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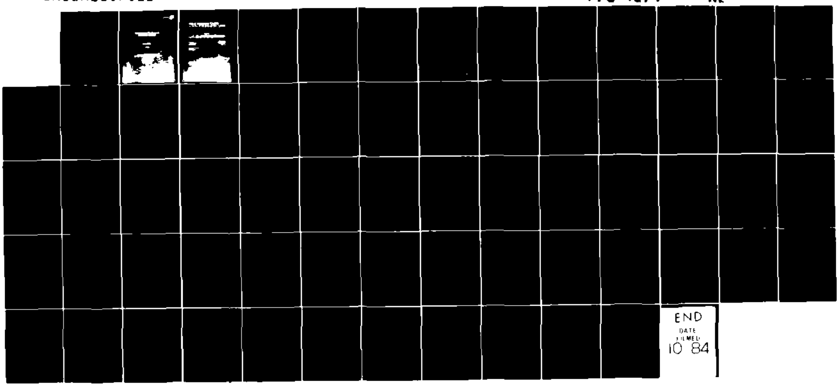

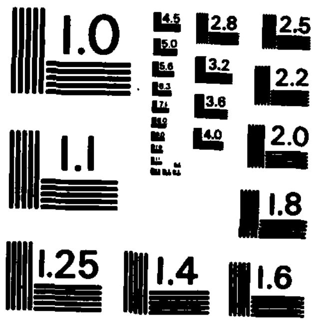

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By means of augmenting the state vector, a stochastic optimal control problem, which is of the standard multivariate "linear-quadratic-Gaussian" type except for a relatively small degree of uncertainty in some of its parameters, is reformulated as a nonlinear stochastic optimal control problem with known parameters, certain of which are small. A perturbation analysis is used to derive an approximation of the optimal control law which is formally accurate to second order in these small parameters..

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CONTENTS

I. INTRODUCTION AND SUMMARY

One important class of problems that arises in applications of automatic control theory is that of adaptive control of a dynamic system. When suitably reformulated in terms of an augmented state vector, such a control problem becomes one of controlling a composite nonlinear system having some static or slowly varying state components, usually called parameters, which are not precisely known or directly observed. Most of the recent work in this area has focused on proving the global stability of control laws. Such proofs tend to be more feasible for less efficient control laws, however, and efficiency is also important for practical applications. This aspect of the problem can be investigated by analyzing it as an optimal control problem with respect to an explicit performance criterion. This approach is pursued here, with all the uncertainties in the (original) system dynamics, available measurements and parameter values formulated as random variables.

Exact solutions to such control problems are typically unobtainable, so consideration is limited here to certain cases in which these uncertainties are small, with the objective of developing asymptotic approximations of optimal control laws in terms of the perturbations induced by departures of the random variables from their mean values. In the absence of parameter uncertainty, the limiting case of small random variations in the measurements and system dynamics reduces the problem under fairly general conditions to solving the deterministic optimal control problem that results from all the random variables assuming their mean values with certainty (which is a relatively easy problem and not examined here), and then solving a stochastic optimal control problem in the (small) perturbations of the state, control, and measurement variables from their values in this optimally controlled deterministic case, where this stochastic problem has linear dynamics and state measurements (in the perturbation variables), a quadratic performance criterion, and Gaussian white noise added to the dynamics and measurements (Reference 1). The dynamics for a scalar state perturbation variable x in a simple case of this sort might have the form

$$\dot{x} = fx + gu + w$$

where u is a control variable, w is noise, and f and g are known parameters.

If f itself, however, is a function of some variable y, say an aerodynamic coefficient, which is not precisely known in advance, it can be expanded in a Taylor series about an average value $\bar{y}$. Truncating this expansion at second order turns out to leave a useful degree of accuracy here, and gives

$$f = f(\bar{y}) + f'(\bar{y})(y - \bar{y}) + \frac{1}{2}f''(\bar{y})(y - \bar{y})^2 .$$

If the typical size of $y - \bar{y}$ is h (assumed suitably small), then a new variable

$$\theta = \frac{y - \bar{y}}{h}$$

can be defined whose typical size is unity, and for which

$$f = f(\bar{y}) + hf'(\bar{y})\theta + h^2 f''(\bar{y})\theta^2 .$$

Thus we are led to consider dynamics of the following form for the adaptive extension of this problem with uncertainty in the value of f:

$$\dot{x} = \bar{f}x + gu + \alpha x\theta + \beta x\theta^2 + w ,$$

where $\bar{f}$, g, α and β are known, θ can be considered another state variable (static or slowly varying) to be estimated along with x, α is of order h, and β is of order h^2. A variety of considerations of this sort for the other parameters (such as g) in the multivariate case of the above stochastic optimal control problem leads to the general problem formulated in Section III, which is the actual subject of this report.

An approximation of an optimal control law is then developed for this problem which is asymptotically accurate to second order in the scaling parameter h as h approaches zero; i.e., as the parameter uncertainty vanishes (first-order approximations are degenerate here). This approximately optimal control law is specified in terms of a finite-dimensional initial-value system of ordinary differential equations. The control consists of generally cubic feedback on the current conditional means of the dynamic system state vector and a parameter vector (which together form a composite state vector in the analysis), and linear feedback on the current conditional cross-covariance matrix for these two vectors. The means and cross-covariance matrix are generated recursively by a finite measurement-driven system of stochastic differential equations, as in a Kalman filter.

The analysis here is only formal, however; e.g., it is assumed without rigorous justification that the neglected error terms are sufficiently "well-behaved" that all quantities that appear to be of higher order are so in some appropriate sense. Also, many of the order-of-magnitude arguments are made as if the (dynamic system) state and parameter vectors were both scalars without explicitly mentioning this at the time. In the multivariate case the validity of these arguments might also depend on some subtle conditions on the dependency structure of the state and parameter components, but such questions are beyond the scope of this report. For these reasons it is important to augment this type of analysis by testing the results on specific numerical examples. One such example is included here and the theory appears to give reasonable and meaningful results in this case.

The results here apply directly only to adaptive control problems in which the parameter uncertainties are relatively small. The hope is, however, that these results could provide some useful clues for constructing more efficient control schemes under more general conditions, even though the design process itself might evolve along more pragmatic lines.

II. NOTATION

Unless otherwise indicated, lower case letters denote (real) column vectors or scalars. Matrices are denoted by capital roman letters. A^T denotes the transpose of a matrix A. If A is square, $|A|$ denotes its determinant, and tr(A) its trace.

It will be convenient to make rather extensive use of three-way matrices, which are always denoted by capital Greek letters here. For continuity of notation, the following definitions are adopted for such a three-way matrix Γ, with vector x and matrices A and B of compatible dimensions, and with repeated indices denoting summation:

$(\Gamma x)_{ij} = \Gamma_{ij\sigma} x_\sigma$	(matrix)
$(Ax^T)_{ijk} = A_{ij} x_k$	(three-way matrix)
$(A\Gamma)_{ijk} = A_{i\sigma} \Gamma_{\sigma jk}$	(three-way matrix)
$(\Gamma B)_{ijk} = \Gamma_{ij\sigma} B_{\sigma k}$	(three-way matrix)
$(\Gamma')_{ijk} = \Gamma_{jki}$ and $(\Gamma^T)_{ijk} = \Gamma_{kji}$	(three-way matrices)
$[\mathrm{tr}(\Gamma)]_i = \Gamma_{\sigma i\sigma}$	(column vector, when applicable)

Γ is called symmetric if $\Gamma = \Gamma' = \Gamma'' = \Gamma^T$.

With these definitions, the expression $AF\Gamma BDxx^T$ is fully associative. Many other consequences are obvious. Some useful but less obvious properties are:

$$\mathrm{tr}(\Gamma' x) = [\mathrm{tr}(\Gamma)]^T$$

$$A\,\mathrm{tr}(\Gamma') = \mathrm{tr}(A\Gamma')$$

$$\mathrm{tr}(A\Gamma) = \mathrm{tr}(\Gamma A)$$

$$(\Gamma B)' = B^T\Gamma'' \quad \text{and} \quad \Gamma'' A^T = (A\Gamma)''$$

$$(A\Gamma B)^T = B^T\Gamma^T A^T$$

$$(\Gamma' x)A^T = (A\Gamma)' x \quad \text{and} \quad (\Gamma'' x)B = (\Gamma B)'' x$$

$$\Gamma \text{ symmetric} \Rightarrow (B^T\Gamma B)' B \quad \text{and} \quad A(A\Gamma A^T)'' \text{ symmetric.}$$

Some use will also be made of four-way matrices, which are denoted by capital roman bold letters here. The following operations involving four-way matrices are likewise defined:

$(\mathbf{A}')_{ijkm} = \mathbf{A}_{jkmi}$	(four-way matrix)
$(\mathbf{A}^T)_{ijkm} = \mathbf{A}_{mjki}$	(four-way matrix)

$[\mathrm{tr}(\mathbf{A})]_{ij} = \mathbf{A}_{\sigma ij\sigma}$ (matrix, when applicable)

$(\mathbf{AA})_{ijkm} = A_{i\sigma}\mathbf{A}_{\sigma jkm}$ (four-way matrix)

$(\mathbf{A}B)_{ijkm} = \mathbf{A}_{ijk\sigma}B_{\sigma m}$ (four-way matrix)

$(\mathbf{A}x)_{ijk} = \mathbf{A}_{ijk\sigma}x_\sigma$ (three-way matrix)

$(x^T\mathbf{A})_{ijk} = \mathbf{A}_{\sigma ijk}x_\sigma$ (three-way matrix)

$(\Gamma\Omega)_{ijkm} = \Gamma_{ij\sigma}\Omega_{\sigma km}$ (four-way matrix)

Analogously, products such as $\Gamma\mathbf{A}$ are occasionally formed by summing over repeated adjacent indices; i.e., a five-way matrix in this case such that

$$(\Gamma\mathbf{A})_{ijkmn} = \Gamma_{ij\sigma}\mathbf{A}_{\sigma kmn} ,$$

and the trace of higher-way matrices is always obtained by summing over repeated outer indices so that, for example,

$$[\mathrm{tr}(\mathbf{B}\Gamma)]_{ijk} = \mathbf{B}_{\sigma ij\lambda}\Gamma_{\lambda k\sigma} .$$

Also, outer products of various kinds of matrices are denoted by $\otimes$, so that

$(A \otimes B)_{ijkm} = A_{ij}B_{km}$ (four-way matrix)

or

$(\Gamma \otimes A)_{ijkmn} = \Gamma_{ijk}A_{mn}$. (five-way matrix)

Parentheses are omitted in this notation if the order of association is immaterial or if the interpretation is unambiguous.

The probability density function of a (vector) random variable x is denoted by $p_x(\cdot)$ and the corresponding expectation operator by $\mathbf{E}_x$. The covariance matrix of x is denoted by cov(x). When the meaning is clear from the context, $p(x)$, $p(x/y)$, and $\mathbf{E}(x)$ are often used as abbreviations for $p_x(x)$, $p_{x/y}(x,y)$, and $\mathbf{E}_x(x)$.

III. BASIC PROBLEM AND APPROACH

In order to include various multivariate extensions of the problem described in the introduction, and also to include the possibility of adding some precisely known components to the motion state x, we consider a situation in which finite-dimensional real-valued state vectors x and θ have the dynamics

$$\dot{x} = Fx + Gu + 2\,\mathrm{tr}(\Gamma\theta x^T) + 2\mathrm{tr}(\Psi\theta u^T) + \mathrm{tr}(\mathbf{F}\theta\theta^T)x + \mathrm{tr}(\mathbf{L}\theta\theta^T)u + (I + \dot{\Sigma}'\theta)^T w, \quad (1)$$

$$\dot{\theta} = Tu + \varepsilon w_2 \ , \quad \varepsilon \text{ a scalar} \ , \tag{2}$$

and for which the measurement vector

$$z = Hx + 2K\theta + 2\,\mathrm{tr}(\Omega\theta x^T) + \mathrm{tr}(\mathbf{H}\theta\theta^T)x + \mathrm{tr}(\bar{\Delta}\theta\theta^T) + v \tag{3}$$

is available at each instant $t > t_0$, where the time argument t of these variables and of the coefficient matrices, which also may be time-varying, is suppressed in the notation, and where

- u is a control vector,
- w, w_2, and v are independent zero-mean Gaussian white noise processes with respective covariance parameters Q, Q_2, and R, which may be time-varying,
- at the initial time t_0, which is specified *a priori*, $x(t_0)$ and $\theta(t_0)$ have the independent Normal prior distributions $N(\hat{x}_0, P_0)$ and $N(0, L_0)$, respectively,
- ε and the components of K and of the three-way matrices Γ, Ψ, $\bar{\Sigma}$, and Ω are approximately infinitesimal, say of order h; the components of T,$\bar{\Delta}$ and of all the four-way matrices are of order h^2; no other quantities, including the components of $P_0{}^{-1}$, $L_0{}^{-1}$ and R^{-1}, are very large compared to unity, and
- $\mathbf{F} = \mathbf{F}^T$, $\mathbf{L} = \mathbf{L}^T$, $\mathbf{H} = \mathbf{H}^T$, and $\bar{\Delta} = \bar{\Delta}^T$ (which imposes no loss of generality).

The problem is to find a feedback control law, specifying the current control $u(t)$ as a function of t and past control and measurement values, which minimizes the (scalar) performance criterion

$$J = \mathbf{E}\Bigg\{ \frac{1}{2} \Big[x_f^T\Big(S_f + \theta_f^T \mathbf{A}\theta_f\Big)x_f + \theta_f^T \mathrm{tr}\Big(\Psi_f x_f x_f^T\Big)\Big]$$
$$+ \int_{t_0}^{t_f} \Bigg(\frac{1}{2}\Big[x^T(A + \theta^T\mathbf{G}\theta)x + u^T(B + \theta^T\mathbf{M}\theta)u\Big]$$
$$+ \Big[\theta^T Z + \mathrm{tr}^T(\Delta\theta\theta^T) + c^T\Big]u + \frac{1}{2}\theta^T\Big[\mathrm{tr}(\Phi x x^T) + \mathrm{tr}(\Theta u u^T)\Big]\Bigg)\, dt\Bigg\}, \tag{4}$$

where $\mathbf{E}$ denotes prior expectation, t_f is an *a priori* specified terminal time, and for all

$$t\,\varepsilon[t_0, t_f]:$$

- S_f and $A(t)$ are positive-semidefinite

- $B(t)$ is positive-definite
- no component of S_f, $A(t)$, $B(t)$, $B^{-1}(t)$, or $c(t)$ is very large compared to unity
- the components of $Ч_f$, $Z(t)$, $\Phi(t)$, and $\Theta(t)$ are of order h; those of $\mathbf{A}$, $\Delta(t)$, $\mathbf{G}(t)$ and $\mathbf{M}(t)$ are of order $h2$
-

$$Ч_f = Ч_f^T \, , \ \Delta(t) = \Delta^T(t) \, , \ \Phi(t) = \Phi^T(t) \, , \ \Theta(t) = \Theta^T(t) \, ,$$

$$\mathbf{A} = \mathbf{A}^T \, , \ \mathbf{A}'' = \mathbf{A}''^T \, , \ \mathbf{G}(t) = \mathbf{G}^T(t) \, , \ \mathbf{G}''(t) = \mathbf{G}''^T(t) \, ,$$

$$\mathbf{M}(t) = \mathbf{M}^T(t) \, , \text{ and } \mathbf{M}''(t) = \mathbf{M}''^T(t) \; .$$

The vector θ is slowly varying and is meant to represent normalized uncertainties in the parameter matrices F, G, Q, H, S_f, A, and B in what would otherwise be a standard linear-quadratic-Gaussian (LQG) optimal control problem with x alone as the state vector. It is also assumed here that this corresponding LQG problem, i.e., with all the three- and four-way matrices zero, is such that the mean and covariance matrix of the conditional distribution of $x(t)$, and also the latter's inverse, remain of order unity, and is such that the prior expected value of $x^T x$ remains of order unity under the well-known optimal control law for this case.

The actual objective here is only to obtain an approximation of an optimal control law for small h. If h were zero, of course, the problem would reduce to the corresponding LQG case just mentioned, for which the exact solution is known to be a linear feedback control on the current conditional mean of x, which in turn can be generated recursively from the incoming measurements by a Kalman filter. Unlike the situation described in Reference 2, however, the departures from this linear solution are all determined by effects of order $h2$ here, basically because the small nonlinear terms in the dynamics and measurements, and the small cubic and quantic terms in the criterion, all involve components of the vector θ, about which no information could be inferred if h were zero (i.e., if the three- and four-way matrices were identically zero). Thus the analysis must be carried out to second order to obtain nontrivial results. Otherwise the basic approach taken here is the same as that in Reference 2–first, recursive equations are developed for the evolution of the parameters of an appropriate approximation to the conditional probability density of the state given the available measurements; then a dynamic programming analysis is used to determine an approximation of an optimal control law. In this case, however, the problem is analyzed in terms of the augmented state vector

$$\begin{bmatrix} x \\ \theta \end{bmatrix} ,$$

and a second-order Edgeworth expansion must be used to approximate the conditional density of this augmented state to order $h2$. Considerable simplification occurs because of the special structure of this case as a nonlinear control problem, though, and only the first, second, and some third central moments of this Edgeworth expansion actually need to be computed for the purpose of determining the largest nontrivial effects of θ-uncertainty on the optimal control, meaning the asymptotic form, as h approaches zero, of the *departures* of the optimal control from that for the corresponding LQG problem. These departures are all of order h^2 when the time $t_t - t_0$ over which the system is being controlled is of order unity, but can attain, at least in formal appearance, magnitudes of order unity when this control interval becomes of order $1/h^2$. The analysis here will not be even formally valid for control intervals of larger order of magnitude. It is necessary, however, to compute fourth central moments (and the other remaining parameters of the Edgeworth expansion) in order to implement an alternative approximation of the optimal control law which has the same formal degree of asymptotic accuracy (in h) but which might perform better for noninfinitesimal h. These fourth central moments are also necessary for determining the conditional state density to an accuracy of order h^2.

IV. MOTION-STATE AND PARAMETER ESTIMATION

To determine the joint conditional density of $x(t)$ and $\theta(t)$ to order h^2, given the measurements up to a generic time t, the estimation process is considered in discrete time increments of size δt, with $\delta t << h^6$ so that $\delta t^{3/2} << h^3 \delta t$. The result is then established by what is basically an induction argument. It is assumed as an induction hypothesis that the joint conditional density $p[x(t), \theta(t)/y(t)]$, where $y(t)$ denotes the measurement history up to time t, has the form of a certain class of Edgeworth expansions except for an error of higher order than h^2. This class of density approximations includes the Normal joint prior of $x(t_0)$ and $\theta(t_0)$ as a special case, and is shown in complete detail for scalar x and θ in the appendix. The induction argument is then completed by calculating the corresponding conditional density at time $t + \delta t$ to a formal accuracy of order $h^2 \delta t$. The calculation shows this latter density to be of the same form, with each parameter having changed by only δt times its hypothesized order in h and the error by an amount of higher order than $h^2 \delta t$. Since this error therefore remains formally small compared to h^2 when accumulated over a time interval of order unity, the desired result is obtained for $t - t_0$ of this size. It is also reasonable to consider this result formally valid for longer elapsed times on the basis that the joint x,θ system is "sufficiently observable" that the effects of the error increments are damped out rapidly enough over time that their cumulative effect always remains small compared to h^2. No investigation has been made of precise conditions required or sufficing to make this happen, however. The resulting equations governing the evolution over time of the parameters of the second-order Edgeworth approximation have the formal appearance of being stable in forward time with time constants of decay which are short enough that these parameters at least would remain of the proper order of magnitude indefinitely under normal circumstances. The control u is treated as a known deterministic time function in this section.

The rest of this section is mostly devoted to a description of induction step calculations and the final result. These calculations are too lengthy to be presented in detail here, so only a rather specific outline of the procedure is given. The calculations are carried out in detail in the appendix for the case of scalar x and θ, however, except for any consideration of the higher-order error terms, which are simply deleted whenever they occur there.

In the calculations of the induction step, the transition of the composite state from

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \text{ to } \begin{bmatrix} x(t+\delta t) \\ \theta(t+\delta t) \end{bmatrix}$$

is approximated to order $h^2\delta t$ by the following sequence of transformations:

1.

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \left[\begin{array}{c|c} I+F\delta t & 2\Psi^{T}u\delta t \\ \hline O & I \end{array}\right] \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} G \\ T \end{bmatrix} u\delta t$$

(where the t argument is suppressed except for $x(t)$ and $\theta(t)$)

2.

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} + \left[\begin{array}{c} 2\mathrm{tr}(\Gamma\theta_1 x_1^T) + \mathrm{tr}(F\theta_1\theta_1^T)x_1 + \mathrm{tr}(L\theta_1\theta_1^T)u \\ \hline O \end{array}\right]\delta t$$

3.

$$\begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} + \sqrt{\delta t}\begin{bmatrix} \bar{w}_1 \\ O \end{bmatrix};$$

$$\bar{w}_1 \text{ is Normal}\left[O, \left(I + \bar{\Sigma}'\theta_2\right)^T Q\left(I + \bar{\Sigma}'\theta_2\right)\right] \text{ given } x_2 \text{ and } \theta_2 .$$

4.

$$\begin{bmatrix} x(t+\delta t) \\ \theta(t+\delta t) \end{bmatrix} = \begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} + \sqrt{\delta t}\begin{bmatrix} O \\ - \\ \bar{w}_2 \end{bmatrix}; \ \bar{w}_2 \text{ is Normal}\left(O, \varepsilon^2 Q_2\right).$$

Starting with the induction hypothesis for $p[x(t), \theta(t)/y(t)]$, the joint density of each of these variables, also conditioned on $y(t)$, is then computed in turn to order $h^2\delta t$ according to the following outline for each variable in the sequence:

1. Since u is a known constant, the joint density of x_1 and θ_1 given $y(t)$ follows easily from standard results for linear transformations of random variables.

