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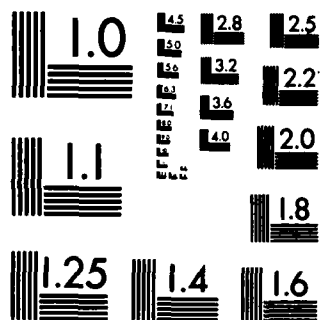
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LOCAL GEOID DETERMINATION IN MOUNTAIN REGIONS

HELMUT MORITZ

DEPARTMENT OF GEODETIC SCIENCE AND SURVEYING
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210

DECEMBER 1983

SCIENTIFIC REPORT NO. 5

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CHRISTOPHER JERELI
Contract Manager


THOMAS P. ROONEY
Chief, Geodesy & Gravity Branch

FOR THE COMMANDER


DONALD H. ECKHARDT
Director
Earth Sciences Division

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determination of height anomalies by integration of vertical deflections along the earth's surface. Extensive consideration is given to the reduction for curvature of the plumb line and to topographic-isostatic reduction of vertical deflections in the classical sense, which is a reduction to sea level, and in the modern sense, where the point remains on the earth's surface. Then the application of collocation to the present problem is discussed. Finally, some results concerning geoid computations for Austria by collocation are outlined.

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FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technical University at Graz and Adjunct Professor, Department of Geodetic Science and Surveying, The Ohio State University, under Air Force Contract No. F19628-82-K-0017, Research Project No. 714255, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Dr. Christopher Jekeli, Contract Manager.

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1. INTRODUCTION

The computation of a detailed geoid, or of a detailed gravity potential field, in limited areas, especially in mountainous regions, has not been very much in the focus of attention recently. There may be various reasons for this.

For two decades now, global geoid determinations, either from satellite data or from a combination of satellite and gravimetric data have been in the center of interest (cf. Lerch, et al., 1979; Reigber, et al., 1983; Rapp, 1981). Even (almost) purely gravimetric global geoids have been successfully computed (cf. March and Chang, 1979).

Over the oceans, the geoid is now known to an accuracy of perhaps ± 1 or ± 2 m, due to satellite altimetry. Unfortunately, satellite altimetry does not work over land areas. The classical method for a detailed geoid determination on the continents is the gravimetric method, in spite of the fact that it is severely handicapped by lack of an adequate gravity coverage (or lack of information on such a coverage). Thus we have the paradoxical situation that on the oceans, long a stepchild of geodesy, the geoid is now in general known much better than on the continents.

Still, the gravimetric method has continued to fascinate theoreticians because it gives rise to very interesting and deep mathematical problems, related to the geodetic boundary-value problem, or problem of Molodensky (cf. Moritz, 1980, Part D). In the recent years, the combination of satellite altimetry on sea and gravimetry on land has led to another interesting boundary-value problem, the altimetry-gravimetry boundary-value problem (cf. Sanso, 1983).

These enormous practical and theoretical developments concerning global satellite and gravimetric gravity field determination have somewhat overshadowed the determination of detailed geoids in smaller areas. Especially in mountainous regions, such local geoid determinations are difficult. The gravimetric method does not work very well in high mountains. The astrogeodetic method, using astronomical observations of

latitude and longitude, does work well there, but is considered time-consuming and somewhat old-fashioned, perhaps also because working during the night is not very popular nowadays. An appropriate use of gravity and astrogeodetic data in high mountains must involve some topographic-isostatic reduction, which is also sometimes considered old-fashioned. Furthermore, the theory behind the astrogeodetic method is not nearly as attractively difficult as the theory of Molodensky's problem. Last but not least, high-mountain areas are exceptional and, apart from such countries as Switzerland and Austria, are frequently regions of little economic interest. For these and similar reasons, the main stream of geodetic practice and theory has flown with grand indifference around high mountains, ignoring such trivial obstacles.

Still, a country such as Switzerland has made a virtue out of necessity and has traditionally been very active in local astrogeodetic geoid determination (Elmiger, 1969; Gurtner, 1978; Gurtner and Elmiger, 1983). Recently, Austria has followed up (Lichtenegger, et al., 1983). It has been found that, even besides the problem of getting the required observations, the underlying theory is not so trivial as one might think and shows quite interesting features.

Concerning measurements, astronomical observations have again proved very feasible in mountains; see the articles by Erker, Bretterbauer, Lichtenegger and Chesi in Chapter 2 of (Lichtenegger, et al., 1983). The main advantages of astrogeodetic versus gravimetric data for local geoid determination in mountain regions may be summarized as follows:

1. It is sufficient to have astrogeodetic deflections of the vertical in the region of geoid determination; no data are needed outside that region as they would be in the gravimetric method.
2. Errors in the topographic height have significantly less influence on deflections than on gravity data. Thus a relatively crude terrain model will be sufficient for the use of astrogeodetic data.

As a matter of fact, the two types of data are not mutually exclusive; an optimal geoid determination will combine astrogeodetic deflections of the vertical, gravity anomalies, and possibly data of other type. A suitable technique for this purpose is least-squares collocation.

From the observational point of view it is interesting to note that inertial surveying techniques will be able to furnish deflections of the vertical and gravity anomalies rapidly and with sufficient accuracy for many purposes.

Let us finally try to give a list of various methods of geoid determination:

- conventional satellite techniques (doppler, laser, etc.)
- satellite-to-satellite tracking
- satellite gradiometry
- satellite altimetry
- aerial gradiometry
- gravimetry
- astrogeodesy

As a general rule, these methods are listed in such a way as to start with the most global and end up with the most local method, that is, according to decreasing globality or increasing locality. In general, going down the list also corresponds to increasing resolution and accuracy.

Again it should be stressed that these methods complement each other and should be combined for best results.

The present report deals primarily with the lower end of the list, providing a detailed theory of local geoid determination in areas with difficult topography. The role (and necessity) of topographic-isostatic reduction is investigated in detail. The computations for Austria give concrete numerical results for questions which have been much discussed theoretically, such as the difference between geoidal heights and height anomalies according to Molodensky (quasigeoidal heights), or the numerical effect of analytical continuation from the earth's surface to sea level.

