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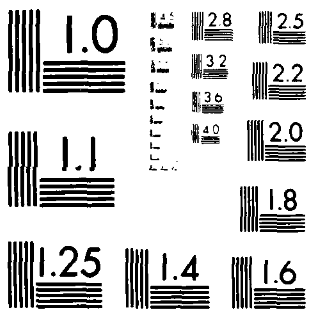

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A TRIVARIATE CLOUGH-TOCHER SCHEME FOR TETRAHEDRAL DATA

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A Trivariate Clough-Tocher Scheme for Tetrahedral Data

Peter Alfeld*

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Abstract

An interpolation scheme is described for values of position, gradient and Hessian at scattered points in three variables. The domain is assumed to have been tesselated into tetrahedra. The interpolant has local support, is globally once differentiable, piecewise polynomial, and reproduces polynomials of degree up to three exactly. The scheme has been implemented in a **FORTRAN** research code.

AMS (MOS) Subject Classification: 65D05

Key Words: Scattered Data, Trivariate Interpolation

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SIGNIFICANCE AND EXPLANATION

This report describes an interpolation scheme for scattered data in three independent variables, as they may arise, e.g. in the desciption of temperature or pressure in a three-dimensional solid. The interpolant will be once differentiable. The user must supply function values at the scattered points and also values of first and second order derivatives. The domain of the interpolant is the convex hull of the given points and is assumed to have been tesselated with tetrahedra. The scheme is piecewise polynomial, and reproduces all polynomials of degree up to three exactly. A non-portable **FORTRAN** research code is available from the author.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A Trivariate Clough-Tocher Scheme for Tetrahedral Data

Peter Alfeld*

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1. Introduction

Multivariate interpolation problems arise in many situations where physical phenomena involving several space variables (and possibly time as well) are modeled, designed, analyzed, or simulated. One application might be temperature as a (trivariate) function of the three spatial variables. The analyst is given a set of four-dimensional data through which he wishes to pass a smooth trivariate function. In a design context, he may wish to alter the data interactively until a satisfactory function has been found. Because the geometric information may be located arbitrarily in four-dimensional space, the interpolation scheme must be able to handle such data. A frequently used approach consists of smoothly piecing together interpolants defined only on parts of the underlying domain. Such schemes have the property of being local, i.e. the value of the interpolant at a given point is dependent only upon a small amount of data close to the point of evaluation. This has two important practical advantages: Firstly, the cost of evaluation of a scheme is largely independent of the amount of data, and, secondly, a change in the data alters the interpolant only in a small region around those points where the change takes place.

We will assume that the domain of interest has been tesselated into tetrahedra, with the vertices of the tetrahedra being the data points. How the tesselation may be accomplished is an important and difficult question which which will not be addressed in this

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paper. However, see Barnhill and Little, 1984, for some answers to this question. An interpolant that is defined piecewise on tetrahedra is called *tetrahedral*.

Important attributes of a scheme include the maximum degree of the derivatives that have to be supplied at the data points and the degree of differentiability of the interpolant. We use the term *C^r scheme for C^s data* to describe a globally r-times differentiable scheme that requires derivatives through s-th order as data. Usually, a user requires his scheme to be at least C^1, but is unable to provide data of degree larger than 0.

Several tetrahedral schemes are known. Alfeld, 1984a, describes a C^1 scheme for C^1 data. Barnhill and Little, 1984, describe a similar scheme that requires some auxilliary data. In another paper, Alfeld, 1984b, describes a class of C^m schemes for C^m data, for arbitrary m and in arbitrarily many variables.

However, all of the existing tetrahedral schemes yield rational interpolants. One would like to have polynomial interpolants for several reasons. Firstly, they are more efficiently evaluated. Secondly, they are more easily integrated. This is useful, for example, in the finite element technique for the solution of partial differential equations or in certain techniques for the generation of missing derivative data (see Alfeld,1984c).

In this paper, we derive a C^1 scheme for C^2 data which yields an interpolant that is piecewise quintic on each tetrahedron. The scheme is modeled after the well-known bivariate Clough-Tocher scheme (Strang and Fix, 1973, p. 82). That scheme is a C^1 scheme for C^1 data. Our scheme requires C^2 data. Attempts to obtain a trivariate Clough-Tocher type scheme for C^1 data have been undertaken by the author and others, but to date have been unsuccessful.

However, even requiring C^2 data is a significant gain over what would be required for straight polynomial interpolation on a tetrahedron. Ženišek, 1973, has shown that such a scheme would require a polynomial of degree at least 9 (with 220 parameters) and at least C^4 data.

The paper is organized as follows: In section 2, an explicit expression for the bivariate Clough-Tocher scheme is derived. The purpose of that section is to introduce the appropriate machinery in a familiar context. Although the bivariate scheme has been used for a long time, to the author's knowledge this is the first time that explicit expressions are given. However, a reader familiar with barycentric coordinates may wish to read only step 0 in section 2 and then skip to section three which introduces the trivariate scheme. Section 4 briefly discusses computational aspects.

2. The Bivariate Clough-Tocher Scheme

The derivation of the trivariate Clough-Tocher scheme is quite involved algebraically, and difficult to illustrate graphically because of the three-dimensional domain. Therefore, in this section, the well-known bivariate Clough-Tocher scheme is derived using precisely the same approach that will also be employed for the derivation of the trivariate scheme. The next section will then amount to a description of the differences between the bivariate and trivariate schemes.

The bivariate Clough-Tocher scheme uses as data position and gradient at the vertices of a general triangle. The triangle is divided about its centroid into three subtriangles. We refer to the overall triangle as the *macrotriangle* and to its subtriangles as *microtri-*

angles. The interpolant is constructed to be a cubic polynomial on each microtriangle. It is differentiable over the macrotriangle. Continuity between macrotriangles is ensured by interpolation to the data. Differentiability between macrotriangles is forced by the requirement that the first order perpendicular cross-boundary derivative along edges be linear (instead of quadratic). The approach is illustrated in figure 1.

First, some concepts and notation have to be introduced. Three ingredients are fundamental to the construction: The Bézier form of a multivariate polynomial, the identification of the appropriate Bézier nets, and the identification of a particular parameter with each of the conditions defining the interpolant.

To define the Bézier form of a bivariate polynomial we consider a general triangle with vertices V_i, where $i = 1, 2, 3$. A general point P in $\Re^2$ is expressed in terms of *barycentric coordinates*, that is

$$P = \sum_{i=1}^{3} b_i V_i \quad \text{where} \quad \sum_{i=1}^{3} b_i = 1.$$

Any polynomial q of degree m, say, can be uniquely expressed in its *Bézier form* as

$$q(P) = \sum_{i+j+k=m} \frac{m!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

(see Farin, 1983). We will have to differentiate q. Let D denote any (unnormalized) directional derivative, i.e. $D = \frac{\partial}{\partial e}$ where e is a vector in $\Re^2$. Then

$$Dq(P) = \sum_{i+j+k=m-1} \frac{(m-1)!}{i!j!k!} \hat{c}_{ijk} b_1^i b_2^j b_3^k \tag{2.1}$$

where

$$\hat{c}_{ijk} = m(c_{i+1,j,k} Db_1 + c_{i,j+1,k} Db_2 + c_{i,j,k+1} Db_3) \tag{2.2}$$

(Subscripts are separated by commas whenever one or more of them cannot be written as just one character).

The c_{ijk} are the *Bézier ordinates* of q. They can be associated with the points $C_{ijk} := (iV_1 + jV_2 + kV_3)/m$. The points $(C_{ijk}, c_{ijk}) \in \Re^3$are said to be the *control points* of the Bézier polynomial, and the set of control points constitutes the *Bézier net* of q. If two polynomials are defined on two neighboring triangles, then the resulting piecewise function is continuous across the interface iff the Bézier ordinates on the interface coincide. Thus Bézier ordinates on the interface can be identified and the union of the two Bézier nets can be considered the Bézier net of the piecewise polynomial function defined on the union of the two triangles. This idea will be expanded below.

We now proceed in 6 steps:

Step 0: *Forcing Continuity on the Macrotriangle.* (This is step 0 because the task is accomplished by choosing suitable notation rather than doing analysis.)

We express the centroid of the macrotriangle by $V_4 = (V_1 + V_2 + V_3)/3$ and denote (internal and external) edges by $e_{ij} := V_j - V_i$ (i.e. e_{ij} is the line-segment from V_i to V_j) where $i, j = 1, 2, 3, 4$. The microtriangle where $b_i = 0$ ($i = 1, 2, 3$) is denoted by T_i.

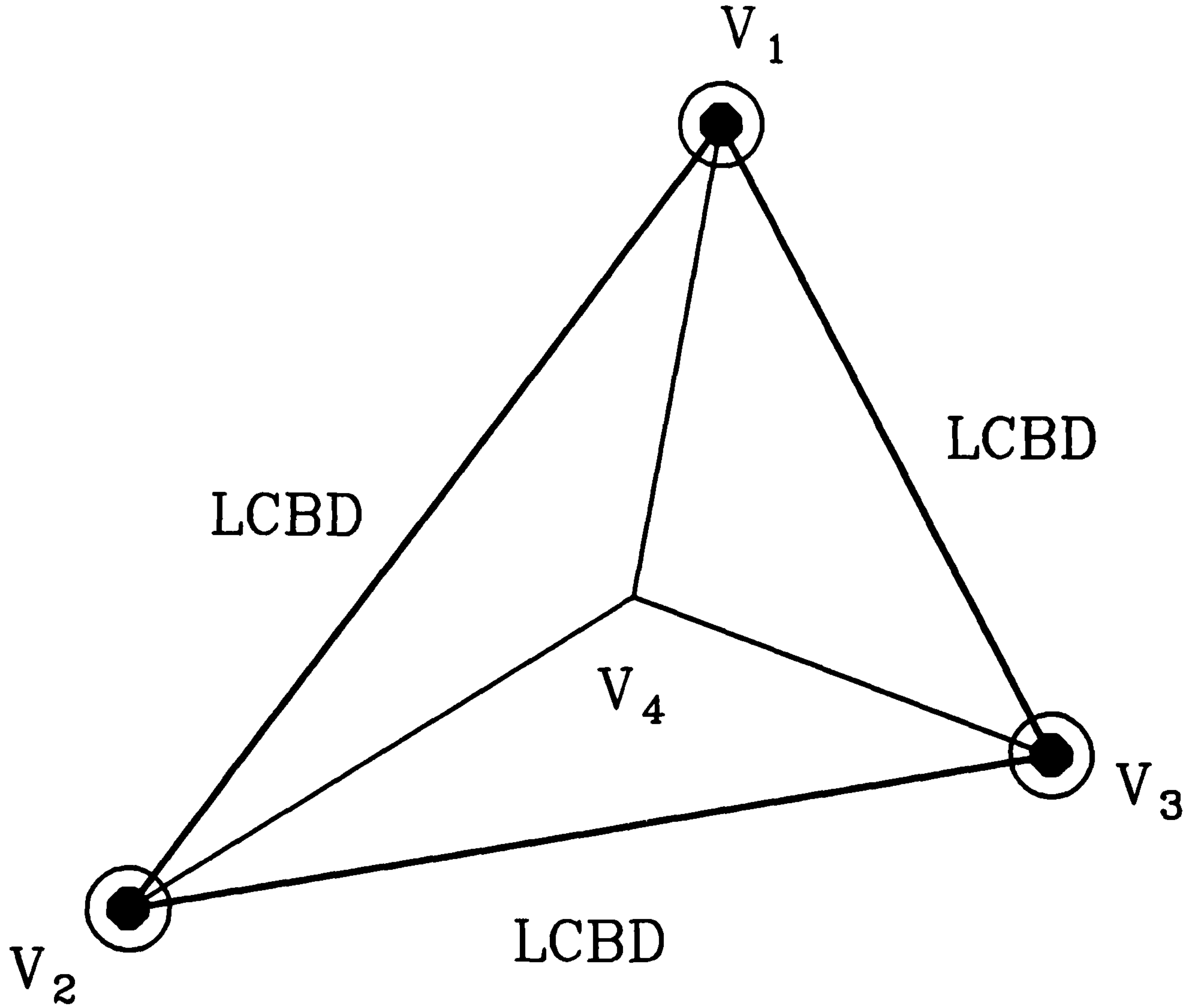

Figure 1
Stencil for Bivariate Scheme

position given
gradient given
LCBD linear cross boundary derivative

We now extend the notion of barycentric coordinates. Any point P in the macrotriangle can be expressed uniquely as