2. From the definitions of x_2 and θ_2, either

$$x_1 = x_2 - \delta t\left[2\mathrm{tr}\left(\Gamma\theta_2 x_2^T\right) + \mathrm{tr}\left(F\theta_2\theta_2^T\right)x_2 + \mathrm{tr}\left(L\theta_2\theta_2^T\right)u\right]$$

to order $h^2\delta t$ or $|x_1| >> 1$ for x_2 and θ_2 of order unity. Also, the Jacobian matrix whose i,j-th element is

$$\partial\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix}_i / \partial\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}_j$$

can be expressed as

$$I + \delta t \left[\begin{array}{c|c} F + 2\Gamma'^T\theta_2 + \mathrm{tr}\left(F\theta_2\theta_2^T\right) & 2\Gamma^T x_2 + 2\mathrm{tr}^T\left(F\theta_2 x_1^T\right) + 2\,\mathrm{tr}^T\left(L\theta_2 u^T\right) \\ \hline 0 & 0 \end{array} \right],$$

whose determinant is $1 + \delta t\,[\mathrm{tr}(F + 2\Gamma\theta_2) + \mathrm{tr}\,\mathrm{tr}\,(F\theta_2\theta_2^T)]$ to order $h^2\delta t$. The joint density of x_2 and θ_2 can then be obtained to this accuracy from that of x_1 and θ_1 for x_2 and θ_2 of order unity by applying the standard formula for the transformation of probability densities, because large values of x_1 and θ_1 have negligible density and $x_1 - x_2$ is otherwise of order $h\delta t$. This approximation is also adequate for large x_2 or θ_2 because x_1 and θ_1 cannot both be of order unity then, and the density is again negligible. This result is converted to the desired Edgeworth form by expanding it in a Taylor series about the mean and covariance matrix of x_2 and θ_2, and deleting terms which are negligible compared to $h^2\delta t$. This mean and covariance matrix can be evaluated directly to this accuracy from the definition of x_2 and what is known about the joint density of x_1 and θ_1.

3. The joint density of x_3 and θ_3 is calculated according to the standard formula

$$p_{s_3}(x,\theta) = \int_{-\infty}^{\infty} p_{s_2}(x - w\sqrt{\delta t},\theta)\,p_{w_1/s_2}(w,x,\theta)\,dw$$

for the sum of two random variables, where the conditioning on $y(t)$ is suppressed in the notation, and where

$$s_2 = \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} \text{ and } s_3 = \begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix}.$$

This integral can be evaluated to an accuracy of order $h^2\delta t$ by "completing the square" for the product of the two exponential factors and using standard results for evaluating the moments of Normal distributions. It is then straightforward to evaluate the mean and covariance matrix of s_3 to order $h^2\delta t$. Expanding the density approximation obtained for s_3 in a Taylor series about these values and deleting terms small compared to $h^2\delta t$. gives the result in the desired Edgeworth form.

4. Obtaining the joint density of $x(t+\delta t)$ and $\theta(t+\Delta)$ (to order $h^2\delta t$) from that of x_3 and θ_3 is conceptually just a repetition of the previous computation, the formula for the sum of the appropriate two random variables this time being

$$p_{s_4}(x,\theta) = \int_{-\infty}^{\infty} p_{s_3}(x,\theta - w\sqrt{\Delta})\,p_{w_2/s_3}(w)\,dw,$$

where

$$s_4 = \begin{bmatrix} x(t+\Delta) \\ \theta(t+\Delta) \end{bmatrix}.$$

Actually, $\bar{w}_2$ is statistically independent of s_3 and the conditioning indicated above is superfluous. Because of this fact and the small variance of $\bar{w}_2$, the details are much simpler than in the previous computation.

Once $p[x(t+\delta t),\theta(t+\delta t)/y(t)]$ has been obtained in this way, the induction step is completed by approximating the effects of the measurements between t and $t+\delta t$ by a random variable $\tilde{z}$ such that $p[\tilde{z}/x(t+\delta t), \theta(t+\delta t)]$ is Normal and conditionally independent of $y(t)$ with

$$\text{mean}\,(\tilde{z}) = H(t+\delta t)x(t+\delta t) + 2K(t+\delta t)\theta(t+\delta t) + 2\,\text{tr}\,[\Omega(t+\delta t)\theta(t+\delta t)x^T(t+\delta t)]$$

$$+\ \text{tr}[\mathbf{H}(t+\delta t)\theta(t+\delta t)\theta^T(t+\delta t)]\,x\,(t+\delta t) + \text{tr}[\,\bar{\Delta}\,(t+\delta t)\theta(t+\delta t)\theta^T(t+\delta t)]$$

and

$$\text{cov}\,(\tilde{z}) = \frac{1}{\delta t}\,R(t+\delta t)$$

which is formally accurate except for terms of order $\delta t^{3/2}$, and therefore accurate to order $h^2\delta t$ since $\delta t << h^4$. This construction allows $p[x(t+\delta t),\theta(t+\delta t)/y(t+\delta t)]$ to be determined by the Bayes rule to this accuracy as

$$p[x(t+\delta t),\theta(t+\delta t)/y(t)]\cdot p[\tilde{z}/x(t+\delta t),\theta(t+\delta t)]$$

divided by its integral over the $x(t+\delta t)$ and $\theta(t+\delta t)$ arguments. Expanding this density in a Taylor series up to terms of order $h^2\delta t$ about its mean and covariance matrix (treating z as a quantity of order $\delta t^{-1/2}$) then gives the final result described above for the induction step. Zero-mean random terms of order $\sqrt{\delta t}$ but of third or higher order in h are also deleted in this last expansion if they are statistically independent for disjoint time increments (of size δt) because it follows from the Chebychev inequality that such terms only contribute effects of order h^3 when accumulated over time intervals of order unity, except perhaps for a set of realizations with negligible prior probability.

The second-order Edgeworth approximation for $p(x,\theta/y)$ which is established by this induction argument has the form of a Normal density multiplied by a polynomial in x and θ whose leading term is unity. The parameters of this approximation are the mean and covariance matrix of the Normal factor, which are also (to order h^2) the mean and covariance matrix of the joint conditional density of x and θ, and a set of quantities determining the coefficients of the polynomial factor, all of which are small compared to unity. These parameters are denoted here as follows, and have the following interpretations in terms of this joint conditional density:

$$\left.\begin{aligned} \hat{x} &= \mathbf{E}(x) \\ \hat{\theta} &= \mathbf{E}(\theta) \end{aligned}\right\} \quad \text{partition of mean vector}$$

$$\left.\begin{aligned} M &= \mathbf{E}[(x-\hat{x})(x-\hat{x})^T] \\ E &= \mathbf{E}[(x-\hat{x})(\theta-\hat{\theta})^T] \\ L &= \mathbf{E}[(\theta-\hat{\theta})(\theta-\hat{\theta})^T] \end{aligned}\right\} \quad \text{partition of covariance matrix}$$

$$\left.\begin{aligned} \Lambda &= \tfrac{1}{2}\mathbf{E}\{[(x-\hat{x})(\theta-\hat{\theta})^T](x-\hat{x})^T\} \\ \bar{\nabla} &= \tfrac{1}{2}\mathbf{E}\{[(\theta-\hat{\theta})(x-\hat{x})^T](\theta-\hat{\theta})^T\} \end{aligned}\right\} \quad \text{two partition components of three-way matrix of third central moments}$$

N, a four-way matrix whose $ijkm$-th component is (to order h^2)

$$\frac{1}{4}\,\mathbf{E}\left[(x-\hat{x})_i(\theta-\hat{\theta})_j(\theta-\hat{\theta})_k(x-\hat{x})_m\right] - \frac{1}{4}M_{im}L_{jk} - \frac{1}{2}E_{ij}E_{mk} \;.$$

Also, the components of E and Λ are of order h, and those of $\bar{\nabla}$ and N are of order h^2. As in the prior density, $\hat{x}$, M and L are of order unity. From Equation 2, θ is the time integral of a control-dependent term of order h^2 (assuming control values are kept to order unity) and a white noise term with variance parameter of order h^2, so θ will be of order unity for the estimation intervals of order $1/h^2$ being considered here. Since the posterior variance L (of θ) is of order unity, this means that the posterior mean $\hat{\theta}$ is also of order unity.

If the calculations of the induction step are carried out according to the preceding outline, the eventual result is a coupled set of recursions for generating these parameters to order h^2 from the incoming state measurements (see the appendix). If z is used to denote the average of the measurement vector over a current discretization time increment, these recursions can be expressed as measurement-driven differential equations in terms of the formally corresponding continuous-time variables. With the time argument t suppressed in the notation and with the definitions

$$\Sigma = \frac{1}{2}\left(Q\bar{\Sigma}^T + \bar{\Sigma}\,Q\right), \tag{5}$$

$$\hat{z} = H\hat{x} + 2K\hat{\theta} + 2\,\mathrm{tr}\left[\Omega\,(\hat{\theta}\hat{x}^T + E^T)\right] + \mathrm{tr}\left[\bar{\Delta}\,(\hat{\theta}\hat{\theta}^T + L)\right] + \mathrm{tr}\left[\mathrm{H}\,(\hat{\theta}\hat{\theta}^T + L)\right]\hat{x}, \tag{6}$$

and

$$D = \frac{1}{2}\left(M - P\right) \tag{7}$$

for the time function P defined by

$$\dot{P} = FP + PF^T + Q - PH^TR^{-1}HP\,;\; P(t_0) = P_0\,, \tag{8}$$

these differential equations become the following:

$$\dot{\hat{x}} = F\hat{x} + Gu + 2\,\mathrm{tr}\left[\Gamma\left(\hat{\theta}\hat{x}^T + E^T\right)\right] + 2\,\mathrm{tr}\left(\Psi\hat{\theta}u^T\right)$$

$$+ \mathrm{tr}\left[\mathbf{F}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\hat{x} + \mathrm{tr}\left[\left(\mathbf{L}'u\right)\left(\hat{\theta}\hat{\theta}^T + L\right)\right] + \Big\{\left(P + 2D\right)\Big[H^T + 2\Omega\hat{\theta}$$

$$+ \mathrm{tr}^T\left(\mathbf{H}\hat{\theta}\hat{\theta}^T\right)\Big] + 2\left[E\left(K^T + \Omega^T\hat{x}\right) + \mathrm{tr}\left(\Lambda'\Omega^T\right) + \mathrm{tr}\left(\Lambda''\Omega\right)\right]\Big\} R^{-1}\left(z - \hat{z}\right);$$

$$\hat{x}(t_0) = \hat{x}_0 \tag{9}$$

$$\dot{\hat{\theta}} = Tu + \Big\{E^T\left(H^T + 2\Omega\hat{\theta}\right) + 2L\Big[K^T + \Omega^T\hat{x} + \overline{\Delta}^T\hat{\theta}$$

$$+ \mathrm{tr}\left(\mathbf{H}'\hat{x}\hat{\theta}^T\right)\Big]\Big\} R^{-1}\left(z - \hat{z}\right); \hat{\theta}\left(t_0\right) = 0 \tag{10}$$

$$\dot{D} = \left(F - PH^TR^{-1}H\right)D + D\left(F^T - H^TR^{-1}HP\right) + \Big[E\left(\Gamma' - \Omega'R^{-1}HP\right)$$

$$+ \left(\Gamma'^T - PH^TR^{-1}\Omega'^T\right)E^T\Big]'\hat{x} + \Big[\left(\Gamma'' - PH^TR^{-1}\Omega''\right)\left(P + 2D\right)$$

$$+ \left(P + 2D\right)\left(\Gamma^{T\prime} - \Omega^{T\prime}R^{-1}HP\right) + \Sigma\Big]'\hat{\theta} + \left(E\Psi' + \Psi'^TE^T\right)'u$$

$$+ \frac{1}{2}\Big\{P\,\mathrm{tr}^T\left[\mathbf{F}\left(\hat{\theta}\hat{\theta}^T + L\right)\right] + \mathrm{tr}\left[\mathbf{F}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]P + \mathrm{tr}\left[\overline{\Sigma}'\left(\hat{\theta}\hat{\theta}^T + L\right)\overline{\Sigma}'^TQ\right]\Big\}$$

$$- PH^TR^{-1}\Big\{2\,tr\left(\Omega\Lambda''\right) + KE^T + \frac{1}{2}\,\mathrm{tr}\left[\mathbf{H}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]P\Big\}$$

$$- \Big\{2\,\mathrm{tr}\left(\Lambda'\Omega^T\right) + \frac{1}{2}P\,\mathrm{tr}^T\left[\mathbf{H}\left(\hat{\theta}\hat{\theta}^T + L\right)\right] + EK^T\Big\}R^{-1}HP$$

$$- 2\left(DH^T + P\Omega\hat{\theta}\right)R^{-1}\left[HP + \left(\Omega^TP\right)'\hat{\theta}\right] + \Big\{\Big[\Lambda'L^{-1}E^T$$

$$+ \left(\Lambda'L^{-1}E^T\right)' + \left(\Lambda'L^{-1}E^T\right)''\Big]H^T + \left(E\Omega^TP + P\Omega E^T\right)' + 2\Lambda\Big(K^T$$

$$+ \Omega^T\hat{x}\Big)\Big\}R^{-1}\left(z - \hat{z}\right); D\left(t_0\right) = 0 \tag{11}$$

$$\dot{E} = \left(F - PH^TR^{-1}H\right)E - 2PH^TR^{-1}KL - 2DH^TR^{-1}\left(HE + KL\right) + 2\Big\{\left(\Gamma L\right)^{\prime\prime}\hat{x}$$

$$- \left[L\Omega' R^{-1}H\left(P + 2D\right)\right]^{\prime}\hat{x} + \Big[E^T\Gamma^{\prime\prime} - E^T\left(H^TR^{-1}\Omega^{\prime\prime} + \Omega^{\prime\prime T}R^{-1}H\right)P$$

$$- L\left(K^TR^{-1}\Omega^{\prime\prime} + \bar{\Delta}' R^{-1}H\right)P\Big]^{\prime}\hat{\theta} + \left(\Psi L\right)^{\prime\prime}u + \mathrm{tr}\left[\left(\mathbf{F}^{\prime\prime\prime} - PH^TR^{-1}\mathbf{H}^{\prime\prime\prime}\right)L\hat{\theta}\hat{x}^T\right]$$

$$+ \mathrm{tr}^T\left[\left(\mathbf{L}L\right)^{\prime}u\hat{\theta}^T\right]\Big\} + 2\Big\{\Lambda\left(H^T + 2\Omega\hat{\theta}\right) + \Big[L\left(\Omega^T + \mathbf{H}\hat{\theta}\right)P$$

$$+ 2L\Omega^TD\Big]^{\prime}\Big\}R^{-1}\left(z - \hat{z}\right);\ E\left(t_0\right) = 0 \qquad (12)$$

$$\dot{L} = \varepsilon^2Q_2 - \left[E^TH^T + 2L\left(K^T + \Omega^T\hat{x}\right)\right]R^{-1}\left[HE + 2KL + 2\left(\Omega L\right)^{\prime\prime}\hat{x}\right] + 2\Big\{\bar{\nabla}' H^T + \Big[E^T\Omega L$$

$$+ L\Omega^TE + L\left(\bar{\Delta} + \left(\mathbf{H}'\hat{x}\right)^{\prime\prime}\right)L\Big]^{\prime}\Big\}R^{-1}\left(z - \hat{z}\right);\ L\left(t_0\right) = L_0 \qquad (13)$$

$$\dot{\Lambda} = \left(F - PH^TR^{-1}H\right)\Lambda + \Lambda\left(F^T - H^TR^{-1}HP\right) + \left(P\Gamma L\right)^{\prime\prime} + \left(L\Gamma^TP\right)^{\prime} - PH^TR^{-1}\left(P\Omega L\right)^{\prime\prime}$$

$$- \left(L\Omega P\right)' R^{-1}HP + L\Sigma^{\prime\prime} + 2\left[\left(\Gamma L\right)^{\prime\prime} - \Lambda H^TR^{-1}H - PH^TR^{-1}\left(\Omega L\right)^{\prime\prime} - P\left(L\Omega^T\right)' R^{-1}H\right]D$$

$$+ 2D\left[\left(L\Gamma^T\right)^{\prime} - H^TR^{-1}H\Lambda - \left(L\Omega^T\right)' R^{-1}HP - H^TR^{-1}\left(\Omega L\right)^{\prime\prime}P\right] + \left[P\left(\mathbf{F}\hat{\theta}\right)^TL\right]^{\prime\prime}$$

$$+ \left[L\left(\mathbf{F}\hat{\theta}\right)P\right]^{\prime} - PH^TR^{-1}\left[P\left(\mathbf{H}\hat{\theta}\right)^TL\right]^{\prime\prime} - \left[L\left(\mathbf{H}\hat{\theta}\right)P\right]' R^{-1}HP - 2P\left(\Omega\hat{\theta}\right)R^{-1}\left(P\Omega L\right)^{\prime\prime}$$

$$- 2\left(L\Omega^TP\right)' R^{-1}\left(\Omega'^T\hat{\theta}\right)P - 2\Big[\Lambda\left(PH^TR^{-1}\Omega^{\prime\prime} + P\Omega^{T\prime}R^{-1}H\right)^{\prime} + \Big(PH^TR^{-1}\Omega^{\prime\prime}\Lambda$$

$$+ P\Omega^{T\prime}R^{-1}H\Lambda\Big)^{\prime\prime}\Big]\hat{\theta} + \frac{1}{2}\Big\{\Big[\left(\bar{\Sigma}\bar{\Sigma}^TQ\right)^{\prime T} + \left(\bar{\Sigma}\bar{\Sigma}^TQ\right)^{\prime\prime T}\Big]L\Big\}^{\prime\prime T\prime\prime}\hat{\theta};\ \Lambda\left(t_0\right) = 0 \qquad (14)$$

$$\dot{\bar{\nabla}} = \left[\left(F - PH^TR^{-1}H\right)\bar{\nabla}^{\prime\prime}\right]^{\prime} + E^T\Gamma L + L\Gamma^TE + \frac{1}{2}\left\{L\left[\mathrm{tr}\left(\mathbf{F}L\right)\hat{x} + \mathrm{tr}\left(\mathbf{L}L\right)u\right]^T\right\}^{\prime\prime}$$

$$- E^T H^T R^{-1}\left(L\Omega^T P\right)' - \left(P\Omega L\right)'' R^{-1} HE - \left[E^T H^T + 2L\left(K^T + \Omega^T \hat{x}\right)\right] R^{-1}\left(\Lambda + L\Omega^T P\right)'$$

$$- \left(\Lambda + P\Omega L\right)'' R^{-1}\left[HE + 2KL + 2\left(\Omega L\right)''\hat{x}\right] - L\left(\bar{\Delta}' R^{-1} HP\right)'' L + 2L\left[\Omega' R^{-1}\left(z - \hat{z}\right)\right]\Lambda'$$

$$+ 2\Lambda''\left[\Omega' R^{-1}\left(z - \hat{z}\right)\right]^T L + 2\left[N H^T R^{-1}\left(z - \hat{z}\right)\right]'; \ \bar{\nabla}\left(t_0\right) = 0 \qquad (15)$$

$$\dot{N} = \left(F - PH^T R^{-1} H\right) N + N\left(F^T - H^T R^{-1} HP\right) + \frac{1}{2}\left[P\left(LFL\right)'' + \left(LF^T L\right)'' P\right.$$

$$\left. - P\left(LHL\right)'' R^{-1} HP - PH^T R^{-1}\left(LH^T L\right)'' P\right] - \frac{1}{4} L \bar{\Sigma}^T Q \bar{\Sigma} L$$

$$- \left[\Lambda H^T + \left(L\Omega^T P\right)'\right] R^{-1}\left[H\Lambda + \left(P\Omega L\right)''\right] - \Lambda H^T R^{-1}\left(P\Omega L\right)''$$

$$- \left(L\Omega^T P\right)' R^{-1} H\Lambda - \left\{\frac{1}{4} L \bar{\Sigma}^T Q \bar{\Sigma} L + \Lambda H^T R^{-1}\left(P\Omega L\right)'' + \left(L\Omega^T P\right)' R^{-1} H\Lambda\right.$$

$$\left. + \left[\Lambda H^T + \left(L\Omega^T P\right)'\right] R^{-1}\left[H\Lambda + \left(P\Omega L\right)''\right]\right\}^{\prime T}; \ N\left(t_0\right) = 0 \qquad (16)$$

If the formalism of these differential equations is interpreted in the sense of Ito equations with z now denoting the original state measurement vector, these equations generate the same parameter values as their discrete-time counterparts to a formal accuracy of order h^2 for time discretization intervals $\delta t << h^4$. These Ito differential equations thus determine the joint conditional density of $x(t)$ and $\theta(t)$, and the corresponding conditional moments indicated earlier, to a formal accuracy of order h^2 in the original continuous-time problem. It also follows from the form of these equations that $P = P^T, D = D^T, \Lambda = \Lambda^T, \bar{\nabla} = \bar{\nabla}^T$, $N = N^T$ and $N'' = N''^T$, which is also necessary if these quantities are to have the conditional moment properties ascribed to them earlier.