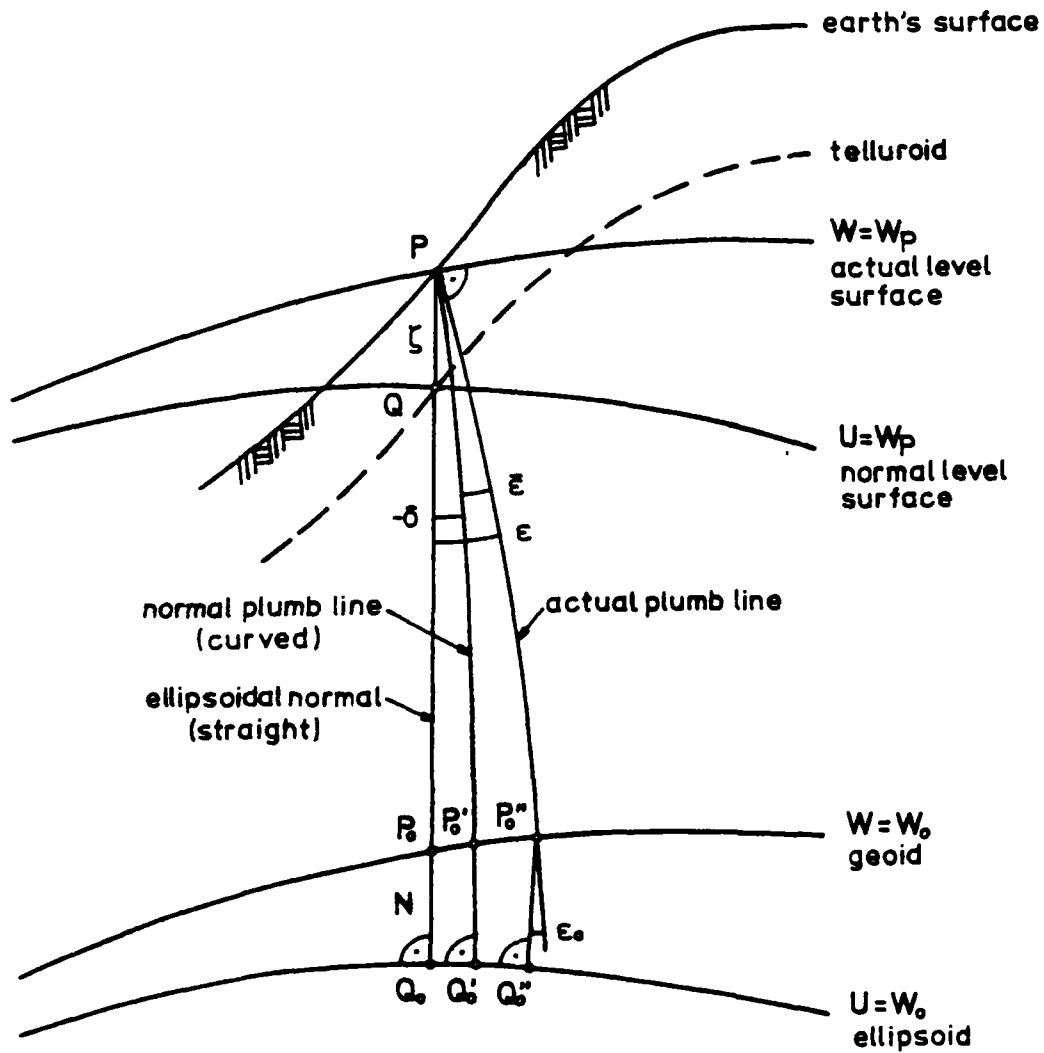
2. Definitions

FIGURE 1. The basic geometry

Fig. 1 illustrates the basic quantities. In the classical theory, the geoid is defined by its deviation N from a reference ellipsoid; N is the geoidal height. The geoid is a level surface $W = W_0 = \text{const.}$ of the gravity potential W ; the ellipsoid is defined to be the level surface $U = U_0 = \text{const.}$ of a normal gravity potential U ; the constants W_0 and U_0 are usually assumed to be equal. Cf. PG, sec. 2-13¹).

For the modern theory according to Molodensky (PG, sec. 8-3), to each point P of the earth's surface we associate a point Q in such a way that Q lies on the straight ellipsoidal normal through P and that

$$U(Q) = W(P) \quad . \quad (1)$$

That is, Q is defined such that its normal potential U equals the actual potential W of P .

This corresponds to the classical relation

$$U_0 = U(Q_0) = W(P_0) = W_0 \quad (2)$$

mentioned above, by which U_0 is taken to be equal to W_0 ; cf. Fig. 1. By the same correspondence, the height anomaly according to Molodensky,

$$\zeta = QP \quad , \quad (3)$$

is the modern equivalent of the classical geoidal height,

$$N = Q_0P_0 \quad . \quad (4)$$

1) By the symbol PG we shall in the sequel denote the book "Physical Geodesy" (Heiskanen and Moritz, 1967).

Using the anomalous potential

$$T = W - U , \quad (5)$$

we have according to Bruns' theorem

$$\zeta = \left(\frac{T}{\gamma} \right)_Q , \quad N = \left(\frac{T}{\gamma} \right)_{Q_0} , \quad (6)$$

γ denoting ellipsoidal normal gravity.

The points P_0 form the geoid, and the points Q_0 constitute the ellipsoid, both being level surfaces (of W and U , respectively). On the other hand, the points P form the earth's surface, and the set of points Q defines an auxiliary surface, denoted as telluroid according to R. A. Hirvonen. As a matter of fact, neither the earth's surface nor the Telluroid are level surfaces, which makes matters more complicated than in the classical situation, where we deal with level surfaces.

Following a suggestion of Molodensky, one could plot the height anomalies ζ as vertical distances from the reference ellipsoid. Thus one obtains a geoid-like surface, the quasi-geoid, and ζ could be considered as quasi-geoidal heights. In contrast to the geoid, however, the quasi-geoid is not a level surface and does not admit of a natural physical interpretation. Therefore, working with height anomalies ζ , it is best to consistently consider them quantities referred to the earth's surface (vertical distances between earth surface and telluroid), rather than using the quasi-geoidal concept.

The classical gravity anomaly Δg_0 at sea level is defined as

$$\Delta g_0 = g(P_0) - \gamma(Q_0) , \quad (7)$$

where g denotes gravity and γ , normal gravity. (So far, $g(P_0)$ denotes the actual gravity on the geoid; we are not yet considering mass-transporting gravity reductions.)

Analogously we have according to Molodensky:

$$\Delta g = g(P) - \gamma(Q) \quad . \quad (8)$$

Generally we shall, as far as feasible, use the subscript "o" to designate quantities referred to sea level, to distinguish them from quantities referred to the earth's surface, which do not carry such a subscript. For instance, Δg_0 refers to sea level and Δg , to the earth's surface.

Regarding plumb-line definition, we must distinguish three lines (Fig. 1):

- (a) The straight ellipsoidal normal $Q_0 P$,
- (b) The actual plumb-line $P'' P$,
- (c) The normal plumb-line $P' P$.

The ellipsoidal normal is geometrically defined as the straight line through P perpendicular to the ellipsoid. The (actual) plumb line is defined by the condition that, at each point of the line, the tangent coincides with the gravity vector $\underline{g}$ at that point; the plumb line is very slightly curved, but its curvature is irregular, being determined by the irregularities of topographic masses. The normal plumb line, at each of its points, is tangent to the normal gravity vector γ ; it possesses a curvature that is even smaller and completely regular.

The points P_0 , P'_0 , and P''_0 coincide within a few decimeters, and we shall not distinguish them in the sequel. The reason is that the distance, in seconds of arc, between P_0 and P''_0 , is much smaller than the effect of plumb line curvature (PG, p. 180-181). The same holds, of course, for Q_0 , Q'_0 , and Q''_0 .

The direction of the gravity vector $\underline{g}$ is the direction of (the tangent to) the plumb line. It is determined by two angles, the astronomical latitude Φ and the astronomical longitude Λ . Let Φ , Λ be referred to the earth surface (to point P) and Φ_0 , Λ_0 to the geoid (strictly speaking, to point P_0''). The differences

$$\delta\Phi = \Phi_0 - \Phi, \quad \delta\Lambda = \Lambda_0 - \Lambda \quad (9)$$

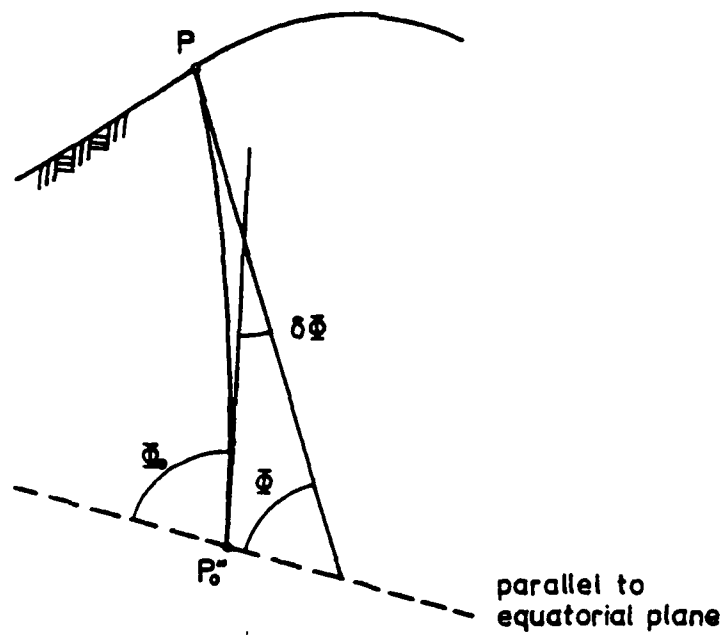


FIGURE 2. Curvature of the plumb line along a north-south profile

express the effect of plumb-line curvature (Fig. 2). Hence we have

$$\phi_0 = \phi + \delta\phi, \quad \Lambda_0 = \Lambda + \delta\Lambda. \quad (10)$$

Knowing the plumb-line curvature ($\delta\phi$, $\delta\Lambda$), we could use these simple formulas to compute the sea-level values ϕ_0 , Λ_0 from the observed surface values ϕ , Λ .

