Similar relations hold when there is loss instead of gain. In particular (5.27) continues to be valid when ΔE_L is replaced by its absolute value. To achieve the limit one must then take m negative and replace it by its absolute value in (5.28).

We now calculate the entropy change due to the synchrotron radiation. For this purpose we may ignore the presence of the laser radiation and introduce the radiation damping of synchrotron oscillations in a phenomenological way. We study the distribution function at $z = z_c$, a point just before the electrons enter the microwave cavity, and consider its change as it makes one turn around the storage ring. That is, we study

$$\Delta F_R \equiv F(J, E, \Omega, z_c + Z) - F(J, E, \Omega, z_c) \quad . \quad (5.29)$$

In (5.28) we have replaced the time by the synchrotron oscillation phase Ω , and have on the basis of arguments given at the beginning of this section assumed that there is no dependence on the action phase θ of the betatron oscillations. An adequate phenomenological Fokker-Planck expression for ΔF_R is

$$\Delta F_R = - \frac{\partial}{\partial E} (\Delta E F) - \frac{\partial}{\partial J} (\Delta J F) - \frac{\partial}{\partial \Omega} (\Delta \Omega F) \quad (5.30)$$

with

$$\Delta E = - 2A_S(E-E_0) - \lambda \Omega$$

$$\Delta \Omega = \mu(E-E_0)$$

$$\Delta J = -2A_J J \quad .$$

Here ΔE , $\Delta \Omega$, and ΔJ represent the change in the variables

E , Ω , J which occur as a result of passage through the microwave cavity, wiggler magnet, and transport around the ring;

λ and μ are constant parameters which determine the synchrotron oscillation frequency, E_0 is the equilibrium energy, and A_S , A_J represent synchrotron oscillation and betatron oscillation damping constants per turn. Noting that $\int F_R dE dJ d\Omega = 0$ we find

$$\begin{aligned} \Delta S_R &= - \int \Delta F_R \ln F dJ dE d\Omega \\ &= \int \left(\Delta E \frac{\partial F}{\partial E} + \Delta J \frac{\partial F}{\partial J} + \Delta \Omega \frac{\partial F}{\partial \Omega} \right) dJ dE d\Omega \\ &= -2(A_S + A_J) \quad . \end{aligned} \quad (5.31)$$

Now applying the Robinson theorem¹³ for plane orbits we have

$$\Delta S_R = -3 \frac{R_S}{E_0} \quad (5.32)$$

where R_S is the synchrotron radiation per turn per particle.

Combining (5.32) with (5.27) we obtain our principal result

$$E_L < \frac{3}{2} \frac{\delta E}{E_0} R_S \quad (5.33)$$

This limit is essentially the same as that found in ref. 1. for the standard one dimensional uniform wiggler and can be shown to hold quite generally for any one dimensional wiggler which satisfies the gain-spread theorem. Because a necessary connection between gain and entropy production has been established by equations (5.13) and (5.16) and because (5.33) has been established in the two extreme cases of no transverse excitation and full gain expansion, it seems very likely that a relation differing from (5.33) only by a numerical factor of order one holds quite generally.

A question which one might raise in connection with this result and which will be addressed in a future publication is whether it would be practical to improve storage ring laser efficiency by increasing $\delta E/E_0$ from the conventional 0.01 usually

assumed to say 0.1. Should this prove to be feasible from the standpoint of storage ring design, the question of whether sufficient laser gain is available under these circumstances to permit laser operations would remain. Transverse gradient wigglers, because of their large bandwidth in energy, may have an important advantage here. In the conventional zero gradient wiggler the gain tends to vary as the square of the reciprocal energy aperture, while for the transverse gradient wiggler designed to operate at the $m = 0$ resonance it falls off only linearly. Gain at the $m = 1$ resonance is not directly affected by the large aperture, but one must take account of the fact that to achieve the limit (5.33) the steady state transverse excitation must be connected to the energy aperture as implied by (5.28). Taking into account the bandwidth in transverse excitation one again finds that the gain tends to vary inversely as the energy aperture. Investigation to determine which of the available options may be most advantageous is in progress.

6.0 SUMMARY AND CONCLUSIONS

In this paper we have proposed an extension of the Madey gain-spread theorem to two-dimensional wigglers and shown it to be quite generally valid. It has the important consequence that an FEL wiggler which yields gain must at the same time generate either energy spread or transverse excitation. Furthermore ~~we have~~ found that in an FEL operating quasi linearly in a storage ring, that laser gain guarantees the production of entropy with every pass through the wiggler. Consequently the laser radiation generated is restricted to be some small fraction of the synchrotron radiation and our analysis suggests that this fraction is of the order of the fractional energy aperture. This leads us to conclude that the achievement of high efficiency steady state storage ring operation in a quasi linear regime is not possible. Nonetheless, it appears that gain expanded wigglers may have superior linear gain as compared with conventional wigglers and thus prove to be useful for application in storage rings with large fractional energy aperture.

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