For state estimation purposes, it is sometimes preferable to work with the quantity M directly. In this case one can, for the same order of formal accuracy, integrate

$$\dot{M} = \left\{F + 2\Gamma''^T\hat{\theta} + \mathrm{tr}\left[F\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\right\} M + 2\left(\Gamma''\hat{x} + \Psi'' u\right) E^T$$

$$+ M\left\{F^T + 2\Gamma\hat{\theta} + \mathrm{tr}^T\left[F\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\right\} + 2E\left(\Gamma^T\hat{x} + \Psi^T u\right) + Q + 2\Sigma'\hat{\theta}$$

$$+ \mathrm{tr}\left[\bar{\Sigma}'\left(L + \hat{\theta}\hat{\theta}^T\right)\bar{\Sigma}'^T Q\right] - \left\{M\left[H^T + 2\Omega\hat{\theta} + \mathrm{tr}^T\left(H\hat{\theta}\hat{\theta}^T + HL\right)\right] + 4\,\mathrm{tr}\left(\Lambda'\Omega^T\right)\right.$$

$$+ 2E\left(K^T + \Omega^T\hat{x}\right)\Big\} R^{-1}\Big\{\Big[H + 2\Omega'^T\hat{\theta} + \mathrm{tr}\left(\mathrm{H}\hat{\theta}\hat{\theta}^T + \mathrm{H}L\right)\Big]M + 4\,\mathrm{tr}\left(\Omega\Lambda''\right)$$

$$+ 2\left(K + \Omega''\hat{x}\right)E^T\Big\} + 2\Big\{\Big[\Lambda'L^{-1}E^T + \left(\Lambda'L^{-1}E^T\right)' + \left(\Lambda'L^{-1}E^T\right)''\Big]H^T$$

$$+ \left(E\Omega^T M + M\Omega E\right)' + 2\Lambda'\left(K^T + \Omega^T\hat{x}\right)\Big\} R^{-1}\left(z - \hat{z}\right);\ M\left(t_0\right) = P_0 \qquad (17)$$

in place of $\dot{P}$ and $\dot{D}$, and replace P by M and D by zero everywhere in Equations 9 and 10 and 12 through 16. It is also sometimes inconvenient to work with appropriately normalized motion-state and parameter vectors x and θ. In the scalar case examined in the appendix, the variables of actual interest are

$$\frac{x}{\sqrt{M}} \text{ and } \frac{\theta}{\sqrt{L}}$$

and the (formal) validity of the approximations made in the derivation really depend on

$$\frac{M}{R},\ \frac{Q}{M} \text{ and } \frac{Q_2}{L}$$

being of order unity,

$$\varepsilon,\ \frac{\theta}{\sqrt{L}},\ \frac{\Gamma}{\sqrt{L}},\ \Omega\sqrt{\frac{L}{M}},\ \frac{Tu}{\sqrt{L}},\ \Psi u\sqrt{\frac{L}{M}},\ \bar{\Sigma}\sqrt{L},\ K\sqrt{\frac{L}{M}} \text{ and } \frac{E}{\sqrt{ML}}$$

being of order h, and

$$FL,\ \frac{\mathrm{L}Lu}{\sqrt{M}},\ \mathrm{H}L \text{ and } \frac{\bar{\Delta}L}{\sqrt{M}}$$

being of order h^2, or these same conditions with M replaced by P. In the multivariate case this validity actually depends on some reasonable multivariate extension of these conditions. The dependency structure of the equations of motion and measurement probably plays an important role here, however, which complicates the situation.

V. CONTROL OPTIMALITY CONDITIONS

The optimal control law approximations here are derived by applying conditions resulting from a standard type of dynamic programming analysis. The first step is to consider the conditional expected "cost-to-go" under an optimal control law at a general time $t \geq t_0$, by which is meant a generalization of the definition of Equation 4 in which the integration is from t to t_f, the expectation is conditioned on the controls and state measurements up to time t, and the control values $u(\tau)$, $t \le \tau \le t_f$, are generated from the measurements available at time τ by an optimal control law. This expected cost-to-go is a function of time and the currently available measurement data (including past control values), but, because of the statistical independence of future process and measurement noise values from the current composite state (x,θ), the cost-to-go depends only on this state's current conditional distribution (Reference 3). This distribution is specified via the multivariate extension of Equation A-1 to order h^2 by the current values of the measurement statistics $\hat{x}$, $\hat{\theta}$, D, E, L, Λ, $\overline{\nabla}$ and N. Assuming now that some arbitrary but reasonable control law has been specified, the noise-induced fluctuations of all these statistics except $\hat{\theta}$ and L from their prior expected values (for this choice of control law) will become essentially independent over time intervals of order unity because each is multiplied by the coefficient matrix $(F - PH^TR^{-1}H)$ in the differential equation generating it from the measurements in the system of Equations 5 through 16. The observability properties being assumed for this problem imply that this coefficient matrix will have the effect of "damping out" the effects of these fluctuations over longer time intervals. In using Equation 13 to generate L, on the other hand, the contributions of these fluctuations and of the last measurement-driven term are effectively integrated over observation intervals up to order $1/h^2$ since Equation 13 must be integrated this long (except for a negligible set of realizations) to change L significantly compared to its typical size of order unity. Thus the departures of L from its prior expected value over an interval of this length will essentially be, in order of magnitude, the sum of $1/h^2$ zero-mean, statistically independent terms, each representing the effect of such contributions over a time interval of order unity minus its prior expected value. From Equation 13 and the orders of magnitude of the quantities involved, each of the terms in this sum is of order h^2, and therefore a random variable whose variance is of order h^4. From a standard result, such a sum is a zero-mean random variable with variance of order h^2. There are fewer terms in this sum for shorter observation intervals, and its variance would then be even smaller. By the Chebychev inequality, this means that the departures of L from its prior expected value are of order h over observation intervals up to order $1/h^2$ (except for a set of realizations with negligible probability, which will be ignored henceforth). Approximating L by its prior expected value, and $\hat{\theta}$ and D by zero in Equation 14 therefore changes Λ by order h^2. Since the "damping coefficient matrix" $(F - PH^TR^{-1}H)$ multiplying Λ in Equation 14 is such that these errors only accumulate over time intervals of order unity in the integration of this equation, the error in approximating Λ in this way is formally only of order h^2. This approximation of Λ is a deterministic time function, and is in fact the prior expected value of Λ (to order h^2, the accuracy of the state estimation analysis). Likewise, approximating both L and Λ by these deterministic time functions in Equations 6, 9 through 12, 15, and 16 changes E by order h^2 and leaves $\hat{x}$, $\hat{\theta}$, D, $\overline{\nabla}$, N formally unchanged to this order of accuracy.

Now consider the possibility that the optimal conditional expected cost-to-go is, to an accuracy of order h^2 except perhaps for a set of realizations with negligibly small probability, a scalar-valued function (also called J here) of the form

$$J = \frac{1}{2}\hat{x}^T\left(S + W\right)\hat{x} + \left(\zeta + \eta\right)^T\hat{x} + \phi^T\hat{\theta} + \mathrm{tr}\left[\left(S + Y\right)D + EV^T\right]$$

$$+ \frac{1}{2}\hat{\theta}^T U\hat{\theta} + \hat{x}^T N\hat{\theta} + \hat{x}^T \mathrm{tr}\left(E^T\Pi'\right) + \hat{\theta}^T \mathrm{tr}\left(\nabla D\right)$$

$$+\frac{1}{2}\hat{\theta}^T \operatorname{tr}\left(Ч\hat{x}\hat{x}^T\right) + \frac{1}{2}\hat{x}^T \operatorname{tr}\left(\Xi'\hat{\theta}\hat{\theta}^T\right) + \frac{1}{2}\operatorname{tr}\left[D \operatorname{tr}\left(\mathbf{R}D\right)\right]$$

$$+\frac{1}{2}\operatorname{tr}\left[E \operatorname{tr}\left(\mathbf{B}E^T\right)\right] + \frac{1}{2}\hat{x}^T \operatorname{tr}\left(\mathrm{I}\hat{\theta}\hat{\theta}^T\right)\hat{x} + \sigma\,, \tag{18}$$

where W, η, ϕ, Y, V, U, N, Π, ∇, $Ч$, Ξ, $\mathbf{R}$, $\mathbf{B}$, I and σ are deterministic time functions and the time argument t is suppressed in the notation, where S and ζ are the time function specified by

$$\dot{S} = -SF - F^TS - A + SGB^{-1}G^TS\,;\; S(t_f) = S_f \tag{19}$$

and

$$\dot{\zeta} = \left(SGB^{-1}G^T - F^T\right)\zeta + SGB^{-1}C\,;\; \zeta\left(t_f\right) = 0\,, \tag{20}$$

where $\hat{x}$, $\hat{\theta}$, D and E are the moment parameters of the conditional state density at the current time t approximated to order h^2 by Equations 5 through 16, and where

σ is of order $1/h^2$,

ϕ is of order $1/h$,

Y, U, $\mathbf{R}$ and $\mathbf{B}$ are of order unity,

V, N, Π, ∇, $Ч$ and η are of order h,

W, Ξ and I are of order h^2,

W, Y and U are symmetric,

$Ч = Ч^T$, $\Xi' = \Xi'^T$, $\nabla = \nabla^T$,

$\mathrm{I} = \mathrm{I}^T$, $\mathrm{I}'' = \mathrm{I}''^T$,

$\mathbf{B} = \mathbf{B}''$, and

$\mathbf{R} = \mathbf{R}'' = \mathbf{R}^T$.

By their definitions, S and P are also symmetric and of order unity, ζ is of order unity, and D is symmetric and of order h. The nonzero first and second partial derivatives of this function J with respect to variables other than t are

$$J_{\hat{x}} = \hat{x}^T\left[S + W + Ч'\hat{\theta} + \operatorname{tr}\left(\mathrm{I}\hat{\theta}\hat{\theta}^T\right)\right] + \zeta^T + \eta^T + \hat{\theta}^TN^T + \operatorname{tr}^T\left(E^T\Pi' + \frac{1}{2}\Xi\hat{\theta}\hat{\theta}^T\right), \tag{21}$$

$$J_{\hat{\theta}} = \Phi^T + \hat{x}^T N + \hat{\theta}^T \left[U + \Xi'' \hat{x} + \mathrm{tr}\left(\Gamma'' \hat{x}\hat{x}^T \right) \right] + \mathrm{tr}^T \left(\frac{1}{2} Ч \hat{x}\hat{x}^T + \nabla D \right) \tag{22}$$

$$J_D = S + Ч + \nabla' \hat{\theta} + \mathrm{tr}\left(\mathbf{R} D \right) , \tag{23}$$

$$J_E = V^T + \Pi'' \hat{x} + \mathrm{tr}\left(\mathbf{B} E^T \right) , \tag{24}$$

$$J_{\hat{x}\hat{x}} = S + W + Ч' \hat{\theta} + \mathrm{tr}\left(\mathbf{I} \hat{\theta}\hat{\theta}^T \right) , \tag{25}$$

$$J_{\hat{\theta}\hat{x}} = N^T + Ч'' \hat{x} + \Xi' \hat{\theta} + 2\, \mathrm{tr}\left(\Gamma' \hat{x}\hat{\theta}^T \right) , \tag{26}$$

$$J_{E\hat{x}} = \Pi'' , \tag{27}$$

$$J_{\hat{\theta}\hat{\theta}} = U + \Xi'' \hat{x} + \mathrm{tr}\left(\Gamma'' \hat{x}\hat{x}^T \right) , \tag{28}$$

$$J_{D\hat{\theta}} = \nabla' , \tag{29}$$

$$J_{DD} = \mathbf{R} , \quad \text{and} \tag{30}$$

$$J_{EE} = \mathbf{B} , \tag{31}$$

where subscripts now denote partial differentiation with the convention that

$$\left(J_E \right)_{ij} = \frac{\partial J}{\partial E_{ji}} ,$$

$$\left(J_{\hat{\theta}\hat{x}} \right)_{ij} = \frac{\partial^2 J}{\partial \hat{\theta}_i \partial \hat{x}_j} ,$$

$$\left(J_{E\hat{x}} \right)_{ijk} = \frac{\partial^2 J}{\partial E_{ji} \partial \hat{x}_k} , \quad \text{and}$$

$$\left(J_{EE} \right)_{ijkm} = \frac{\partial^2 J}{\partial E_{im} \partial E_{kj}} .$$

Since the other second partials are zero, the usual invariant imbedding formalism of dynamic programming (Reference 4), applied to a very small time increment δt and using $\hat{x}$,

$\hat{\theta}$, D, and E as state variables, shows that the Bellman equation for the optimal expected cost-to-go function reduces to the following equation for J to second order in h, at least for all but a set of realizations with negligibly small probability:

$$\begin{aligned}
-J_t = \min_u \mathrm{E}_{/\mathrm{data}(t)} \Bigg\{ & \frac{1}{2} x^T\left(A + \Phi'\theta + \theta^T G\theta\right)x + \frac{1}{2} u^T\Big(B \\
& + \Theta'\theta + \theta^T M\theta\Big)u + \left[\theta^T Z + \mathrm{tr}^T\left(\Delta\theta\theta^T\right) + c^T\right]u + J_{\hat{x}}\frac{\delta\hat{x}}{\delta t} \\
& + J_{\hat{\theta}}\frac{\delta\hat{\theta}}{\delta t} + \mathrm{tr}\Bigg[J_D\frac{\delta D}{\delta t} + \frac{1}{2}J_{\hat{x}\hat{x}}\frac{\delta\hat{x}\delta\hat{x}^T}{\delta t} + \frac{1}{2}\frac{\delta D}{\delta t}\,\mathrm{tr}\left(J_D \delta D\right) \\
& + \frac{1}{2}\frac{\delta E}{\delta t}\,\mathrm{tr}\left(J_{EE}\delta E^T\right)\Bigg] + \mathrm{tr}\left(J_E\frac{\delta E}{\delta t} + J_{\hat{\theta}\hat{x}}\frac{\delta\hat{x}\delta\hat{\theta}^T}{\delta t}\right) \\
& + \mathrm{tr}\left(\frac{\delta E}{\delta t}J_{E\hat{x}}\delta\hat{x} + \frac{\delta D}{\delta t}J_{D\hat{\theta}}\delta\hat{\theta}\right) + \frac{1}{2}\,\mathrm{tr}\left(J_{\hat{\theta}\hat{\theta}}\frac{\delta\hat{\theta}\delta\hat{\theta}^T}{\delta t}\right)\Bigg\} . \qquad (32)
\end{aligned}$$

In Equation 32, $\delta\hat{x}$ denotes the increment in $\hat{x}$ that occurs during the interval $[t, t+\delta t]$ for the control value u, which is presumed to be constant during this interval with negligible loss of generality, and likewise for $\delta\hat{\theta}$, δD and δE. At the final time t_f, the integral term of Equation 4 vanishes in the definition of the cost-to-go function, so J must satisfy the condition

$$J\left(t_f,\hat{x}_f,\hat{\theta}_f,D_f,E_f\right) = \mathrm{E}_{/\mathrm{data}(t_f)}\left\{\frac{1}{2}x_f^T\left(S_f + \theta_f^T A\theta_f\right)x_f + \frac{1}{2}\theta_f^T\left(Ч_f x_f x_f^T\right)\right\}$$

to second order in h for all but a negligibly improbable set of the realizable values of the final-time statistics $\hat{x}_f$, $\hat{\theta}_f$, D_f and E_f. From the moment properties and orders-of-magnitude of these statistics, the boundary condition of the Bellman equation for J therefore becomes

$$\begin{aligned}
J\left(t_f,\hat{x},\hat{\theta},D,E\right) = & \frac{1}{2}\hat{x}^T\left[S_f + \mathrm{tr}\left(AL_f\right)\right] + \frac{1}{2}\,\mathrm{tr}\Big[S_f\Big(P_f + 2D\Big) \\
& + P_f\,\mathrm{tr}\left(AL_f\right)\Big] + \frac{1}{2}\hat{\theta}^T\,\mathrm{tr}\left[Ч_f\left(\hat{x}\hat{x}^T + P_f\right)\right] \\
& + \hat{x}^T\,\mathrm{tr}\left(E^T Ч_f'\right) + \mathrm{tr}\,\mathrm{tr}\left(Ч_f\Lambda_f\right) \qquad (33)
\end{aligned}$$

to order h^2, where L and Λ now denote the deterministic approximations described at the beginning of this section.

The limiting process here is such that δt can be regarded as arbitrarily small compared to h, so that terms of order $\delta t^{3/2}$ are negligible compared to those of order $h^3 \delta t$. Therefore, since Equations 9 through 12 are to be interpreted as the formally corresponding Ito equations, and since the measurement noise v in the interval $(t, t + \delta t)$ is independent of the composite state (x, θ) at time t,

$$\mathbf{E}(z - \hat{z}) = 0$$

and

$$\mathbf{E}[(z - \hat{z})(z - \hat{z})^T \delta t] = R(t)$$

conditioned on the data available at time t, and the following conditional expected increments can be evaluated from these equations to a formal accuracy of order $h^2 \delta t$ for any given $u(t)$:

$$\mathbf{E}\left(\frac{\delta \hat{x}}{\delta t}\right) = F\hat{x} + Gu + 2\,\mathrm{tr}\left[\Gamma\left(\hat{\theta}\hat{x}^T + E^T\right)\right] + 2\,\mathrm{tr}\left(\Psi\hat{\theta}u^T\right)$$
$$+ \,\mathrm{tr}\left[\mathbf{F}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\hat{x} + \mathrm{tr}\left[\left(\mathbf{L}'u\right)\left(\hat{\theta}\hat{\theta}^T + L\right)\right] \quad (34)$$

$$\mathbf{E}\left(\frac{\delta\hat{\theta}}{\delta t}\right) = Tu \quad (35)$$

$$\mathbf{E}\left(\frac{\delta D}{\delta t}\right) = \left(F - PH^TR^{-1}H\right)D + D\left(F^T - H^TR^{-1}HP\right) + \left[E\left(\Gamma' - \Omega' R^{-1}HP\right)\right.$$
$$+ \left.\left(\Gamma'^T - PH^TR^{-1}\Omega'^T\right)E^T\right]'\hat{x} + \left[\left(\Gamma'' - PH^TR^{-1}\Omega''\right)\left(P + 2D\right) + \left(P + 2D\right)\left(\Gamma^{T\prime}\right.\right.$$
$$\left.\left. - \,\Omega^{T\prime}R^{-1}HP\right) + \Sigma\right]'\hat{\theta} + \left(E\Psi' + \Psi'^TE^T\right)'u + \frac{1}{2}P\,\mathrm{tr}^T\left[\mathbf{F}\left(\hat{\theta}\hat{\theta}^T + L\right)\right.$$
$$+ \,\mathrm{tr}\left[\mathbf{F}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]P + \frac{1}{2}\,\mathrm{tr}\left[\bar{\Sigma}'\left(\hat{\theta}\hat{\theta}^T + L\right)\bar{\Sigma}'^TQ\right] - PH^TR^{-1}\left\{KE^T\right.$$
$$+ \,2\,\mathrm{tr}\left(\Omega\Lambda''\right)P + \frac{1}{2}\,\mathrm{tr}\left[\mathbf{H}\left(\hat{\theta}\hat{\theta}^T + L\right)P\right]\left.\right\} - \left\{EK^T + 2P\,\mathrm{tr}\left(\Lambda'\Omega^T\right)\right.$$
$$+ \left.\frac{1}{2}P\,\mathrm{tr}\left[\mathbf{H}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\right\}R^{-1}HP - 2\left(DH^T + P\Omega\hat{\theta}\right)R^{-1}\left[HP + \left(\Omega^TP\right)''\hat{\theta}\right] \quad (36)$$

$$\mathbf{E}\left(\frac{\delta E}{\delta t}\right) = \left(F - PH^TR^{-1}H\right)E - 2PH^TR^{-1}KL - 2DH^TR^{-1}\left(HE + KL\right)$$

$$+ 2\left(\Gamma L\right)''\hat{x} - 2\left[L\Omega' R^{-1}H\left(P + 2D\right)\right]'\hat{x} + 2\left[E^T\Gamma'' - E^T\left(H^TR^{-1}\Omega''\right.\right.$$

$$\left.\left. + \Omega''^TR^{-1}H\right)P - L\left(K^TR^{-1}\Omega'' + \bar{\Delta}'R^{-1}H\right)P\right]'\hat{\theta} + 2\,\mathrm{tr}\left[\left(\mathbf{F}'''\right.\right.$$

$$\left.\left. - PH^TR^{-1}\mathbf{H}'''\right)L\hat{\theta}\hat{x}^T\right] + 2\left(\Psi L\right)''u + 2\,\mathrm{tr}^T\left[\left(\mathbf{L}L\right)'\hat{\theta}^T\right] \tag{37}$$

$$\mathbf{E}\left(\frac{\delta\hat{x}\delta\hat{x}^T}{\delta t}\right) = \left\{\left(P + 2D\right)\left[H^T + 2\Omega\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}\hat{\theta}\hat{\theta}^T\right)\right] + 2\left[E\left(K^T + \Omega^T\hat{x}\right)\right.\right.$$

$$\left.\left. + \mathrm{tr}\left(\Lambda'\Omega^T\right) + \mathrm{tr}\left(\Lambda''\Omega\right)\right]\right\} R^{-1}\left\{\left(P + 2D\right)\left[H^T + 2\Omega\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}\hat{\theta}\hat{\theta}^T\right)\right]\right.$$

$$\left. + 2\left[E\left(K^T + \Omega^T\hat{x}\right) + \mathrm{tr}\left(\Lambda'\Omega^T\right) + \mathrm{tr}\left(\Lambda''\Omega\right)\right]\right\}^T \tag{38}$$

$$\mathbf{E}\left(\frac{\delta\hat{x}\delta\hat{\theta}^T}{\delta t}\right) = 2\left\{\left(P + 2D\right)\left[H^T + 2\Omega\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}\hat{\theta}\hat{\theta}^T\right)\right] + 2\left[E\left(K^T + \Omega^T\hat{x}\right)\right.\right.$$

$$\left.\left. + \mathrm{tr}\left(\Lambda'\Omega^T\right) + \mathrm{tr}\left(\Lambda''\Omega\right)\right]\right\}R^{-1}\left[K + \Omega''\hat{x} + \bar{\Delta}''\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}'\hat{x}\hat{\theta}^T\right)\right]L \tag{39}$$

$$\mathbf{E}\left(\frac{\delta E\delta\hat{x}^T}{\delta t}\right) = 2\left\{\Lambda\left(H^T + 2\Omega\hat{\theta}\right) + \left[L\left(\Omega^T + \mathbf{H}\hat{\theta}\right)P + 2L\Omega^TD\right]'\right\}R^{-1}\left\{\left(P\right.\right.$$

$$\left. + 2D\right)\left[H^T + 2\Omega\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}\hat{\theta}\hat{\theta}^T\right)\right] + 2\left[E\left(K^T + \Omega^T\hat{x}\right) + \mathrm{tr}\left(\Lambda'\Omega^T\right)\right.$$

$$\left.\left. + \mathrm{tr}\left(\Lambda''\Omega\right)\right]\right\}^T \tag{40}$$

$$\mathbf{E}\left(\frac{\delta\hat{\theta}\delta\hat{\theta}^T}{\delta t}\right) = 4L\left[K^T + \Omega^T\hat{x} + \bar{\Delta}^T\hat{\theta} + \mathrm{tr}\left(\mathbf{H}'\hat{x}\hat{\theta}^T\right)\right]R^{-1}\left[K + \Omega''\hat{x} + \bar{\Delta}''\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}'\hat{x}\hat{\theta}^T\right)\right]L \tag{41}$$

$$\mathbf{E}\left(\frac{\delta D\delta\hat{\theta}^T}{\delta t}\right) = 2\left\{\left[\Lambda' L^{-1}E^T + \left(\Lambda' L^{-1}E^T\right)' + \left(\Lambda' L^{-1}E^T\right)''\right]H^T + \left(E\Omega^T P + P\Omega E^T\right)'\right.$$

$$\left. + 2\Lambda\left(K^T + \Omega^T\hat{x}\right)\right\}R^{-1}\left[K + \Omega''\hat{x} + \bar{\Delta}''\hat{\theta} + \mathrm{tr}^T\left(\mathbf{H}'\hat{x}\hat{\theta}^T\right)\right]L \tag{42}$$

$$\mathbf{E}\left(\frac{\delta D\otimes\delta D}{\delta t}\right) = \left\{\left[\Lambda' L^{-1}E^T + \left(\Lambda' L^{-1}E^T\right)' + \left(\Lambda' L^{-1}E^T\right)''\right]H^T + \left(E\Omega^T P + P\Omega E^T\right)'\right.$$

$$\left. + 2\Lambda\left(K^T + \Omega^T\hat{x}\right)\right\}R^{-1}\left\{H\left[EL^{-1}\Lambda'' + \left(EL^{-1}\Lambda''\right)' + \left(EL^{-1}\Lambda''\right)''\right]\right.$$

$$\left. + \left(E\Omega^T P + P\Omega E^T\right)'' + 2\left[K\Lambda + \left(\Omega\Lambda\right)''\hat{x}\right]\right\} \tag{43}$$

$$\mathbf{E}\left(\frac{\delta E\otimes\delta E^T}{\delta t}\right) = 4\left\{\Lambda\left(H^T + 2\Omega\hat{\theta}\right) + \left[L\left(\Omega^T + \mathbf{H}\hat{\theta}\right)P + 2L\Omega^T D\right]'\right\}R^{-1}\left\{\left(H\right.\right.$$

$$\left.\left. + 2\Omega'^T\hat{\theta}\right)\Lambda + \left[L\left(\Omega^T + \mathbf{H}\hat{\theta}\right)P + 2L\Omega^T D\right]'^T\right\} \tag{44}$$