In the same way as ϕ , Λ are related to the actual plumb line, the geodetic latitude ϕ and the geodetic longitude λ refer to the straight ellipsoidal normal. The quantities

$$\xi = \phi - \phi, \quad \eta = (\Lambda - \lambda) \cos\phi \quad (11)$$

are the components of the deflection of the vertical in a north-south and an east-west direction. For an arbitrary azimuth α , the vertical deflection ϵ is given by

$$\epsilon = \xi \cos\alpha + \eta \sin\alpha. \quad (12)$$

These quantities ξ , η , ϵ refer to the earth's surface. Cf. Fig. 1, which shows ϵ .

Similarly we have for the geoid

$$\xi_0 = \phi_0 - \phi, \quad \eta_0 = (\Lambda_0 - \lambda) \cos\phi, \quad (13)$$

$$\epsilon_0 = \xi_0 \cos\alpha + \eta_0 \sin\alpha. \quad (14)$$

See again Fig. 1 for ϵ_0 , noting that we do not distinguish the normals in O_0 and Q'_0 as we have mentioned above.

In addition, we need the normal direction of the plumb line at the surface point P ; it is defined as the tangent to the normal plumb line at P ; the corresponding latitude and longitude will be denoted by $\bar{\phi}$, $\bar{\lambda}$. Hence we have

$$\phi = \bar{\phi} + \delta\phi, \quad \lambda = \bar{\lambda} + \delta\lambda, \quad (15)$$

where $\delta\phi$, $\delta\lambda$ express the normal plumb-line curvature. These equations are the "normal equivalent" to (10): the "normal surface values" $\bar{\phi}$, $\bar{\lambda}$ correspond to the "actual surface values" ϕ , λ , and the ellipsoidal values ϕ , λ correspond to the geoidal values ϕ_0 , λ_0 . (To make the analogy complete, we should replace $\phi = \phi(P_0)$ by $\phi(P'_0)$, but we have consistently been neglecting such differences.)

In marked contrast to the actual plumb-line curvature, it is very easy to compute the normal curvature of the plumb line: by PG, p. 196 we have

$$\delta\phi = -0.17'' h_{km} \sin 2\phi, \quad \delta\lambda = 0, \quad (16)$$

h_{km} denoting elevation in kilometers.

Since the ellipsoidal normal and hence ϕ , λ are geometrically defined, we may call the quantities (11) "geometric deflections of the vertical" at the earth's surface. On the other hand, the normal plumb line is physically (or dynamically) defined by means of the external gravity field of an equipotential ellipsoid. Hence also $\bar{\phi}$, $\bar{\lambda}$ are dynamically defined, and we may call the quantities obtained by replacing ϕ , λ by $\bar{\phi}$, $\bar{\lambda}$:

$$\bar{\xi} = \phi - \bar{\phi}, \quad \bar{\eta} = (\lambda - \bar{\lambda}) \cos \phi, \quad (17)$$

"dynamical deflections of the vertical" at the earth's surface. By (15) and (16) we have

$$\bar{\xi} = \xi + \delta\phi, \quad \bar{\eta} = \eta \quad (18)$$

since $\delta\lambda = 0$. For an azimuth α we accordingly have

$$\bar{\epsilon} = \bar{\xi}\cos\alpha + \bar{\eta}\sin\alpha. \quad (19)$$

Compare ϵ and $\bar{\epsilon}$ in Fig. 1, and note that in this figure, δ denotes the curvature of the normal plumb line for the azimuth α , given by the analogous formula

$$\delta = \delta\phi\cos\alpha + (\delta\lambda\cos\phi)\sin\alpha = \delta\phi\cos\alpha. \quad (20)$$

3. Astronomical Leveling

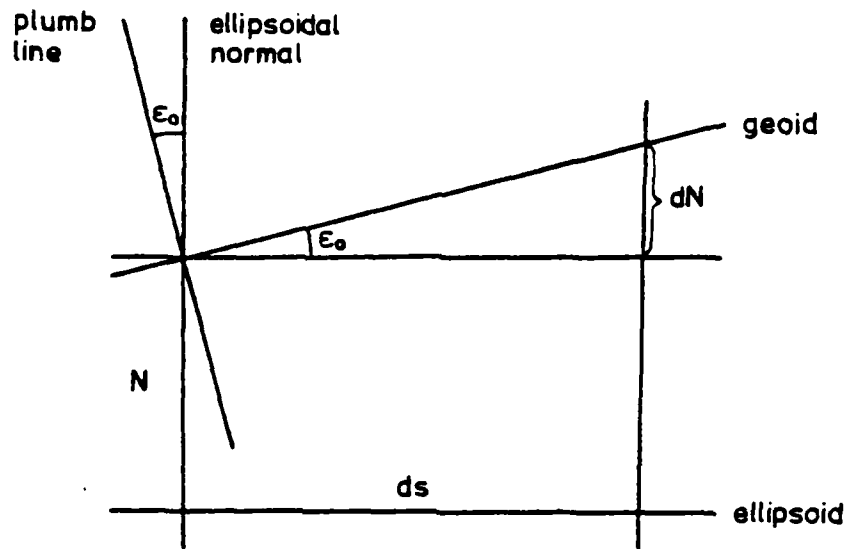


FIGURE 3. Astronomical Leveling according to Helmert

From Fig. 3 we take the well-known differential relation

$$dN = -\epsilon_0 ds \quad , \quad (21)$$

ϵ_0 denoting the deflection of the vertical at the geoid. Integration between two points A and B yields the difference between their geoidal heights:

$$N_B - N_A = - \int_A^B \epsilon_0 ds \quad , \quad (22)$$

or approximately,

$$N_B - N_A = - \frac{\epsilon_{0A} + \epsilon_{0B}}{2} s_{AB} \quad , \quad (23)$$

where s_{AB} denotes the horizontal distance between A and B. The minus sign is conventional.

