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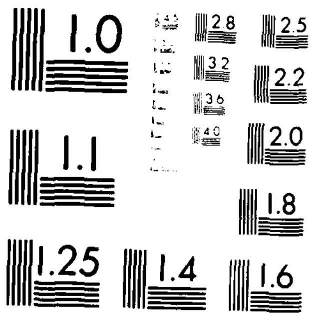


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MRC Technical Summary Report #2704

# PARTITIONS OF UNITY AND APPROXIMATION

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## SIGNIFICANCE AND EXPLANATION

Spaces of piecewise polynomial functions on regular partitions have important applications to finite elements, surface fitting, etc. However, the structure of such spaces is rather complicated when some global smoothness is demanded of the pp functions. For example, the order of approximation to smooth functions is not known even when the partition is very simple.

Moreover, so far there is no general criterion for deciding when such spaces can be used to approximate arbitrary functions. We have conjectured in [BDH] that a necessary and sufficient condition for this is that the space contain a stable and local partition of unity. In this work, we prove this result for a general class of univariate function spaces which abstract the properties of pp function spaces. This should set the world of approximation theory on its ears.

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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

# PARTITIONS OF UNITY AND APPROXIMATION

C. de Boor[1] and R. DeVore[1,2]

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ABSTRACT

We show that for certain translation invariant spaces $S$, a necessary and sufficient condition for the eventual denseness of the corresponding scaled spaces $S_h$ is that $S$ contain a stable and locally supported partition of unity.

These results have been motivated by recent work on approximation by multivariate piecewise polynomials on regular meshes.

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PARTITIONS OF UNITY AND APPROXIMATION

C. de Boor[1] and R. DeVore[1,2]

**1. Introduction.** We are interested in the existence of partitions of unity in certain translation invariant spaces. We are motivated by recent investigations of approximation by multivariate piecewise polynomial functions on regular partitions. For example, in [BD], two partitions were singled out. The first is the partition $\Sigma$ of $\mathbb{R}^2$ into squares of unit side length whose vertices are the integer lattice points. The other is the triangulation of $\mathbb{R}^2$ obtained by inserting the northeast diagonal into each square. These partitions have acquired the names "two-direction" and "three-direction" mesh, respectively, since the lines generating them have two, respectively, three distinct directions. The "four-direction" or "criss-cross" mesh, obtained from the "three-direction" mesh by drawing in the other diagonals has also been studied [DM].

It has become clear from these investigations that the description of the approximation properties of pp functions, even on these simple meshes, is much more complicated than in the univariate case.

We shall say that a partition $\tau$ of $\mathbb{R}^s$ is regular if $\tau$ is invariant under translation by any $\alpha \in \mathbb{Z}^s$ ; i.e., $\tau + \alpha = \tau$ for such $\alpha$ . We denote by $\pi_k^\rho(\tau)$ the space of pp functions $f$ with subdivision $\tau$ of total degree $k$ and global smoothness $\rho$ , that is $f \in C^\rho(\mathbb{R}^s)$ . A problem of particular interest has been to determine the approximation order to smooth functions (say $C^\infty$ ) from the spaces $\pi_k^\rho(h\tau)$ . So far this problem has been completely settled only when $\tau = \Sigma$ as above [BD] or for certain special choices of $k$ and $\rho$ for the three- or four-direction mesh [BD], [$BH_{1-2}$], [DM], [J]. These investigations show that the approximation order depends in a subtle way on the smoothness $\rho$ and the number of distinct directions. For example, in contrast to the univariate case, an increase in $\rho$ usually decreases the approximation order (and not just the coefficient); also, an increase in the number of distinct directions may increase the order.

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Lower bounds for the approximation order are usually obtained with the aid of explicit methods of approximation, known as **quasi-interpolants**. These use special splines with local support known as **box** splines. It is not necessary to go into the properties of these splines here. We only mention the fact that it is easy to make up a partition of unity consisting of box splines and this is the important property used in the construction of quasi-interpolants.

It has not been clear to us whether this represents the only viable approach to the construction of suitable approximants. For example, in [BDH], we have asked the following:

(1.1) Is $\bigcup_{h>0} \pi_k^\rho(h\tau)$ dense in $C_0(\mathbb{R}^s)$ if and only if $\pi_k^\rho(\tau)$ contains a good partition of unity?

By a **good** partition of unity we mean a (countable) collection $\Phi$ of functions with the properties that

(1.2)
(i) $\sup_{\varphi\in\Phi} \operatorname{diam} \operatorname{supp} \varphi < \infty$ ;
(ii) $\Sigma_{\varphi\in\Phi} \varphi = 1$
(iii) $\Sigma_{\varphi\in\Phi} |\varphi| < \infty$ .