The moment properties of $\hat{x}$, $\hat{\theta}$, $P + 2D$, E, L and Λ can be used to evaluate terms not involving these increments in the right-hand side of Equation 32 to order h^2 as

$$\frac{1}{2}\hat{x}^T\left[A + \mathrm{tr}\left(\mathbf{G}L\right)\right]\hat{x} + \frac{1}{2}\hat{x}^T\,\mathrm{tr}\left(\mathbf{G}\hat{\theta}\hat{\theta}^T\right)\hat{x} + \mathrm{tr}\left(AD\right)$$

$$+ \frac{1}{2}\hat{\theta}^T\,\mathrm{tr}\left(\mathbf{G}''P\right)\hat{\theta} + \frac{1}{2}\hat{\theta}^T\,\mathrm{tr}\left[\Phi\left(\hat{x}\hat{x}^T + 2D\right)\right] + \frac{1}{2}\,\mathrm{tr}^T\left(\Phi P\right)\hat{\theta}$$

$$+ \hat{x}^T\,\mathrm{tr}\left(E^T\Phi'\right) + \frac{1}{2}u^T\left\{B + \Theta'\hat{\theta} + \mathrm{tr}\left[\mathbf{M}\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\right\}u + \left\{c^T + \hat{\theta}^T Z\right.$$

$$\left. + \mathrm{tr}^T\left[\Delta\left(\hat{\theta}\hat{\theta}^T + L\right)\right]\right\}u + \frac{1}{2}\,\mathrm{tr}\left(AP\right) + \mathrm{tr}\,\mathrm{tr}\left(\Phi\Lambda\right). \tag{45}$$

The order of accuracy of Equations 34 through 45 is retained if L and Λ are regarded as the deterministic approximations mentioned earlier. Furthermore, Equation 35 is actually accurate except for terms of order $\sqrt{\delta t}$ which are negligible to order h^3. This is because

$$\theta\left(t+\delta t\right)=\theta(t)+Tu\delta t+\omega\,;\quad \omega=\int_t^{t+\delta t}\varepsilon(t)w_2\,\mathrm{d}t$$

except for discretization errors of order δt^2. Since ω is a zero-mean random variable that is independent of $\theta(t)$, $E[\theta(t+\delta t)/y(t)] = \hat{\theta}(t)+Tu\delta t$ to this accuracy by definition of $\hat{\theta}$ and elementary properties of the expectation operator. So, since $\hat{\theta}(t+\delta t)$ is, conditioned on $y(t)$, equivalent (except for errors of order $\delta t^{3/2}$) to the prior expected value of $\theta(t+\delta t)$ conditioned on $\mathcal{Y}$ (as defined in the previous section), which is just $E[\theta(t+\delta t)/y(t)]$ by decomposition of expectation.

Since expectation is a linear operation, Equations 34 through 45 can be substituted for the random variables in the right-hand side of Equation 32 to eliminate the conditional expectation there. Since Equation 35 is accurate to order h^3, this substitution is accurate to order h^2 even though $J_{\hat{\theta}}$ is of order $1/h$ by assumption. It would be needlessly tedious to display the entire result of this substitution, but the terms depending on u are of the form

$$\frac{1}{2}u^TXu+\gamma_u^T$$

where

$$X=B+\Theta'\hat{\theta}+\mathrm{tr}\left[\mathbf{M}\left(\hat{\theta}\hat{\theta}^T+L\right)\right]$$

and

$$\gamma=c+Z^T\hat{\theta}+\mathrm{tr}\left[\Delta\left(\hat{\theta}\hat{\theta}^T+L\right)\right]+\left\{G^T+2\Psi\hat{\theta}+\mathrm{tr}^T\left[\mathrm{L}\left(\hat{\theta}\hat{\theta}^T+L\right)\right]\right\}J_{\hat{x}}^T$$

$$+\,T^TJ_{\hat{\theta}}^T+2\,\mathrm{tr}\left(E\Psi'J_D\right)+2\,\mathrm{tr}\left(L\Psi'J_E^T\right)+2\,\mathrm{tr}\left(\mathrm{L}''J_E^T\right)L\hat{\theta}\ .$$

Setting the u-derivative of this expression to zero gives the minimizing u as $-X^{-1}\gamma$. Expanding X^{-1} to second order in h and substituting from Equations 21 through 24 for the partials in γ gives this value of u to order h^2 as

$$u=u_0+u_1+u_2\ , \tag{46}$$

where

$$u_0=-B^{-1}\left[G^T\left(S\hat{x}+\zeta\right)+c\right]\ , \tag{47}$$

$$u_1=-B^{-1}\left\{G^T\eta+T^T\phi+\left[Z^T+G^TN+2\Psi^T\zeta-\Theta B^{-1}\left(G^T\zeta+c\right)\right]\hat{\theta}\right.$$

$$\left.+\,\mathrm{tr}\left[\left(\mathrm{Y}G+2S\Psi''-SGB^{-1}\Theta\right)\hat{x}\hat{\theta}^T\right]\right\} \tag{48}$$

and

$$
\begin{aligned}
u_2 = & -B^{-1}\Big(\mathrm{tr}\Big[\Big(\Delta + 2\Psi' V\Big)L\Big] + \mathrm{tr}^T\Big(\mathbf{L}L\Big)\zeta - \mathrm{tr}\Big(\mathbf{M}L\Big)B^{-1}\Big(G^T\zeta + c\Big) \\
& + \Big\{G^T W + \Big[\mathrm{tr}^T\Big(\mathbf{L}L\Big) - \mathrm{tr}\Big(\mathbf{M}L\Big)B^{-1}G^T\Big]S + 2\,\mathrm{tr}\Big(\Psi'\Pi' L\Big)\Big\}\hat{x} \\
& + \Big(2\Psi'^T\eta - \Theta B^{-1}T^T\phi + T^T U\Big)\hat{\theta} + 2\,\mathrm{tr}\Big\{E^T\Big[\Big(S + Y\Big)\Psi'^T \\
& + \mathrm{tr}^T\Big(\mathbf{B}L\Psi\Big) + \frac{1}{2}\Big(G^T\Pi\Big)^{\cdot}\Big]\Big\} + \mathrm{tr}\Big[\Big\{\Delta + \frac{1}{2}\Big(G^T\Xi\Big)' \\
& + 2\Psi' N - \Theta B^{-1}\Big(Z^T + G^T N + 2\Psi^{T\prime}\zeta\Big) + \Big[\mathbf{L}''\zeta \\
& + \Big(\mathbf{M} - \Theta'' B^{-1}\Theta'\Big)^{\cdot} B^{-1}\Big(G^T\zeta + c\Big)\Big]^{\cdot}\Big\}\hat{\theta}\hat{\theta}^T\Big] + \mathrm{tr}\Big[\Big\{2\Psi'Ч^{\cdot} \\
& + \Big[\Gamma G + S\mathbf{L}''^T - \Big(ЧG + 2S\Psi''\Big)B^{-1}\Theta \\
& + SGB^{-1}\Big(\Theta'' B^{-1}\Theta' - \mathbf{M}\Big)^{\cdot}\Big]^{\cdot}\Big\}\hat{\theta}\hat{\theta}^T\Big]\hat{x}\Big) \; . \qquad (49)
\end{aligned}
$$

From the orders of magnitude being assumed for the parameters here, u_i is of order h^i, $i = 0,1,2$. Using this value of u to eliminate the minimization in Equation 32 replaces the u-dependent terms from the previous substitution (for the expected increments) by

$$
-\frac{1}{2}\Big(u_0 + u_1 + u_2\Big)^T\Big\{B + \Theta'\hat{\theta} + \mathrm{tr}\Big[\mathbf{M}\Big(\hat{\theta}\hat{\theta}^T + L\Big)\Big]\Big\}\Big(u_0 + u_1 + u_2\Big) \; . \qquad (50)
$$

Substituting from Equations 47 through 49 for u_0, u_1 and u_2 here, substituting from Equations 22 through 31 for the partial derivatives of J in the other terms of the previous substitution in Equation 32, using the fact that J_t is just the right-hand side of Equation 18 with dots over the deterministic time functions, retaining only terms up to order h^2, and equating coefficients of like powers in $\hat{x}$, $\hat{\theta}$, D and E, shows that Equation 32 with boundary condition (Equation 33) would be satisfied to order h^2 by the function J of Equation 18 if σ obeyed a certain ordinary differential equation whose details aren't important here, and if the other coefficients obeyed the following equations:

$$\dot{W} = W\left(GB^{-1}G^TS - F\right) + \left(SGB^{-1}G^T - F^T\right)W - S\,\mathrm{tr}\left(\mathbf{F}L\right)$$

$$- \mathrm{tr}^T\left(\mathbf{F}L\right)S - \mathrm{tr}\left(\mathbf{G}L\right) + 2\,\mathrm{tr}\left[L\left(\Omega' R^{-1}HP - \Gamma'\right)\Pi'\right.$$

$$\left. + \Pi'^T\left(PH^TR^{-1}\Omega'^T - \Gamma'^T\right) - \Omega' R^{-1}HP Ч' L - LUL\Omega' R^{-1}\Omega'^T\right]$$

$$- 2\,\mathrm{tr}\left(Ч' L\Omega' R^{-1}HP\right) - 2\,\mathrm{tr}\left(\Omega'^T LUL\Omega' R^{-1}\right)$$

$$+ SGB^{-1}\left[\mathrm{tr}^T\left(\mathbf{L}L\right)S + 2\,\mathrm{tr}\left(\Psi'\Pi' L\right)\right] + \left[S\,\mathrm{tr}\left(\mathbf{L}L\right)\right.$$

$$\left. + 2\,\mathrm{tr}\left(\Pi' L\Psi'\right)\right]B^{-1}G^TS - SGB^{-1}\mathrm{tr}\left(\mathbf{M}L\right)B^{-1}G^TS;\; W\left(t_f\right) = 0 \qquad (51)$$

$$\dot{\eta} = \left(SGB^{-1}G^T - F^T\right)\eta + SGB^{-1}\left[T^T\phi + \mathrm{tr}\left(\Delta L + 2\Psi' VL\right) + \mathrm{tr}^T\left(\mathbf{L}L\right)\zeta\right]$$

$$- \mathrm{tr}\left(\mathbf{F}L\right)\zeta + 2\,\mathrm{tr}\left\{\left[\Omega' R^{-1}HP\left(V - N\right) + K^TR^{-1}HP\left(\Pi - Ч\right)' - \Gamma' V\right.\right.$$

$$\left.\left. - 2UL\Omega' R^{-1}K\right]L\right\} + \left[WG + S\,\mathrm{tr}\left(\mathbf{L}L\right) + 2\,\mathrm{tr}\left(\Pi' L\Psi'\right)\right.$$

$$\left. - SGB^{-1}\mathrm{tr}\left(\mathbf{M}L\right)\right]B^{-1}\left(G^T\zeta + c\right);\; \eta\left(t_f\right) = 0 \qquad (52)$$

$$\dot{\phi} = \mathrm{tr}\left[\frac{1}{2}P\left(-\Phi - H^TR^{-1}HPЧ\right) + 2PH^TR^{-1}\Omega'' PY - \left(S + Y\right)\left(\Sigma + 2\Gamma'' P\right)\right]$$

$$+ \left(N^TG + Z + 2\Psi'\zeta\right)B^{-1}\left[G^T\left(\zeta + \eta\right) + T^T\phi + c\right] + \left(UT + 2\Psi'\eta\right)B^{-1}\left(G^T\zeta + c\right)$$

$$- \frac{1}{2}\,\mathrm{tr}\left[B^{-1}\Theta B^{-1}\left(G^T\zeta + c\right)\left(\zeta^TG + 2\eta^TG + 2\phi^TT + c\right)\right];\; \phi\left(t_f\right) = \frac{1}{2}\,\mathrm{tr}\left(Ч_fP_f\right) \qquad (53)$$

$$\dot{Y} = Y\left(PH^TR^{-1}H - F\right) + \left(H^TR^{-1}HP - F^T\right)Y - SGB^{-1}G^TS;\; Y\left(t_f\right) = 0 \qquad (54)$$

$$\dot{V} = \left(H^T R^{-1} HP - F^T\right) V + 2YPH^T R^{-1} K + \mathrm{tr}^T\left(K^T R^{-1} HP\mathbf{B}L\right)$$
$$- H^T R^{-1}\left(HPN + 2KLU\right) - 2\Gamma^{T'}\zeta + \left[2\left(S + Y\right)\Psi'' + \Pi^T G\right.$$
$$\left. + 2\,\mathrm{tr}'^T\left(\mathbf{B}L\Psi'\right)\right] B^{-1}\left(G^T\zeta + c\right)\;;\;\; V\left(t_f\right) = 0 \tag{55}$$

$$\dot{U} = -\,\mathrm{tr}\left(\mathbf{G}''P\right) + \mathrm{tr}\left\{\nabla\left[2\left(PH^T R^{-1}\Omega'' - \Gamma''\right)P - \Sigma\right]\right.$$
$$+ \left[2\left(PH^T R^{-1}\Omega'' - \Gamma''\right)P - \Sigma\right]\nabla - \Gamma'' PH^T R^{-1} HP - 2PH^T R^{-1}\Omega'' P\text{Ч}$$
$$- 2\text{Ч}PH^T R^{-1}\Omega'' P - 2\left(S + Y\right)P\mathbf{F}'' - \frac{1}{2}\left[\left(S + Y\right)\bar{\Sigma}\,Q\,\bar{\Sigma}^T + Q\,\bar{\Sigma}^T\left(S + Y\right)\bar{\Sigma}\right]$$
$$\left. + 2P\left(\mathbf{H}'' R^{-1} H + \Omega^{T'} R^{-1}\Omega''\right)PY\right\} + 2\,\mathrm{tr}\left(\Omega'' PYP\Omega^{T'} R^{-1}\right)$$
$$+ \left\{2\Delta' + 2N^T\Psi'' + 2\left(\Psi^{T'} N\right)' + \left(\mathbf{L}' + \mathbf{L}'^T\right)'\zeta - \left(N^T G + Z + 2\Psi'\zeta\right)B^{-1}\Theta + \frac{1}{2}\left(\Xi'' + \Xi^T\right)\right.$$
$$\left. - \left[\Theta B^{-1}\left(G^T N + Z^T + 2\Psi^{T'}\zeta\right)\right]'\right\} B^{-1}\left(G^T\zeta + c\right) + \left(N^T G + Z + 2\Psi'\zeta\right)B^{-1}\left(G^T N\right.$$
$$\left. + Z^T + 2\Psi^{T'}\zeta\right) + \mathrm{tr}\left\{B^{-1}\left[\frac{1}{2}\left(\Theta B^{-1}\Theta\right)' + \frac{1}{2}\left(\Theta B^{-1}\Theta\right)'^T - \mathbf{M}\right] B^{-1}\left(G^T\zeta\right.\right.$$
$$\left.\left. + c\right)\left(\zeta^T G + c^T\right)\right\}\;;\;\; U\left(t_f\right) = \mathrm{tr}\left(\mathbf{A}'' P_f\right) \tag{56}$$

$$\dot{N} = \left(SGB^{-1}G^T - F^T\right)N - 2\Gamma^{T'}\left(\zeta + \eta\right) + SGB^{-1}\left[Z^T + T^T U + 2\Psi'^T\left(\zeta + \eta\right)\right]$$
$$+ \left(\text{Ч}G + 2S\Psi'' - SGB^{-1}\Theta\right)B^{-1}\left[G^T\left(\zeta + \eta\right) + T^T\phi + \mathrm{c}\right]\;;\;\; N\left(t_f\right) = 0 \tag{57}$$

$$\dot{\Pi} = \left(SGB^{-1}G^{T} - F^{T}\right)\Pi + \Pi\left(PH^{T}R^{-1}H - F\right) - \Phi - 2\left(S\Gamma'' + \Gamma^{T'}S\right)$$

$$- \left[\text{Ч}PH^{T} + 2\left(UL\Omega^{T}\right)'\right]R^{-1}H + 2\,\mathrm{tr}\left[L\left(\Omega' R^{-1}HP - \Gamma'\right)\mathbf{B}\right]$$

$$+ 2\left(PH^{T}R^{-1}\Omega'' - \Gamma''\right)^{T}Y + 2SGB^{-1}\left[\Psi^{T'}\left(S + Y\right) + \mathrm{tr}'\left(\mathbf{B}L\Psi'\right)\right] \;;\; \Pi\left(t_f\right) = \text{Ч}_f \quad (58)$$

$$\dot{\nabla} = \left(H^{T}R^{-1}HP - F^{T}\right)\nabla + \nabla\left(PH^{T}R^{-1}H - F\right) - \Phi - 2\left(S\Gamma'' + \Gamma^{T'}S\right) - \text{Ч}PH^{T}R^{-1}H$$

$$- H^{T}R^{-1}HP\text{Ч} + 2Y\left[P\left(H^{T}R^{-1}\Omega'' + \Omega^{T'}R^{-1}H\right) - \Gamma''\right] + 2\left[\left(\Omega^{T'}R^{-1}H + H^{T}R^{-1}\Omega''\right)P\right.$$

$$\left. - \Gamma^{T'}\right]Y + \mathrm{tr}'\left[2\left(PH^{T}R^{-1}\Omega'' - \Gamma''\right)P\mathbf{R} - \Sigma\mathbf{R}\right] \;;\; \nabla\left(t_f\right) = \Upsilon_f \quad (59)$$

$$\dot{\text{Ч}} = \left(SGB^{-1}G^{T} - F^{T}\right)\text{Ч} + \text{Ч}\left(GB^{-1}G^{T}S - F\right) - \Phi + 2S\left(\Psi''B^{-1}G^{T}S - \Gamma''\right)$$

$$+ 2\left(SGB^{-1}\Psi^{T'} - \Gamma^{T'}\right)S - SGB^{-1}\Theta B^{-1}G^{T}S \;;\; \text{Ч}\left(t_f\right) = \text{Ч}_f \quad (60)$$

$$\dot{\Xi} = \frac{1}{2}\left(SGB^{-1}G^{T} - F^{T}\right)\left(\Xi + \Xi^{T'}\right) - 2N^{T}\Gamma'^{T} - 2\left(\Gamma'N\right)' - 2\mathbf{F}''\zeta + SGB^{-1}\left\{2\left[\Delta'\right.\right.$$

$$\left. + N^{T}\Psi'' + \left(\Psi^{T'}N\right)''\right] + \left(\mathbf{L}' + \mathbf{L}'^{T}\right)'\zeta + \left[\left(\Theta B^{-1}\Theta\right)'' + \left(\Theta B^{-1}\Theta\right)^{T'} - 2\mathbf{M}''\right]''B^{-1}\left(c\right.$$

$$\left.\left. + G^{T}\zeta\right)\right\}' + \left[\dot{\Gamma}G + S\mathbf{L}''^{T} + 2\left(\Psi'\text{Ч}'\right)'' - \left(\text{Ч}G + 2S\Psi''\right)B^{-1}\Theta\right]B^{-1}\left(G^{T}\zeta + c\right)$$