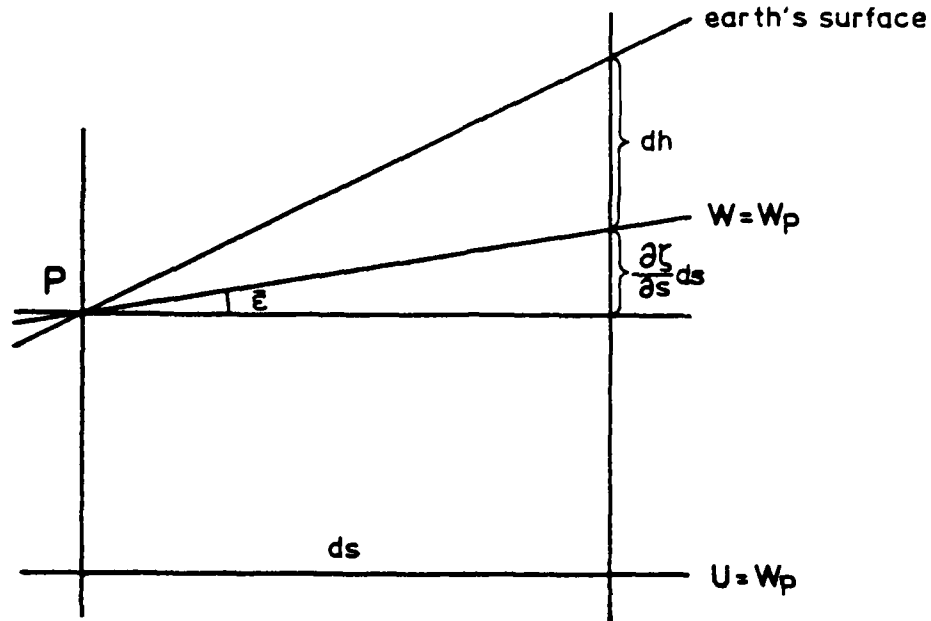


FIGURE 4. Astronomical leveling according to Molodensky

A corresponding relation to height anomalies according to Molodensky is found as follows (Molodensky, et al., 1962, p. 125):

$$d\zeta = \frac{\partial \zeta}{\partial s} ds + \frac{\partial \zeta}{\partial h} dh, \quad (24)$$

notations following Fig. 4. Since the earth's surface is not a level surface, we also have a vertical part ($\partial \zeta / \partial h$) h in addition to the usual horizontal part ($\partial \zeta / \partial s$) ds . The vertical part arises from change in height and is usually smaller than the horizontal part.

In analogy to (21), the horizontal part is given by

$$\frac{\partial \zeta}{\partial s} = -\bar{\epsilon}, \quad (25)$$

$\bar{\epsilon}$ denoting the dynamical deflection of the vertical at the earth's surface; cf. (19) and Fig. 1. For the vertical part we have by (6);

$$\frac{\partial \zeta}{\partial h} = \frac{\partial}{\partial h} \left(\frac{T}{Y} \right) = \frac{1}{Y} \left(\frac{\partial T}{\partial h} - \frac{1}{Y} \frac{\partial Y}{\partial h} T \right) \quad (26)$$

or

$$\frac{\partial \zeta}{\partial h} = - \frac{\Delta g}{Y} = - \frac{g - \gamma}{Y} \quad (27)$$

according to the fundamental equation of physical geodesy (PG, p. 298, eq. 8-20)).

Hence (24) becomes

$$d\zeta = -\bar{\epsilon} ds - \frac{g - \gamma}{Y} dh \quad (28)$$

On integrating this relation we get the difference of the height anomaly ζ :

$$\zeta_B - \zeta_A = - \int_A^B \bar{\epsilon} ds - \int_A^B \frac{\Delta g}{Y} dh \quad ; \quad (29)$$

The gravity anomaly Δg refers to the earth's surface according to (8). The first term on the right-hand side represents the Helmert integral (22) of the surface deflection $\bar{\epsilon}$, and the second term is Molodensky's correction to the Helmert integral, necessary to obtain height anomalies. This correction depends on the gravity g at the earth's surface.

4. Reduction for Plumb-Line Curvature

For Helmert's formula (22), the deflection component ϵ_c refers to the geoid. In fact, by (13) and (14) we have

$$\epsilon_o = (\phi_o - \phi) \cos \alpha + (\Lambda_o - \lambda) \cos \phi \sin \alpha, \quad (30)$$

where the astronomical coordinates ϕ_o , Λ_o are taken at the geoid and given by (10):

$$\phi_o = \phi + \delta\phi, \quad \Lambda_o = \Lambda + \delta\Lambda. \quad (31)$$

Thus the astronomically measured surface values ϕ , Λ must be reduced to sea level by applying corrections $\delta\phi$, $\delta\Lambda$ for plumb-line curvature.

These corrections may be expressed by

$$\delta\phi = \frac{\partial(OC)}{\partial x}, \quad \delta\Lambda \cos \phi = \frac{\partial(OC)}{\partial y}, \quad (32)$$

where OC denotes the orthometric correction in leveling and the local axes x and y are horizontal and directed towards north and east, respectively. Thence follows (PG, p. 195):

$$\delta\phi = -\frac{H}{\bar{g}} \frac{\partial \bar{g}}{\partial x} + \frac{\bar{g} - g}{\bar{g}} \tan \beta_1, \quad \delta\Lambda \cos \phi = -\frac{H}{\bar{g}} \frac{\partial \bar{g}}{\partial y} + \frac{\bar{g} - g}{\bar{g}} \tan \beta_2. \quad (33)$$

Here, H denotes the orthometric height (the length of the curved plumb line segment P_oP in Fig. 1), $\bar{g}$ is the mean value of gravity along the plumb line between P_o and P , and β_1 and β_2 designate the angles of inclination of a terrain profile in a north-south and an east-west direction.

Using these formulas, we thus have first to differentiate the orthometric correction in a horizontal direction (by (32)) and then to integrate by the Helmert formula (22). As a matter of fact, differentiation and subsequent integration should give back the original quantity, but they will in general fail to do so exactly because of inaccuracies inherent in the processes of numerical differentiation and integration.

Thus it is preferable to apply the orthometric correction directly to the geoidal height difference: by PG, pp. 200-201 we have

$$N_B - N_A = - \int_A^B \epsilon ds - OC_{AB} , \quad (34)$$

where OC_{AB} denotes the orthometric correction along the profile AB and is the "geometric deflection of the vertical" at the earth's surface given by (11) and (12); cf. a corresponding remark at the end of Sec. 1.

Eq. (34) thus represents a classical analogue of the Molodensky formula (29). This analogy is particularly conspicuous if we write (34) in differential form, using (32) and (33):

$$dN = -\epsilon ds + \frac{H}{\bar{g}} d\bar{g} - \frac{g-\bar{g}}{\bar{g}} dH . \quad (35)$$

The comparison of this expression with its analogue (28) shows as the main difference the fact that (35) contains mean gravity $\bar{g}$. Now, $\bar{g}$ is the average of all values of gravity along a plumb-line between sea level and earth's surface, and gravity inside the earth cannot be measured, nor can it be computed rigorously because the rock density ρ inside the earth is unknown. Thus $\bar{g}$ cannot be determined with complete rigor.

This is the point where Molodensky's criticism of the classical theories of physical geodesy enters. This criticism is fully justified from a conceptual point of view and has been extremely fruitful for the development of modern theoretical geodesy. On the other hand, from a practical point of view, we may say that, with a reasonably realistic

density model, mean gravity $\bar{g}$ and hence orthometric heights H and geoidal heights N can be computed quite well with satisfactory accuracy; cf. the simple estimates in PG, p. 169.

5. Approximate Reduction for Plumb-Line Curvature

A sufficiently precise determination of plumb-line curvature according to (33) or of the orthometric correction in (34) is rather cumbersome, however.

An approximate procedure (Elmiger, 1969; Gurtner, 1978) uses an analogy to classical gravity reduction, applied to the direction of the gravity vector: gravity reduction is applied to the magnitude g of the gravity vector $\underline{g}$.