$$P = \sum_{i=1}^{4} b_i V_i \quad \text{where} \quad \sum_{i=1}^{4} b_i = 1, \ \prod_{i=1}^{3} b_i = 0, \quad \text{and} \quad b_i \geq 0, \ i = 1,2,3,4.$$

The b_i can be thought of as piecewise linear cardinal functions on the triangulation of the macrotriangle into microtriangles. On each microtriangle, at least one of b_1, b_2, or b_3, will be zero, and the other coordinates will be the ordinary barycentric coordinates from that microtriangle that contains P.

We will have to differentiate on the microtriangles in the direction of (internal and external) edges. Consider a general triangle spanned by the set $\{V_a, V_b, V_c\}$ and suppose that $\{i,j,k\} \subseteq \{a,b,c\}$. Then it follows from the cardinal properties of barycentric coordinates that

$$\frac{\partial b_k}{\partial e_{ij}} = \delta_{kj} - \delta_{ki} \tag{2.3}$$

where δ is the Kronecker delta.

The piecewise cubic interpolant (on a general macrotriangle) can now be written as

$$q(P) = \sum_{i+j+k+l=3} \frac{3!}{i!j!k!l!} c_{ijkl} b_1^i b_2^j b_3^k b_4^l \tag{2.4}$$

where, by convention, $0^0 = 1$.

The resulting generalized Bézier net is illustrated in figure 2. The function q defined in (2.4) is a continuous piecewise cubic function on the macrotriangle with 19 parameters c_{ijkl}.

Note that internal continuity (i.e. continuity everywhere on the macrotriangle) has been obtained simply by the judicious choice of a notation that makes Bézier control points coincide on internal edges.

Step 1: *Forcing Internal Differentiability.* In this step, we impose conditions on the Bézier ordinates that will ensure that the piecewise cubic function is differentiable everywhere on the macrotriangle. The standard way of forcing differentiability between triangles is to require that certain groups of 4 Bézier control points each be colinear (see Farin, 1983). We rederive the conditions for the present special case because the technique is generally useful, and because it is needed here in a slightly more general context (i.e. differentiability at the centroid).

Consider for example edge e_{14}. Any cross-boundary derivative on that edge must be independent of whether it be evaluated on the microtriangle T_2 or the microtriangle T_3. It is sufficient to consider any one particular cross-boundary derivative, because any other one can be expressed as a linear combination of the considered one and the tangential derivative (in the direction of edge e_{14}) which is continuous because the function under construction has been forced to be continuous everywhere on the macrotriangle. So consider for example the derivative in the direction of edge e_{34}.

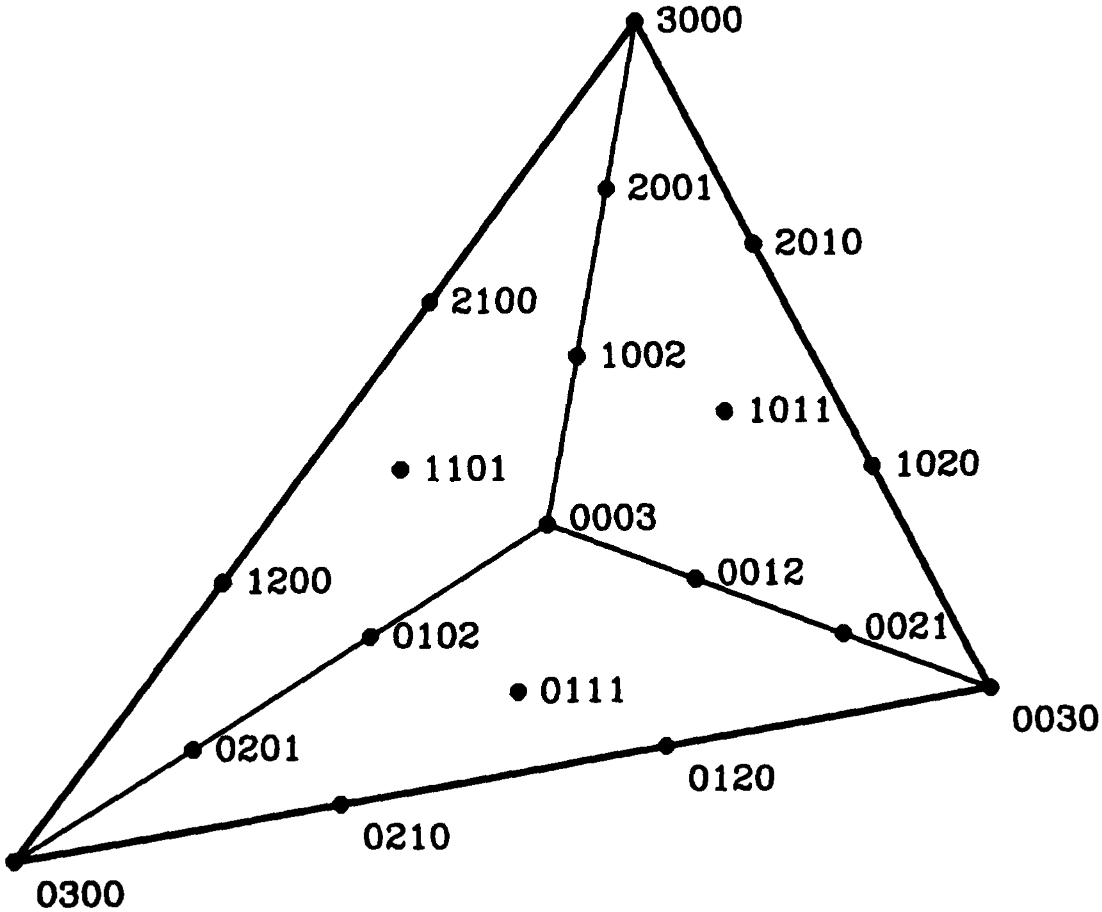

Figure 2

The generalized Bézier Net

In order to obtain an appropriate condition we carry out the differentiation on both T_2 and T_3, restrict both of the resulting expressions to edge e_{14}, and equate the coefficients of the resulting quadratic.

We differentiate on T_2 directly using (2.1-3) and on T_3 by expressing $e_{34} = e_{41} + e_{42}$, and then using (2.1-3). Carrying out this plan yields the equations

$$\begin{aligned} c_{0003} &= (c_{1002} + c_{0102} + c_{0012})/3 \\ c_{1002} &= (c_{1101} + c_{1011} + c_{2001})/3 \\ c_{2001} &= (c_{2100} + c_{2010} + c_{3000})/3 \end{aligned}$$

Proceeding similarly on the other two internal edges yields the conditions:

$$\begin{aligned} c_{0102} &= (c_{1101} + c_{0111} + c_{0201})/3 \\ c_{0201} &= (c_{1200} + c_{0300} + c_{0210})/3 \\ c_{0012} &= (c_{1011} + c_{0111} + c_{0021})/3 \\ c_{0021} &= (c_{1020} + c_{0120} + c_{0030})/3 \end{aligned}$$

Thus, in the present special case, the differentiability conditions simply require that Bézier ordinates on internal edges (in the interior of the macrotriangle) be the averages of the neighboring Bézier ordinates in the directions given by the internal edges! Also notice that in each of the above conditions one of the Bézier ordinates plays a special role. Each equation can be thought of as expressing the parameter on the left in terms of the right hand side which involves parameters that are still to be determined.

At this stage, q is a piecewise polynomial function that has 12 parameters still available and that is continuously differentiable on the macrotriangle.

The C^1 conditions have a remarkable consequence: At the centroid, the function q is actually C^2! This fact was first observed by (Farin, 1983) but the following argument will be useful in the trivariate context: Consider for example the microtriangles T_1 and T_3. We have to show that three independent second order derivatives are continuous when going through the centroid from one microtriangle to the other. So consider the derivatives

$$\frac{\partial^2}{\partial e_{24}\partial e_{14}}, \quad \frac{\partial^2}{\partial e_{24}\partial e_{34}}, \text{ and } \frac{\partial^2}{\partial e_{14}\partial e_{34}}.$$

Of these, the first two are tangential in the direction of e_{24} which is shared by T_1 and T_3. They are hence continuous. Similarly, the third derivative is continuous between the microtriangles T_1 and T_2 since it is tangential in the direction e_{34}. It is also continuous between T_2 and T_3 since it is tangential in the direction of e_{14}. (Obviously, all mixed partial derivatives commute). Hence it is continuous between T_1 and T_3. This completes the argument.

Step 2: *Interpolation to Vertex Data*. The key observation is that we enforced differentiability at the vertices of the macrotriangle and that hence we need to interpolate to each datum only on any one particular microtriangle that is convenient for that datum. With this observation, interpolation is straightforward. For convenience, we think of the data as being given in terms of some primitive function F. Proceeding similarly as in (Barnhill and Farin, 1981) we obtain:

$$
\begin{aligned}
c_{3000} &= F(V_1) \\
c_{2100} &= (\frac{\partial F}{\partial e_{12}}(V_1) + 3c_{3000}))/3 \\
c_{2010} &= (\frac{\partial F}{\partial e_{13}}(V_1) + 3c_{3000}))/3 \\
c_{0300} &= F(V_2) \\
c_{1200} &= (\frac{\partial F}{\partial e_{21}}(V_2) + 3c_{0300}))/3 \\
c_{0210} &= (\frac{\partial F}{\partial e_{23}}(V_2) + 3c_{0300}))/3 \\
c_{0030} &= F(V_3) \\
c_{1020} &= (\frac{\partial F}{\partial e_{31}}(V_3) + 3c_{0030}))/3 \\
c_{0120} &= (\frac{\partial F}{\partial e_{32}}(V_3) + 3c_{0030}))/3
\end{aligned}
$$

At this stage, we have obtained an interpolant that is differentiable on the macrotriangle. The construction so far has not taken into account smoothness requirements between macrotriangles, and there are still three parameters at our disposal.

