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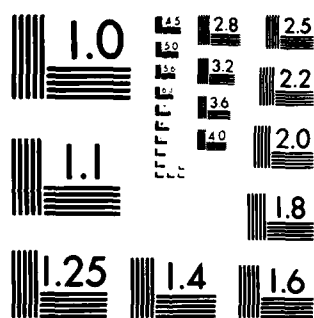
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AN INTEGRAL INEQUALITY WITH APPLICATIONS
TO ORDER STATISTICS

by

Philip J. Boland and Frank Proschan

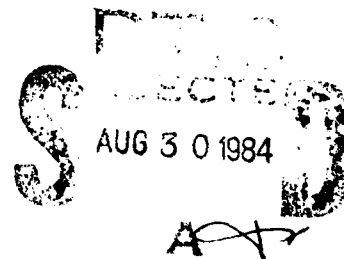
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An integral inequality is proved giving sufficient conditions on functions ψ and ϕ in order to ensure that whenever $G_i \geq F_i$ for $i=1, \dots, n$, then

$$\int_0^{\infty} \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt \leq \int_0^{\infty} \psi(t) \phi(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt.$$

Applications in reliability theory and order statistics are given.



AN INTEGRAL INEQUALITY WITH APPLICATIONS
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ABSTRACT

We say the life distribution function G majorizes the life distribution function F (written $G \supset F$) if

$$\int_x^\infty \overline{G}(t) dt \geq \int_x^\infty \overline{F}(t) dt \quad \text{for all } x \geq 0$$

$$\text{and} \quad \int_0^\infty \overline{G}(t) dt = \int_0^\infty \overline{F}(t) dt < +\infty.$$

An integral inequality is proved giving sufficient conditions on functions ψ and ϕ in order to ensure that whenever $G_i \supset F_i$ for $i=1, \dots, n$, then

$$\int_0^\infty \psi(t) \phi(\overline{G}_1(t), \dots, \overline{G}_n(t)) dt \leq \int_0^\infty \psi(t) \phi(\overline{F}_1(t), \dots, \overline{F}_n(t)) dt.$$

Applications in reliability theory and order statistics are given.

1. Introduction.

For given life distribution functions F and G , the respective survival functions are $\bar{F} = 1-F$ and $\bar{G} = 1-G$. We define the partial ordering $\overset{m}{\succ}$ on the class of life distributions with finite means by $G \overset{m}{\succ} F$ (m for majorization) if

$$(1.1) \quad \int_x^\infty \bar{G}(t)dt \geq \int_x^\infty \bar{F}(t)dt \quad \text{for all } x \geq 0$$

and

$$(1.2) \quad \mu_G = \int_0^\infty \bar{G}(t)dt = \int_0^\infty \bar{F}(t)dt = \mu_F < +\infty.$$

If X and Y are nonnegative random variables with respective distribution functions F and G , then Ross [11] says " Y is more variable than X " (written $Y \geq_v X$ or $G \geq_v F$) if (1.1) holds. Stoyan [14] equivalently defines Y to be "larger in mean residual life" than X (written $G \geq_c F$ or in previous publications $G \overset{(2)}{\geq} F$) if (1.1) holds. Bessler and Veinott [3] use the terminology " Y is stochastically larger in mean than X ." The notation of Stoyan (c for convex) is suggested by the following characterization:

$$G \geq_c F$$

$$\Leftrightarrow \int_0^\infty \Psi(t)dG(t) \geq \int_0^\infty \Psi(t)dF(t)$$

holds for all increasing (that is nondecreasing) convex functions Ψ , provided the integrals exist.

For life distribution functions F and G , $G \overset{m}{\succ} F$ if and only if $G \geq_c F$ (or $G \geq_v F$) and G and F have equal finite means ($\mu_F = \mu_G$). For distribution functions with finite means, the following useful characterization of $G \overset{m}{\succ} F$ (see for example Ross [11] or Stoyan [14]) is an immediate corollary of Theorem 2.1:

$$G \overset{m}{\succ} F$$

$\Leftrightarrow$

$$\int_0^\infty \Psi(t)dG(t) \geq \int_0^\infty \Psi(t)dF(t)$$

holds for all convex functions Ψ , provided the integrals exist.