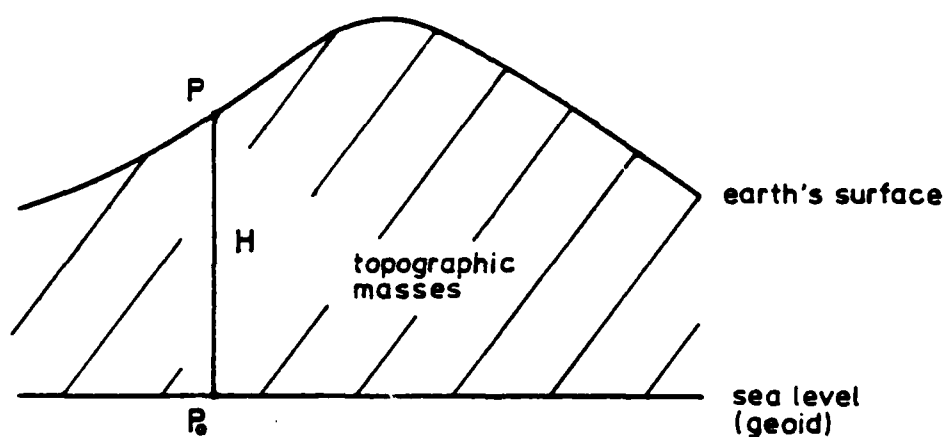


FIGURE 5. The geometry in gravity reduction

Consider first the gravity reduction of Poincare-Prey (PG, p. 165): cf. Fig. 5. It consists of the following three steps:

1. The topographic masses, i.e., the masses above the geoid, are removed computationally by subtracting its attraction A_T from the observed gravity value g at the surface point P .

2. The reduced gravity value $g - A_T$ so obtained at P is transferred to point P_0 at sea level by adding the free-air reduction F .

3. The topographic masses are restored by adding its attraction A_T^0 at P_0 .

The result of these three steps,

$$g_0 = g - A_T + F + A_T^0 \quad (36)$$

gives actual gravity g_0 on the geoid. A weak point of the procedure (apart from errors in A_T and A_T^0 due to imperfect knowledge of density ρ) is the computation of the free-air reduction¹ F by the formula

$$F = - \frac{\partial \gamma}{\partial h} H, \quad (37)$$

replacing the vertical gravity gradient of the actual gravity field (after removal of the topographic masses) by the normal gravity gradient $\partial \gamma / \partial h$.

The astronomical coordinates ϕ and λ (defining the direction of the gravity vector $\underline{g}$) can be treated in complete analogy. Let ξ_T be the ξ component of the deflection of the vertical at P as computed from the topographic masses only. Let further be ξ_T^0 the corresponding topographic deflection of the vertical at P_0 . In complete analogy to the three steps mentioned above we have now:

1) So called because after the removal of the topographic masses, the point P lies "in free air".

measured at P	ϕ
1. removal of topography, at P	ξ_T
2. "free-air reduction" $P \rightarrow P_0$ = normal plumb-line curvature	$\delta\phi$
3. restoration of topography, at P_0	ξ_T^0

The result of these three steps,

$$\phi_0 = \phi - \xi_T + \delta\phi + \xi_T^0, \quad (38)$$

thus gives the astronomical latitude ϕ_0 at the geoid. This formula is completely analogous to (36).

The comparison of (38) with (10) yields, for the actual plumb-line curvature $\delta\phi$, the approximate expression

$$\delta\phi = -\xi_T + \xi_T^0 + \delta\phi, \quad (39)$$

where $\delta\phi$ denotes the normal plumb-line curvature (16). By (10), (11), and (13) we thus get for the vertical deflection at the geoid:

$$\xi_0 = \phi_0 - \phi = \phi - \phi + \delta\phi = \xi + \delta\phi, \quad (40)$$

with an analogous equation for the component η_0 .

Within the accuracy of this procedure we thus obtain actual vertical deflections on the geoid, in the same way as the Poincare-Prey reduction gives actual gravity on the geoid. Hence we may apply to ξ_0 , η_0 the Helmert formula (22) to get differences of the geoidal height N .

If we have a reasonably realistic density model for the topographic masses, we can compute ξ_T , η_T and ξ_T^0 , η_T^0 with an accuracy which might be satisfactory for many purposes. The weak point of the

procedure, in the same way as for gravity reduction, is the computation of the "free-air reduction" of the plumb line by applying the normal plumb-line curvature.

Physically this implies the assumption that, after the removal of the topographic masses, the earth's crust is "regularized" to such an extent that its external gravity field will then be approximately equal to the normal ellipsoidal gravity field. This assumption would mean that the co-geoid of the Bouguer reduction (PG, sec. 3-3) coincides with the reference ellipsoid. In fact, however, these co-geoid heights are, by an order of magnitude, larger than the actual geoid heights (Helmert, 1884, pp. 354-57; PG, sec. 3-6). It is true that these large cogenital heights have a smooth and regional behavior and thus affect the local plumb-line curvature to a lesser degree, but the assumption of a normal plumb-line curvature is nevertheless rather questionable.

Topographic-isostatic reduction. A considerably better "regularization" of the earth's crust may be expected from the application of a topographic-isostatic reduction. The large regional features of the co-geoid are essentially reduced in this way. The important step is to take some isostatic model, and it is not so essential which isostatic model is taken; a conventional Airy-Heiskanen model with $T = 30\text{km}$ (PG, Sec. 3-5) may be satisfactory in many cases.

Physically, a topographic-isostatic reduction means that the topographic masses (above the geoid) are computationally removed and used to fill the mass deficiencies which exist below sea level according to the theory of isostasy. If the isostatic compensation were exact, the earth's crust would become completely homogeneous after such an isostatic reduction, so that the irregularities of the geoid would disappear after reduction: the isostatic co-geoid would, theoretically, coincide with the reference ellipsoid. In reality, of course, no isostatic model exactly corresponds to the actual geological situation, so that, rather than coinciding with the ellipsoid, the isostatic co-geoid will deviate less and more smoothly from the ellipsoid than the geoid does. Thus the topographic-isostatic reduction achieves a considerable smoothing of geoidal heights and, especially, of deflections of the vertical; cf. Sec. 10.

For the present purpose it is of particular interest that also the curvature of the plumb-line will be smoothed by the topographic-isostatic reduction. In this way it will certainly become closer to normal plumb-line curvature than with a mere removal of the topographic masses.

Therefore it is appropriate to modify the reduction procedure given at the beginning of the present section, in such a way that in Step 1 the topography is not merely removed completely but that it is transported into the interior of the geoid in order to make up the isostatic mass deficiencies. This procedure has to be reversed in Step 3 in order to restore the original topography. By this we achieve that the free-air reduction by means of the normal plumb-line curvature (Step 2) appears to be better justified.

For our computation this means that the vertical deflections ξ_T , η_T , computed from the topographic masses, are to be replaced by deflections ξ_{TI} , η_{TI} representing the combined effect of the topographic masses and their isostatic compensation. The numerical computation is done by the same formulas as for ξ_T , η_T by dividing the topographic and compensating masses into vertical prisms, if possible using a digital terrain model¹. The underlying isostatic model should be reasonably realistic and, above all, simple and well defined, such as the Airy-Heiskanen model.

Then eq. (39) is to be replaced by

$$\delta\phi = -\xi_{TI} + \xi_{TI}^0 + \delta\phi, \quad (41)$$

1) Relevant formulas can be found in (Forsberg and Tscherning, 1981) with references to (MacMillan, 1958, pp. 78-79) and (Jung, 1961, p. 167).

where ξ_{TI} and ξ_{TI}^0 express the deflection effect of the topographic and compensating masses at points P and P_0 . By (40) we then have

$$\xi_0 = \xi - \xi_{TI} + \xi_{TI}^0 + \delta\phi, \quad \eta_0 = \eta - \eta_{TI} + \eta_{TI}^0, \quad (42)$$

as the normal curvature component $\delta\lambda$ is zero.

Let us repeat once more that ξ , η denote the "geometrical" surface deflections (11) and that ξ_0 , η_0 denote the deflections of the vertical on the geoid (and not on the isostatic co-geoid!). Eq. (22) will then give differences of geoidal heights N .

It should be noted, however, that even (41) is only an approximate expression for the actual plumb-line curvature, an expression which is better than (39) but nevertheless affected by a certain error.