Step 3: *Forcing Global Smoothness.* We wish to apply the Clough-Tocher scheme on each macrotriangle in a given triangulation such that the overall function is differentiable everywhere in the domain covered by the triangulation. Along an external edge of a macrotriangle, the interpolant reduces to a univariate cubic which is uniquely determined by the four data at the endpoints of the edge. At this stage, the scheme is therefore globally continuous (but not differentiable).

To force differentiability it is again sufficent to consider one particular cross-boundary derivative across an edge joining two macrotriangles. However, the only meaningful direction is perpendicular across the edge because that edge is the only object common to the triangles. Since the interpolant is piecewise cubic the cross-boundary derivative restricted to the edge will in general be quadratic. The data given at the edge impose only two conditions. Therefore, a third one must be imposed consistently on both macrotriangles. It is natural to require that the perpendicular cross-boundary derivative be linear. This choice maintains the maximum degree of precision (quadratic) that is attainable.

We proceed as follows: Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. A normal to e_{jk} can then be written as

$$n_i = e_{j4} + \gamma_i e_{jk} \tag{2.5}$$

where γ_i is determined by the requirement that $n_i \circ e_{jk} = 0$. i.e. $\gamma_i = -(e_{j4} \circ e_{jk})/(e_{jk} \circ e_{jk})$. We then differentiate q in the direction of n_i using (2.5) and (2.1-3), restrict the derivative to edge e_{jk}, and set the leading coefficient of the resulting quadratic equal to zero. This yields the conditions

$$
\begin{aligned}
c_{0111} = &\big(\gamma_1(-c_{0300} + 3c_{0210} - 3c_{0120} + c_{0030}) \\
&+ (-c_{0300} + 2c_{0210} - c_{0120} + c_{0021} + c_{0201})\big)/2 \\
c_{1011} = &\big(\gamma_2(-c_{0030} + 3c_{1020} - 3c_{2010} + c_{3000}) \\
&+ (-c_{0030} + 2c_{1020} - c_{2010} + c_{2001} + c_{0021})\big)/2 \\
c_{1101} = &\big(\gamma_3(-c_{3000} + 3c_{2100} - 3c_{1200} + c_{0300}) \\
&+ (-c_{3000} + 2c_{2100} - c_{1200} + c_{2001} + c_{0201})\big)/2
\end{aligned}
$$

At this stage, we have achieved our objective: a piecewise cubic polynomial interpolant that is gobally differentiable. The construction used up all parameters, i.e. the interpolant is unique. In the trivariate case, however, there will be a family of smooth interpolants. In that case, disposing of the remaining parameters will constitute step 5.

The above equations for the coefficients of q describe a linear system. However, in this case, the system happens to be triangular! Thus the coefficients can be computed by Forward Elimination, e.g. in the following sequence: $c_{3000}, c_{2010}, c_{2100}, c_{0300}, c_{1200}, c_{0210}, c_{0030}, c_{1020}, c_{0120}, c_{2001}, c_{0201}, c_{0021}, c_{1101}, c_{1011}, c_{0111}, c_{1002}, c_{0102}, c_{0012}, c_{0003}$.

Each of the parameters is naturally associated with a particular condition, and we can think of any condition as eliminating the corresponding parameter. The sequence of evaluation, and the type of condition corresponding to a parameter is illustrated in figure 3.

3. Construction of the Trivariate Scheme

In this section, we follow the template provided by the preceding one. We assume that we are given values of position, gradient, and Hessians at scattered points in $\Re^3$ (see figure 4. for the stencil). We also assume that the domain has been tesselated into tetrahedra, and only consider the interpolation problem on a general tetrahedron.

The system of equations determining the interpolant has been derived and solved using the symbol manipulation language **REDUCE** (A.C. Hearn, 1983). Writing down all the equations and their solution in the text would be tedious and error prone. Instead, the **REDUCE** output of the solution has been reproduced photographically in the appendix. For a thorough understanding of the material in this section, the reader should familiarize himself with the organization of the appendix, and keep refering to it when reading this section. For a less thorough understanding, this section may be read by itself.

3.0 Notation and Internal Continuity

We consider a general macrotetrahedron with vertices V_1, V_2, V_3 and V_4, and denote the centroid by $V_5 = (V_1 + V_2 + V_3 + V_4)/4$. The location of a general point P is expressed as

$$P = \sum_{i=1}^{5} b_i V_i \quad \text{where} \quad \sum_{i=1}^{5} b_i = 1, \ \prod_{i=1}^{4} b_i = 0, \quad \text{and} \quad b_i \geq 0, \ i = 1,2,3,4,5$$

and the function to be constructed is of the form

$$q(P) = \sum_{i+j+k+l+m=5} \frac{5!}{i!j!k!l!m!} c_{ijklm} b_1^i b_2^j b_3^k b_4^l b_5^m$$

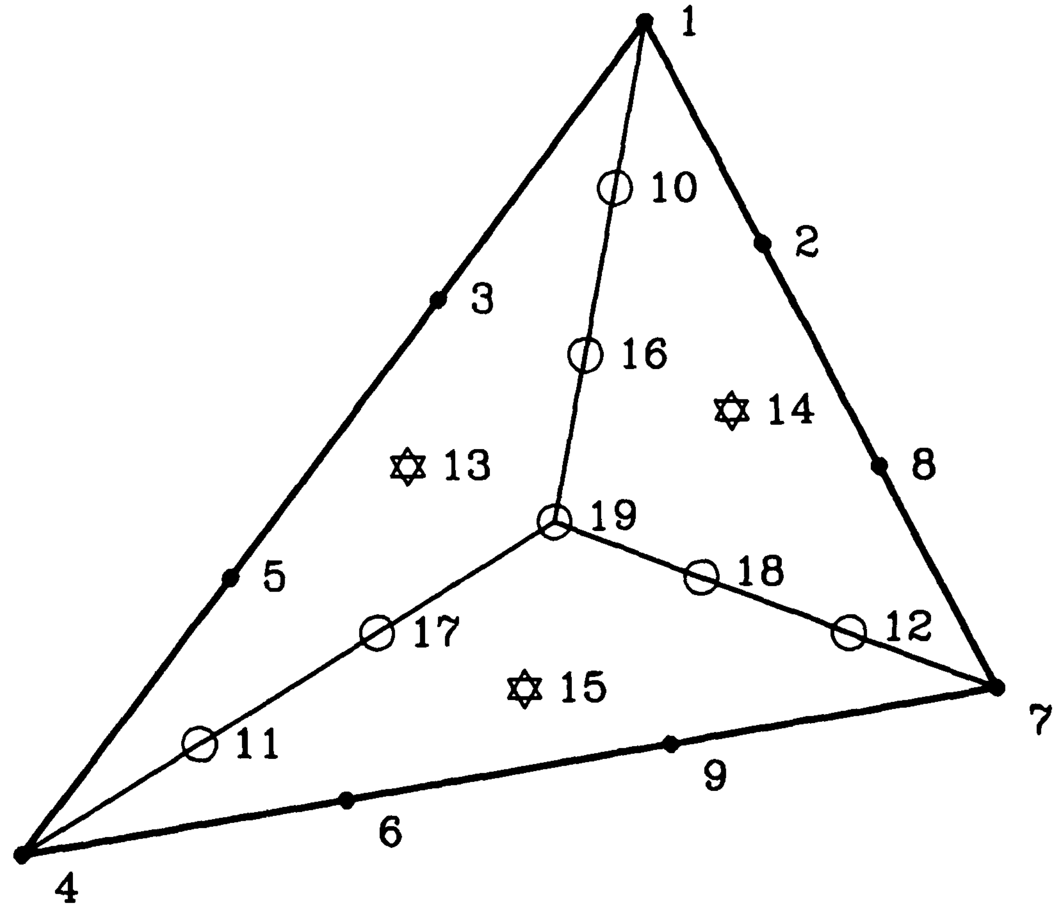

Figure 3
Role of Coefficients and
Sequence of Evaluation

• interpolation
○ internal smoothness
✡ external smoothness

Note that our interpolant will be piecewise quintic. Since the data include second order derivatives, on any edge there are six degrees of freedom (position and first and second order tangential derivatives at each vertex). Thus, using any degree less than 5 would imply that data could not be supplied independently at the vertices. Also, since an arbitrary number of edges may emanate from a given vertex, the specification of derivative data — which usually is a necessary preprocessing step — could not be carried out on an individual tetrahedron, or a set of points close to the given vertex, but rather has to take into account the data at all other points, making derivative generation a process. As was pointed out in the introduction, it would have been preferable to build a C^1 scheme for C^1 data, which could have utilized piecewise *cubic* functions, but attempts to do so have not succeeded to date.

At this stage, q is continuous on the macrotetrahedron, and contains 121 free parameters.

3.1 Internal Differentiability

As in the bivariate case, we obtain the conditions that Bézier ordinates on the interfaces between microtetrahedra should equal the average of their three neighbors (in the directions of the internal edges). For details see equations no. 53-78, 91-111, and 116-121 in the appendix.

At this stage, q is differentiable everywhere on the macrotetrahedron, and contains $121 - 53 = 68$ free parameters. However, there is a subtle difficulty: We contemplate interpolation to vertex data, including second order derivatives. As in the bivariate case, we would like to interpolate to data only on suitable microtetrahedra, and enforce interpolation on the other ones by smoothness. Thus we have to enforce second order differentiability at the vertices of the macrotetrahedron, although the scheme under construction is only C^1.

Remarkably, it turns out that the scheme is already C^2 at the vertices, in spite of our not having made any effort to achieve this. The crucial fact is that three (rather than two) microtetrahedra meet at each vertex. The argument is similar to the bivariate case where we obtained serendipitous C^2 smoothness at the centroid of the macrotriangle:

Consider, for example, vertex V_1 and continuity of second order derivatives between the microtetrahedra T_2 and T_3. We need to find six independent second order derivatives that are continuous at V_1. The interface between T_2 and T_3 is the triangle spanned by $\{V_1, V_4, V_5\}$. Because the scheme is already C^1, all second order derivatives tangential in that triangle will be continuous. Thus for example

$$\frac{\partial^2}{\partial e_{14} \partial e_{14}}, \frac{\partial^2}{\partial e_{14} \partial e_{15}}, \frac{\partial^2}{\partial e_{15} \partial e_{15}}, \frac{\partial^2}{\partial e_{12} \partial e_{14}}, \frac{\partial^2}{\partial e_{12} \partial e_{15}}$$

are all continuous at V_1. We need one more derivative. This is given by

$$\frac{\partial^2}{\partial e_{25} \partial e_{35}}$$

which is continuous because it is tangential between T_2 and T_4 and also between T_3 and T_4, and hence between T_2 and T_4. This completes the argument.