$$- SGB^{-1}\Theta B^{-1}\left(G^{T}N + Z^{T} + 2\Psi^{T'}\zeta\right) + \left\{\left[\Gamma G + S\mathbf{L}''^{T} + 2\left(\Psi'\text{Ч}'\right)''\right.\right.$$

$$\left.\left. - \left(\text{Ч}G + 2S\Psi''\right)B^{-1}\Theta\right]B^{-1}\left(G^{T}\zeta + c\right) - SGB^{-1}\Theta B^{-1}\left(G^{T}N + Z^{T} + 2\Psi^{T'}\zeta\right)\right\}^{T'}$$

$$+ \left\{\left(N^{T}G + Z + 2\Psi'\zeta\right)B^{-1}\left(\text{Ч}G + 2S\Psi''\right)' + \left(G^{T}\text{Ч} + 2S\Psi^{T'}\right)B^{-1}\left(G^{T}N + Z^{T} + 2\Psi^{T'}\zeta\right)\right\}'' \;;$$

$$\Xi\left(t_f\right) = 0 \quad (61)$$

$$\dot{\mathbf{R}} = \left(H^T R^{-1} H P - F^T\right)\mathbf{R} + \mathbf{R}\left(P H^T R^{-1} H - F\right) + \left[\left(H^T R^{-1} H P - F^T\right)\mathbf{R}\right.$$

$$\left. + \mathbf{R}\left(P H^T R^{-1} H - F\right)\right]^{\cdot} + Y \otimes H^T R^{-1} H + H^T R^{-1} H \otimes Y$$

$$+ \left(Y \otimes H^T R^{-1} H + H^T R^{-1} H \otimes Y\right)^T ; \quad \mathbf{R}\left(t_f\right) = 0 \qquad (62)$$

$$\dot{\mathbf{B}} = \left(H^T R^{-1} H P - F^T\right)\mathbf{B} + \left[\left(H^T R^{-1} H P - F^T\right)\mathbf{B}\right]^{\cdot} - \frac{1}{2}\left(U \otimes H^T R^{-1} H\right)^T$$

$$- \frac{1}{2}\left(U \otimes H^T R^{-1} H\right)^{T^{\cdot}} ; \quad B\left(t_f\right) = 0 \qquad (63)$$

$$\dot{\mathbf{I}} = -\mathbf{G} - \left(F^T \boldsymbol{\Gamma} + \boldsymbol{\Gamma} F + \mathbf{F}^{\cdot} S + S \mathbf{F}^{\cdot T}\right)^{\cdot} - \text{Ч}\,\boldsymbol{\Gamma}'^{T} - \Gamma' \text{Ч}' - \left(\Gamma' \text{Ч}\right)^{T^{\cdot}}$$

$$- \left(\text{Ч}\boldsymbol{\Gamma}^{\cdot}\right)^{\cdot} + \frac{1}{2}\left(\left\{S G B^{-1}\left[\left(\Theta B^{-1} \Theta\right)^{\cdot} + \left(\Theta B^{-1} \Theta\right)^{T^{\cdot}} - 2\mathbf{M}\right] + \boldsymbol{\Gamma} G + S \mathbf{L}^{\cdot T}\right.\right.$$

$$\left.\left. + 2\left(\Psi' \text{Ч}'\right)^{\cdot}\right\} B^{-1} G^T S\right)^{\cdot} + \frac{1}{2}\left\{\left[\boldsymbol{\Gamma} G + S \mathbf{L}^{\cdot T} + 2\left(\Psi' \text{Ч}'\right)^{\cdot}\right] B^{-1} G^T S\right\}^{\cdot T}$$

$$+ \frac{1}{2}\left\{\left[\boldsymbol{\Gamma} G + S \mathbf{L}^{\cdot T} + 2\left(\Psi' \text{Ч}'\right)^{\cdot}\right] B^{-1} G^T S\right\}^{T^{\cdot}} + \frac{1}{2}\left\{\left[\boldsymbol{\Gamma} G + S \mathbf{L}^{\cdot T}\right.\right.$$

$$\left.\left. + 2\left(\Psi' \text{Ч}'\right)^{\cdot}\right] B^{-1} G^T S\right\}^{T^{\cdot T}} + \left(\frac{1}{2} \text{Ч} G + S \Psi^{\cdot}\right)^{\cdot T} B^{-1}\left(\text{Ч} G + 2 S \Psi^{\cdot}\right)^{\cdot}$$

$$+ \left[\left(\frac{1}{2} \text{Ч} G + S \Psi^{\cdot}\right)^{\cdot T} B^{-1}\left(\text{Ч} G + 2 S \Psi^{\cdot}\right)^{\cdot}\right]^T ; \quad \mathbf{I}\left(t_f\right) = \mathbf{A} \qquad (64)$$

Of course, L and Λ denote the prior expected value approximations throughout Equations 33 through 64. Since these approximations are deterministic time functions (whose computation is described in the next section), so are the cost-function coefficients determined by integration of Equations 51 through 64 in reverse time. From the form of these equations, these coefficients also have the symmetry properties and orders of magnitude postulated for them in Equation 18. The latter follows from the fact that each of them except U, ϕ, and σ is multiplied in the differential equation generating it by a coefficient matrix which, from the assumed observability and controllability properties of the corresponding LQG problem, make that equation stable in reverse time with fast enough damping that the effects of the driving terms can only accumulate over time intervals of order unity. Hence, the orders of magnitude of these quantities are the same as those of the driving terms in the corresponding differential equations. The equations for $\dot{U}$, $\dot{\Phi}$ (and also the one not shown for $\dot{\sigma}$) have no such damping, so their orders of magnitude are in general $1/h^2$ times those of their driving terms for integration intervals (i.e., $t_f - t_0$) of this duration. From inspection of the driving terms in Equations 51 through 64, and also those in the corresponding equation for $\dot{\sigma}$, the previously assumed order of magnitude is obtained in each case.

As a result, the function J of Equation 18 satisfies to order h^2 the Bellman equation, Equation 32, and the boundary condition, Equation 33, for the optimal conditional expected cost-to-go function when the coefficients are determined by Equations 19 and 20, 51 through 64, and the similar ordinary differential equation not shown for $\sigma(t)$. As a further consequence of the Principle of Optimality of dynamic programming which leads to the Bellman equation, the corresponding control law specified by Equations 46 through 49 approximates the optimal control law asymptotically to order h^2. These conclusions are only of a formal nature, however, and were reached with the tacit assumption that the context here is sufficiently regular and well-posed that (1) the higher-order (than h^2) errors introduced by the approximations of the Bellman equation made here only change the solutions, and the corresponding optimal control, to higher order in some appropriate sense; (2) for $h \rightarrow 0$ the optimal value function is a neighboring function, again in some appropriate sense, of that of the corresponding LQG problem (which is well known); and (3) there is only one neighboring solution of the (exact) Bellman equation for the problem at hand for $h \rightarrow 0$ (i.e., the neighboring solution obtained here).

Actually, the derivation of this section would remain formally valid for any specific application as long as the quantities deleted as being of higher order than h^2 remain small compared to those retained, and, of course, the state estimation results are valid as described at the end of Section IV. These deletions are too numerous to list all the resulting conditions for validity explicitly. In particular, however, when Equation 50, via Equations 46 through 48 for u_0, u_1 and u_2, was substituted for the u-dependent terms arising from the Bellman equation, the terms $u_1^T B u_2$ and $u_2^T B u_2$ were deleted whereas $u_0^T B u_0$ was retained. Since B is positive-definite, this means that these conditions must certainly include the requirement that the prior expected values of $u_1^T u_2$ and $u_2^T u_2$ remain small compared to that of $u_0^T u_0$, which is essentially a condition on the perturbation terms in the asymptotically optimal control law being small compared to the dominant terms.

VI. OPTIMAL CONTROL LAW APPROXIMATIONS

In order to determine (to order h^2) the control law which satisfies the optimality conditions, it still remains to find the prior expected values of L and Λ (as given by Equations 13 and 14) when this control law is used. Since the boundary conditions of these equations are specified at t_0 and those of Equations 51 through 64 are specified at t_f, solving these equations might seem to involve the solution of a two-point boundary value problem. Because of the structure of these equation systems and the orders of magnitude of the variables involved, however, it is possible to evaluate these prior expected values to the desired degree of asymptotic accuracy by solving only initial value systems of (ordinary) differential equations. To do this, first define the following prior expected values under the optimal control law:

$$\bar{x} \triangleq \mathbf{E}(\hat{x}) = \mathbf{E}(x) \tag{65}$$

$$\bar{E} \triangleq \mathbf{E}(E) \tag{66}$$

$$\hat{X} \triangleq \mathbf{E}\left[\left(\hat{x} - \bar{x}\right)\left(\hat{x} - \bar{x}\right)^T\right] = \mathbf{E}\left(\hat{x}\hat{x}^T\right) - \bar{x}\bar{x}^T \tag{67}$$

$$\hat{\Phi} \triangleq \mathbf{E}\left[\left(E - \bar{E}\right)\left(\hat{x} - \bar{x}\right)^T\right] = \mathbf{E}\left(E\hat{x}^T\right) - \bar{E}\bar{x}^T \tag{68}$$

$$\hat{\mathbf{E}} \triangleq \mathbf{E}\left[\left(E - \bar{E}\right)\otimes\left(E - \bar{E}\right)\right] = \mathbf{E}\left(E\otimes E\right) - \bar{E}\otimes\bar{E} \; . \tag{69}$$

These are all deterministic time functions with the time argument t suppressed in the notation.

By Equations 46 and 47, the optimal control law gives the control as

$$u = u_0 = -B^{-1}\left[G^T\left(S\hat{x} + \zeta\right) + c\right] \tag{70}$$

to order unity, so from Equation 9 for a generic time t and the fact noted earlier that $\mathbf{E}(z - \hat{z}) = 0$ there conditioned on the data up to time t,

$$\mathbf{E}_{/data(t)}\left\{\hat{x}\left(t + \delta t\right)\right\} = \hat{x} + \left[\left(F - GB^{-1}G^TS\right)\hat{x} - GB^{-1}\left(G^T\zeta + c\right)\right]\delta t \tag{71}$$

to order δt for a very short time increment δt, where the quantities on the right-hand side are all evaluated at time t. Taking prior expectations of both sides of Equation 71 and using definitions 65 through 69 gives, in the continuous-time limit,

$$\dot{\bar{x}} = \left(F - GB^{-1}G^TS\right)\bar{x} - GB^{-1}\left(G^T\zeta + c\right) \; ; \; \bar{x}\left(t_0\right) = \hat{x}_0 \tag{72}$$

to order unity. Similarly,

$$\dot{\bar{E}} = \left(F - PH^TR^{-1}H\right)\bar{E} - 2PH^TR^{-1}KL + 2\left[\left(L\Gamma' - L\Omega'R^{-1}HP\right)' - \left(SGB^{-1}\Psi L\right)'\right]\bar{x} - 2\left(\Psi L\right)'B^{-1}\left(G^T\zeta + c\right) \; ; \; \bar{E}\left(t_0\right) = 0 \tag{73}$$

to order h, where, for this accuracy, L can be regarded as the prior expected value approximation being sought. Since, as noted earlier,

$$\mathbf{E}\left[\left(z-\hat{z}\right)\left(z-\hat{z}\right)^T\delta t\right] = R\left(t\right)$$

in Equations 9 through 12 in this context, it also follows from Equations 9 and 70 that

$$\mathbf{E}_{/data(t)}\left[\hat{x}\left(t+\delta t\right)\hat{x}^T\left(t+\delta t\right)\right] = \hat{x}\hat{x}^T + \left\{\left[\left(F - GB^{-1}G^TS\right)\hat{x}\right.\right.$$

$$\left. - GB^{-1}\left(G^T\zeta + c\right)\right]\hat{x}^T + \hat{x}\left[\hat{x}^T\left(F^T - SGB^{-1}G^T\right)\right.$$

$$\left.\left. - \left(\zeta^T G + c^T\right)B^{-1}G^T\right] + PH^TR^{-1}HP\right\}\delta t \tag{74}$$

to order unity. Taking prior expectations, using definitions 65 through 69, taking the continuous-time limit and subtracting

$$\frac{d}{dt}\left(\bar{x}\bar{x}^T\right)$$

using Equation 72 gives

$$\dot{\hat{X}} = \left(F - GB^{-1}G^TS\right)\hat{X} + \hat{X}\left(F^T - SGB^{-1}G^T\right) + PH^TR^{-1}HP;\ \hat{X}\left(t_0\right) = 0 \tag{75}$$

to order unity. From analogous derivations,

$$\dot{\hat{\Phi}} = \left(F - PH^TR^{-1}H\right)\hat{\Phi} + \hat{\Phi}\left(F^T - SGB^{-1}G^T\right) + 2\left[\left(L\Gamma' - L\Omega'R^{-1}HP\right)'\right.$$

$$\left. - \left(SGB^{-1}\Psi L\right)'\right]\hat{X} + 2\left[\Lambda H^T + \left(L\Omega^TP\right)'\right]R^{-1}HP\ ;\ \hat{\Phi}\left(t_0\right) = 0 \tag{76}$$

to order h, and

$$\dot{\hat{\mathbf{E}}} = \left(F - PH^TR^{-1}H\right)\hat{\mathbf{E}} + \hat{\mathbf{E}}\left(F^T - H^TR^{-1}HP\right) + 2\left[\left(L\Gamma' - L\Omega'R^{-1}HP\right)'\right.$$

$$\left. - \left(SGB^{-1}\Psi L\right)'\right]\hat{\Phi}^T + 2\hat{\Phi}\left[\left(\Gamma'^TL - PH^TR^{-1}\Omega'^TL\right)' - \left(L\Psi^TB^{-1}G^TS\right)'\right]$$

$$4\left[\Lambda H^T + \left(L\Omega^TP\right)'\right]R^{-1}\left[H\Lambda + \left(P\Omega L\right)'\right]\ ;\ \hat{\mathbf{E}}\left(t_0\right) = 0 \tag{77}$$

to order h^2, where L again denotes the prior expected value approximation and Λ now denotes the approximation (accurate to order h) obtained from Equation 14 with this approximation for L and with D and $\hat{\theta}$ replaced by zero.

Taking prior expectations in Equation 13 and using definitions 65 through 69 then shows that the prior expected value approximation L satisfies the deterministic differential equation

$$\dot{L} = \varepsilon^2 Q_2 - \left(\bar{E}^T H^T + 2LK^T \right) R^{-1} \left(H\bar{E} + 2KL \right) - \operatorname{tr}\left(H^T R^{-1} H \hat{\mathrm{E}} \right)$$

$$- 2L \operatorname{tr}\left[\Omega^{T'} R^{-1} H \left(\bar{E}\bar{x}^T + \hat{\Phi} \right) \right] - 2\operatorname{tr}\left[\left(\bar{E}\bar{x}^T + \hat{\Phi} \right) \Omega^{T'} R^{-1} H \right] L$$

$$- 4\left[L\left(K^T R^{-1} \Omega'^T + \Omega' R^{-1} K \right) L \right]' \bar{x} - 4L \operatorname{tr}\left[\Omega^{T'} R^{-1} \Omega'' \left(\hat{X} + \bar{x}\bar{x}^T \right) \right] L \; ; \; L\left(t_0\right) = L_0 \quad (78)$$

to order h^2. Over control intervals $t_f - t_0$ of order $1/h^2$, therefore, this equation generates (the prior expected of) L to within an error which is formally of order h with virtual certainty. This is the same order of accuracy with which this prior expectation approximates the sample values of L generated by integrating equation system 5 through 16.

Consequently, an approximation of the optimal control law which has the formal appearance of being asymptotically accurate to second order in h (as h goes to zero) can be computed and implemented as follows:

1. **Integrate Equation 8 for $P(t)$ in forward time from t_0 to t_f, and Equations 19 and 20 for $S(t)$ and $\zeta(t)$ in reverse time from t_f to t_0.**

2. **Using the results of Step 1, integrate Equations 72, 73, and 75 through 78 for**

   $$\bar{x}(t),\ \bar{E}(t),\ \hat{X}(t),\ \hat{\Phi}(t),\ \hat{\mathrm{E}}(t),\ and\ L(t)$$

   in forward time from t_0 to t_f together with the approximate equation

   $$\dot{\Lambda} = \left(F - PH^T R^{-1} H \right)\Lambda + \Lambda\left(F^T - H^T R^{-1} HP \right) + \left(P\Gamma L \right)'' + \left(L\Gamma^T P \right)'$$

   $$- PH^T R^{-1} \left(P\Omega L \right)'' - \left(L\Omega P \right)' R^{-1} HP + L\Sigma'' \; ; \; \Lambda\left(t_0\right) = 0 \quad (79)$$

   for $\Lambda(t)$.

3. **Using the results of Steps 1 and 2, integrate Equations 51 through 64 for the value function coefficients in reverse time from t_f to t_0.**

4. **Implement the control law by using these coefficients and L from Step 2 in Equations 46 through 49 for the control value u, while integrating equation system 9 through**

16 in real time to generate $\hat{x}$, $\hat{\theta}$, D, and E (which are also needed to evaluate Equations 46 through 49) from the incoming measurements z.

Actually, the same order of asymptotic accuracy can be obtained by using the result of Equations 78 and 79 for L and Λ in the generation of $\hat{x}$, $\hat{\theta}$, D and E in Step 4. In this case, only Equations 9 through 12 need be integrated in real time to implement the control law, together, of course, with the real-time evaluation of Equations 46 through 49 for u. However, the results here are of greatest significance when h is as large as meaningfully possible, in which case these two alternative implementations may have markedly different behaviors. The second one might be less responsive since it uses prior expectations of L and Λ in place of their actual sample values. On the other hand, it might be more robust, in the sense of working better for larger values of h, since L^{-1} is used in generating D or M via Equation 11 or 17.

It is possible, by retaining a higher order of asymptotic accuracy in an analysis similar to that leading to Equations 72 through 78, to evaluate the average sizes and effects of the higher-order terms in the optimal control law. The analogous equations become considerably more complicated, however, and such an analysis is beyond the scope of this report.

VII. INCLUSION OF PRECISELY KNOWN STATE VECTOR COMPONENTS

In some applications of interest, the motion-state vector x can be partitioned as

$$\begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}$$

such that the dynamics of Equation 1 reduce to the partition equations

$$\dot{x}_1 = F_1x_1 + F_2x_2 + G_1u + 2\,\mathrm{tr}\left(\Gamma_1\theta x_1^T\right) + 2\,\mathrm{tr}\left(\Gamma_2\theta x_2\right)$$

$$+ 2\,\mathrm{tr}\left(\Psi_1\theta u^T\right) + \mathrm{tr}\left(\mathbf{F}_1\theta\theta^T\right)x_1 + \mathrm{tr}\left(\mathbf{F}_2\theta\theta^T\right)x_2 + \left(I + \bar{\Sigma}'_1\theta\right)w_1 \qquad (80)$$

and

$$\dot{x}_2 = F_3x_2 + G_3u + w_3\,, \qquad (81)$$

the measurement equation is

$$z = H_1x_1 + H_2x_2 + 2K\theta + 2\,\mathrm{tr}\left(\Omega_1\theta x_1^T\right) + 2\,\mathrm{tr}\left(\Omega_2\theta x_2^T\right)$$

$$+ \mathrm{tr}\left(\mathbf{H}_1\theta\theta^T\right)x_1 + \mathrm{tr}\left(\mathbf{H}_2\theta\theta^T\right)x_2 + \mathrm{tr}\left(\bar{\Delta}\theta\theta^T\right) + v\,, \qquad (82)$$

and where the controller knows the current value of x_2 exactly. The process noise partitions w_1 and w_3 are uncorrelated, the covariance-matrix parameter R of the measurement noise v is positive-definite as before, and the order-of-magnitude restrictions on the parameters is the same as before.

Since x_2 is known precisely, the joint conditional probability distribution of x and θ is singular, and the analysis of Section IV does not apply directly to this case. The same procedure can be applied to Equations 80, 2, and 82, however, to derive similar recursive equations for generating the same type of second-order Edgeworth approximation to $p(x_1,\theta/y)$ that is asymptotically accurate to order h^2. Since the current value of x_2 is available as well as that of u, the transition of the reduced composite state (x_1,θ) from t to $t + \delta t$ can be approximated by a sequence of four transformations of the same form as those described for (x,θ) in Section IV. Furthermore, Equation 82 is equivalent to

$$\left(z - H_2 x_2\right) = H_1 x_1 + 2\left(K + \Omega_2^{\cdot} x_2\right)\theta + 2\,\mathrm{tr}\left(\Omega_1 \theta x_1\right)$$

$$+ \mathrm{tr}\left(\mathbf{H}_1 \theta\theta^T\right)x_1 + \mathrm{tr}\left[\left(\bar{\Delta} + \mathbf{H}_2' x_2\right)\theta\theta^T\right] + v \; .$$

Since x_2 can be treated as a known parameter here, this is a measurement equation for x_1 and θ that has the same form as Equation 3 for x and θ, with $(z - H_2 x_2)$ playing the role of the measurement variable. Thus the derivations of Section IV can be applied in this context with no change other than a redefinition of variables.

Not surprisingly, however, the result reduces to what Equations 5 through 17 would be if constructed for the entire composite state vector

$$\begin{bmatrix} x \\ \hline \theta \end{bmatrix}$$

except for the following modifications:

1. The differential equations are not integrated for components of the conditional moment parameters involving components of x_2.

2. In the other differential equations, all occurrences of conditional mean components of x_2 are replaced by the corresponding (known) component of x_2 (i.e., $\hat{x}_2$ is replaced by x_2); all occurrences of the higher conditional (central) moment components referring to components of x_2 are replaced by zero.

As a result, the derivations of Sections V and VI for the asymptotic approximation of the optimal control law all go through for this case with these modifications.