We note in particular that if $G \supset F$, then

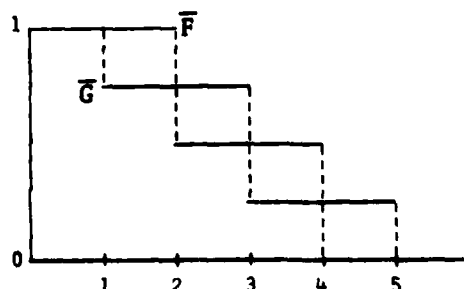
$$\sigma_G^2 = \int_0^{\infty} (t - \mu_G)^2 dG(t) \geq \int_0^{\infty} (t - \mu_F)^2 dF(t) = \sigma_F^2.$$

Hence $G \supset F$ implies that the life distribution represented by G is 'more dispersed' than that represented by F around their common mean.

For life distribution functions F and G with a common mean, $G \supset F$ is a more general relationship than $G \geq F$ (F is star shaped with respect to G). When F and G are continuous life distributions (where $F(0) = G(0) = 0$, F and G have interval support and G is strictly increasing on its support), then $G \geq F$ if $G^{-1}F(x)$ is star-shaped (that is $\frac{G^{-1}F(x)}{x}$ is increasing for $x > 0$). If $G \geq F$ and F and G have a common mean, then $\bar{F}(x)$ crosses $\bar{G}(x)$ once and from above as $x:0 \rightarrow \infty$, so that in particular $G \supset F$ (see Barlow and Proschan [2]). For a continuous life distribution function F with mean μ , let us define $G(x) = 1 - e^{-x/\mu}$ to be the exponential distribution with the same mean. Then F is IFRA (increasing failure rate average) $\Leftrightarrow G \geq F$, and F is HNBUE (harmonic new better than used in expectation) $\Leftrightarrow G \supset F$. See Klefsjö [6] for further properties of HNBUE distributions.

If F and G are two life distribution functions with common mean and $\bar{F}(x)$ crosses $\bar{G}(x)$ once and from above as $x:0 \rightarrow \infty$, then $G \supset F$, however the converse is clearly not true. For example let F and G be defined as follows:

$$F(x) = \begin{cases} 0 & x < 2 \\ 1/2 & 2 \leq x < 4 \\ 1 & 4 \leq x \end{cases} \quad G(x) = \begin{cases} 0 & x < 1 \\ 1/4 & 1 \leq x < 3 \\ 3/4 & 3 \leq x < 5 \\ 1 & 5 \leq x \end{cases}.$$



Then $G \supset F$ and G 'crosses' F three times.

A vector $\underline{b} = (b_1, \dots, b_n)$ majorizes the vector $\underline{a} = (a_1, \dots, a_n)$ if

$$\sum_{i=k}^n b_{[i]} \geq \sum_{i=k}^n a_{[i]} \quad \text{for } k=2, \dots, n$$

$$\text{and} \quad \sum_{i=1}^n b_{[i]} = \sum_{i=1}^n a_{[i]},$$

where the $b_{[i]}$'s and $a_{[i]}$'s are the components of $\underline{b}$ and $\underline{a}$ respectively in ascending order. When $\underline{b}$ majorizes $\underline{a}$ we write $\underline{b} \stackrel{m}{>} \underline{a}$.

Suppose now that $\underline{b}$ and $\underline{a}$ are n dimensional vectors with nonnegative components such that $\underline{b} \stackrel{m}{>} \underline{a}$. If G and F are respectively the distribution functions for the uniform distributions on the components of $\underline{b}$ and $\underline{a}$, then $G \stackrel{m}{>} F$. This is our motivation for using the letter m for our partial ordering on the family of life distribution functions with finite means.

2. An Integral Inequality.

The following theorem is a variant of an integral inequality obtained by Fan and Lorentz [4].

Theorem 2.1. Let $\phi = [0,1]^n \rightarrow [0,\infty)$ be a continuous increasing function, and assume that for $i=1, \dots, n$, F_i and G_i are life distribution functions where $G_i \stackrel{m}{>} F_i$.