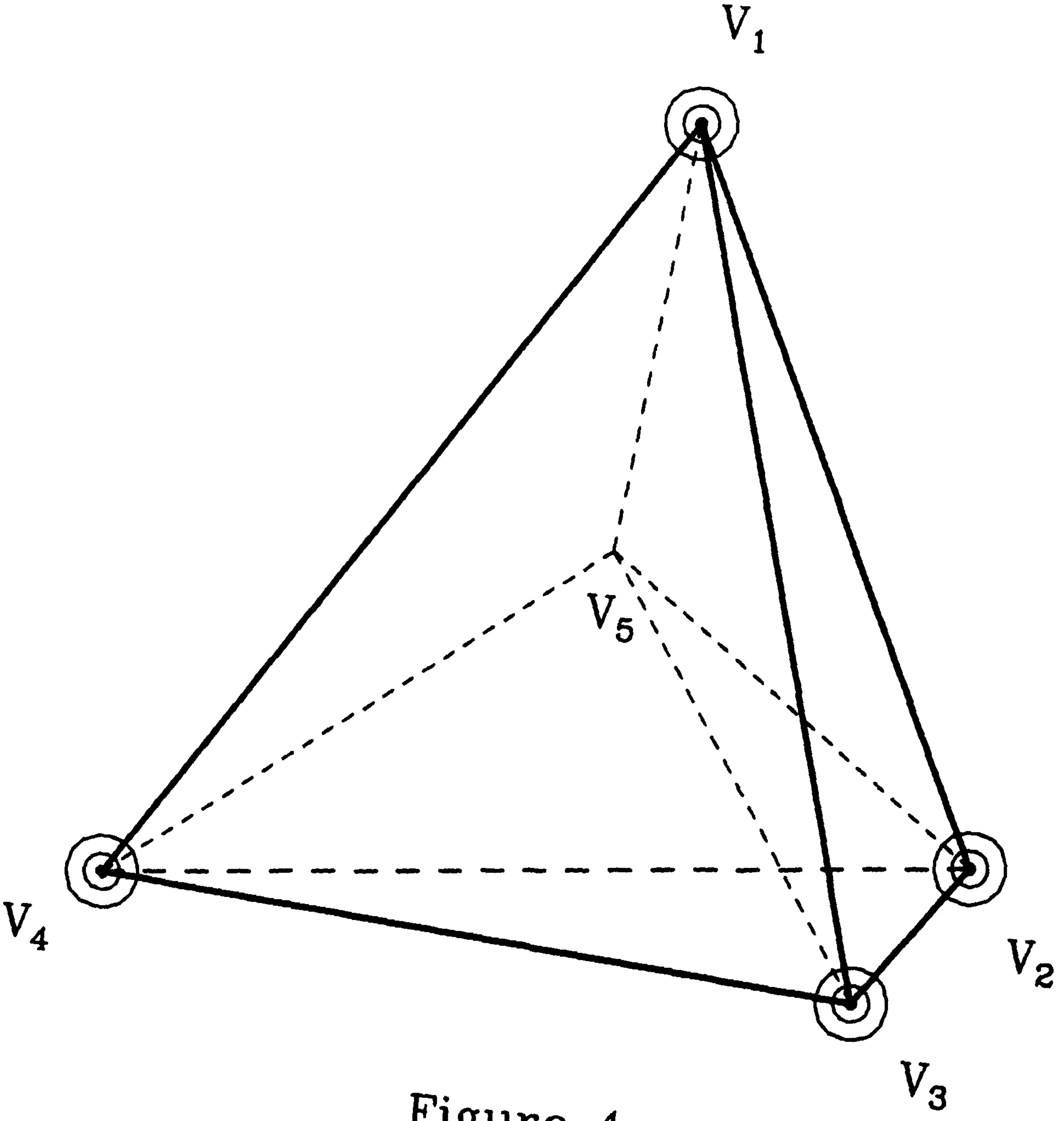

Figure 4
The Stencil for the Trivariate Scheme

 position, gradient, and Hessian given

It is noteworthy, although not relevant to our construction, that by a similar procedure the function q can be shown to be C^3 at the centroid (where all four microtetrahedra meet).

3.2 Interpolation to Vertex Data

Having forced the appropriate degree of internal smoothness, interpolation to the vertex data is again straightforward. 40 parameters are required, see equations 1-40 in the appendix. At this stage, the scheme does interpolate and is internally C^1. We have to use some of the remaining 28 parameters to force external smoothness.

3.3 Intertetrahedral Smoothness

External smoothness is built up starting at the edges of the macrotetrahedron and then proceeding to its faces.

Continuity across edges. This has already been achieved, since along any edge the function q reduces to a univariate quintic polynomial which is determined uniquely by the six data given at the vertices of the edge.

Differentiability across edges. Any cross-boundary derivative across an edge will in general be univariate quadratic. However, the vertex data imply only four conditions. The resulting ambiguity is removed by requiring that any first order derivative perpendicularly across an edge be in fact cubic. This condition is in terms of the edge and data on the edge only, and nence forces differentiability across the edges.

We exemplify the analysis by considering edge e_{12}. We need to consider derivatives in two independent directions perpendicularly across the edge, and construct one direction in each of the two microtetrahedra joining along the edge. For example, consider the microtetrahedron T_4. We express a normal to e_{12} as

$$n_{123} = e_{13} + \gamma_{123} e_{12}$$

where

$$n_{123} \circ e_{12} = 0, \quad \text{i.e.} \quad \gamma_{123} = -\frac{e_{13} \circ e_{12}}{e_{12} \circ e_{12}}.$$

Requiring that $\partial q / \partial n_{123}$ be cubic yields a condition for c_{22100}. Similarly, considering e_{12} a part of T_3 yields a condition for c_{22010}. Ten more similar conditions are obtained by considering the other five edges of the macrotetrahedron. See equations 41-52 in the appendix for details.

Continuity across faces. This is implied by our course of action along the edges. On each face, the interpolant reduces to a specific bivariate quintic interpolant (Q_{18}, see Barnhill and Farin, 1981) which is determined uniquely by the vertex data.

Differentiability across faces. Consider a derivative perpendicular to a face. In general, this will be a bivariate quartic polynomial, having 15 degrees of freedom. Of these, 9 have been removed by vertex interpolation and three have been used in forcing the derivative to be cubic along edges. The last three degrees of freedom are removed by requiring the derivative on the face to equal a reduced cubic which interpolates to the nine data on the face (and is cubic along edges), and where the coefficient corresponding to the centroid of the face has been chosen so as to minimize the sum of the squares of the third derivatives in the directions of the edges.

Specifically, if the cross-boundary derivative on a face spanned by V_i, V_j, V_k is expressed as

$$\sum_{\alpha+\beta+\gamma=3} \frac{3!}{\alpha!\beta!\gamma!} d_{\alpha\beta\gamma} b_i^{\alpha} b_j^{\beta} b_k^{\gamma}$$

then

$$d_{111} = \big(3(d_{210} + d_{201} + d_{120} + d_{102} + d_{021} + d_{012}) - 2(d_{003} + d_{030} + d_{300})\big)/12$$

This choice maintains quadratic precision of the derivative (and cubic precision of the overall interpolant q).

We need an expression for a direction perpendicular across a face. Consider for example face 1. The normal is expressed as

$$n_1 = e_{25} + \alpha_1 e_{23} + \beta_1 e_{34}$$

where α_1 and β_1 are determined by the linear equations

$$n_1 \circ e_{23} = n_1 \circ e_{34} = 0.$$

This process yields three conditions on each face. Detailed expressions are listed in equations 79-90 of the appendix.

At this stage, we have an internally and externally differentiable interpolant which, however, still contains four free parameters.

3.4 Condensation of Parameters

We still have to dispose of the four parameters c_{01112}, c_{10112}, c_{11012}, and c_{11102}. In choosing them, it is desirable to meet two objectives:

(1) The scheme should reproduce exactly polynomials of a degree as high as possible (i.e. cubics).

(2) The interpolant should be independent of a relabeling of the vertices.

These objectives are achieved by picking the remaining parameters so as to minimize $\left(D_i^4 q(V_5)\right)^2$ for each T_i where, for example

$$D_i = \left(\frac{\partial}{\partial e_{25}} + \frac{\partial}{\partial e_{35}} + \frac{\partial}{\partial e_{45}}\right)$$

See equations 112 through 115 in the appendix for the detailed conditions.

3.5 Solution of the Linear System

The conditions derived in the preceding subsections constitute a linear system of 121 equations. In the bivariate case, the corresponding system happened to be triangular. In this case, it turns out that only 106 equations can be solved by Forward Substitution, and that the remaining 15 equations form a full matrix. The overall system was solved using **REDUCE**, and the results are listed in the appendix.

3.6 Precision of the Trivariate Scheme

An important attribute of any interpolation scheme is its degree of precision, i.e. the maximum degree of a polynomial that is reproduced exactly by the scheme if the data are generated by the polynomial. Tracing back through the construction it is easy to see that our scheme is precise for cubic polynomials. On the other hand, by construction, derivatives perpendicular across a face are reduced cubics with only quadratic precision. Therefore, the overall scheme cannot have more than cubic precision. A parameter count might suggest that a higher degree is possible: There are 40 vertex data, and a trivariate quartic has only 35 coefficients. However, the quartic $b_1 b_2 b_3 b_4$ generates homogeneous data at the vertices. Hence any linear interpolation process will yield the zero function as the interpolant. Thus our scheme has the maximum attainable degree of precision.

4. Computational Aspects

In this section, some practical aspects are discussed as they have been incorporated into a (non-portable) **FORTRAN** research code that can be obtained from the author.

1. Construction of the Code. The present scheme shares with most other local multivariate schemes the drawback of great computational complexity. The feasibility of programming such schemes from printed formulas is called into question. Instrumental in the writing of the code was a feature of the symbol manipulation language **REDUCE** that allows it to write formulas in **FORTRAN** syntax after deriving them from much simpler ingredients.

2. Labeling of coefficients. Having coefficients with five subscripts is convenient for the analysis but not suitable for a computer implementation. One would like to use a linearly indexed array for storing the coefficients. We address the problem somewhat more generally:

Let I denote the set of all k-tuples of non-negative integers that add up to m. There are $M := \binom{m+k-1}{k-1}$ elements in I. A bijective correspondence between I and $\{1, 2, \cdots, M\}$ is defined in terms of binomial coefficients by:

$$N(i_1, \cdots, i_k) = 1 + \binom{i_{k-1} + 0}{1} + \binom{i_{k-2} + i_{k-1} + 1}{2} + \cdots + \binom{i_1 + i_2 + \cdots i_{k-1} + k - 2}{k-1}$$

For the tetrahedral Clough-Tocher scheme, $m = k = 5$, $M = 126$, and the indices $N(1,1,1,1,2) = 50$, $N(1,1,1,2,1) = 99$, $N(1,1,2,1,1) = 98$, $N(1,2,1,1,1) = 95$, and $N(2,1,1,1,1) = 85$ are unused.

3. Identification of the microtetrahedron. Suppose the macrotetrahedron in which the point of evaluation, P, say, resides has been identified. (This is done by evaluating the barycentric coordinates with respect to the tetrahedron and checking that they are all non-negative.) Now the microtetrahedron containing P must be identified. Denote the barycentric coordinates by b_i, b_j, b_k, b_l where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and assume

$b_i = min\{b_i, b_j, b_k, b_l\}$. Then P is contained in T_i. Denoting the barycentric coordinates of P with respect to T_i by d_5, d_j, d_k, d_l it is easy to see that $d_5 = 4b_i$, $d_j = b_j - b_i$, $d_k = b_k - b_i$, $d_l = b_l - b_i$.