Actually, it is easy to see that the existence of a good partition of unity is sufficient for denseness. Namely, if $x_\varphi \in \operatorname{supp} \varphi$ , then

$$Lf := \Sigma_{\varphi\in\Phi} f(x_\varphi)\varphi$$

gives an operator $L$ whose dilate $L_h := \sigma_{1/h} L \sigma_h$ , with

$$(\sigma_h f)(x) := f(hx) ,$$

provides an approximation $L_h f$ in $\pi_k^\rho(h\tau)$ which converges to $f$ as $h \longrightarrow 0$ .

It turns out, but is much more difficult to prove, that the existence of a good partition of unity is necessary for denseness. Here we shall present a proof of this fact for the univariate case but for much more general spaces $S$ than just pp ones, in an attempt to extract the essential features of pp function spaces on *regular partitions*.

The ideas of the proof seem to carry over to the multivariate case, but the constructions involved are much more involved. For this reason, we feel that it is beneficial to present here the basic ideas in their simplest setting and report elsewhere on their multivariate analogues.

2. **The univariate case.** We let $S$ be a space of functions on $\mathbb{R}$ with the properties:

(2.1)
(i) $S$ is translation invariant, i.e., $f \in S \implies f(\cdot \pm 1) \in S$ ;

(ii) $\mu := \dim S|_{[0,1)} < \infty$ ;

(iii) $\dim S = \infty$ ;

(iv) $S$ is closed under uniform convergence on compact sets.

We note for later reference that (i) and (iii) imply

(v) $\dim S|_{[0,\infty)} = \infty$ .

These are the essential properties of pp spaces over regular meshes. However, it is easy to give examples of such spaces $S$ which are not pp. For example, if $\Phi$ is a finite collection of compactly supported functions, then the closure under uniform convergence on compact sets of the span of the translates $\varphi_j := \varphi(\cdot - j)$ , $j \in \mathbb{Z}$ , $\varphi \in \Phi$ , has these properties. The results of this section will show that any space $S$ satisfying (2.1) can be obtained this way modulo some finite dimensional space.

Associated with $S$ we have the dilated spaces $S_h := \sigma_{1/h} S$ .

**Theorem 1.** <u>If</u> $\bigcup_{h>0} S_h$ <u>is dense in</u> $C_0(\mathbb{R})$ , <u>then</u> $S$ <u>contains a good partition of unity</u>.

In preparation for the proof, we now construct certain compactly supported functions in $S$ . Let $S^-$ denote the subspace of $S$ consisting of those functions which vanish on $(-\infty,0]$. It will follow later that $S^-$ is nontrivial. For now, we prove:

**Lemma 2.** <u>For each integer</u> $n > 0$ , <u>there is an</u> $R_n$ <u>such that any</u> $\varphi \in S$ <u>vanishing on</u> $[-R_n,0]$ <u>agrees on</u> $[0,n]$ <u>with some function from</u> $S^-$ .

**Proof.** For $r$ , $j > 0$ , we let $L_{r,j}$ denote *the space of functions defined on* $[0,j)$ which are restrictions of functions in $S$ which vanish on $[-r,0]$ . Then $L_{r,j} \subseteq L_{r',j}$ whenever $r > r'$ . Hence $(\dim L_{r,j})_{r=1}^{\infty}$ is a decreasing sequence of (nonnegative) integers, therefore is eventually constant. This implies that we can find an increasing sequence $(R_j)$ such that

$$L_{r,j} = L_j := L_{R_j,j} \quad \text{for all} \quad r \geq R_j \ , \ j=1,2, \ldots \ .$$

Now given $n > 0$ and $\varphi \in L_n$, we define

$$n_i := n + i \quad , \quad r_i := \max \{R_{n_i}, n_i\} \ , \ i=0,1, \ldots \ .$$

Then, since $r_1 \geq R_{n_1} \geq R_n$, there is an $f_1 \in S$ which vanishes on $[-r_1,0]$ and agrees with $\varphi$ on $[0,n]$. Similarly, $r_2 \geq R_{n_2} \geq R_{n_1}$ and so there is an $f_2 \in S$ which vanishes on $[-r_2,0]$ and agrees with $f_1$ on $[0,n_1]$. In this manner, we obtain a sequence $(f_j)$ such that $f_j$ vanishes on $[-r_j,0]$ and agrees with $f_i$ on $[0,n_i]$ for all $i < j$. Hence $f_j$ converges uniformly on compact sets to some $f$, necessarily in $S^-$ because of (2.1.iv), with the desired properties. |||