From a conceptual point of view let us emphasize once more that the geoidal values ξ_0 , η_0 as computed from (42) correspond to actual gravity anomalies $g_0 - \gamma$ at the geoid, where g_0 denotes actual gravity at the geoid (inside the masses) as provided by the reduction of Poincaré and Prey (PG, p. 146). Topographic-isostatic reduction has only been an auxiliary device to compute ξ_0 , η_0 in a better way. In the next section we shall, however, use isostatic reduction in a way that is analogous to classical geoid determination by gravity reduction.

6. Determination of the Co-geoid

The application of topographic-isostatic reduction, described in the preceding section, admits of an interesting alternative variant. In the preceding section, the isostatic co-geoid was mentioned to provide a conceptual background, but it was not used explicitly since ξ_0 , η_0 were applied to determine the geoid directly. The alternative approach to be described now determines first the isostatic co-geoid, from which the geoid is obtained by taking into account the indirect effect. This approach fully corresponds to classical gravimetric geoid determination by isostatic gravity reduction (PG, Secs. 3-6 and 8-2).

In the three-step approach described in the preceding section we will only perform Step 1 -- removal of topography plus isostatic compensation in their effect at P -- and Step 2 -- free-air reduction from P to P_0 by applying the normal plumb-line curvature $\delta\phi$. The third step -- restitution of topography -- will be omitted. The result is thus

$$\xi_0^C = \xi - \xi_{TI} + \delta\phi, \quad \eta_0^C = \eta - \eta_{TI}; \quad (43)$$

it gives boundary values since after the removal of topographic masses, sea level represents a boundary of the solid earth. More exactly, this boundary surface is the isostatic co-geoid. The deflections ξ_0^C , η_0^C as given by (43) are the precise equivalent of isostatic gravity anomalies $\Delta g_0^C = \Delta g_I$ (PG, p. 140) and likewise refer to the co-geoid.

Then it is possible to apply the Helmert formula (22) to ξ_0^C , η_0^C to obtain differences of co-geoid heights N^C :

$$N_B^C - N_A^C = - \int_A^B \xi_0^C ds, \quad (44)$$

with

$$\epsilon_0^C = \xi_0^C \cos \alpha + \eta_0^C \sin \alpha \quad . \quad (45)$$

From the co-geoid obtained in this way we get the actual geoid by applying the indirect effect; cf. PG, p. 141. Thus, the geoidal height N follows from

$$N = N^C + \delta N \quad . \quad (46)$$

For the indirect effect δN , an application of the Bruns formula (6) gives

$$\delta N = \frac{T_{TI}^0}{\gamma} \quad , \quad (47)$$

where T_{TI}^0 denotes the potential of the topographic masses together with their isostatic compensation, taken at the geoidal point P_0 . This potential is computed similarly as the effect of the topographic-isostatic masses on the gravity anomaly and on the deflection of the vertical.

If we have applied this procedure correctly, we should theoretically get the same result for the geoidal height N as if we had used the Helmert integration of the vertical deflections ξ_0 , η_0 given by (42). Thus the indirect effect δN must satisfy the relation

$$\delta N_B - \delta N_A = - \int_A^B (\xi_{TI}^0 \cos \alpha + \eta_{TI}^0 \sin \alpha) ds \quad . \quad (48)$$

For the present method, this result is of little use since δN is computed directly from (47); perhaps it could be used for checking purposes.

An advantage of the present procedure is the fact that the isostatically reduced vertical deflections ξ_0^C , η_0^C , being much smaller and smoother, can be interpolated better than the surface values ξ , η and also better than the geoidal values ξ_0 , η_0 . Being only an essentially equivalent modification of the procedure of Sec. 5, it shares, however, its main drawback: the application of the normal plumb-line curvature reduction $\delta\phi$ is problematical.

7. Topographic-Isostatic Reduction from a Modern Point of View

For the reasons mentioned at the end of the preceding section, it is natural to try and find a way which makes use of the clear advantages of the topographic-isostatic reduction but avoids the problems inherent in a free-air reduction from P to P_0 .

For this purpose let us once more consider eqs. (43) but interpret them differently. We shall put

$$\xi^C = \xi - \xi_{TI} + \delta\phi, \quad \eta^C = \eta - \eta_{TI}. \quad (49)$$

By means of (18) this may be written

$$\xi^C = \bar{\xi} - \xi_{TI}, \quad \eta^C = \bar{\eta} - \eta_{TI}. \quad (50)$$

The interpretation of (50), however, is clear, simple, and rigorous: from the dynamic deflections of the vertical at P , which are the very quantities $\bar{\xi}$ and $\bar{\eta}$, we subtract the effect of the topographic-isostatic masses, ξ_{TI} and η_{TI} , likewise at P . The vertical deflections so obtained, ξ^C and η^C , thus do not really refer to the (co-)geoid; in reality, they refer to the earth's surface!

But what, then, means the normal plumb-line curvature $\delta\phi$ in (49)? Does it not mean a reduction from the earth's surface to sea level? No, in eqs. (18) it only denotes the transformation between the geometrical and the dynamical deflection of the vertical, both referred to the point P of the earth's surface. This is also clear from Fig. 1, which illustrates the formula

$$\bar{\epsilon} = \epsilon + \delta, \quad (51)$$

extending (18) to an arbitrary azimuth, δ being defined by (20).

This interpretation of (49) or (50) as isostatically reduced deflections of the vertical at the earth's surface is exact, whereas the interpretation of (43) as deflections at the co-geoid was only approximate. Therefore we now have written ξ^C , η^C instead of our former notation ξ_0^C , η_0^C . This is the desired rigorous interpretation of our isostatically reduced vertical deflections.

This interpretation exactly corresponds to the modern view of gravity reduction according to the theory of Molodensky as described in PG, sec. 8-11. According to this view, the isostatically (or in some other way) reduced gravity anomalies continue to refer to the earth's surface. The classical gravity reduction (PG, sec. 8-2) had comprised two procedures: mass transport and shift $P \rightarrow P_0$; the new view of gravity reduction only considers the mass transport; the problematic shift $P \rightarrow P_0$ is avoided.

Formally, a "normal free-air reduction":

$$F = - \frac{\partial \gamma}{\partial h} h \quad (52)$$

may be said to occur also in Molodensky's theory: normal gravity γ in the new definition (8) of the gravity anomaly, where it refers to the telluroid point Q , is computed by

$$\gamma = \gamma_0 + \frac{\partial \gamma}{\partial h} h, \quad (53)$$

with $h = Q_0Q$ denoting the normal height of P . But instead of reducing actual gravity g downward, from P to P_0 , now normal gravity is reduced upward. Whereas for the first process the use of the normal gradient $\partial \gamma / \partial h$ is problematic, it is fully justified for the second process.

In a similar way, we might, if we wish, interpret $\delta \phi$ as a reduction of ϕ for normal curvature of the plumb-line upwards, say, from P_0 to P . This is possible because in (15), ϕ could be said to refer to P'_0 (because P_0 and P'_0 practically coincide), and because $\bar{\phi}$ denotes the latitude of the tangent to the normal plumb line at P . This interpretation is instructive because of the analogy with gravity reduction, though regarding ϕ and $\bar{\phi}$ as geometric and dynamic latitude of the same point P appears more natural.

As pointed out above, the present interpretation of ξ^C and η^C as isostatically reduced deflections of the vertical at the earth's surface is conceptually rigorous and therefore also practically more accurate, but this decisive advantage implies a computational drawback if integration along a profile is used; since this integration must now be performed along the earth's surface and not along a level surface such as the geoid, computation will be more complicated. Instead of the simple Helmert formula (22) we now must use the Molodensky formula (29):

$$\zeta_B^C - \zeta_A^C = - \int_A^B \epsilon^C ds - \int_A^B \frac{g^C - \gamma}{\gamma} dh \quad (54)$$

with

$$\epsilon^C = \xi^C \cos \alpha + \eta^C \sin \alpha, \quad (55)$$

g^C being the isostatically reduced surface value of gravity (measured value g minus attraction of the topographic-isostatic masses).