4. *Recursive Evluation of Interpolant.* The Bézier form of a polynomial allows for a particularly simple recursive form of the evaluation. For details see Farin, 1983.

5. *Efficient evaluation of coefficients.* Coefficients are evaluated only on macrotetrahedra where they are needed. The user has the option of storing them after evaluation, or reevaluating them when needed.

6. *Verification of Code.* The pilot code developed by the author has been tested by the analyzing tool **MICROSCOPE** (Alfeld and Harris, 1984) and it was verified that the code does indeed possess the smoothness, interpolation, and precision properties that are implied by the mathematical construction.

Conclusions

The scheme developed here is the first explicitly given piecewise polynomial C^1 interpolant for tetrahedral data. It has the drawback of requiring C^2 data.

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References

P. Alfeld (1984a) A Discrete C^1 interpolant for Tetrahedral Data, Rocky Mountain Journal of Mathematics, v. 14, No.1 pp.5-16

P. Alfeld (1984b), Multivariate Perpendicular Interpolation, to appear in the SIAM Journal on Numerical Analysis

P. Alfeld (1984c), Multivariate Scattered Data Derivative Generation by Functional Minimization, submitted for publication.

P. Alfeld and B. Harris (1984), **MICROSCOPE**: A Software System for Multivariate Analysis, submitted for publication

R.E. Barnhill and G.Farin (1981), C^1 Quintic Interpolation over Triangles: Two Explicit Representations, Int. J. for Num. Meth. in Eng. 17, pp. 1763-1778.

R.E. Barnhill and F.F. Little (1984), Three- and Four- Dimensional Surfaces, Rocky Mountain Journal of Mathematics, v. 14, Winter 1984, pp. 77-102

G. Farin (1983), Smooth Interpolation to Scattered 3D Data, in R.E. Barnhill and W. Boehm (ed.), Surfaces in Computer Aided Geometeric Design, North-Holland.

A.C. Hearn (1983), **REDUCE** User's Manual, Version 3.0, Rand Publication CP78 (4/83), The Rand Corporation, Santa Monica, CA 90406

D.E Knuth (1984), The TEXbook, Addison Wesley

G. Strang and G.J. Fix (1973), An Analysis of the Finite Element Method, Prentice Hall, 1973

A. Ženíšek, Polynomial Approximation on Tetrahedrons in the Finite Element Method, J. Approx. Theory, v. 7, pp. 334-351

Appendix: Coefficients of Interpolant

In this appendix we list the formulas as they have been computed by **REDUCE**. Partial derivatives of the Function F have been denoted by subscripts, for example

$$F_{14}(V_1) = \frac{\partial F}{\partial e_{14}}(V_1)$$

and

$$F_{1413}(V_1) = \frac{\partial^2 F}{\partial e_{13} \partial e_{14}}(V_1)$$

1: $c_{50000} = F(V_1)$
2: $c_{40010} = (5c_{50000} + F_{14}(V_1))/5$
3: $c_{40100} = (5c_{50000} + F_{13}(V_1))/5$
4: $c_{41000} = (5c_{50000} + F_{12}(V_1))/5$
5: $c_{30020} = (-20c_{50000} + 40c_{40010} + F_{1414}(V_1))/20$
6: $c_{30200} = (-20c_{50000} + 40c_{40100} + F_{1313}(V_1))/20$
7: $c_{32000} = (-20c_{50000} + 40c_{41000} + F_{1212}(V_1))/20$
8: $c_{30110} = (-20c_{50000} + 20c_{40100} + 20c_{40010} + F_{1413}(V_1))/20$
9: $c_{31010} = (-20c_{50000} + 20c_{41000} + 20c_{40010} + F_{1412}(V_1))/20$
10: $c_{31100} = (-20c_{50000} + 20c_{41000} + 20c_{40100} + F_{1312}(V_1))/20$
11: $c_{05000} = F(V_2)$
12: $c_{04010} = (5c_{05000} + F_{24}(V_2))/5$
13: $c_{04100} = (5c_{05000} + F_{23}(V_2))/5$
14: $c_{14000} = (5c_{05000} + F_{21}(V_2))/5$
15: $c_{03020} = (-20c_{05000} + 40c_{04010} + F_{2424}(V_2))/20$
16: $c_{03200} = (-20c_{05000} + 40c_{04100} + F_{2323}(V_2))/20$
17: $c_{23000} = (40c_{14000} - 20c_{05000} + F_{2121}(V_2))/20$
18: $c_{03110} = (-20c_{05000} + 20c_{04100} + 20c_{04010} + F_{2423}(V_2))/20$
19: $c_{13010} = (20c_{14000} - 20c_{05000} + 20c_{04010} + F_{2421}(V_2))/20$
20: $c_{13100} = (20c_{14000} - 20c_{05000} + 20c_{04100} + F_{2321}(V_2))/20$
21: $c_{00500} = F(V_3)$
22: $c_{00410} = (5c_{00500} + F_{34}(V_3))/5$
23: $c_{01400} = (5c_{00500} + F_{32}(V_3))/5$
24: $c_{10400} = (5c_{00500} + F_{31}(V_3))/5$
25: $c_{00320} = (-20c_{00500} + 40c_{00410} + F_{3434}(V_3))/20$
26: $c_{02300} = (40c_{01400} - 20c_{00500} + F_{3232}(V_3))/20$

27: $c_{20300} = (40c_{10400} - 20c_{00500} + F_{3131}(V_3))/20$

28: $c_{01310} = (20c_{01400} - 20c_{00500} + 20c_{00410} + F_{3432}(V_3))/20$

29: $c_{10310} = (20c_{10400} - 20c_{00500} + 20c_{00410} + F_{3431}(V_3))/20$

30: $c_{11300} = (20c_{10400} + 20c_{01400} - 20c_{00500} + F_{3231}(V_3))/20$

31: $c_{00050} = F(V_4)$

32: $c_{00140} = (5c_{00050} + F_{43}(V_4))/5$

33: $c_{01040} = (5c_{00050} + F_{42}(V_4))/5$

34: $c_{10040} = (5c_{00050} + F_{41}(V_4))/5$

35: $c_{00230} = (40c_{00140} - 20c_{00050} + F_{4343}(V_4))/20$

36: $c_{02030} = (40c_{01040} - 20c_{00050} + F_{4242}(V_4))/20$

37: $c_{20030} = (40c_{10040} - 20c_{00050} + F_{4141}(V_4))/20$

38: $c_{01130} = (20c_{01040} + 20c_{00140} - 20c_{00050} + F_{4342}(V_4))/20$

39: $c_{10130} = (20c_{10040} + 20c_{00140} - 20c_{00050} + F_{4341}(V_4))/20$

40: $c_{11030} = (20c_{10040} + 20c_{01040} - 20c_{00050} + F_{4241}(V_4))/20$

41: $c_{22100} = (\gamma_{123}(c_{50000} - 5c_{41000} + 10c_{32000} - 10c_{23000} + 5c_{14000} - c_{05000}) + c_{50000} - 4c_{41000} - c_{40100} + 6c_{32000} + 4c_{31100} - 4c_{23000} + c_{14000} + 4c_{13100} - c_{04100})/6$

42: $c_{22010} = (\gamma_{124}(c_{50000} - 5c_{41000} + 10c_{32000} - 10c_{23000} + 5c_{14000} - c_{05000}) + c_{50000} - 4c_{41000} - c_{40010} + 6c_{32000} + 4c_{31010} - 4c_{23000} + c_{14000} + 4c_{13010} - c_{04010})/6$

43: $c_{21200} = (\gamma_{132}(c_{50000} - 5c_{40100} + 10c_{30200} - 10c_{20300} + 5c_{10400} - c_{00500}) + c_{50000} - c_{41000} - 4c_{40100} + 4c_{31100} + 6c_{30200} - 4c_{20300} + 4c_{11300} + c_{10400} - c_{01400})/6$

44: $c_{20210} = (\gamma_{134}(c_{50000} - 5c_{40100} + 10c_{30200} - 10c_{20300} + 5c_{10400} - c_{00500}) + c_{50000} - 4c_{40100} - c_{40010} + 6c_{30200} + 4c_{30110} - 4c_{20300} + c_{10400} + 4c_{10310} - c_{00410})/6$

45: $c_{21020} = (\gamma_{142}(c_{50000} - 5c_{40010} + 10c_{30020} - 10c_{20030} + 5c_{10040} - c_{00050}) + c_{50000} - c_{41000} - 4c_{40010} + 4c_{31010} + 6c_{30020} - 4c_{20030} + 4c_{11030} + c_{10040} - c_{01040})/6$

46: $c_{20120} = (\gamma_{143}(c_{50000} - 5c_{40010} + 10c_{30020} - 10c_{20030} + 5c_{10040} - c_{00050}) + c_{50000} - c_{40100} - 4c_{40010} + 4c_{30110} + 6c_{30020} - 4c_{20030} + 4c_{10130} + c_{10040} - c_{00140})/6$

47: $c_{12200} = (\gamma_{231}(c_{05000} - 5c_{04100} + 10c_{03200} - 10c_{02300} + 5c_{01400} - c_{00500}) - c_{14000} + 4c_{13100} + 4c_{11300} - c_{10400} + c_{05000} - 4c_{04100} + 6c_{03200} - 4c_{02300} + c_{01400})/6$

48: $c_{02210} = (\gamma_{234}(c_{05000} - 5c_{04100} + 10c_{03200} - 10c_{02300} + 5c_{01400} - c_{00500}) + c_{05000} - 4c_{04100} - c_{04010} + 6c_{03200} + 4c_{03110} - 4c_{02300} + c_{01400} + 4c_{01310} - c_{00410})/6$

49: $c_{12020} = (\gamma_{241}(c_{05000} - 5c_{04010} + 10c_{03020} - 10c_{02030} + 5c_{01040} - c_{00050}) - c_{14000} + 4c_{13010} + 4c_{11030} - c_{10040} + c_{05000} - 4c_{04010} + 6c_{03020} - 4c_{02030} + c_{01040})/6$

50: $c_{02120} = (\gamma_{243}(c_{05000} - 5c_{04010} + 10c_{03020} - 10c_{02030} + 5c_{01040} - c_{00050}) + c_{05000} - c_{04100} - 4c_{04010} + 4c_{03110} + 6c_{03020} - 4c_{02030} + 4c_{01130} + c_{01040} - c_{00140})/6$

51: $c_{10220} = (\gamma_{341}(c_{00500} - 5c_{00410} + 10c_{00320} - 10c_{00230} + 5c_{00140} - c_{00050}) - c_{10400} + 4c_{10310} + 4c_{10130} - c_{10040} + c_{00500} - 4c_{00410} + 6c_{00320} - 4c_{00230} + c_{00140})/6$