Now let

$$R := R_1$$

be the constant of Lemma 2. We improve Lemma 2 in the following way:

**Lemma 3.** If $f \in S$ vanishes on $[-R,0]$, then $\varphi := f\chi_{[0,\infty)}$ is in $S^-$.

**Proof.** From Lemma 2, we know an $f_0 \in S^-$ for which $f - f_0$ vanishes on $[-R,1]$. Hence there is some $f_1$ with $f_1(\cdot+1) \in S^-$, therefore also $f_1 \in S^-$, for which $f - f_0 - f_1$ vanishes on $[-R,2]$. In this manner, we construct a sequence $(f_j)$ in $S^-$ for which $f_0 + \ldots + f_j$ converges to $\varphi$. Since $S^-$ is closed under uniform convergence on compact sets (by (2.1.iv)), $\varphi \in S^-$. |||

Next, we construct a basis of compactly supported functions for $S^-$. Recall the definition $\mu := \dim S|_{[0,1]}$ and set

$$M := \mu R \ .$$

**Lemma 4.** If $f \in S^-$, then there is a function $\varphi \in S$ which is supported on $[0,M]$ and agrees with $f$ on $[0,1]$.

**Proof.** Set $J := [M,M+R]$ and consider the functions $f_j := f(\cdot-j)$, $j=0,\ldots,M$. Since $\dim S|_J \leq M$, these functions are linearly dependent on $J$. Hence there are numbers $c_j$ , not all zero, such that

$$\psi_0 := \Sigma\, c_j\, f_j$$

vanishes on $J$ . Let $i$ be the smallest value of $j$ for which $c_j \neq 0$ . Then $\psi := \psi_0(\cdot - i)/c_i$ agrees with $f$ on $[0,1]$ and vanishes on $a + [-R,0]$ with $a := M+R-i$ . Therefore, by Lemma 3, $\psi_1 := \psi(\cdot + a)\chi_{[0,\infty)}$ is in $S^-$ and so $\varphi := \psi - \psi_1(\cdot - a)$ has the desired properties. |||

It follows from Lemma 4 that there is a <u>finite</u> collection $B$ of functions supported on $[0,M]$ whose restrictions to $[0,1]$ form a basis for $(S^-)|_{[0,1]}$ . We use the abbreviation

$$b_j := b(\cdot - j)\ ,\ b \in B\ ,\ j \in \mathbb{Z}\ .$$

The peeling off argument of Lemma 3 shows that the functions

$$b_j,\ b \in B\ ,\ j=0,1,2,\ \ldots$$

form a basis for $S^-$ . But, in order to insure local linear independence of this basis, we need to choose $B$ with some care. Precisely, we choose $B$ so that

$$\text{length}\ (B) := \Sigma_{b\in B}\ \text{length}(b)$$

is minimal. Here, we mean by length(b) the smallest integer $r$ for which $\operatorname{supp} b \subseteq [i,i+r]$ for some integer $i$ .

**Lemma 5.** <u>For any interval</u> $J := [r,r+R]$ , <u>with</u> $r$ <u>an arbitrary integer and</u> $R$ <u>as before, the collection</u>

$$\Phi_J := \{b_j : \operatorname{supp} b_j \cap J \neq \emptyset\ \}$$

<u>is linearly independent over</u> $J$ .

**Proof.** Suppose that we have $f := \Sigma_{\varphi\in\Phi_J} c_\varphi \varphi = 0$ on $J$ . Then, by Lemma 3, we can find some $g \in S^-$ so that $f_1 := f - g(\cdot - (r+R))$ vanishes on $[r,\infty)$ . If not all the coefficients $c_\varphi$ were zero, then we could choose some $\varphi := b_j$ with $c_\varphi =: c_{b_j} \neq 0$ and so that $j$ is as small as possible. Since $B$ is linearly independent over $[0,1]$ , $(f_1)|_{[j,j+1]} \neq 0$ , hence we must have $j < r$ . Since $f_1$ vanishes outside $[j,r]$ , while $\operatorname{supp} b_j$ intersects $[j,j+1]$ as well as $J$ , this would imply that

$$\mathrm{length}(f_1) < r-j < \mathrm{length}(b_j)$$

while $f_2 := f_1(\cdot+j) \in S^-$ and, on $[0,1]$, is in the span of $B$ with the coefficient of $b$ not zero. But then $B' := (B\backslash\{b\})\cup\{f_2\}$ would also supply a basis and $\mathrm{length}(B') <$ $\mathrm{length}(B)$, contradicting our choice of a basis of minimal length. |||

Suppose now that $f = \Sigma\, c_\varphi\, \varphi$ with the sum over all $\varphi = b_j$ with $j \in \mathbb{Z}$ and $b \in B$. Then the local linear independence just proved allows the conclusion that

(2.2) $$|c_{b_j}| \le \mathrm{const}\, \|f_{|[j,j+R]}\|_\infty$$

for some constant independent of $b$ and $j$.