From the isostatic height anomalies ξ^C obtained in this way, we then get the actual height anomalies ζ by applying the indirect effect:

$$\zeta = \xi^C + \delta\zeta \quad (56)$$

with

$$\delta\zeta = \frac{T_{TI}}{\gamma} \quad (57)$$

This is completely analogous to (46) and (47), but now T_{TI} is the potential of the topographic-isostatic masses at the surface point P . As a matter of fact, normal gravity in (47) refers to the ellipsoid, and in (57), to the telluroid, but the difference is generally small.

For higher mountains, the isostatic reduction procedure described in the present section is preferable in practice to a direct application of Molodensky's formula (29) because the isostatically reduced vertical deflections are much smoother and easier to interpolate. It is, however, extremely laborious from a computational point of view since the integration must be performed along the earth's surface (or, what is practically the same, along the telluroid).

The procedure described in Sec. 6 is easier and less laborious since the integration is performed along a level surface. It is, however, not rigorous theoretically and less accurate practically. Now we can also understand the error inherent in the procedure of Sec. 6: the isostatically reduced deflections ξ_0^C , η_0^C at sea level are by (43) simply put equal to the corresponding surface deflections (49). In reality, however, ξ^C , η^C and ξ_0^C , η_0^C are related by analytical continuation and by no

means identical. For the concept of analytical continuation cf. (PG, sec. 8-10 and p. 324); for the magnitude of the effect of analytical continuation on ζ see Sec. 10.

Finally, we remark that the computational drawback of the present method, the Molodensky integration along the earth's surface, can be completely avoided if we perform our computations in space: instead of integrating along a surface, we perform collocation in space. This modern procedure, to be described in the next section, permits a simple and computationally convenient use of surface deflections and also their combination with gravimetric and other data.

8. Least-Squares Collocation

The principle of collocation is very simple. The anomalous potential T outside the earth is a harmonic function, that is, it satisfies Laplace's differential equation

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 . \quad (58)$$

An approximate analytical representation of the external potential T is obtained by

$$T(P) \cong f(P) = \sum_{k=1}^q b_k \phi_k(P) , \quad (59)$$

that is, by a linear combination f of suitable base functions ϕ_1 , ϕ_2 , . . . , ϕ_q with appropriate coefficients b_k . All these are functions of the space point P under consideration.

As T is harmonic outside the earth's surface, it is natural to choose base functions ϕ_k which are likewise harmonic, so that

$$\Delta \phi_k = 0 , \quad (60)$$

in correspondence to (58).

There are many simple systems of functions satisfying the harmonicity condition (60), and thus we have many possibilities for a suitable choice of base functions ϕ_k . We might, for instance, choose spherical harmonics or potentials of suitably distributed point masses, depending on whether we emphasize global or local applications.

The coefficients b_k may be chosen such that the given observational values are reproduced exactly, for instance, all deflections of the vertical in a given area. This means that the assumed approximating function f in (59) gives the same deflections of the vertical at the observation stations as the actual potential, and hence may well be considered a suitable approximation for T .

Let us now try and put these ideas into a mathematical form.

Interpolation. Let errorless values of T be given at q spatial points $P_1, P_2, \dots, P_q$; these points may lie on the earth's surface or in space above the earth's surface. We put

$$T(P_i) = f_i, \quad i = 1, 2, \dots, q, \quad (61)$$

and postulate that in approximating $T(P)$ by $f(P)$, the observations (61) will be reproduced exactly. The condition for this is

$$\sum_{k=1}^q b_k \phi_k(P_i) = T(P_i) = f_i, \quad (62)$$

whence the system of linear equations,

$$\sum_{k=1}^q A_{ik} b_k = f_i \text{ with } A_{ik} = \phi_k(P_i) \quad (63)$$

or in matrix notation,

$$\underline{A} \underline{b} = \underline{f}. \quad (64)$$

If the square matrix $\underline{A}$ is regular, then the coefficients b_k are uniquely determined by

$$\underline{b} = \underline{A}^{-1} \underline{f} . \quad (65)$$

This model is suitable, for instance, for a determination of the geoid by satellite altimetry, since this method, rather directly, yields geoidal heights N_i and hence, by Bruns' theorem (6), $T(P_i) = \gamma_i N_i$. For the astrogeodetic geoid determination we must generalize this model, which leads us to

Collocation. Here we wish to reproduce, by means of the approximation (59), q measured values which again are assumed to be errorless (this assumption is not essential and will be dropped later). These measured values are assumed to be so-called linear functionals $L_1 T$, $L_2 T$, . . . , $L_q T$ of the anomalous potential T .

In fact, deflections of the vertical,

$$\xi = - \frac{1}{\gamma} \frac{\partial T}{\partial x} , \quad \eta = - \frac{1}{\gamma} \frac{\partial T}{\partial y} , \quad (66)$$

but also gravity anomalies,

$$\Delta g = - \frac{\partial T}{\partial z} - \frac{2}{R} T , \quad (67)$$

are such linear functionals; here, xyz denotes a local coordinate system in which the z -axis is vertical upwards and the x and y axes are directed towards north and east, and $R = 6371\text{km}$ is a mean radius of the earth. Eq. (66) is a consequence of (25), with $\partial s = \partial x$ or ∂y ; normal gravity γ may be considered constant with respect to horizontal derivation. Eq. (67) is the well-known fundamental equation of physical geodesy in spherical approximation (PG, p. 298 and 88); the equations (66) and (67) refer to the earth's surface.

By saying that deflections of the vertical and gravity anomalies are linear functionals of T , we simply indicate the fact that $\bar{\xi}$, $\bar{\eta}$,

Δg depend on T by the expressions (66) and (67) which clearly are linear; they are, of course, the linear terms of a Taylor expansion, neglecting quadratic and higher terms. In the above notation $L_i T$, the symbol L_i denotes, for instance, the operation

$$L_i = \frac{1}{\gamma} \frac{\partial}{\partial x} \quad (68)$$

applied to T at some point.

Putting

$$L_i f = L_i T = \lambda_i \quad (69)$$

and substituting (59) we get

$$\sum_{i=1}^q B_{ik} b_k = \lambda_i \text{ with } B_{ik} = L_i \phi_k. \quad (70)$$

$L_i \phi_k$ denotes the number obtained by applying the operation L_i to the base function ϕ_k ; the coefficient B_{ik} obtained in this way does not depend on the measured values. Eq. (70) is a linear system of q equations for q unknowns, which is quite similar to (63). This method of fitting an analytical approximating function to a number of given linear functionals is called collocation and is frequently used in numerical mathematics.

It is clear that interpolation is a simple special case of collocation, in which

$$L_i f = f(P_i) \quad (71)$$

is the "evaluation functional", giving the value of f at a point P_i . Thus we see that in both interpolation and collocation, the coefficients b_k require the solution of a linear system of equations (which in general will not be symmetric).

Least-squares interpolation. Let us consider a function

$$K = K(P, Q) \quad , \quad (72)$$

in which two points P and Q are the independent variables. Let this function K be

- symmetric with respect to P and Q ,
- harmonic with respect to both points, everywhere outside a certain sphere, and
- positive-definite

(the positive definiteness of a function is defined similarly as in the case of a matrix). Then the function $K(P, Q)$ is called a (harmonic) kernel function; cf, (Moritz, 1980, p. 205). A kernel function $K(P, Q)$ may serve as "building material" from which we can construct base functions. Taking for the base functions the form

$$\phi_k(P) = K(P, P_k) \quad , \quad (73)$$

where P denotes the variable point and P_k is a fixed point in space, we obtain least-squares interpolation.