52: $c_{01220} = (\gamma_{342}(c_{00500} - 5c_{00410} + 10c_{00320} - 10c_{00230} + 5c_{00140} - c_{00050}) - c_{01400} + 4c_{01310} + 4c_{01130} - c_{01040} + c_{00500} - 4c_{00410} + 6c_{00320} - 4c_{00230} + c_{00140})/6$

53: $c_{40001} = (c_{50000} + c_{41000} + c_{40100} + c_{40010})/4$

54: $c_{31001} = (c_{41000} + c_{32000} + c_{31100} + c_{31010})/4$

55: $c_{22001} = (c_{32000} + c_{23000} + c_{22100} + c_{22010})/4$

56: $c_{13001} = (c_{23000} + c_{14000} + c_{13100} + c_{13010})/4$

57: $c_{04001} = (c_{14000} + c_{05000} + c_{04100} + c_{04010})/4$

58: $c_{30101} = (c_{40100} + c_{31100} + c_{30200} + c_{30110})/4$

59: $c_{03101} = (c_{13100} + c_{04100} + c_{03200} + c_{03110})/4$

60: $c_{20201} = (c_{30200} + c_{21200} + c_{20300} + c_{20210})/4$

61: $c_{02201} = (c_{12200} + c_{03200} + c_{02300} + c_{02210})/4$

62: $c_{10301} = (c_{20300} + c_{11300} + c_{10400} + c_{10310})/4$

63: $c_{01301} = (c_{11300} + c_{02300} + c_{01400} + c_{01310})/4$

64: $c_{00401} = (c_{10400} + c_{01400} + c_{00500} + c_{00410})/4$

65: $c_{30011} = (c_{40010} + c_{31010} + c_{30110} + c_{30020})/4$

66: $c_{03011} = (c_{13010} + c_{04010} + c_{03110} + c_{03020})/4$

67: $c_{00311} = (c_{10310} + c_{01310} + c_{00410} + c_{00320})/4$

68: $c_{20021} = (c_{30020} + c_{21020} + c_{20120} + c_{20030})/4$

69: $c_{02021} = (c_{12020} + c_{03020} + c_{02120} + c_{02030})/4$

70: $c_{00221} = (c_{10220} + c_{01220} + c_{00320} + c_{00230})/4$

71: $c_{10031} = (c_{20030} + c_{11030} + c_{10130} + c_{10040})/4$

72: $c_{01031} = (c_{11030} + c_{02030} + c_{01130} + c_{01040})/4$

73: $c_{00131} = (c_{10130} + c_{01130} + c_{00230} + c_{00140})/4$

74: $c_{00041} = (c_{10040} + c_{01040} + c_{00140} + c_{00050})/4$

75: $c_{30002} = (c_{40001} + c_{31001} + c_{30101} + c_{30011})/4$

76: $c_{03002} = (c_{13001} + c_{04001} + c_{03101} + c_{03011})/4$

77: $c_{00302} = (c_{10301} + c_{01301} + c_{00401} + c_{00311})/4$

78: $c_{00032} = (c_{10031} + c_{01031} + c_{00131} + c_{00041})/4$

79: $c_{02111} = (\alpha_1(23c_{05000} - 61c_{04100} - 38c_{04010} + 59c_{03200} + 86c_{03110} + 21c_{03020} - 43c_{02300} - 48c_{02210} - 21c_{02120} - 22c_{02030} + 31c_{01400} + 2c_{01310} - 15c_{01220} + 24c_{01130} + 9c_{01040} - 9c_{00500} - 2c_{00410} + 15c_{00320} - 2c_{00230} - 9c_{00140}) + \beta_1(23c_{04100} - 23c_{04010} - 38c_{03200} + 38c_{03020} + 21c_{02300} + 27c_{02210} - 27c_{02120} - 21c_{02030} - 22c_{01400} + 22c_{01310} - 22c_{01130} + 22c_{01040} + 9c_{00500} - 7c_{00410} - 17c_{00320} + 17c_{00230} + 7c_{00140} - 9c_{00050}) + 23c_{05000} - 38c_{04100} - 38c_{04010} - 23c_{04001} + 21c_{03200} + 48c_{03110} + 38c_{03101} + 21c_{03020} + 38c_{03011} - 22c_{02300} - 21c_{02201} - 22c_{02030} - 21c_{02021} + 9c_{01400} + 2c_{01310} + 22c_{01301} - 15c_{01220} + 2c_{01130} + 9c_{01040} + 22c_{01031} - 9c_{00401} - 2c_{00311} + 15c_{00221} - 2c_{00131} - 9c_{00041})/48$

80: $c_{01211} = (\alpha_1(9c_{05000} - 31c_{04100} + 2c_{04010} + 43c_{03200} - 2c_{03110} - 15c_{03020} - 59c_{02300} + 48c_{02210} + 15c_{02120} + 2c_{02030} + 61c_{01400} - 86c_{01310} + 21c_{01220} - 24c_{01130} + 9c_{01040} - 23c_{00500} + 38c_{00410} - 21c_{00320} + 22c_{00230} - 9c_{00140}) + \beta_1(9c_{04100} - 9c_{04010} - 22c_{03200} + 24c_{03110} - 2c_{03020} + 21c_{02300} - 21c_{02210} - 15c_{02120} + 15c_{02030} - 38c_{01400} + 86c_{01310} -$

$48c_{01220} + 2c_{01130} - 2c_{01040} + 23c_{00500} - 61c_{00410} + 59c_{00320} - 43c_{00230} + 31c_{00140} - 9c_{00050}) + 9c_{05000} - 22c_{04100} + 2c_{04010} - 9c_{04001} + 21c_{03200} + 22c_{03101} - 15c_{03020} - 2c_{03011} - 38c_{02300} + 48c_{02210} - 21c_{02201} + 2c_{02030} - 15c_{02021} + 23c_{01400} - 38c_{01310} + 38c_{01301} + 21c_{01220} - 22c_{01130} + 9c_{01040} - 2c_{01031} - 23c_{00401} + 38c_{00311} - 21c_{00221} + 22c_{00131} - 9c_{00041})/48$

81: $c_{01121} = (\alpha_1(9c_{05000} - 7c_{04100} - 22c_{04010} - 17c_{03200} + 22c_{03110} + 21c_{03020} + 17c_{02300} + 27c_{02120} - 38c_{02030} + 7c_{01400} - 22c_{01310} - 27c_{01220} + 23c_{01040} - 9c_{00500} + 22c_{00410} - 21c_{00320} + 38c_{00230} - 23c_{00140}) + \beta_1(9c_{04100} - 9c_{04010} + 2c_{03200} - 24c_{03110} + 22c_{03020} - 15c_{02300} + 15c_{02210} + 21c_{02120} - 21c_{02030} + 2c_{01400} - 2c_{01310} + 48c_{01220} - 86c_{01130} + 38c_{01040} + 9c_{00500} - 31c_{00410} + 43c_{00320} - 59c_{00230} + 61c_{00140} - 23c_{00050}) + 9c_{05000} + 2c_{04100} - 22c_{04010} - 9c_{04001} - 15c_{03200} - 2c_{03101} + 21c_{03020} + 22c_{03011} + 2c_{02300} + 15c_{02201} + 48c_{02120} - 38c_{02030} - 21c_{02021} + 9c_{01400} - 22c_{01310} - 2c_{01301} + 21c_{01220} - 38c_{01130} + 23c_{01040} + 38c_{01031} - 9c_{00401} + 22c_{00311} - 21c_{00221} + 38c_{00131} - 23c_{00041})/48$

82: $c_{20111} = (\alpha_2(23c_{40100} - 23c_{40010} - 38c_{30200} + 38c_{30020} + 21c_{20300} + 27c_{20210} - 27c_{20120} - 21c_{20030} - 22c_{10400} + 22c_{10310} - 22c_{10130} + 22c_{10040} + 9c_{00500} - 7c_{00410} - 17c_{00320} + 17c_{00230} + 7c_{00140} - 9c_{00050}) + \beta_2(-23c_{50000} + 38c_{40100} + 61c_{40010} - 21c_{30200} - 86c_{30110} - 59c_{30020} + 22c_{20300} + 21c_{20210} + 48c_{20120} + 43c_{20030} - 9c_{10400} - 24c_{10310} + 15c_{10220} - 2c_{10130} - 31c_{10040} + 9c_{00410} + 2c_{00320} - 15c_{00230} + 2c_{00140} + 9c_{00050}) + 23c_{40100} - 23c_{40001} - 38c_{30200} - 38c_{30110} + 38c_{30101} + 38c_{30011} + 21c_{20300} + 48c_{20210} - 21c_{20201} + 21c_{20120} - 21c_{20021} - 22c_{10400} + 22c_{10301} - 22c_{10130} + 22c_{10031} + 9c_{00500} + 2c_{00410} - 9c_{00401} - 15c_{00320} - 2c_{00311} + 2c_{00230} + 15c_{00221} + 9c_{00140} - 2c_{00131} - 9c_{00041})/48$

83: $c_{10211} = (\alpha_2(9c_{40100} - 9c_{40010} - 22c_{30200} + 24c_{30110} - 2c_{30020} + 21c_{20300} - 21c_{20210} - 15c_{20120} + 15c_{20030} - 38c_{10400} + 86c_{10310} - 48c_{10220} + 2c_{10130} - 2c_{10040} + 23c_{00500} - 61c_{00410} + 59c_{00320} - 43c_{00230} + 31c_{00140} - 9c_{00050}) + \beta_2(-9c_{50000} + 22c_{40100} + 7c_{40010} - 21c_{30200} - 22c_{30110} + 17c_{30020} + 38c_{20300} - 27c_{20210} - 17c_{20030} - 23c_{10400} + 27c_{10220} + 22c_{10130} - 7c_{10040} + 23c_{00410} - 38c_{00320} + 21c_{00230} - 22c_{00140} + 9c_{00050}) + 9c_{40100} - 9c_{40001} - 22c_{30200} + 2c_{30110} + 22c_{30101} - 2c_{30011} + 21c_{20300} - 21c_{20201} - 15c_{20120} + 15c_{20021} - 38c_{10400} + 48c_{10310} + 38c_{10301} + 2c_{10130} - 2c_{10031} + 23c_{00500} - 38c_{00410} - 23c_{00401} + 21c_{00320} + 38c_{00311} - 22c_{00230} - 21c_{00221} + 9c_{00140} + 22c_{00131} - 9c_{00041})/48$

84: $c_{10121} = (\alpha_2(9c_{40100} - 9c_{40010} + 2c_{30200} - 24c_{30110} + 22c_{30020} - 15c_{20300} + 15c_{20210} + 21c_{20120} - 21c_{20030} + 2c_{10400} - 2c_{10310} + 48c_{10220} - 86c_{10130} + 38c_{10040} + 9c_{00500} - 31c_{00410} + 43c_{00320} - 59c_{00230} + 61c_{00140} - 23c_{00050}) + \beta_2(-9c_{50000} - 2c_{40100} + 31c_{40010} + 15c_{30200} + 2c_{30110} - 43c_{30020} - 2c_{20300} - 15c_{20210} - 48c_{20120} + 59c_{20030} - 9c_{10400} + 24c_{10310} - 21c_{10220} + 86c_{10130} - 61c_{10040} + 9c_{00410} - 22c_{00320} + 21c_{00230} - 38c_{00140} + 23c_{00050}) + 9c_{40100} - 9c_{40001} + 2c_{30200} - 22c_{30110} - 2c_{30101} + 22c_{30011} - 15c_{20300} + 15c_{20201} + 21c_{20120} - 21c_{20021} + 2c_{10400} - 2c_{10301} + 48c_{10220} - 38c_{10130} + 38c_{10031} + 9c_{00500} - 22c_{00410} - 9c_{00401} + 21c_{00320} + 22c_{00311} - 38c_{00230} - 21c_{00221} + 23c_{00140} + 38c_{00131} - 23c_{00041})/48$