VIII. EXAMPLE

The use of this theory can be illustrated by applying it to an idealization of a planar intercept problem in which a constant-speed interceptor can accelerate perpendicular to its flight path and receives noisy angular measurements of the line of sight to a maneuvering target. Although this example is meant to resemble the guidance of a homing missile, this resemblance is so highly simplified and distorted that the results obtained for it should not be taken as valid for guidance law design in any actual application of this sort. Rather, these results are meant to indicate the qualitative type of information that should be available from a reasonably realistic analysis of such a situation and the basic form this analysis might take, without having to deal with the extra complexity that would be required. In summary, the result of these simplifications is the following problem, in which all the variables are scalars.

Dynamics:

$$\begin{cases} \dot{x} = \left(t_f - t + \theta\right)\left(u \cos\gamma + w_c\right) \\ \dot{\theta} = \dfrac{t_f - t}{V}\left(u \sin\gamma + w_i\right) \end{cases}$$

Measurements:

$$z = \left(1 - \frac{\theta}{t_f - t}\right)x + n$$

Criterion:

$$J = \frac{1}{2}\mathrm{E}\left[a x_f^2 + \int_{t_0}^{t_f} u^2 \mathrm{d}t\right] \;;\; a > 0$$

In this formulation, w_c, w_i, and n are jointly a zero-mean Gaussian white noise process with covariance matrix parameter

$$\begin{bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & R \end{bmatrix}$$

and

$$R = 2\sqrt{qr}(t_f - t)^2$$

$$q = A\bar{a}^2$$

V	=	closing speed (assumed large)	Parameters of nominal intercept, in which both vehicles move in straight lines at constant speed and γ is constant.
t_f	=	time of intercept	
γ	=	angle of interceptor's velocity from its line of sight to the target	

r = variance parameter of angle measurements being approximated (assumed to have the characteristics of radar glint noise)

$\bar{a}$ = r.m.s. target (linear) acceleration – assumed isotropic

A = average time between target acceleration changes

$\theta(t)$ = $T(t) - t_f$; $T(t)$ = "projected" time of closest approach (i.e., if neither vehicle accelerated after time t)

$x(t)$ = projected distance of closest approach at time t

Although this problem might not appear to have exactly the form of those problems treated above, this problem essentially does have the same form because V is typically large (compared to velocity changes resulting from interceptor and target maneuvers) so that a new parameter variable $\theta^* = \sqrt{V}\,\theta$ could be used in place of θ, in which case the dynamics and measurement equations would all have the proper form, except for very small values of $t_f - t$. Then, however, the form is unimportant because the interceptor is so close to the target that it no longer has much control over the outcome. The use of θ rather than θ^* here basically has the effect of making θ/τ itself a quantity of order h, rather than some of the other parameters. Although the noise in the (linear) measurements of x here has been defined to approximate the effects of radar glint noise in the angular measurements of a homing missile, the interceptor cannot measure current target range in this example, which might correspond to the radar being jammed. The choice of performance criterion (for any reasonable value of a) is such that if this range were precisely known, the optimal control law would reduce to the standard proportional navigation law (with an effective navigation gain of 3) operating on a conditional-mean estimate of the line-of-sight rotation rate, except very near the beginning and end of the encounter. In the idealized context of this example, therefore, it is reasonable to view the midcourse portion of the optimal control law as the way that proportional navigation guidance should be modified for the absence of homing range data when the line-of-sight measurements (or their equivalent) are noise-corrupted.

For simplicity, only the case of $\gamma = \hat{x}(t_0) = 0$ is considered in further detail. In the notation of the preceding sections, the variables F, Γ, $\mathbf{F}$, $\mathbf{L}$, T, K, $\mathbf{H}$, $\bar{\boldsymbol{\Delta}}$, $\mathbf{A}$, Ψ_f, $\mathbf{Z}$, Δ, $\mathbf{c}$, $\mathbf{G}$, $\mathbf{M}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\Theta}$ are all zero in this case, and

$$H = B = 1,$$

$$G = t_f - t,$$

$$Q = \varepsilon^2 V^2 Q_2 = q(t_f - t)^2,$$

$$R = 2(t_f - t)^2\sqrt{qr},$$

$$\Psi = 1/2\,,$$

$$\overline{\Sigma} = \frac{1}{t_f - t}\,,\ \text{and}$$

$$\Omega = -\,\frac{1}{2(t_f - t)}\,.$$

As a result, the cost-to-go function coefficients η, ζ, V, N and Ξ are zero, and ϕ is not used. All the variables in this case are scalars, of course, and the equations which must be solved in real time to implement the (asymptotically) optimal control law, called a guidance law in this application, become the following when τ is used to denote $t_f - t$:

$$\hat{z} = \hat{x} - (\hat{x}\hat{\theta} + E)/\tau \tag{83}$$

$$\dot{\hat{x}} = \left(\tau + \hat{\theta}\right)u + \left[\left(P + 2D\right)\left(1 - \hat{\theta}/\tau\right) - \left(\hat{x}E + 2\lambda\right)/\tau\right]\left(\frac{z - \hat{z}}{2\tau^2\sqrt{qr}}\right) \tag{84}$$

$$\dot{\hat{\theta}} = \left[E - \left(\hat{\theta}E + \hat{x}L\right)/\tau\right]\left(\frac{z - \hat{z}}{2\tau^2\sqrt{qr}}\right) \tag{85}$$

$$\begin{aligned}\dot{D} = {} & Eu + \frac{P}{\tau^2\sqrt{qr}}\left[\frac{1}{2}\left(\hat{\theta}P + \hat{x}E + 2\lambda\right)/\tau - \left(1 - \hat{\theta}/\tau\right)D - \left(D - \frac{1}{2}\hat{\theta}P/\tau\right)^2\right] \\ & + \frac{1}{2}q\left(\hat{\theta}^2 + L + 2\tau\hat{\theta}\right) + \left[3\lambda E/L - \left(PE + \lambda\hat{x}\right)/\tau\right]\left(\frac{z - \hat{z}}{2\tau^2\sqrt{qr}}\right)\end{aligned} \tag{86}$$

$$\begin{aligned}\dot{E} = {} & Lu + \left(\frac{1}{\tau^2\sqrt{qr}}\right)\left\{PE\hat{\theta}/\tau - \left(\frac{1}{2}P + D\right)\left(E - \hat{x}L/\tau\right) + \left[\lambda\left(1 - \hat{\theta}/\tau\right)\right.\right. \\ & \left.\left. - \left(\frac{1}{2}P + D\right)L/\tau\right]\left(z - \hat{z}\right)\right\}\end{aligned} \tag{87}$$

$$u = -\left[\left(S + \text{Ч}\hat{\theta}\right)\left(\tau + \hat{\theta}\right) + \left(W + \mathrm{I}\hat{\theta}^2\right)\tau + \Pi L\right]\hat{x} - \left(S + Y + \Pi\tau + \mathbf{B}L\right)E\ . \tag{88}$$

Equations 84 through 87 are integrated (with substitution from Equation 83 for $\hat{z}$) from time t_0 with zero initial conditions, and the remaining variables would be obtained from the

stored solution of a determinisitic ordinary differential equation system for P, λ, L, $\hat{X}$, $\hat{\Phi}$, $\hat{\mathbf{E}}$, S, W, Y, u, Π, ∇, Ч, **R**, **B**, and **I**, as described in Section VI.

MIDCOURSE GUIDANCE APPROXIMATION

This last equation system is not shown in detail because the optimal guidance law in this particular case is quite well approximated, except very near the initial and final times, by one which can be determined much more simply. A common practice in such cases is to use this midcourse approximation for the entire engagement. The equation for S can be integrated analytically to give

$$S = 1/\left(1/a + \tau^3/3\right), \tag{89}$$

which reduces approximately to

$$S = 3/\tau^3 \tag{90}$$

during this midcourse phase and, after considerable manipulation and use of the fact that $\hat{\theta} << \tau$, leads to further approximations

$$u = \left[\frac{3}{2} U\tau/V^2 - 3/\left(\tau + \hat{\theta}\right)^2\right]\hat{x}\ , \tag{91}$$

$$P = \sqrt{2q\sqrt{qr}\,\tau^2}\ , \tag{92}$$

$$\lambda = 2q\sqrt{3r\tau^5}/V\ , \tag{93}$$

and

$$L = \sqrt{6\sqrt{2}\,q}\,(r/q)^{1/8}\tau^2/V \tag{94}$$

then. These approximations allow the optimal control law to be specified by Equations 83 through 87 and 91 through 94 except for determining U in Equation 91. The equation for $\dot{U}$, converted to reverse time τ, approaches

$$\frac{dU}{d\tau} = 3q/\tau^3 \tag{95}$$

during the midcourse phase, which implies that the midcourse approximation for $U(t)$ is simply a constant. The value of this constant (with respect to t or τ) is a function of the parameters specifying the problem, however, and must be found by numerical integration, which reduces to integrating

$$\dot{P} = q\tau^2 - \frac{P^2}{2\tau^2\sqrt{qr}}\ ;\ P\left(t_1\right) \text{ from Equation 92}$$

in forward time to t_f from some t_1, well within the midcourse region and then, using the stored result and Equation 89 for S, integrating the following system of equations in reverse time from t_f to t_1:

$$\dot{Y} = \frac{PY}{\tau^2\sqrt{qr}} - S^2\tau^2 \, ; \quad Y\left(t_f\right) = 0$$

$$\dot{\text{Ч}} = 2S\tau\left(S + \text{Ч}\tau\right) \, ; \quad \text{Ч}\left(t_f\right) = 0$$

$$\dot{\mathbf{I}} = 2S\tau\left(\text{Ч} + \mathbf{I}\tau\right) + \left(S + \text{Ч}\tau\right)^2 \, ; \quad \mathbf{I}\left(t_f\right) = 0$$

$$\dot{\mathbf{R}} = \frac{2}{\tau^2\sqrt{qr}}\left(P\mathbf{R} + Y\right) \, ; \quad \mathbf{R}\left(t_f\right) = 0$$

$$\dot{\nabla} = \frac{P}{\tau^2\sqrt{qr}}\left(\nabla - \text{Ч} - 2Y/\tau - \frac{1}{2}P\mathbf{R}/\tau\right) - 2q\mathbf{R}\tau \, ; \quad \nabla\left(t_f\right) = 0$$

$$\dot{U} = \frac{P^2}{2\tau^4\sqrt{qr}}\left(Y + 2\text{Ч}\tau - 2\nabla\tau - \mathbf{I}\tau^2\right) - q\left(S + Y + 2\nabla\tau\right) \, ; \quad U\left(t_f\right) = 0 \, .$$

It is clear from this description that the desired value $U(t_1)$ depends only on the parameters a, q and r, and the constraints of dimensional analysis imply that this dependency (if reasonably well-behaved) can be expressed in the form

$$\frac{U\left(t_1\right)}{qa^{2/3}} = f\left(\frac{ra^{4/3}}{q}\right) \, .$$

It is a simple matter to determine f empirically, and the result is shown in Figure 1.

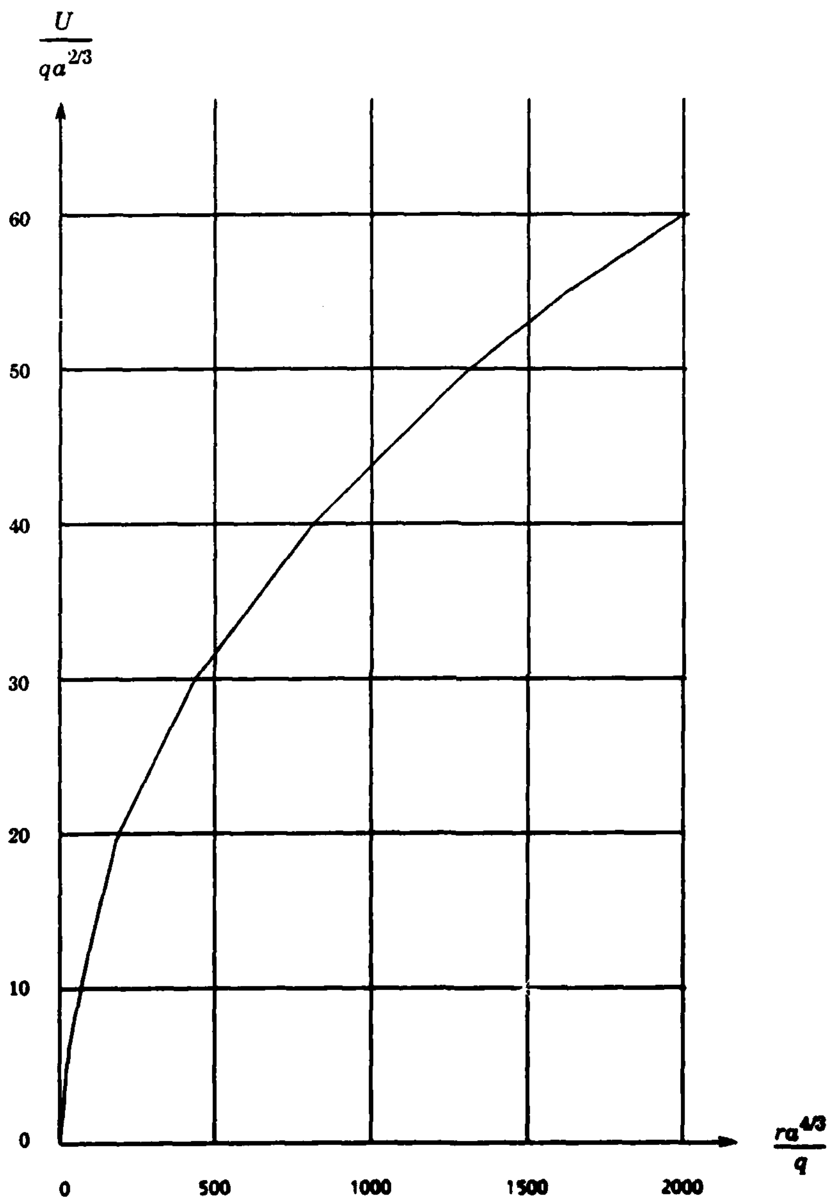

FIGURE 1. Empirical Determination of *f*.

As a final practical point of implementation, this midcourse guidance law approximation would probably be more robust if all occurrences of τ in Equations 83 through 87 and 91 through 94, except within the "$(\tau + \hat{\theta})$" factors of Equations 84 and 91, were replaced by $\tau + \hat{\theta}$, since $\hat{\theta} << \tau$ and since $\tau + \hat{\theta}$ is the current estimate of the time-to-go for closest approach whereas τ is only the prior estimate. It is also interesting to note that the line-of-sight rotation rate ω is given by

$$\omega = \frac{x}{V(\tau + \theta)^2}$$

for small ω (and the purpose of the guidance is to keep it small). Since $\hat{\theta} << \tau$ and $\theta << \tau$, this means that Equation 91 is approximately equivalent to

$$u = \left[\frac{3}{2}U(\tau + \hat{\theta})^3/V - 3V\right]\hat{\omega} , \qquad (96)$$

where $\hat{\omega}$ is the conditional-mean estimate of ω. This is a form of proportional navigation where the navigation gain is modified by a range-dependent term if $U \neq 0$.

LIMITS OF VALIDITY

The midcourse guidance law approximation clearly breaks down unless $\hat{\theta} << \tau$, $L << \tau^2$ (or $\theta << \tau$), and the supposedly dominant terms in Equation 88 for the control are large compared to the perturbation terms, and these appear to be the only restrictions. Since the decomposition of the coefficient matrix for the $\hat{x}_i\hat{x}_j$ terms of the cost-to-go function Equation 18 into $S + W$ was made arbitrarily for convenience in the analysis of Section V, the dominant terms in Equation 88 should really be taken as $(S + W)\tau\hat{x}$. These happen to be the only nonzero terms remaining in the midcourse approximation of Equation 91, however, if $\hat{\theta}$ is ignored as relatively small, so the last restriction mentioned on validity is basically that the coefficient of $\hat{x}$ in Equation 91 remain nonzero. From Figure 1, U is generally positive, so this condition will be violated when τ becomes larger than some critical value τ^*. Disregarding $\hat{\theta}/\tau$ as negligible in Equation 91 shows that

$$\tau^* = \left(\frac{2V^2}{U}\right)^{1/3} \qquad (97)$$

This equation suggests that for engagements such that

$$t_f - t_0 > \left(\frac{2V^2}{U}\right)^{1/3}$$

the "perturbations" in this analysis aren't relatively small any more and the optimal control assumes a drastically different form. The navigation gain of Equation 96 becomes negative when this condition is reached, meaning that the interceptor would deliberately steer away from the target (in relative motion space) instead of homing on it. This may indicate that it becomes to the interceptor's advantage under these conditions to approach the target on a highly curved course to create a kind of synthetic parallax on which to base estimates of current target range. This tactic has been proposed before (Reference 5) and has been shown to enhance an interceptor's ability to estimate range from angle data, although the utility of this enhancement was not analyzed. The results here indicate that it can be advantageous even when its only purpose is to enable better estimates of x (or equivalently the line-of-sight rate ω) to be made. This is because $U \rightarrow 0$ when the measurement noise intensity r approaches zero (see Figure 1), and hence Equation 96 reduces to standard proportional navigation since ω can be estimated exactly under these conditions.

For parameter values typical of a radar-guided homing missile, however, values of τ^* can occur which are within its engagement envelope, within the region of validity of the midcourse guidance approximation, and in the range where glint is normally the predominant source of angle-measurement noise. Thus there is good reason to hope that this sort of analysis could be applied to a reasonably realistic problem formulation to provide some meaningful information about the guidance of such missiles, at least to the extent of indicating conditions under which radical departures from "proportional navigation" guidance might be advantageous.

Appendix A.

INDUCTION STEP COMPUTATIONS FOR SCALAR STATE ESTIMATION

For the case of scalar x and θ, the second-order Edgeworth form assumed by their joint conditional density (to order h^2) reduces to

$$
\begin{aligned}
p(x,\theta) = & \left\{ \frac{ e^{-\frac{1}{2}\left[x-\hat{x} \mid \theta-\hat{\theta}\right]\left[\begin{array}{c|c} M & E \\ \hline E & L \end{array}\right]^{-1}\left[\begin{array}{c} x-\hat{x} \\ \hline \theta-\hat{\theta} \end{array}\right]} }{2\pi\sqrt{ML-E^2}} \right\} \Bigg\{ 1 + \\
& \left(\frac{\lambda}{M^2L}\right)\left((\theta-\hat{\theta})\left[(x-\hat{x})^2-M\right] - 2\frac{E}{L}(x-\hat{x})\left[(\theta-\hat{\theta})^2-L\right] - 2(x-\hat{x})E\right) \\
& + \left(\frac{\xi}{ML^2}\right)(x-\hat{x})\left[(\theta-\hat{\theta})^2-L\right] + \left(\frac{\rho}{M^2L^2}\right)\left[(x-\hat{x})^2-M\right]\left[(\theta-\hat{\theta})^2-L\right] \\
& + \frac{1}{2}\left(\frac{\lambda}{M^2L}\right)^2(\theta-\hat{\theta})^2\left[(x-\hat{x})^4-6M(x-\hat{x})^2+3M^2\right] \Bigg\} \qquad \text{(A-1)}
\end{aligned}
$$

at a generic time t, where the conditioning on the measurement history available at that time is suppressed in the notation, and where E and λ are of order h, ξ and ρ are of order h^2, and $\hat{\theta}$, M, L, and $\hat{x}$ are of order unity. ξ and ρ here take the place of $\overline{\nabla}$ and N in the notation for the multivariate case. Under the hypothesis that x and θ have the joint density (Equation A-1) to order h^2, the joint density is computed to order $h^2\delta t$ for the result of applying each of the four transformations described in Section IV to x and θ, and for conditioning x and θ on the measurements (also restricted to the scalar case) in a time interval $(t, t + \delta t)$. The result in each case will be another density of the form of Equation A-1, so the result for the entire induction step will be that the joint density remains of this form, with parameter changes that can be computed to order $h^2\delta t$ as the composition of the changes resulting from each of these individual transformation and conditioning operations. The effects of any higher-order (than h^2) error in the density approximation Equation A-1 are not considered in these computations, and are tacitly assumed to be of higher order in h than $h^2\delta t$ for the sort of reasons described in Reference 6. Also, the time increment δt is denoted by Δ for convenience in these computations.