- a) If ψ is nonnegative decreasing, ϕ is convex in each variable separately and ϕ satisfies the following property:

$$(2.1) \quad \phi(u_i + h, u_j + k) - \phi(u_i + h, u_j) - \phi(u_i, u_j + k) + \phi(u_i, u_j) \geq 0$$

for all $i \neq j$, $0 \leq u_i \leq u_i + h \leq 1$, $0 \leq u_j \leq u_j + k \leq 1$

(where we have used the notational simplification of omitting those arguments of ϕ which are the same in a given formula),

then providing the integrals exist,

$$(2.2) \quad \int_0^{\infty} \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt \leq \int_0^{\infty} \psi(t) \phi(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt.$$

b) If ψ is nonnegative increasing, ϕ is concave in each variable separately and ϕ satisfies the following property:

$$(2.3) \quad \phi(u_i + h, u_j + k) - \phi(u_i + h, u_j) - \phi(u_i, u_j + k) + \phi(u_i, u_j) \leq 0$$

for all $i \neq j$, $0 \leq u_i \leq u_i + h \leq 1$, $0 \leq u_j \leq u_j + k \leq 1$,

then providing the integrals exist

$$(2.4) \quad \int_0^\infty \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt \geq \int_0^\infty \psi(t) \phi(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt.$$

Proof: We prove only a), the proof of b) following in a similar fashion.

(i) Initially we show that it suffices to prove the result for the case when $F_1, G_1, \dots, F_n, G_n$ all have finite support. In turn to establish this we show that if the inequality is valid whenever F_1 and G_1 have finite support, then it is true in general.

Suppose now that $F_1, G_1, \dots, F_n, G_n$ are arbitrary life distributions where $G_i \stackrel{m}{>} F_i$ for $i = 1, \dots, n$. Given $\epsilon > 0$, we can find S so that

$$\int_S^\infty \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt < \epsilon.$$

Now define F_1' and G_1' by

$$\begin{aligned} \bar{F}_1'(t) &= \bar{F}_1(t) & t < S \\ &0 & t \geq S \\ \bar{G}_1'(t) &= \bar{G}_1(t) & t < S \\ &\bar{G}_1(S) & S \leq t \leq S + \frac{\int_0^S \bar{F}_1(t) dt - \int_0^S \bar{G}_1(t) dt}{\bar{G}_1(S)} \\ &0 & \text{otherwise,} \end{aligned}$$

(if $\bar{G}_1(S) = 0$, then both G_1 and F_1 have finite support). Then $G_1' \stackrel{m}{>} F_1'$, and

$$\begin{aligned} \int_0^\infty \psi(t) \phi(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_n(t)) dt &\geq \int_0^\infty \psi(t) \phi(\bar{F}_1'(t), \bar{F}_2(t), \dots, \bar{F}_n(t)) dt \\ &\geq \int_0^\infty \psi(t) \phi(\bar{G}_1'(t), \bar{G}_2(t), \dots, \bar{G}_n(t)) dt \\ &\geq \int_0^\infty \psi(t) \phi(\bar{G}_1(t), \bar{G}_2(t), \dots, \bar{G}_n(t)) dt - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, the conclusion follows.

(ii) It now remains to show that

$$\int_0^\infty \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt \leq \int_0^\infty \psi(t) \phi(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt$$

whenever $G_i \stackrel{m}{>} F_i$ for all $i = 1, \dots, n$, and where the support of F_i and $G_i \subseteq [0, S]$ for all $i = 1, \dots, n$.

Let $\epsilon > 0$ be given. As ϕ is continuous, there exists a $\delta > 0$ such that whenever $u, v \in [0, 1]^n$ and $\|u - v\| = \max_{i=1, \dots, n} |u_i - v_i| < \delta$, then $|\phi(u) - \phi(v)| < \epsilon/2S\psi(0)$.