Since $S_{|[-R,0]}$ is finite dimensional, we can choose a finite dimensional subspace $T$ of $S$ which is carried faithfully onto $S_{|[-R,0]}$ by the restriction map $f \longmapsto f_{|[-R,0]}$. For $s \in S$, let $s_T$ be the unique element of $T$ which agrees with $s$ on $[-R,0]$. Then

$$(s-s_T)\chi_{[0,\infty]} \in S^-$$

by Lemma 3, hence on $[0,\infty)$,

(2.3) $$s = s_T + \Sigma_{j\ge 0}\, \Sigma_{b\in B}\, c_{b_j}\, b_j$$

for suitable coefficients $c_{b_j}$. Since $S_{|[0,\infty)}$ is infinite dimensional by (2.1.v), this shows that $B \ne \emptyset$ and that $S^-$ is also infinite dimensional. Further, since the linear space of all linear maps from $T$ to $S_{|[-R,0]}$ has dimension $(\dim T)^2$, the maps

$$T \longrightarrow S_{|[-R,0]}: f \longmapsto f(\cdot - a)_{|(-R,0]}\ , \quad a \in A$$

must be linearly dependent whenever $A \subset \mathbb{Z}$ and $|A| > (\dim T)^2$. This together with Lemma 3 implies that, for any $A \subset \mathbb{Z}$ with more than $(\dim T)^2$ elements, there exist coefficients $c_a$, $a\in A$ so that

(2.4) $$\max c_a = \max |c_a| = 1\ , \text{ and}$$
$$\chi_{[0,\infty)}\, \Sigma_{a\in A}\, c_a f(\cdot - a) \in S^- \text{ for all } f \in T\ .$$

**Proof of Theorem 1.** Let $\tilde{g}$ be the continuous piecewise linear function with integer breakpoints which has the value $1$ at $x = 1, 2, 3$ and vanishes at all other integers. Since $\tilde{g} \in C_0(\mathbb{R})$, there is $\tilde{s}_h \in S_h$ such that $\tilde{s}_h$ converges to $\tilde{g}$ uniformly on $\mathbb{R}$.

From now on, we consider only $h \in 1/\mathbb{N}$. We write $s_h := \sigma_h \tilde{s}_h$ and $g_h := \sigma_h \tilde{g}$. Then $\|g_h - s_h\| \longrightarrow 0$ as $h \longrightarrow 0$. Since $s_h \in S$, we have $s_h = t_h + s_h^-$ on $[0,\infty)$, with $t_h \in T$ and $s_h^- \in S^-$. Now choose $A \subset 8\mathbb{N}/h$ as above. Then the functions $g_h(\cdot - a)$, $a \in A$, are supported in $[0,\infty)$ and have disjoint supports. We define $G_h := \Sigma_{a\in A}\, c_a\, g_h(\cdot - a)$, with the $c_a$ as given by (2.4). Then $f_h := \Sigma_{a\in A}\, c_a\, s_h^-(\cdot - a)$ is in $S^-$, and $\|f_h - G_h\| \longrightarrow 0$ as $h \longrightarrow 0$.

Now, because of (2.4), $G_h$ is identically one on an interval of the form $[m_h - 1/h,\, m_h + 1/h]$. Hence, with $F_h := f_h(\cdot + m_h)$, we have

$$\|F_h - 1\|_{[-1/h,1/h]} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0 .$$

Since $f_h \in S^-$, we can write

$$F_h =: \Sigma_{j\in\mathbb{Z}}\, \Sigma_{b\in B}\, c_{b_j}(h)\, b_j .$$

In view of (2.2),

(2.5) $$|c_{b_j}(h)| \le \text{const} \quad \text{for all } b,\ j,\ h .$$

We can take therefore a subsequence $h'$ of $h$ so that, for all $b$ and $j$, $c_{b_j}(h')$ converges as $h' \longrightarrow 0$ to some $c_{b_j}$. It follows that

$$\Sigma_{j\in\mathbb{Z}}\, \Sigma_b\, c_{b_j}\, b_j$$

converges uniformly on compact sets to some function $F \in S$. Since we also have that $F_h$ converges to $F$ uniformly on compact sets, we must have $F = 1$. Thus $(c_{b_j} b_j)_{b\in B,\, j\in\mathbb{Z}}$ is a partition of unity in $S$. It is good because of (2.5). |||

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