Without loss of generality at this point, only the case of $\hat{x} = \hat{\theta} = 0$ is treated in these induction step computations. In the case of nonzero $\hat{x}$ or $\hat{\theta}$ at any particular point in time, the dynamics and measurement equations for $(x - \hat{x})$ and $(\theta - \hat{\theta})$ reduce to the same form as those for x and θ. Hence the same induction step computations can be applied to $p(x - \hat{x}, \theta - \hat{\theta})$ at

that time instant with a redefinition of the parameters. The desired result for the increment in $p(x,\theta)$ can then be obtained by merely shifting the mean of the density by $(\hat{x},\hat{\theta})$

TRANSFORMATION 1

$$\text{Let} \begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} 1+F\Delta & 2\Psi u\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} + \begin{bmatrix} Gu\Delta \\ Tu\Delta \end{bmatrix}, \tag{A-2}$$

where u is a known quantity (the current control chosen by the controller), and let $p(x,\theta)$ be given by Equation A-1 (with $\hat{x} = \hat{\theta} = 0$). Inverting this transformation gives

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+F\Delta} & \dfrac{2\Psi u\Delta}{1+F\Delta} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y - Gu\Delta \\ \eta - Tu\Delta \end{bmatrix} \tag{A-3}$$

Since this inverse is single-valued, the standard result for transforming probability densities gives

$$p_{y,\eta}(\alpha,\beta) = \frac{p_{x,\theta}[c(\alpha,\beta),d(\alpha,\beta)]}{\left\| \begin{matrix} 1+F\Delta & 2\Psi u\Delta \\ 0 & 1 \end{matrix} \right\|} \tag{A-4}$$

where (c,d) is obtained by applying the inverse transformation A-3 to (α,β). Thus,

$$p_{y,\eta}(\alpha,\beta) = \frac{1}{1+F\Delta} p_{x,\theta}\left[\frac{\alpha - Gu\Delta + 2\Psi u\Delta(\beta - Tu\Delta)}{1+F\Delta}, \beta - Tu\Delta\right] . \tag{A-5}$$

Straightforward substitution of Equation A-1, which is really for $p_{x,\theta}(x,\theta)$, for $p_{x,\theta}$ (with different arguments) in Equation A-5 shows that $p(y,\eta)$ is given by Equation A-1 with arguments y and η, and with different parameter values $\hat{x}^*$, $\hat{\theta}^*$, M^*, and E^* in place of their unstarred counterparts. Except for terms of order Δ^2, these parameter changes are:

$$\left.\begin{aligned} \hat{x}^* &= Gu\Delta \\ \hat{\theta}^* &= Tu\Delta \\ M^* &= M + (2FM + 4\Psi Eu)\Delta \\ E^* &= E + (FE + 2\Psi Lu)\Delta \end{aligned}\right\} \tag{A-6}$$

TRANSFORMATION 2

Let $y = x + (2\Gamma\theta x + F\theta^2 x + L\theta^2 u)\Delta$ with $p(x,\theta)$ as given by Equation A-1. It follows from the discussion of this transformation in Section IV that the standard formula for the transformation of probability densities can be applied here to give

$$p(y,\theta) = \frac{p_{x,\theta}\left[\dfrac{y - L\theta^2 u\Delta}{1 + 2\Gamma\theta\Delta + F\theta^2\Delta}, \theta\right]}{1 + 2\Gamma\theta\Delta + F\theta^2\Delta} \qquad \text{(A-7)}$$

to order $h^2\Delta$. Now let

$$\hat{y} = (2\Gamma E + L L u)\Delta$$

and

$$V = M + (8\Gamma\lambda + 2FLM)\Delta$$

Then the exponential factor of

$$p_{x,\theta}\left(\frac{y - L\theta^2 u\Delta}{1 + 2\Gamma\theta\Delta + F\theta^2\Delta}, \theta\right)$$

can be determined from Equation A-1 as

$$\frac{e^{-\frac{1}{2}\left[y + a\Delta \;\vdots\; \theta\right]\begin{bmatrix} V - b\Delta & E \\ E & L \end{bmatrix}^{-1}\begin{bmatrix} y + a\Delta \\ \theta \end{bmatrix}}}{2\pi\sqrt{(V - b\Delta)L - E^2}} \qquad \text{(A-8)}$$

except for terms of order Δ^2, where

$$a = -2\Gamma(y\theta - E) - L(\theta^2 - L)u - Fy(\theta^2 - L) - FLy$$

and

$$b = 8\Gamma\lambda + 2FLM$$

Expanding expression A-8 to order $h^2\Delta$ gives this exponential factor as

$$\left\{\frac{e^{-\frac{1}{2}\left[y \;\vdots\; \theta\right]\begin{bmatrix} V & E \\ E & L \end{bmatrix}^{-1}\begin{bmatrix} y \\ \theta \end{bmatrix}}}{2\pi\sqrt{VL - E^2}}\right\}\left\{1 - \frac{b\Delta}{2V^2}\left[(y - \hat{y})^2 - V\right] + \frac{a\Delta}{VL}\left[E\theta + L(y - \hat{y})\right]\right\} \qquad \text{(A-9)}$$

Dividing the polynomial factor of Equation A-9 by $(1 + 2\Gamma\theta\Delta + \mathrm{F}\theta^2\Delta)$ and using the definitions of a, b and c gives

$$1 + \frac{2\Gamma\Delta}{M}\left\{\theta\left[\left(y-\hat{y}\right)^2 - V\right] - 2\frac{E}{L}\left(y-\hat{y}\right)\left(\theta^2 - L\right) - 2\left(y-\hat{y}\right)E\right\}$$
$$+ \frac{\mathrm{F}\Delta}{M}\left[\left(y-\hat{y}\right)^2 - V\right]\left[\theta^2 - L\right] + \frac{\Delta}{M}\left(\mathrm{F}\hat{y} + \mathrm{L}u + 2\Gamma\frac{E}{L}\right)\left(y-\hat{y}\right)\left[\theta^2 - L\right]$$
$$- 4\frac{\Gamma\lambda\Delta}{M^2}\left[\left(y-\hat{y}\right)^2 - V\right] \qquad \text{(A-10)}$$

to order $h^2\Delta$. Expanding the polynomial factor of

$$p_{x,\theta}\left(\frac{y - \mathrm{L}\theta^2 u\Delta}{1 + 2\Gamma\theta\Delta + F\theta^2\Delta}, \theta\right)$$

to order $h^2\Delta$ gives the remaining factor in Equation A-7 to this accuracy as

$$1 + \left(\frac{\lambda}{V^2 L}\right)\left\{\theta\left[\left(y-\hat{y}\right)^2 - V\right] - 2\frac{E}{L}\left(y-\hat{y}\right)\left[\theta^2 - L\right] - 2\left(y-\hat{y}\right)E\right\}$$
$$+ \left(\frac{\xi}{VL^2}\right)\left(y-\hat{y}\right)\left[\theta^2 - L\right] + \left(\frac{\rho}{V^2 L^2}\right)\left[\left(y-\hat{y}\right)^2 - V\right]\left[\theta^2 - L\right]$$
$$+ \frac{1}{2}\left(\frac{\lambda}{V^2 L}\right)^2\theta^2\left[\left(y-\hat{y}\right)^4 - 6V\left(y-\hat{y}\right)^2 + 3V^2\right]$$
$$- 4\Gamma\left(\frac{\lambda}{M^2 L}\right)\Delta\left\{\left[\left(y-\hat{y}\right)^2 - V\right]\left[\theta^2 - L\right] + L\left[\left(y-\hat{y}\right)^2 - V\right]\right.$$
$$\left. + V\left[\theta^2 - L\right] + LV\right\} \qquad \text{(A-11)}$$

To order $h^2\Delta$, therefore, the joint density $p(y,\theta)$ is the product of the exponential factor in Equation A-9 and a polynomial which is itself the product of the polynomials A-10 and A-11. Multiplying these two polynomials gives the polynomial factor of $p(y,\theta)$ to order $h^2\Delta$ as

$$1+\left(\frac{\lambda}{V^2L}+\frac{2\Gamma}{V}\Delta\right)\left\{\theta\left[\left(y-\hat{y}\right)^2-V\right]-2\frac{E}{L}\left(y-\hat{y}\right)\left[\theta^2-L\right]-2\left(y-\hat{y}\right)E\right\}$$

$$+\left[\frac{\xi}{VL^2}+\frac{\Delta}{V}\mathrm{L}u\right]\left(y-\hat{y}\right)\left[\theta^2-L\right]$$

$$+\left[\frac{\rho}{V^2L^2}+\left(\frac{4\Gamma\lambda}{V^2L}+\frac{F}{V}\right)\Delta\right]\left[\left(y-\hat{y}\right)^2-V\right]\left[\theta^2-L\right]$$

$$+\frac{1}{2}\left(\frac{\lambda}{V^2L}+\frac{2\Gamma}{V}\Delta\right)^2\theta^2\left[\left(y-\hat{y}\right)^4-6V\left(y-\hat{y}\right)^2+3V^2\right]$$

Comparing this with the exponential factor of A-9 shows that their product $p(y,\theta)$ is the same as $p_{x,\theta}(y,\theta)$ to order $h^2\Delta$ except that $\hat{x}^*$, E^*, M^*, λ^*, ξ^* and ρ^* replace the corresponding unstarred parameters, where

$$\left.\begin{aligned}
\hat{x}^* &= (2\Gamma E + \mathrm{L}Lu)\Delta \\
E^* &= E \\
M^* &= M + (8\Gamma\lambda + 2\mathrm{F}LM)\Delta \\
\lambda^* &= \lambda + 2\Gamma ML\Delta \\
\xi^* &= \xi + L^2\mathrm{L}u\Delta \\
\rho^* &= \rho + (4L\Gamma\lambda + \mathrm{F}ML^2)\Delta
\end{aligned}\right\}\qquad \text{(A-12)}$$

to this order of accuracy.

TRANSFORMATION 3

Let $y = x + w\sqrt{\Delta}$ with $p(x,\theta)$ given by Equation A-1 and with w conditioned on x and θ having a zero-mean Normal distribution with variance $(1 + \sigma\theta)^2Q$, where σ is of order h. Then, as noted in Section IV,

$$p(y,\theta) = \int_{-\infty}^{\infty} p_{x,\theta}(y - w\sqrt{\Delta},\theta)p_{w/x,\theta}(w,x,\theta)dw\ . \qquad \text{(A-13)}$$

The integrand in Equation A-13 is the product of two exponential factors and a polynomial in y, w, and θ. The product of the exponential factors alone can be written as

$$\sqrt{\Delta}\left(\frac{e^{-\frac{1}{2}\left[y-w\sqrt{\Delta}\,\vdots\,\theta\right]\left[\begin{matrix} M & E \\ E & L\end{matrix}\right]^{-1}\left[\frac{y-w\sqrt{\Delta}}{\theta}\right]}}{2\pi\sqrt{ML-E^2}}\right)\left(\frac{e^{-\frac{w^2\Delta}{2Q(1+\sigma\theta)^2\Delta}}}{\sqrt{2\pi Q(1+\sigma\theta)^2\Delta}}\right),$$

by multiplying and dividing by $\sqrt{\Delta}$. Completing the square in the exponent shows that this product is also equal to

$$\left(\frac{e^{-\frac{1}{2}\left[y\,\vdots\,\theta\right]\left[\begin{matrix} M+Q(1+\sigma\theta)^2\Delta & E \\ E & L\end{matrix}\right]^{-1}\left[\frac{y}{\theta}\right]}}{2\pi\sqrt{M+Q(1+\sigma\theta)^2L\Delta-E^2}}\right)\left(\frac{e^{-\frac{\left\{w-\frac{Q(1+\sigma\theta)^2}{ML-E^2}\left(Ly-E\theta\right)\sqrt{\Delta}\right\}^2}{2Q(1+\sigma\theta)^2\left[1-\frac{Q(1+\sigma\theta)^2L\Delta}{ML-E^2}\right]}}}{\sqrt{2\pi Q(1+\sigma\theta)^2\left[1-\frac{Q(1+\sigma\theta)^2L\Delta}{ML-E^2}\right]}}\right) \qquad \text{(A-14)}$$

except for negligible terms of order $\Delta^{3/2}$. If this expression is substituted for the two exponential factors in the integrand of Equation A-13, the first factor of Equation A-14 can be brought outside the integration, which then becomes

$$\int_{-\infty}^{\infty}\frac{e^{-\frac{[w-a(y,\theta)\sqrt{\Delta}]^2}{2Q(1+\sigma\theta)^2+b(\theta)\Delta}}}{\sqrt{2\pi[Q(1+\sigma\theta)^2+b(\theta)\Delta]}}\left\{1+P(y-w\sqrt{\Delta},\theta\right\}dw\ , \qquad \text{(A-15)}$$

where

$$a(y,\theta)=\frac{Q(1+\sigma\theta)^2}{ML-E^2}\left[Ly-E\theta\right],$$

$$b(\theta)=-\frac{Q^2L}{ML-E^2}(1+\sigma\theta)^4\ ,$$

and where $P(x,\theta)$ denotes all terms but the leading one in the polynomial factor of density (Equation A-1). Except for terms of order $\Delta^{3/2}$, a Maclaurin series expansion of $P(y-w\sqrt{\Delta},\theta)$ in w gives

$$1 + \mathrm{P}\left(y - w\sqrt{\Delta}, \theta\right) = 1 + \left(\frac{\lambda}{M^2L}\right)\left[\theta\left(y^2 - M\right) - 2\frac{E}{L}y\left(\theta^2 - L\right)\right.$$

$$\left. - 2yE\right] + \left(\frac{\xi}{ML^2}\right)y\left[\theta^2 - L\right] + \left(\frac{\rho}{M^2L^2}\right)\left(y^2 - M\right)\left(\theta^2 - L\right)$$

$$+ \frac{1}{2}\left(\frac{\lambda}{M^2L}\right)^2\theta^2\left(y^4 - 6My^2 + 3M^2\right) - \left[2\left(\frac{\lambda}{M^2L}\right)\left(y\theta - E\right)\right.$$

$$+ \left(\frac{\xi}{M^2L} - 2\frac{\lambda E}{M^2L^2}\right)\left(\theta^2 - L\right) + 2\left(\frac{\rho}{M^2L^2}\right)y\left(\theta^2 - L\right)$$

$$\left. + 2\left(\frac{\lambda}{M^2L}\right)^2\theta^2\left(y^3 - 3My\right)\right]w\sqrt{\Delta} + \left[\frac{\lambda}{M^2L}y + \frac{\rho}{M^2L^2}\left(\theta^2 - L\right)\right.$$

$$\left. + 3M\left(\frac{\lambda}{M^2L}\right)^2\theta^2\left(y^3 - 3My\right)\right]w^2\Delta \quad \text{(A-16)}$$

Substituting this equation makes Equation A-15 *the integral of the right-hand side of* Equation A-16 as a polynomial in w multiplied by a Normal w-density with

$$\text{mean} = a(y,\theta)\sqrt{\Delta}$$

and

$$\text{variance} = Q(1 + \sigma\theta)^2 + b(\theta)\Delta \; .$$

This integral can therefore be evaluated by applying standard results for the first and second moments of Normal random variables, the result being simply the right-hand side of Equation A-16 with $a(y,\theta)\Delta$ substituted for $w\sqrt{\Delta}$ and $Q(1+\sigma\theta)^2\Delta$ substituted for $w^2\Delta$, except for terms of order Δ^2. Using the definition of $a(y,\theta)$ in this substitution, retaining only terms which are significant to order $h^2\Delta$, and rearranging terms gives the integral (Equation A-15) as

$$1 + \left(\frac{\lambda}{M^2L}\right)\left[\theta\left(y^2 - M\right) - 2\frac{E}{L}y\left(\theta^2 - L\right) - 2yE\right]$$

$$+ \left(\frac{\xi}{ML^2}\right)y\left(\theta^2 - L\right) + \left(\frac{\rho}{M^2L^2}\right)\left(y^2 - M\right)\left(\theta^2 - L\right)$$

$$+ \frac{1}{2}\left(\frac{\lambda}{M^2L}\right)^2\theta^2\left(y^4 - 6My^2 + 3M^2\right)$$

$$-Q\Delta\left\{\frac{2}{M}\left(\frac{\lambda}{M^2L}\right)^2\theta^2\left(y^4-6My^2+3M^2\right)\right.$$

$$+\left[3\left(\frac{\lambda}{M^2L}\right)^2+\frac{2}{M}\left(\frac{\rho}{M^2L^2}\right)+4\frac{\sigma}{M}\left(\frac{\lambda}{M^2L}\right)\right]\left(y^2-M\right)\left(\theta^2-L\right)$$

$$+\frac{2}{M}\left(\frac{\lambda}{M^2L}\right)\left[\theta\left(y^2-M\right)-2\frac{E}{L}y\left(\theta^2-L\right)-2yE\right]$$

$$+\frac{\xi}{M^2L^2}y\left(\theta^2-L\right)+\frac{\lambda}{M^2}\left(\frac{\lambda}{ML}-4\frac{\sigma}{M}\right)\left(y^2-M\right)$$

$$\left.+\left(\frac{\rho}{M^2L^2}+2\frac{\lambda\sigma}{M^2L}\right)\left(\theta^2-L\right)+\left(\frac{\lambda}{M^2L}\right)\theta+2\frac{\sigma\lambda}{M^2}\right\} . \qquad \text{(A-17)}$$

Now define

$$V = M+Q(1+\sigma^2L)\Delta , \qquad \text{(A-18)}$$

so that

$$M = V-Q(1+\sigma^2L)\Delta . \qquad \text{(A-19)}$$

Substituting Equation A-19 into Equation A-17 and expanding to order $\hbar^2\Delta$ gives the integral of Equation A-15 to this accuracy as

$$1+\left(\frac{\lambda}{V^2L}\right)\left[\theta\left(y^2-V\right)-2\frac{E}{L}y\left(\theta^2-L\right)-2yE\right]$$

$$+\left(\frac{\xi}{VL^2}\right)y\left(\theta^2-L\right)+\left(\frac{\rho}{V^2L^2}\right)\left(y^2-V\right)\left(\theta^2-L\right)$$

$$+\frac{1}{2}\left(\frac{\lambda}{V^2L}\right)^2\theta^2\left(y^4-6Vy^2+3V^2\right)$$

$$-Q\Delta\left[\frac{4\sigma}{M}\left(\frac{\lambda}{M^2L}\right)\left(y^2-V\right)\left(\theta^2-L\right)\right.$$

$$\left.+4\frac{\sigma L}{M}\left(\frac{\lambda}{M^2L}\right)\left(y^2-V\right)+2\frac{\sigma\lambda}{M^2L}\left(\theta^2+L\right)+2\frac{\sigma\lambda}{M^2}\right] . \qquad \text{(A-20)}$$

Using Equation A-19 to expand the first exponential factor of Equation A-14 to order $h^2\Delta$ gives

$$\left(\frac{e^{-\frac{1}{2}\left[y \mid \theta\right]\begin{bmatrix} V & E \\ E & L \end{bmatrix}^{-1}\begin{bmatrix} y \\ \theta \end{bmatrix}}}{2\pi\sqrt{VL-E^2}}\right)\left\{1+\frac{Q\sigma\Delta}{M^2}\left[\theta\left(y^2-V\right)\right.\right.$$

$$\left.\left. -2\frac{E}{L}y\left(\theta^2-L\right)-2yE+\frac{\sigma}{2}\left(y^2-V\right)\left(\theta^2-L\right)\right]\right\} \quad . \qquad \text{(A-21)}$$

The joint density $p(y,\theta)$ is the product of expressions A-20 and A-21 to order $h^2\Delta$. Carrying out this multiplication to this order of accuracy and rearranging terms shows that $p(y,\theta)$ is the same as $p_{x,\theta}(y,\theta)$ with the parameters M, λ and ρ changed respectively to

$$\left.\begin{aligned} M^* &= M + Q\left(1+\sigma^2 L\right)\Delta \\ \lambda^* &= \lambda + QL\sigma\Delta, \text{ and} \\ \rho^* &= \rho + \tfrac{1}{2}QL^2\sigma^2\Delta \quad . \end{aligned}\right\} \qquad \text{(A-22)}$$

TRANSFORMATION 4

This is a special case ($\sigma = 0$) of Transformation 3 with the roles of x and θ interchanged and with Q replaced by $\varepsilon^2 Q_2$, which is of order h^2. The same method can be applied, but most of the terms are zero or negligible to order $h^2\Delta$ in this case and the only change in the joint density is that the L parameter changes to

$$L^* = L+\varepsilon^2 Q_2 \Delta \quad . \qquad \text{(A-23)}$$

CONDITIONING

Let $p(x,\theta)$ have the joint density (Equation A-1) and let

$$z = x+2K\theta+2\Omega\theta x+\mathrm{H}\theta^2 x+D\theta^2+\upsilon \qquad \text{(A-24)}$$

where K and Ω are of order h, H and D are of order h^2, and where υ is an independent (of x and θ) zero-mean Normal random variable with variance r/Δ, with r of order unity. For simplicity, D is used in place of $\bar{\Delta}$ here and H has been taken as unity (which represents no loss of generality in this scalar case). By the Bayes rule,

$$p(x,\theta/z) \propto p(z/x,\theta)p(x,\theta) \qquad \text{(A-25)}$$

as a function of x and θ. From Equation A-24 and the density of υ,

$$p(z/x,\theta) = \frac{e^{-\frac{\Delta}{2r}\left[z-\left(x+2K\theta+2\Omega\theta x+\mathrm{H}\theta^2 x+D\theta^2\right)\right]^2}}{\sqrt{2\pi r/\Delta}} \qquad \text{(A-26)}$$