85: $c_{21011} = (\alpha_3(-23c_{50000} + 38c_{41000} + 61c_{40010} - 21c_{32000} - 86c_{31010} - 59c_{30020} + 22c_{23000} + 21c_{22010} + 48c_{21020} + 43c_{20030} - 9c_{14000} - 24c_{13010} + 15c_{12020} - 2c_{11030} - 31c_{10040} + 9c_{04010} + 2c_{03020} - 15c_{02030} + 2c_{01040} + 9c_{00050}) + \beta_3(23c_{50000} - 61c_{41000} - 38c_{40010} + 59c_{32000} + 86c_{31010} + 21c_{30020} - 43c_{23000} - 48c_{22010} - 21c_{21020} - 22c_{20030} + 31c_{14000} + 2c_{13010} - 15c_{12020} + 24c_{11030} + 9c_{10040} - 9c_{05000} - 2c_{04010} + 15c_{03020} -$

$2c_{02030} - 9c_{01040}) + 23c_{40010} - 23c_{40001} - 38c_{31010} + 38c_{31001} - 38c_{30020} + 38c_{30011} + 21c_{22010} - 21c_{22001} + 48c_{21020} - 21c_{20030} - 21c_{20021} - 22c_{13010} + 22c_{13001} - 22c_{10040} + 22c_{10031} + 9c_{04010} - 9c_{04001} - 2c_{03020} - 2c_{03011} - 15c_{02030} + 15c_{02021} + 2c_{01040} - 2c_{01031} + 9c_{00050} - 9c_{00041})/48$

86: $c_{12011} = (\alpha_3(-9c_{50000} - 22c_{41000} + 7c_{40010} - 21c_{32000} - 22c_{31010} + 17c_{30020} + 38c_{23000} - 27c_{22010} - 17c_{20030} - 23c_{14000} + 27c_{12020} + 22c_{11030} - 7c_{10040} + 23c_{04010} - 38c_{03020} + 21c_{02030} - 22c_{01040} + 9c_{00050}) + \beta_3(9c_{50000} - 31c_{41000} + 2c_{40010} + 43c_{32000} - 2c_{31010} - 15c_{30020} - 59c_{23000} + 48c_{22010} + 15c_{21020} + 2c_{20030} + 61c_{14000} - 86c_{13010} + 21c_{12020} - 24c_{11030} + 9c_{10040} - 23c_{05000} + 38c_{04010} - 21c_{03020} + 22c_{02030} - 9c_{01040}) + 9c_{40010} - 9c_{40001} - 22c_{31010} + 22c_{31001} + 2c_{30020} - 2c_{30011} + 21c_{22010} - 21c_{22001} - 15c_{20030} + 15c_{20021} - 38c_{13010} + 38c_{13001} + 48c_{12020} + 2c_{10040} - 2c_{10031} + 23c_{04010} - 23c_{04001} - 38c_{03020} + 38c_{03011} + 21c_{02030} - 21c_{02021} - 22c_{01040} + 22c_{01031} + 9c_{00050} - 9c_{00041})/48$

87: $c_{11021} = (\alpha_3(-9c_{50000} - 2c_{41000} + 31c_{40010} + 15c_{32000} + 2c_{31010} - 43c_{30020} - 2c_{23000} - 15c_{22010} - 48c_{21020} + 59c_{20030} - 9c_{14000} + 24c_{13010} - 21c_{12020} + 86c_{11030} - 61c_{10040} + 9c_{04010} - 22c_{03020} + 21c_{02030} - 38c_{01040} + 23c_{00050}) + \beta_3(9c_{50000} - 7c_{41000} - 22c_{40010} - 17c_{32000} + 22c_{31010} + 21c_{30020} + 17c_{23000} + 27c_{21020} - 38c_{20030} + 7c_{14000} - 22c_{13010} - 27c_{12020} + 23c_{10040} - 9c_{05000} + 22c_{04010} - 21c_{03020} + 38c_{02030} - 23c_{01040}) + 9c_{40010} - 9c_{40001} + 2c_{31010} - 2c_{31001} - 22c_{30020} + 22c_{30011} - 15c_{22010} + 15c_{22001} + 21c_{20030} - 21c_{20021} + 2c_{13010} - 2c_{13001} + 48c_{11030} - 38c_{10040} + 38c_{10031} + 9c_{04010} - 9c_{04001} - 22c_{03020} + 22c_{03011} + 21c_{02030} - 21c_{02021} - 38c_{01040} + 38c_{01031} + 23c_{00050} - 23c_{00041})/48$

88: $c_{21101} = (\alpha_4(23c_{50000} - 61c_{41000} - 38c_{40100} + 59c_{32000} + 86c_{31100} + 21c_{30200} - 43c_{23000} - 48c_{22100} - 21c_{21200} - 22c_{20300} + 31c_{14000} + 2c_{13100} - 15c_{12200} + 24c_{11300} + 9c_{10400} - 9c_{05000} - 2c_{04100} + 15c_{03200} - 2c_{02300} - 9c_{01400}) + \beta_4(23c_{41000} - 23c_{40100} - 38c_{32000} + 38c_{30200} + 21c_{23000} + 27c_{22100} - 27c_{21200} - 21c_{20300} - 22c_{14000} + 22c_{13100} - 22c_{11300} + 22c_{10400} + 9c_{05000} - 7c_{04100} - 17c_{03200} + 17c_{02300} + 7c_{01400} - 9c_{00500}) + 23c_{50000} - 38c_{41000} - 38c_{40100} - 23c_{40001} + 21c_{32000} + 48c_{31100} + 38c_{31001} + 21c_{30200} + 38c_{30101} - 22c_{23000} - 21c_{22001} - 22c_{20300} - 21c_{20201} + 9c_{14000} + 2c_{13100} + 22c_{13001} - 15c_{12200} + 2c_{11300} + 9c_{10400} + 22c_{10301} - 9c_{04001} - 2c_{03101} + 15c_{02201} - 2c_{01301} - 9c_{00401})/48$

89: $c_{12101} = (\alpha_4(9c_{50000} - 31c_{41000} + 2c_{40100} + 43c_{32000} - 2c_{31100} - 15c_{30200} - 59c_{23000} + 48c_{22100} + 15c_{21200} + 2c_{20300} + 61c_{14000} - 86c_{13100} + 21c_{12200} - 24c_{11300} + 9c_{10400} - 23c_{05000} + 38c_{04100} - 21c_{03200} + 22c_{02300} - 9c_{01400}) + \beta_4(9c_{41000} - 9c_{40100} - 22c_{32000} + 24c_{31100} - 2c_{30200} + 21c_{23000} - 21c_{22100} - 15c_{21200} + 15c_{20300} - 38c_{14000} + 86c_{13100} - 48c_{12200} + 2c_{11300} - 2c_{10400} + 23c_{05000} - 61c_{04100} + 59c_{03200} - 43c_{02300} + 31c_{01400} - 9c_{00500}) + 9c_{50000} - 22c_{41000} + 2c_{40100} - 9c_{40001} + 21c_{32000} + 22c_{31001} - 15c_{30200} - 2c_{30101} - 38c_{23000} + 48c_{22100} - 21c_{22001} + 2c_{20300} + 15c_{20201} + 23c_{14000} - 38c_{13100} + 38c_{13001} + 21c_{12200} - 22c_{11300} + 9c_{10400} - 2c_{10301} - 23c_{04001} + 38c_{03101} - 21c_{02201} + 22c_{01301} - 9c_{00401})/48$

90: $c_{11201} = (\alpha_4(9c_{50000} - 7c_{41000} - 22c_{40100} - 17c_{32000} + 22c_{31100} + 21c_{30200} + 17c_{23000} + 27c_{21200} - 38c_{20300} + 7c_{14000} - 22c_{13100} - 27c_{12200} + 23c_{10400} - 9c_{05000} + 22c_{04100} - 21c_{03200} + 38c_{02300} - 23c_{01400}) + \beta_4(9c_{41000} - 9c_{40100} + 2c_{32000} - 24c_{31100} + 22c_{30200} - 15c_{23000} + 15c_{22100} + 21c_{21200} - 21c_{20300} + 2c_{14000} - 2c_{13100} + 48c_{12200} - 86c_{11300} + 38c_{10400} + 9c_{05000} - 31c_{04100} + 43c_{03200} - 59c_{02300} + 61c_{01400} - 23c_{00500}) + 9c_{50000} + 2c_{41000} - 22c_{40100} - 9c_{40001} - 15c_{32000} - 2c_{31001} + 21c_{30200} + 22c_{30101} + 2c_{23000} +$

$15c_{22001} + 48c_{21200} - 38c_{20300} - 21c_{20201} + 9c_{14000} - 22c_{13100} - 2c_{13001} + 21c_{12200} -$
$38c_{11300} + 23c_{10400} + 38c_{10301} - 9c_{04001} - 22c_{03101} - 21c_{02201} + 38c_{01301} - 23c_{00401})/48$

91: $c_{21002} = (c_{31001} + c_{22001} + c_{21101} + c_{21011})/4$

92: $c_{12002} = (c_{22001} + c_{13001} + c_{12101} + c_{12011})/4$

93: $c_{20102} = (c_{30101} + c_{21101} + c_{20201} + c_{20111})/4$

94: $c_{02102} = (c_{12101} + c_{03101} + c_{02201} + c_{02111})/4$

95: $c_{10202} = (c_{20201} + c_{11201} + c_{10301} + c_{10211})/4$

96: $c_{01202} = (c_{11201} + c_{02201} + c_{01301} + c_{01211})/4$

97: $c_{20012} = (c_{30011} + c_{21011} + c_{20111} + c_{20021})/4$

98: $c_{02012} = (c_{12011} + c_{03011} + c_{02111} + c_{02021})/4$

99: $c_{00212} = (c_{10211} + c_{01211} + c_{00311} + c_{00221})/4$

100: $c_{10022} = (c_{20021} + c_{11021} + c_{10121} + c_{10031})/4$

101: $c_{01022} = (c_{11021} + c_{02021} + c_{01121} + c_{01031})/4$

102: $c_{00122} = (c_{10121} + c_{01121} + c_{00221} + c_{00131})/4$

103: $c_{20003} = (c_{30002} + c_{21002} + c_{20102} + c_{20012})/4$

104: $c_{02003} = (c_{12002} + c_{03002} + c_{02102} + c_{02012})/4$

105: $c_{00203} = (c_{10202} + c_{01202} + c_{00302} + c_{00212})/4$

106: $c_{00023} = (c_{10022} + c_{01022} + c_{00122} + c_{00032})/4$

107: $c_{10004} = (255c_{40001} + 680c_{31001} + 680c_{30101} + 680c_{30011} - 3060c_{30002} + 1020c_{22001} +$
$1020c_{21101} + 1020c_{21011} - 4032c_{21002} + 1020c_{20201} + 1020c_{20111} - 4032c_{20102} +$
$1020c_{20021} - 4032c_{20012} + 8838c_{20003} + 680c_{13001} + 1020c_{12101} + 1020c_{12011} -$
$4032c_{12002} + 1020c_{11201} + 1020c_{11021} + 680c_{10301} + 1020c_{10211} - 4032c_{10202} +$
$1020c_{10121} + 680c_{10031} - 4032c_{10022} + 251c_{04001} + 664c_{03101} + 664c_{03011} - 3012c_{03002} +$
$996c_{02201} + 972c_{02111} - 4032c_{02102} + 996c_{02021} - 4032c_{02012} + 8478c_{02003} + 664c_{01301} +$
$972c_{01211} - 4032c_{01202} + 972c_{01121} + 664c_{01031} - 4032c_{01022} + 251c_{00401} + 664c_{00311} -$
$3012c_{00302} + 996c_{00221} - 4032c_{00212} + 8478c_{00203} + 664c_{00131} - 4032c_{00122} + 251c_{00041} -$
$3012c_{00032} + 8478c_{00023})/1008$