There exist only a finite number of points r in $[0, S]$ where at least one of $F_1, G_1, \dots, F_n, G_n$ has a jump discontinuity with jump $> \delta/2$. Hence we can find an integer N large enough so that

$$(1) \quad \frac{\psi(0)4rS}{\epsilon} \sup |\phi| < N$$

and (2) on all but at most r of the N intervals $\left[0, \frac{S}{N}\right]$,

$$\left[\frac{S}{N}, \frac{S+1}{N}\right] \dots, \left[\frac{(N-1)S}{N}, \frac{NS}{N}\right],$$

$$\max_i \left[\bar{F}_i \left(\frac{jS}{N} \right) - \bar{F}_i \left(\frac{(j+1)S}{N} \right) \right] < \delta \quad \text{and} \quad \max_i \left[\bar{G}_i \left(\frac{jS}{N} \right) - \bar{G}_i \left(\frac{(j+1)S}{N} \right) \right] < \delta.$$

Hence for each $i = 1, \dots, n$, we define the following simple survival functions:

$$\bar{F}_i''(t) = \left(\int_{jS/N}^{(j+1)S/N} \bar{F}_i(t) dt \right) / S/N$$

and

$$\bar{G}_i''(t) = \left(\int_{jS/N}^{(j+1)S/N} \bar{G}_i(t) dt \right) / S/N$$

when $t \in \left[\frac{jS}{N}, \frac{(j+1)S}{N} \right)$ for some $j = 0, \dots, N-1$, and zero otherwise.

Note that $G_i'' \stackrel{m}{\geq} F_i''$ for all $i = 1, \dots, n$.

Moreover,

$$\begin{aligned} & \left| \int_0^S \psi(t) \phi(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt - \int_0^S \psi(t) \phi(\bar{F}_1''(t), \dots, \bar{F}_n''(t)) dt \right| \\ &= \left| \sum_{j=0}^{N-1} \int_{jS/N}^{(j+1)S/N} \psi(t) \left[\phi(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt - \phi(\bar{F}_1''(t), \dots, \bar{F}_n''(t)) \right] dt \right| \end{aligned}$$

$$< \psi(0) 2r \sup |\phi| \frac{S}{N} + \frac{\epsilon}{2S} N \left(\frac{S}{N} \right)$$

$$< \epsilon.$$

$$\text{Similarly, } \left| \int_0^S \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt - \int_0^S \psi(t) \phi(\bar{G}_1''(t), \dots, \bar{G}_n''(t)) dt \right| < \epsilon.$$

Therefore, it suffices to prove (2.2) for the case when all F_i, G_i are step functions which are constant on $\left[\frac{jS}{N}, \frac{(j+1)S}{N} \right)$, $j = 0, \dots, N-1$. Furthermore, without loss of generality we may assume that ψ is constant on each interval of the form $\left[\frac{jS}{N}, \frac{(j+1)S}{N} \right)$ for $j = 0, \dots, N-1$.

(iii) Assume now that $G_i \stackrel{m}{>} F_i$ for $i = 1, \dots, n$ and that all $2n$ functions have support in $[0, S)$ and are constant on each interval $\left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$ for $j=0, \dots, N-1$. We also assume ψ is constant on each of these intervals and use the notational simplification $\psi(j) = \psi\left(\frac{jS}{N}\right)$ for $j=0, \dots, N-1$.

Each $\bar{G}_i$ may be transformed into $\bar{F}_i$ by a finite succession of transformations τ of the following type (see Hardy, Littlewood and Pólya [5]). τ changes the value v_{ji} of $\bar{G}_i$ on the interval $\left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$ into $v_{ji} + h$ and the value v_{ki} of $\bar{G}_i$ on $\left[\frac{kS}{N}, \frac{(k+1)S}{N}\right)$ into $v_{ki} - h$ where $j < k$ and

$$0 \leq v_{ki} - h \leq v_{ki} \leq v_{ji} \leq v_{ji} + h \leq 1.$$

Letting Δ_τ denote the change in the integral $\int_0^S \psi(t) \phi(\bar{G}_1(t), \dots, \bar{G}_n(t)) dt$ resulting from such a transformation τ , we complete the proof by showing that $\Delta_\tau \geq 0$. Without loss of generality $i=1$, and hence