Expanding the exponent about the value indicated and then using the Maclaurin series expansion of the exponential function gives

$$p(z/x,\theta) = \frac{e^{-\frac{\Delta}{2r}(z-x-2K\theta)^2}}{\sqrt{2\pi r/\Delta}}\left\{1+\frac{\Delta}{r}\left[2\Omega\theta x+\mathrm{H}\theta^2x+D\theta^2\right]\left(z-x-2K\theta\right)\right\} \qquad \text{(A-27)}$$

except for zero-mean random terms of order Δ and other terms of order $\Delta^{3/2}$ in the polynomial factor. Forming the product of Equation A-25 by multiplying Equations A-27 and A-1, and using standard results of Kalman filter theory to complete the square in the resulting exponential factor, gives

$$\begin{aligned} p(x,\theta/z) \propto\, & N\left(\begin{bmatrix}\bar{x}\\ \bar{\theta}\end{bmatrix},\begin{bmatrix}V & \eta\\ \eta & \Phi\end{bmatrix}\right)\Bigg\{1+\frac{\lambda}{M^2L}\bigg[\theta\left(x^2-M\right) \\ & -2\frac{E}{L}x\left(\theta^2-L\right)-2xE\bigg]+\frac{\xi}{ML^2}x\left(\theta^2-L\right) \\ & +\frac{\rho}{M^2L^2}\left(x^2-M\right)\left(\theta^2-L\right)+\frac{1}{2}\left(\frac{\lambda}{M^2L}\right)^2\theta^2\Big(x^4 \\ & -6Mx^2+3M^2\Big)+\frac{2\Delta}{r}\Omega\theta\left(z-x\right)\bigg[1+\frac{\lambda}{M^2L}\theta\left(x^2-M\right) \\ & +\frac{\Delta}{r}\left(\mathrm{H}x+D\right)\theta^2\left(z-x\right)\bigg]\Bigg\} \end{aligned} \qquad \text{(A-28)}$$

to order $h^2\Delta$ divided by the order of the integral of Equation A-28 over x and θ, where $N(a,A)$ denotes the Normal density with mean a and covariance matrix A, and where

$$\bar{x} = \frac{\Delta}{r}\left(M+2KE\right)z$$

$$\bar{\theta} = \frac{\Delta}{r}\left(E+2KL\right)z$$

$$V = M-\frac{\Delta}{r}\left(M^2+4KME\right)$$

$$\eta = E-\frac{\Delta}{r}\left(ME+2KML\right)$$

$$\phi = L - \frac{\Delta}{r}\left(E + 2KL\right)^2 .$$

The integral of Equation A-28 can be evaluated by applying standard results for moments of a Normal distribution, which gives

$$1 + \frac{\Delta}{r}\left\{\left[2\Omega\left(\bar{x}\bar{\theta} + \eta\right) + DL\right]z - 2\Omega\left(\bar{x}\eta + \bar{\theta}V + 2K\bar{x}\phi\right) - \mathbf{H}LV\right\} \qquad \text{(A-29)}$$

to order $h^2\Delta$. Since this integral is of order unity, $p(x,\theta/z)$ is given to order $h^2\Delta$ by dividing the right-hand side of Equation A-28 by Equation A-29. Expressing the polynomial factor of Equation A-28 in terms of $\bar{x}$, $\bar{\theta}$, V, η, and ϕ to order $h^2\Delta$ and dividing by Equation A-29 gives $1 + P_K$, where, to order $h^2\Delta$,

$$P_K = \frac{\lambda}{V^2\phi}\left\{\left(\theta - \bar{\theta}\right)\left[\left(x - \bar{x}\right)^2 - V\right] - 2\frac{\eta}{\phi}\left(x - \bar{x}\right)\left[\left(\theta - \bar{\theta}\right)^2 - \phi\right] - 2\left(x - \bar{x}\right)\eta\right.$$

$$+ \frac{\xi}{V\phi^2}\left(x - \bar{x}\right)\left[\left(\theta - \bar{\theta}\right)^2 - \phi\right] + \frac{\rho}{V^2\phi^2}\left[\left(x - \bar{x}\right)^2 - V\right]\left[\left(\theta - \bar{\theta}\right)^2 - \phi\right]$$

$$+ \frac{1}{2}\left(\frac{\lambda}{V^2\phi}\right)^2\left(\theta - \bar{\theta}\right)^2\left[\left(x - \bar{x}\right)^4 - 6V\left(x - \bar{x}\right)^2 + 3V^2\right]$$

$$+ \frac{z}{r}\Delta\left\{2\Omega\left[\bar{\theta} + 2V\phi\left(\frac{\lambda}{V^2\phi}\right)\right]\left(x - \bar{x}\right) + 2\Omega\bar{x}\left(\theta - \bar{\theta}\right)\right.$$

$$+ \left[\eta + 2\left(K + \Omega\bar{x}\right)\phi\right]\left(\frac{\lambda}{V^2\phi}\right)\left[\left(x - \bar{x}\right)^2 - V\right] + 2\left(\Omega + \frac{\lambda}{V\phi}\right)\left[\left(x - \bar{x}\right)\left(\theta - \bar{\theta}\right) - \eta\right]$$

$$+ \left(\mathbf{H}\bar{x} + D + \frac{\xi}{\phi^2} - 2\frac{\eta\lambda}{V\phi^2}\right)\left[\left(\theta - \bar{\theta}\right)^2 - \phi\right]$$

$$+ \frac{2}{V\phi}\left(\frac{\rho}{\phi} + 2\Omega\lambda\right)\left(x - \bar{x}\right)\left[\left(\theta - \bar{\theta}\right)^2 - \phi\right] + 2\Omega\bar{x}\left(\frac{\lambda}{V^2\phi}\right)\left[\left(x - \bar{x}\right)^2 - V\right]\left[\left(\theta - \bar{\theta}\right)^2 - \phi\right]$$

$$\left.\left. + 2\left(\frac{\lambda}{V^2\phi}\right)\left(\Omega + \frac{\lambda}{V\phi}\right)\left(\theta - \bar{\theta}\right)^2\left[\left(x - \bar{x}\right)^3 - 3V\left(x - \bar{x}\right)\right]\right\}\right\}$$

$$- \frac{\Delta}{r}\left\{\left[D\phi + 2\Omega\left(\bar{x}\bar{\theta} + 2\eta + 2K\phi\right)\right]\left(x - \bar{x}\right) + 2\Omega V\left(\theta - \bar{\theta}\right)\right.$$

$$+\left[H\phi+2\Omega\left(\bar\theta+3\frac{\lambda}{V}\right)\right]\left[\left(x-\bar x\right)^2-V\right]+2\Omega\bar x\left[\left(x-\bar x\right)\left(\theta-\bar\theta\right)-\eta\right]$$

$$+\left(HV+4K\Omega\bar x\right)\left[\left(\theta-\bar\theta\right)^2-\phi\right]+2\left(\Omega+\frac{\lambda}{V\phi}\right)\left[\left(\theta-\bar\theta\right)\left((x-\bar x)^2-V\right)\right.$$

$$\left.-2\frac{\eta}{\phi}\left(x-\bar x\right)\left((\theta-\bar\theta)^2-\phi\right)-2\left(x-\bar x\right)\eta\right]+\left[D+\frac{\xi}{\phi^2}+4\Omega\left(K+\frac{\eta}{\phi}\right)\right.$$

$$\left.+2V\left(2K+\frac{\eta}{\phi}\right)\left(\frac{\lambda}{V^2\phi}\right)\right]\left(x-\bar x\right)\left[\left(\theta-\bar\theta\right)^2-\phi\right]+\left(H+\frac{2\rho}{V\phi^2}\right.$$

$$\left.+6\frac{\Omega\lambda}{V\phi}\right)\left[\left(x-\bar x\right)^2-V\right]\left[\left(\theta-\bar\theta\right)^2-\phi\right]+2\Omega\bar x\left(\frac{\lambda}{V^2\phi}\right)\left(\theta-\bar\theta\right)^2\left[\left(x-\bar x\right)^3\right.$$

$$\left.\left.-3V\left(x-\bar x\right)\right]+2\left(\frac{\lambda}{V^2\phi}\right)\left(\Omega+\frac{\lambda}{V\phi}\right)\left(\theta-\bar\theta\right)^2\left[\left(x-\bar x\right)^4-6V\left(x-\bar x\right)^2+3V^2\right]\right\}\ . \qquad \text{(A-30)}$$

The first and second moments of $p(x,\theta/z)$ can now be evaluated by applying standard results for moments of Normal distributions. The results are, to order $h^2\Delta$,

$$E(x)\triangleq\tilde x=\bar x+2\Omega\left(V\bar\theta+\eta\bar x+2\lambda\right)\frac{z\Delta}{r}-\frac{V}{r}\left[D\phi+2\Omega\left(\bar x\bar\theta+3\eta+2K\phi\right)\right]\Delta \qquad \text{(A-31)}$$

$$E(\theta)\triangleq\tilde\theta=\bar\theta+2\Omega\frac{\phi}{r}\left(\bar x z-V\right)\Delta \qquad \text{(A-32)}$$

$$\mathrm{var}(x)\triangleq t=V+2\frac{V}{r}\left[2\Omega\eta+\left(2\Omega\phi\bar x+2K\phi+3\eta\right)\frac{\lambda}{V\phi}\right]z\Delta$$

$$-2\frac{V}{r}\left[HV\phi+2\Omega\left(\eta\bar x+V\bar\theta+3\lambda\right)\right]\Delta \qquad \text{(A-33)}$$

$$\mathrm{cov}\ (x,\theta)\triangleq\alpha=\eta+\frac{2}{r}\left(\Omega V\phi+\lambda\right)z\Delta-2\frac{V}{r}\Omega\phi\bar x\Delta \qquad \text{(A-34)}$$

$$\mathrm{var}(\theta)\triangleq\omega=\phi+\frac{2}{r}\left[\xi+2\Omega\eta\phi+\left(H\bar x+D\right)\phi^2\right]z\Delta-2\frac{\phi}{r}\left\{HV\phi+2\Omega\bar x\left[\eta+\left(2K+\Omega\bar x\right)\phi\right]\right\}\Delta \qquad \text{(A-35)}$$

Now let

$$\beta = \tilde{x} - \bar{x}, \; \sigma = \tilde{\theta} - \bar{\theta}, \; \varepsilon = t - V, \; s = a - \eta$$

and $u = \omega - \phi$. Using Equations A-31 through A-35, and the fact that

$$\left(\frac{z\Delta}{r}\right)^2 = \frac{\Delta}{r}$$

except for (negligible) zero-mean random quantities of order Δ, gives the following result to order $h^2\Delta$:

$$\beta = 2\frac{\Delta}{r}\Omega\left(t\tilde{\theta} + a\tilde{x} + 2\lambda\right)z - \frac{\Delta}{r}t\left[D\omega + 2\Omega\left(\tilde{x}\tilde{\theta} + 3a + 2K\omega + 4\Omega\omega\tilde{x} + 2\frac{\lambda}{t}\tilde{x}\right)\right] \quad \text{(A-36)}$$

$$\sigma = 2\frac{\Delta}{r}\Omega\omega\left(\tilde{x}z - t\right) \quad \text{(A-37)}$$

$$\varepsilon = 2\frac{\Delta}{r}t\left[2\Omega a + \left(2\Omega\omega\tilde{x} + 2K\omega + 3a\right)\frac{\lambda}{t\omega}\right]z - 2\frac{\Delta}{r}t\Bigg\{\mathrm{H}t\omega + 2\Omega\left(a\tilde{x} + t\tilde{\theta}\right) + t\omega\left[4\Omega^2 + 16\Omega\left(\frac{\lambda}{t\omega}\right) + 6\left(\frac{\lambda}{t\omega}\right)^2\right]\Bigg\} \quad \text{(A-38)}$$

$$s = 2\frac{\Delta}{r}t\omega\left[\left(\Omega + \frac{\lambda}{t\omega}\right)z - \Omega\tilde{x}\right] \quad \text{(A-39)}$$

$$u = 2\frac{\Delta}{r}\left[\xi + 2\Omega a\omega + \left(\mathrm{H}\tilde{x} + D\right)\omega^2\right]z - 2\frac{\Delta}{r}\omega^2\left[\mathrm{H}t + 4\Omega^2 t + 2\Omega^2\tilde{x}^2 + 4\mathrm{K}\Omega\tilde{x} + 2\frac{a}{\omega}\Omega\tilde{x} + 4\Omega\frac{\lambda}{\omega}\right] \quad \text{(A-40)}$$

By construction,

$$N\left(\begin{bmatrix}\bar{x}\\ \bar{\theta}\end{bmatrix}, \begin{bmatrix}V & \eta\\ \eta & \phi\end{bmatrix}\right) = \frac{e^{-\frac{1}{2}\left[m+\beta \,|\, n+\sigma\right]\left[\begin{array}{c|c} t-\varepsilon & a-s \\ \hline a-s & \omega-u\end{array}\right]^{-1}\left[\begin{array}{c} m+\beta \\ \hline n+\sigma\end{array}\right]}}{2\pi\sqrt{(t-\varepsilon)(\omega-u)-(a-s)^2}} \quad \text{(A-41)}$$

to order $h^2\Delta$ (for all z of order $\Delta^{-1/2}$), where

$$m \triangleq x - \tilde{x}$$

and

$$n \triangleq \theta - \tilde{\theta} \ .$$

A Taylor series expansion of the right-hand side of Equation A-41 shows that

$$N\left(\begin{bmatrix} \bar{x} \\ \bar{\theta} \end{bmatrix}, \begin{bmatrix} V & \eta \\ \eta & \phi \end{bmatrix}\right) = N\left(\begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix}, \begin{bmatrix} t & \alpha \\ \alpha & \omega \end{bmatrix}\right)\left(1 + P_G\right) \qquad \text{(A-42)}$$

to order $h^2\Delta$, where

$$P_G = \frac{1}{2}\left(\frac{s}{t\omega}\right)^2\left(m^2 - t\right)\left(n^2 - \omega\right) + \frac{s\sigma}{t\omega^2}m\left(n^2 - \omega\right) + \frac{1}{t}\left(\frac{\alpha s}{t\omega} - \frac{\varepsilon}{2t}\right)\left(m^2 - t\right) - \frac{s}{t\omega}\left(mn - \alpha\right)$$

$$+ \left[\frac{1}{2}\left(\frac{\sigma}{\omega}\right)^2 + \frac{1}{\omega}\left(\frac{\alpha s}{t\omega} - \frac{u}{2\omega}\right)\right]\left(n^2 - \omega\right) + \frac{1}{t}\left(\frac{\alpha\sigma}{\omega} - \beta\right)m - \frac{\sigma}{\omega}n \ . \qquad \text{(A-43)}$$

In substituting from Equations A-36 through A-40 in Equation A-43 it suffices for order $h^2\Delta$ accuracy to use the approximations

$$\left(\frac{s}{t\omega}\right)^2 = \frac{4}{r}\left(\Omega + \frac{\lambda}{t\omega}\right)^2\Delta \ ,$$

$$\frac{s\sigma}{t\omega^2} = \frac{4}{r}\left(\Omega + \frac{\lambda}{t\omega}\right)\Omega\tilde{x}\Delta$$

and

$$\left(\frac{\sigma}{\omega}\right)^2 = \frac{4}{r}\Omega^2\tilde{x}^2\Delta \ ,$$

and the result of such substitution is

$$P_G = -\frac{\Delta}{r}\left\{\left[2\omega\left(K + \Omega\tilde{x}\right) + \alpha\right]\left(\frac{\lambda}{t^2\omega}\right)\left(m^2 - t\right) + 2\left(\Omega + \frac{\lambda}{t\omega}\right)\left(mn - \alpha\right)\right.$$

$$\left. + \left(\mathrm{H}\tilde{x} + D + \frac{\xi}{\omega^2} - 2\frac{\alpha\lambda}{\omega^2 t}\right)\left(n^2 - \omega\right) + 2\Omega\left(\tilde{\theta} + 2\frac{\lambda}{t}\right)m + 2\Omega\tilde{x}n\right\} z$$

$$+ \frac{\Delta}{r}\left\{2\left(\Omega + \frac{\lambda}{t\omega}\right)^2\left(m^2 - t\right)\left(n^2 - \omega\right) + 4\left(\Omega + \frac{\lambda}{t\omega}\right)\Omega\tilde{x}m\left(n^2 - \omega\right) + \left[\mathrm{H}\omega + 2\Omega\tilde{\theta}\right.\right.$$

$$+\omega\left(4\Omega^2+16\Omega\left(\frac{\lambda}{t\omega}\right)+6\left(\frac{\lambda}{t\omega}\right)^2\right)\Big]\left(m^2-t\right)+2\Omega\tilde{x}\left(mn-\alpha\right)+\Big[\mathrm{H}t+4\Omega\left(\frac{\lambda}{\omega}+K\tilde{x}+\Omega\tilde{x}^2\right.$$

$$\left.+\Omega t\right)\Big]\left(n^2-\omega\right)+\Big[D\omega+2\Omega\left(\tilde{x}\tilde{\theta}+2\alpha+2K\omega+4\Omega\omega\tilde{x}+2\frac{\lambda}{t}\tilde{x}\right)\Big]m+2\Omega tn\Big\} \quad . \qquad \text{(A-44)}$$

By definition,

$$x-\bar{x} = m+\beta \ ,$$

$$\theta-\bar{\theta} = n+\sigma \ ,$$

$$V = t-\varepsilon \ ,$$

$$\eta = \alpha-s \ ,$$

$$\phi = \omega-u \ ,$$

$$\bar{x} = \tilde{x}-\beta$$

and

$$\bar{\theta} = \tilde{\theta}-\sigma \quad .$$

Making these substitutions in Equation A-30 and using Equations A-36 through A-40 shows that, to order $h^2\Delta$,

$$P_K = \frac{\lambda}{t^2\omega}\Big[n\left(m^2-t\right)-2\frac{\alpha}{\omega}m\left(n^2-\omega\right)-2\alpha m\Big]+\frac{\xi}{t\omega^2}m\left(n^2-t\right)$$

$$+\frac{\rho}{t^2\omega^2}\left(m^2-t\right)\left(n^2-\omega\right)+\frac{1}{2}\left(\frac{\lambda}{t^2\omega}\right)^2n^2\left(m^4-6tm^2+3t^2\right)$$

$$+\frac{z}{r}\Delta\Big\{\Big[2\Omega\left(\bar{\theta}+4\frac{\lambda}{t}\right)+4\frac{\lambda^2}{t^2\omega}\Big]m+2\Omega\tilde{x}n$$

$$+\Big[\alpha+2\omega\left(K+2\Omega\tilde{x}\right)\Big]\left(\frac{\lambda}{t^2\omega}\right)\left(m^2-t\right)+2\left(\Omega+\frac{\lambda}{t\omega}\right)\left(mn-\alpha\right)$$

$$+\left(\mathrm{H}\tilde{x}+D+\frac{\xi}{\omega^2}-2\frac{\alpha\lambda}{t\omega^2}\right)\left(n^2-\omega\right)+\Big[\frac{2\rho}{t\omega^2}+4\frac{\lambda}{t\omega}\left(2\Omega+\frac{\lambda}{t\omega}\right)m\left(n^2-\omega\right)$$

$$+2\Omega\tilde{x}\left(\frac{\lambda}{t^2\omega}\right)\left(m^2-t\right)\left(n^2-\omega\right)+2\left(\frac{\lambda}{t^2\omega}\right)\left(\Omega+\frac{\lambda}{t\omega}\right)n^2\left(m^3-3mt\right)\Big)\Big\}$$

$$+\frac{\Delta}{r}\Big\{\omega\left(2\Omega\tilde{x}\right)^2+4t\omega\left(\Omega+\frac{\lambda}{t\omega}\right)^2-\Big[D\omega+2\Omega\left(\widetilde{x\theta}+2\alpha+2K\omega\right)\Big]m$$

$$-2\Omega tn-\left[H\omega+2\Omega\left(\tilde{\theta}+5\frac{\lambda}{t}\right)+2\frac{\lambda^2}{t^2\omega}\right]\left(m^2-t\right)-2\Omega\tilde{x}\left(mn-\alpha\right)$$

$$+\left[4\left(\Omega\omega t+\lambda\right)\frac{\lambda}{t\omega^2}-4K\Omega\tilde{x}-Ht\right]\left(n^2-\omega\right)-2\left(\Omega+\frac{\lambda}{t\omega}\right)\Big[n\left(m^2-t\right)$$

$$-2\frac{\alpha}{\omega}m\left(n^2-\omega\right)-2\alpha m\Big]-\Big[D+\frac{\xi}{\omega^2}+4\Omega\left(K+\frac{\alpha}{\omega}\right)+2t\left(2K+\frac{\alpha}{\omega}\right.$$

$$\left.+2\Omega\tilde{x}\right)\left(\frac{\lambda}{t^2\omega}\right)\Big]m\left(n^2-\omega\right)-\left(H+\frac{2\rho}{t\omega^2}+6\Omega\frac{\lambda}{t\omega}\right)\left(m^2-t\right)\left(n^2-\omega\right)$$

$$-2\Omega\tilde{x}\left(\frac{\lambda}{t^2\omega}\right)n^2\left(m^3-3mt\right)-2\left(\frac{\lambda}{t^2\omega}\right)\left(\Omega+\frac{\lambda}{t\omega}\right)n^2\left(m^4-6tm^2+3t^2\right)\Big\}\ . \qquad \text{(A-45)}$$

By construction

$$p(x,\theta/z)=N\left(\begin{bmatrix}\tilde{x}\\ \tilde{\theta}\end{bmatrix}\begin{bmatrix}t & \alpha\\ \alpha & \omega\end{bmatrix}\right)\left(1+P_G\right)\left(1+P_K\right) \qquad \text{(A-46)}$$

to order $h^2\Delta$. Substituting from Equations A-44 and A-45 for P_G and P_K and carrying out the indicated polynomial multiplication in Equation A-46 to this accuracy (meaning also that $z^2\Delta$ is replaced by r) results in a density of the form of Equation A-1 for $p(x,\theta/z)$ with parameters

$$\hat{x}^* = \left(M + 2KE + 4\Omega\lambda\right)\frac{z\Delta}{r} - M\left(DL + 2\Omega E\right)\frac{\Delta}{r}$$

$$\hat{\theta}^* = \left(E + 2KL\right)\frac{z\Delta}{r}$$

$$M^* = t = M + 4\left[\Omega ME + \lambda\left(K + \frac{3}{2}\frac{E}{L}\right)\right]\frac{z\Delta}{r}$$

$$- \left[M^2 + 2M\left(\mathrm{H}ML + 2KE + 4\Omega\lambda\right)\right]\frac{\Delta}{r}$$

$$E^* = \alpha = E + 2\left(\Omega ML + \lambda\right)\frac{z}{r}\Delta - M\left(E + 2KL\right)\frac{\Delta}{r}$$

$$L^* = \omega = L + 2\left(\xi + 2\Omega LE + DL^2\right)\frac{z\Delta}{r} - \left(E + 2KL\right)^2\frac{\Delta}{r} \qquad \text{(A-47)}$$

$$\lambda^* = \lambda - 2\left(\lambda M + \Omega M^2 L\right)\frac{\Delta}{r}$$

$$\xi^* = \xi + 2\left(\rho + 2\Omega\lambda L\right)\frac{z}{r}\Delta - \left\{ML\left(DL + 2\Omega E\right) + M\xi\right.$$

$$\left. + 2\left(E + 2KL\right)\left(\lambda + \Omega ML\right)\right\}\frac{\Delta}{r}$$

$$\rho^* = \rho - \left[2M\rho + \mathrm{H}M^2L^2 + 2\left(\lambda + \Omega ML\right)^2 + 4\lambda\Omega ML\right]\frac{\Delta}{r}$$

The composition of the parameter changes for the four transformations and conditioning is just the sum of the incremental changes Equations A-6, A-12, A-22, A-23, and A-47 to order $h^2\Delta$.

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