108: $c_{01004} = (-392c_{40001} - 680c_{31001} - 1228c_{30101} - 1228c_{30011} + 4704c_{30002} - 1020c_{22001} -$
$1020c_{21101} - 1020c_{21011} + 4032c_{21002} - 1842c_{20201} - 2664c_{20111} + 4032c_{20102} -$
$1842c_{20021} + 4032c_{20012} - 21168c_{20003} - 680c_{13001} - 1020c_{12101} - 1020c_{12011} +$
$4032c_{12002} - 1020c_{11201} - 1020c_{11021} - 1228c_{10301} - 2664c_{10211} + 4032c_{10202} -$
$2664c_{10121} - 1228c_{10031} + 4032c_{10022} + 35532c_{10004} - 114c_{04001} - 116c_{03101} -$
$116c_{03011} + 1368c_{03002} - 174c_{02201} + 672c_{02111} + 4032c_{02102} - 174c_{02021} + 4032c_{02012} +$
$3852c_{02003} - 116c_{01301} + 672c_{01211} + 4032c_{01202} + 672c_{01121} - 116c_{01031} + 4032c_{01022} -$
$251c_{00401} - 664c_{00311} + 3012c_{00302} - 996c_{00221} + 4032c_{00212} - 8478c_{00203} - 664c_{00131} +$
$4032c_{00122} - 251c_{00041} + 3012c_{00032} - 8478c_{00023})/34524$

109: $c_{00104} = (-392c_{40001} - 1792c_{31001} - 116c_{30101} - 1228c_{30011} + 4704c_{30002} - 2688c_{22001} -$
$1020c_{21101} - 4356c_{21011} + 4032c_{21002} - 174c_{20201} + 672c_{20111} + 4032c_{20102} - 1842c_{20021} +$
$4032c_{20012} - 21168c_{20003} - 1792c_{13001} - 1020c_{12101} - 4356c_{12011} + 4032c_{12002} -$
$1020c_{11201} - 4356c_{11021} - 116c_{10301} + 672c_{10211} + 4032c_{10202} + 672c_{10121} - 1228c_{10031} +$

$4032c_{10022} + 35532c_{10004} - 392c_{04001} - 116c_{03101} - 1228c_{03011} + 4704c_{03002} - 174c_{02201} + 672c_{02111} + 4032c_{02102} - 1842c_{02021} + 4032c_{02012} - 21168c_{02003} - 116c_{01301} + 672c_{01211} + 4032c_{01202} + 672c_{01121} - 1228c_{01031} + 4032c_{01022} + 35532c_{01004} + 27c_{00401} + 448c_{00311} - 324c_{00302} + 672c_{00221} + 4032c_{00212} - 16542c_{00203} - 448c_{00131} + 4032c_{00122} - 251c_{00041} + 3012c_{00032} - 8478c_{00023})/70056$

110: $c_{00014} = (-14c_{40001} - 64c_{31001} - 64c_{30101} + 16c_{30011} + 168c_{30002} - 96c_{22001} - 216c_{21101} + 24c_{21011} + 144c_{21002} - 96c_{20201} + 24c_{20111} + 144c_{20102} + 24c_{20021} + 144c_{20012} - 756c_{20003} - 64c_{13001} - 216c_{12101} + 24c_{12011} + 144c_{12002} - 216c_{11201} + 24c_{11021} - 64c_{10301} + 24c_{10211} + 144c_{10202} + 24c_{10121} - 16c_{10031} - 144c_{10022} + 1269c_{10004} - 14c_{04001} - 64c_{03101} + 16c_{03011} + 168c_{03002} - 96c_{02201} + 24c_{02111} - 144c_{02102} + 24c_{02021} + 144c_{02012} - 756c_{02003} - 64c_{01301} - 24c_{01211} + 144c_{01202} - 24c_{01121} + 16c_{01031} + 144c_{01022} + 1269c_{01004} - 14c_{00401} + 16c_{00311} - 168c_{00302} + 24c_{00221} + 144c_{00212} - 756c_{00203} + 16c_{00131} + 144c_{00122} + 1269c_{00104} + 6c_{00041} - 72c_{00032} + 1044c_{00023})/3771$

111: $c_{00005} = (c_{10004} + c_{01004} + c_{00104} + c_{00014})/4$

112: $c_{01112} = (-9c_{40001} - 24c_{31001} - 24c_{30101} - 24c_{30011} + 108c_{30002} - 36c_{22001} - 36c_{21101} - 36c_{21011} + 54c_{21002} - 36c_{20201} - 36c_{20111} + 54c_{20102} - 36c_{20021} + 54c_{20012} - 486c_{20003} - 24c_{13001} - 36c_{12101} - 36c_{12011} + 54c_{12002} - 36c_{11201} - 36c_{11021} - 24c_{10301} - 36c_{10211} + 54c_{10202} - 36c_{10121} - 24c_{10031} + 54c_{10022} - 972c_{10004} + c_{04001} + 16c_{03101} + 16c_{03011} - 12c_{03002} + 24c_{02201} - 84c_{02111} - 36c_{02102} + 24c_{02021} - 36c_{02012} + 54c_{02003} + 16c_{01301} + 84c_{01211} - 36c_{01202} + 84c_{01121} + 16c_{01031} - 36c_{01022} - 108c_{01004} + c_{00401} + 16c_{00311} - 12c_{00302} + 24c_{00221} - 36c_{00212} + 54c_{00203} + 16c_{00131} - 36c_{00122} - 108c_{00104} + c_{00041} - 12c_{00032} + 54c_{00023} - 108c_{00014} - 162c_{00005})/180$

113: $c_{10112} = (-2c_{40001} - 24c_{31001} + 4c_{30101} + 4c_{30011} + 24c_{30002} - 36c_{22001} - 36c_{21101} - 36c_{21011} + 54c_{21002} + 6c_{20201} + 48c_{20111} - 9c_{20102} + 6c_{20021} - 9c_{20012} - 108c_{20003} - 24c_{13001} - 36c_{12101} - 36c_{12011} + 54c_{12002} - 36c_{11201} - 36c_{11021} + 4c_{10301} + 48c_{10211} - 9c_{10202} + 48c_{10121} + 4c_{10031} - 9c_{10022} + 216c_{10004} - 6c_{04001} - 12c_{03101} - 12c_{03011} + 72c_{03002} - 18c_{02201} + 27c_{02102} - 18c_{02021} + 27c_{02012} - 324c_{02003} - 12c_{01301} + 27c_{01202} - 54c_{01112} - 12c_{01031} + 27c_{01022} + 648c_{01004} + c_{00401} + 16c_{00311} - 12c_{00302} + 24c_{00221} - 36c_{00212} + 54c_{00203} + 16c_{00131} - 36c_{00122} - 108c_{00104} + c_{00041} - 12c_{00032} + 54c_{00023} - 108c_{00014} - 162c_{00005})/126$

114: $c_{11012} = (-2c_{40001} - 8c_{31001} - 12c_{30101} + 4c_{30011} + 24c_{30002} - 12c_{22001} - 36c_{21101} + 12c_{21011} + 18c_{21002} - 18c_{20201} + 27c_{20102} + 6c_{20021} - 9c_{20012} - 108c_{20003} - 8c_{13001} - 36c_{12101} + 12c_{12011} + 18c_{12002} - 36c_{11201} + 12c_{11021} - 12c_{10301} + 27c_{10202} - 54c_{10112} + 4c_{10031} - 9c_{10022} + 216c_{10004} - 2c_{04001} - 12c_{03101} + 4c_{03011} + 24c_{03002} - 18c_{02201} + 27c_{02102} + 6c_{02021} - 9c_{02012} - 108c_{02003} - 12c_{01301} + 27c_{01202} - 54c_{01112} + 4c_{01031} - 9c_{01022} + 216c_{01004} - 3c_{00401} + 36c_{00302} - 162c_{00203} + 324c_{00104} + c_{00041} - 12c_{00032} + 54c_{00023} - 108c_{00014} - 162c_{00005})/72$

115: $c_{11102} = (-c_{40001} - 4c_{31001} - 4c_{30101} + 12c_{30002} - 6c_{22001} - 12c_{21101} + 9c_{21002} - 6c_{20201} + 9c_{20102} - 54c_{20003} - 4c_{13001} - 12c_{12101} + 9c_{12002} - 12c_{11201} - 27c_{11012} - 4c_{10301} + 9c_{10202} - 27c_{10112} + 108c_{10004} - c_{04001} - 4c_{03101} + 12c_{03002} - 6c_{02201} + 9c_{02102} - 54c_{02003} - 4c_{01301} + 9c_{01202} - 27c_{01112} + 108c_{01004} - c_{00401} + 12c_{00302} - 54c_{00203} + 108c_{00104} - 81c_{00005})/9$

116: $c_{11003} = (c_{21002} + c_{12002} + c_{11102} + c_{11012})/4$

117: $c_{10103} = (c_{20102} + c_{11102} + c_{10202} + c_{10112})/4$

118: $c_{01103} = (c_{11102} + c_{02102} + c_{01202} + c_{01112})/4$

119: $c_{10013} = (c_{20012} + c_{11012} + c_{10112} + c_{10022})/4$

120: $c_{01013} = (c_{11012} + c_{02012} + c_{01112} + c_{01022})/4$

121: $c_{00113} = (c_{10112} + c_{01112} + c_{00212} + c_{00122})/4$

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An interpolation scheme is described for values of position, gradient and Hessian at scattered points in three variables. The domain is assumed to have been tesselated into tetrahedra. The interpolant has local support, is globally once differentiable, piecewise polynomial, and reproduces polynomials of degree up to three exactly. The scheme has been implemented in a FORTRAN research code.

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