$$\begin{aligned} \Delta_\tau &= \frac{S}{N} \{ \psi(j) [\phi(v_{j1} + h, v_{j2}, \dots, v_{jn}) - \phi(v_{j1}, v_{j2}, \dots, v_{jn})] \\ &\quad - \psi(k) [\phi(v_{k1}, v_{k2}, \dots, v_{kn}) - \phi(v_{k1} - h, v_{k2}, \dots, v_{kn})] \} \\ &\geq \frac{S}{N} \{ \psi(j) [\phi(v_{j1} + h, v_{j2}, \dots, v_{jn}) - \phi(v_{j1}, v_{j2}, \dots, v_{jn})] \\ &\quad - (\phi(v_{j1} + h, v_{k2}, \dots, v_{kn}) - \phi(v_{j1}, v_{k2}, \dots, v_{kn})) \} \end{aligned}$$

(since ϕ is convex in each variable separately)

$$\begin{aligned}
 &= \psi(k) \frac{S}{N} \{ [\phi(v_{j1} + h, v_{k2} + h_2, \dots, v_{kn} + h_n) - \phi(v_{j1}, v_{k2} + h_2, \dots, v_{kn} + h_n)] \\
 &\quad - \phi(v_{j1} + h, v_{k2} + h_2, \dots, v_{kn}) + \phi(v_{j1}, v_{k2} + h_2, \dots, v_{kn})] \\
 &\quad + \dots \\
 &\quad + [\phi(v_{j1} + h, v_{k2} + h_2, v_{k3}, \dots, v_{kn}) - \phi(v_{j1}, v_{k2} + h_2, v_{k3}, \dots, v_{kn})] \\
 &\quad - \phi(v_{j1} + h, v_{k2}, \dots, v_{kn}) + \phi(v_{j1}, v_{k2}, \dots, v_{kn})] \} \\
 &\geq 0
 \end{aligned}$$

(since ϕ satisfies property (2.1) and ψ is nonnegative).

Here $h_i = v_{ji} - v_{ki}$ for $i = 2, \dots, n$.

Corollary 2.2. Let G and F be life distribution functions with finite means. Then $G \stackrel{m}{>} F$ if and only if

- a) For all nonnegative increasing continuous convex ϕ and nonnegative decreasing ψ ,

$$\int_0^{\infty} \psi(t) \phi(\bar{G}(t)) dt \leq \int_0^{\infty} \psi(t) \phi(\bar{F}(t)) dt$$

and

- b) For all nonnegative increasing continuous concave ϕ and nonnegative increasing ψ ,

$$\int_0^{\infty} \psi(t) \phi(\bar{G}(t)) dt \geq \int_0^{\infty} \psi(t) \phi(\bar{F}(t)) dt,$$

provided the integrals exist.

Proof. The only if part follows immediately from Theorem 2.1. Assume

now a) and b) hold. Letting $\phi(u) = u$ and $\psi_x(t) = \chi_{[x, +\infty)}$ (that is the

characteristic function of the interval $[x, +\infty)$) it follows from b) that

$$\int_x^{\infty} \bar{G}(t) dt \geq \int_x^{\infty} \bar{F}(t) dt \text{ for all } x \geq 0. \text{ Taking } \psi(t) \equiv 1, \text{ it follows from a) that}$$

$$\mu_F = \mu_G.$$

Corollary 2.3. If G and F are life distributions with finite means, then

$$G \stackrel{m}{>} F$$

$\Leftrightarrow$

$$(2.5) \quad \int_0^{\infty} \Psi(t) dG(t) \geq \int_0^{\infty} \Psi(t) dF(t)$$

holds for all convex functions Ψ , provided the integrals exist.

Proof. The if part of the result is immediate. Now suppose $G \stackrel{m}{>} F$. It suffices to prove (2.5) for the case where Ψ has derivative ψ and $\Psi(0)=0$.

Then

$$\begin{aligned} \int_0^{\infty} \Psi(t) dG(t) &= \int_0^{\infty} \psi(t) \bar{G}(t) dt \\ &= \int_0^{\infty} [\psi(t) - \psi(0)] \bar{G}(t) dt + \psi(0) \mu_G \\ &\geq \int_0^{\infty} [\psi(t) - \psi(0)] \bar{F}(t) dt + \psi(0) \mu_F \quad (\text{by Theorem 2.1}) \\ &= \int_0^{\infty} \Psi(t) dF(t). \end{aligned}$$

Remark 2.4. Another approach to (2.5) in the proof of Corollary 2.3 is as follows. Suppose $G \stackrel{m}{>} F$. Let Z_G and Z_F be the random variables with respective densities $\frac{1}{\mu_G} \int_0^x \bar{G}(t) dt$ and $\frac{1}{\mu_F} \int_0^x \bar{F}(t) dt$. Then $Z_G \stackrel{st}{\geq} Z_F$ (Z_G is stochastically larger than Z_F) and hence (see for example Ross [11]) $E(\psi(Z_G)) \geq E(\psi(Z_F))$ for all increasing ψ . But

$$\int_0^{\infty} \psi(t) \bar{G}(t) dt = E(\psi(Z_G)) \geq E(\psi(Z_F)) = \int_0^{\infty} \psi(t) \bar{F}(t) dt.$$

3. Applications.

Theorem 3.1. Let $X_1, \dots, X_n, Y_1, \dots, Y_n$ be independent nonnegative random variables where $X_i \sim F_i$ and $Y_i \sim G_i$ for $i=1, \dots, n$, and let $X_{[1]}, \dots, X_{[n]}$ and $Y_{[1]}, \dots, Y_{[n]}$ be respectively the X (Y) observations in increasing order.

Assume that $G_i \stackrel{m}{>} F_i$ for $i=1, \dots, n$. Then

$$a) \int_x^\infty P[Y_{[n]} + \dots + Y_{[k]} > t] dt \geq \int_x^\infty P[X_{[n]} + \dots + X_{[k]} > t] dt$$

for all $x \geq 0$ and $k = 1, 2, \dots, n$.

$$b) (EY_{[1]}, \dots, EY_{[n]}) \stackrel{m}{>} (EX_{[1]}, \dots, EX_{[n]}).$$

Proof. b) follows immediately from a). In what follows $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ will denote any vector whose components are zeroes or ones. For $i = 1, \dots, n$, we define $\phi_i: [0, 1]^n \rightarrow [0, +\infty)$ by

$$\phi_i(u_1, \dots, u_n) = \sum_{\substack{\underline{\epsilon}, \epsilon_1 + \dots + \epsilon_n \geq n-i+1}} u_1^{\epsilon_1} \dots u_n^{\epsilon_n} (1-u_1)^{1-\epsilon_1} \dots (1-u_n)^{1-\epsilon_n}.$$

We note that $EX_{[i]} = \int_0^\infty \phi_i(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt$ for $i = 1, \dots, n$. Now for $k = 1, \dots, n$ we define

$$\begin{aligned} \phi_k(u_1, \dots, u_n) &= \sum_{i=k}^n \phi_i(u_1, \dots, u_n) \\ &= \sum_{i=k}^n \sum_{\substack{\underline{\epsilon}, \epsilon_1 + \dots + \epsilon_n \geq n-i+1}} u_1^{\epsilon_1} \dots u_n^{\epsilon_n} (1-u_1)^{1-\epsilon_1} \dots (1-u_n)^{1-\epsilon_n} \\ &= \sum_{j=1}^n \min(j, n-k+1) \sum_{\substack{\underline{\epsilon}, \epsilon_1 + \dots + \epsilon_n = j}} u_1^{\epsilon_1} \dots u_n^{\epsilon_n} (1-u_1)^{1-\epsilon_1} \dots (1-u_n)^{1-\epsilon_n}. \end{aligned}$$

$$\text{Since } \int_x^\infty P[X_{[n]} + \dots + X_{[k]} > t] dt = \int_x^\infty \phi_k(\bar{F}_1(t), \dots, \bar{F}_n(t)) dt,$$

it suffices by Theorem 2.1 b to show that each ϕ_k satisfies (2.3) and is concave increasing in each variable separately.

$$\text{Now } \frac{\partial}{\partial u_1} \phi_k(u_1, \dots, u_n) = \sum_{j=0}^{n-k} \sum_{\substack{\underline{\epsilon}_1, \epsilon_2 + \dots + \epsilon_n = j}} u_2^{\epsilon_2} \dots u_n^{\epsilon_n} (1-u_2)^{1-\epsilon_2} \dots (1-u_n)^{1-\epsilon_n}$$

where $\underline{\epsilon}_1$ represents an $n-1$ component vector of zeroes and ones.

As $\phi_k(u_1, \dots, u_n)$ is symmetric in $u_1, \dots, u_n$, it follows that ϕ_k is an increasing function linear (and hence concave) in each variable separately. For a continuously twice differentiable function ϕ on $[0,1]^n$, it is easy to verify that the following conditions are equivalent (see Lorentz [7]):

$$(3.1) \quad \phi(u_i + h, u_j + k) - \phi(u_i + h, u_j) - \phi(u_i, u_j + k) + \phi(u_i, u_j) \geq 0$$

for all $i \neq j$, $0 \leq u_i \leq u_i + h \leq 1$, $0 \leq u_j \leq u_j + k \leq 1$.

$$(3.2) \quad \phi(u_i + h, u_j + h) - \phi(u_i + h, u_j) - \phi(u_i, u_j + h) + \phi(u_i, u_j) \geq 0$$

for all $i \neq j$, $0 \leq u_i \leq u_i + h \leq 1$, $0 \leq u_j \leq u_j + h \leq 1$.

$$(3.3) \quad \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \phi(u_1, \dots, u_n) \geq 0$$

for all $i \neq j$.

Therefore, due to the symmetry of ϕ_k and the above equivalence, it suffices to note that

$$\frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \phi_k(u_1, \dots, u_n) = - \sum_{\substack{\epsilon_3, \dots, \epsilon_n \\ \epsilon_{12}, \epsilon_3 + \dots + \epsilon_n = n-k}} u_3^{\epsilon_3} \dots u_n^{\epsilon_n} (1-u_3)^{1-\epsilon_3} \dots (1-u_n)^{1-\epsilon_n} \leq 0$$

(where ϵ_{12} represents an $n-2$ component vector of zeroes and ones).

Remark 3.2. Let $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be random samples of size n from populations with life distribution functions F and G respectively. Barlow and Proschan [1] show that if $G \geq F$ where G and F have common mean, then

$$(EY_{[1]}, \dots, EY_{[n]}) \stackrel{m}{\geq} (EX_{[1]}, \dots, EX_{[n]}).$$

Shaked [13] proves the same result under the more general assumption that $G \stackrel{m}{>} F$. His proof uses the characterization of Corollary 2.3 together with the fact that

$$\psi_k(t_1, \dots, t_n) = t_{[n]} + \dots + t_{[k]}$$

is (separately) convex for each k . It follows that

$$\begin{aligned} EY_{[n]} + \dots + EY_{[k]} &= \int_0^\infty \psi_k(t_1, \dots, t_n) dG(t_1) \dots dG(t_n) \\ &\geq \int_0^\infty \psi_k(t_1, \dots, t_n) dF(t_1) \dots dF(t_n) \\ &= EX_{[n]} + \dots + EX_{[k]}. \end{aligned}$$

Remark 3.3 Suppose that for each $a \in A$, $F^{(a)}$ is distribution function on R , and that γ is a probability measure defined on a σ -field of subsets of A . One may define the n -variate distribution function (assuming appropriate measurability conditions on $F^{(a)}$)

$$F(x_1, \dots, x_n) = \int_A F^{(a)}(x_1) \dots F^{(a)}(x_n) d\gamma(a).$$

If random variables $X_1, \dots, X_n$ have such a joint distribution function, they are said to be 'positively dependent by mixture'. Given $X_1, \dots, X_n$ positively dependent by mixture, let $Y_1, \dots, Y_n$ be independent random variables where Y_i is distributed as X_i for $i = 1, \dots, n$. Shaked [12] (See also Marshall and Olkin [9] and Proschan [10]) has shown that in this case

$$(EY_{[1]}, \dots, EY_{[n]}) \stackrel{m}{>} (EX_{[1]}, \dots, EX_{[n]}).$$

Remark 3.4 Theorem 3.1 shows that if $G_i \stackrel{m}{>} F_i$ for all $i=1, \dots, n$, then for any k $\sum_{i=k}^n Y_{[i]}$ is "more variable" than $\sum_{i=k}^n X_{[i]}$ (in the terminology of Ross [11]) or that $\sum_{i=k}^n Y_{[i]}$ is "larger in mean residual life" than $\sum_{i=k}^n X_{[i]}$

(in the terminology of Stoyan [14]). Since $\Psi_k(t_1, \dots, t_n) = t_{[n]} + \dots + t_{[k]}$ is convex, this also follows by using the result that if $X_1, \dots, X_n, Y_1, \dots, Y_n$ are independent and Y_i is "more variable" than X_i for $i=1, \dots, n$, then $\Psi_k(Y_1, \dots, Y_n)$ is "more variable" than $\Psi_k(X_1, \dots, X_n)$ (see Ross [11]).

Remark 3.5 If $X_1, \dots, X_n$ are independent HNBUE random variables, then Theorem 3.1 b could be useful in constructing bounds on the expected order statistics $EX_{[1]}, \dots, EX_{[n]}$.

Example 3.6 Let us consider the following problem of general interest. n components are to be purchased in order to form a coherent system (for example a k out of n system), and all of the components are to be purchased from either company A or company B. Let us suppose that each company makes the claim that components of type i have mean life μ_i ($i=1, \dots, n$), but that company B is known to be 'more variable' than company A in the production of any type of component. If we wish to maximize the mean life of the system, from which company should we buy?

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be random variables representing the lifetimes of the components from A and B respectively. If we can assume that the components function independently within the system and that Y_i is more variable than X_i in the sense that $G_i \stackrel{m}{>} F_i$ (where $X_i \sim F_i$ and $Y_i \sim G_i$) for all $i=1, \dots, n$, then we know that

$$(EY_{[1]}, \dots, EY_{[n]}) \stackrel{m}{>} (EX_{[1]}, \dots, EX_{[n]}).$$

In particular $EY_{[1]} - EX_{[1]} \leq 0$ and $EY_{[n]} - EX_{[n]} \geq 0$. Therefore if our system is a series system we would buy from A, while if it is parallel we would buy from B. This result was observed by Marshall and Proschan [8].

For a more general k out of n system, we would be interested in the expected order statistics $EX_{[n-k+1]}$ and $EY_{[n-k+1]}$ in order to compare companies A and B. Although

$$(EY_{[1]}, \dots, EY_{[n]}) \stackrel{m}{>} (EX_{[1]}, \dots, EX_{[n]}),$$

$EY_{[i]} - EX_{[i]}$ may theoretically at least undergo many sign changes as $i:1 \rightarrow n$ even in the case when $F_i = F$ and $G_i = G$ for all $i=1, \dots, n$. However under the assumption that $G \stackrel{m}{>} F$ where G and F are continuous, G is strictly increasing on its interval support and $G(0)=F(0)=0$, one may show that the number of sign changes in $EY_{[i]} - EX_{[i]}$ is no greater than the number of sign changes in $\bar{G}(x) - \bar{F}(x)$ as $x:0 \rightarrow \infty$. Since $\binom{n-1}{i-1} F^{i-1}(t) \bar{F}^{n-i}(t)$ is totally positive of order ∞ in i and t , this follows using the variation diminishing property of totally positive functions and the identity

$$EY_{[i]} - EX_{[i]} = \int_0^{\infty} n(G^{-1}F(t) - t) \binom{n-1}{i-1} F^{i-1}(t) \bar{F}^{n-i}(t) dt$$

(see Barlow and Proschan [1]). In particular if $\bar{F}$ crosses $\bar{G}$ once then there exists a constant C (depending on n , F and G) such that

$$EY_{[i]} - EX_{[i]} \leq 0 \quad \text{for } i < C$$

and

$$EY_{[i]} - EX_{[i]} \geq 0 \quad \text{for } i > C.$$

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