This name originates from the statistical interpretation of the kernel function as a covariance function; then least-squares interpolation has some minimum properties (least variance, similarly as in least-squares adjustment). This interpretation is not essential, however; one may also work with arbitrary analytical kernel functions, considering the procedure as a purely analytical mathematical approximation technique. Normally one tries to combine both aspects in a reasonable way.

Substituting (73) into (63) we get

$$A_{ik} = K(P_i, P_k) = C_{ik} ; \quad (74)$$

this square matrix now is symmetric (in the general case, A_{ik} is not symmetric!) and positive definite (because of the corresponding properties of the function $K(P, Q)$). Then the coefficients b_k follow from (65) and may be substituted into (59). With the notation

$$\phi_k(P) = K(P, P_k) = C_{Pk} , \quad (75)$$

the result may be written in the form

$$f(P) = [C_{P1} \ C_{P2} \ \dots \ C_{Pq}] \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1q} \\ C_{21} & C_{22} & \dots & C_{2q} \\ \vdots & \vdots & & \vdots \\ C_{q1} & C_{q2} & \dots & C_{qq} \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \end{bmatrix} , \quad (76)$$

known from least-squares interpolation of gravity (PG, p. 268).

Least-squares collocation. Here we again derive the base functions from a kernel function $K(P, Q)$, but in a way slightly different from (73): we put

$$\phi_k(P) = L_k^Q K(P, Q) , \quad (77)$$

where L_k^Q means that the functional L_k is applied to the variable Q ; the result no longer depends on Q (since the application of a functional results in a definite number). Thus we must in (70) put

$$B_{ik} = L_i^P L_k^Q K(P, Q) = C_{ik} , \quad (78)$$

which gives a matrix which again is symmetric. Solving (70) for b_k and substituting into (59) gives with

$$\phi_k(P) = L_k^Q K(P, Q) = C_{Pk} \quad (79)$$

the formula

$$f(P) = [C_{P1} \ C_{P2} \ \dots \ C_{Pq}] \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1q} \\ C_{21} & C_{22} & \dots & C_{2q} \\ \vdots & & & \vdots \\ C_{q1} & C_{q2} & \dots & C_{qq} \end{bmatrix}^{-1} \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_q \end{bmatrix} . \quad (80)$$

This is formally the same expression as (76), but with f_i replaced by ℓ_i and with "covariances" C_{ik} and C_{pi} defined by (78) and (79).

In the statistical interpretation, $f(P)$ is an optimal estimate (in the sense of least variance) for the anomalous potential T and hence for the height anomaly $\zeta = T/\gamma$ on the basis of arbitrary measuring data. For geoid determination in mountain areas, relevant measuring data primarily are ξ , η , and Δg . The covariances C_{ik} and C_{pi} are given by known analytical expressions (Moritz, 1980, sec. 15). A general computer program for collocation is described in (Sunkel, 1980).

Least-squares collocation may easily be generalized to observational data affected by random errors; systematic effects may also be taken into consideration. In addition to the estimated quantities (f in our present case) we may also compute their standard error by a formula similar to (80). A comprehensive presentation of a least-squares collocation may be found in (Moritz, 1980).

9. Application of Collocation to Geoid Determination

It is well known that the direct interpolation of free-air gravity anomalies (which essentially are surface gravity anomalies (8)) in high mountains, e.g. by least-squares interpolation, leads to relatively poor results, because of the correlation of the free-air anomalies with elevation (PG, sec. 7-10). This correlation with elevation constitutes a considerable trend, which must be removed before the interpolation. Bouguer anomalies take care of the dependence on the local irregularities of elevation; isostatic anomalies are, in addition, also largely independent on the regional features of topography (PG, p. 285).

In exactly the same way we must remove the main trend of the vertical deflections ξ , η and the gravity anomalies Δg by an isostatic reduction, before applying collocation. Thus isostatic reduction, pragmatically regarded as trend removal, is essential for the practical application of least-squares collocation in mountainous regions (Forsberg and Tscherning, 1981).

Physically speaking, we transport the topographic masses to the interior of the geoid in such a way that the isostatic mass deficiencies are filled. The observation point P remains in its position on the earth's surface. In this way, not only the harmonic character of the anomalous potential T outside the earth's surface is preserved, but, in addition, the computational removal of the topographic masses above sea level makes the function T harmonic down to sea level. Hence, the collocation formula (80) can be applied also at sea level, giving co-geoid heights

N^C . By applying the inverse reduction (the indirect effect) to the computed height anomalies ζ^C and co-geoid heights N^C we get actual ζ and N . It can be expected that errors in the isostatic model used (e.g., an Airy-Heiskanen model) will largely cancel in this combined procedure of reduction and "anti-reduction."

The procedure is theoretically optimal and well suited for computer use. The integrability conditions, which in Helmert integration are

represented by the closures of the individual triangles, are automatically taken into account. The fact that the deflections of the vertical are given only in a certain region has the effect that the geoid can only be computed in that region. Since even by collocation, differences in geoidal height between two neighboring stations A and B depend essentially only on the deflections in those two stations, the lack of data outside the region under consideration will hardly cause a noticeable distortion. Note, however, that the addition of a constant to all geoidal heights N will not affect the deflections of the vertical; hence, astrogeodetic data determine the geoidal heights only up to an additive constant. This constant may be chosen such that the average value of the computed N is zero, and the result of collocation comes near to this case.

To get immediately nearly geocentric geoidal heights it is appropriate to take into consideration a global trend which mainly affects ζ and N itself, by subtracting the effect of a suitable global gravity field, say the gravity earth model (given as a spherical-harmonic expansion up to degree $180^\circ \times 180^\circ$) of Rapp (1981), following Sunkel (1983). This will be described in the next section; in the present section we shall limit ourselves to the isostatic reduction.

Computational procedure. It consists of the following steps:

1. Transformation of the astrogeodetic surface deflections ξ , η from the local datum used for the geocentric Geodetic Reference System 1980 by the well-known differential formulas of Vening Meinesz (PG, sec. 5-9). This is necessary since collocation requires a reference system which is as realistic as possible.
2. Application of the normal plumb line curvature (16) to the "geometric" surface deflections ξ , η gives the "dynamic" surface deflections $\bar{\xi}$, $\bar{\eta}$ by (15).
3. Computation of the gravity anomalies Δg , also referred to the earth's surface according to (8).
4. The topographic-isostatic reduction of $\bar{\xi}$, $\bar{\eta}$, Δg by (50) and PG, eq. (8-94) gives values ξ^C , η^C , Δg^C which, of course, continue to refer to the surface point P .

5. The application of collocation to ξ^c , η^c , Δg gives height anomalies ζ^c and co-geoid height N^c , by simply varying the elevation parameter (H and zero, respectively) in the collocation program.

6. By applying the indirect effect (57) and (47) we get actual height anomalies ζ and geoidal heights N .

10. Geoid Computation for Austria

Sunkel (1983) has used least-squares collocation to calculate the geoid for the main part of Austria. In addition to the isostatic reduction according to Airy-Heiskanen ($T = 30\text{km}$), he also removes a global trend by means of a earth gravity model, represented by a spherical-harmonic expansion up to a certain degree N . In particular, he used the model of Rapp (1981) with $N = 180$.

After removing the topographic-isostatic trend T_{TI} and this global trend T_{EM}^N (EM denotes earth model), there remains a residual anomalous potential δT , given by

$$\delta T = T - T_{TI} - T_{EM}^N + T_{TI}^N . \quad (81)$$

Since the earth model potential T_{EM}^N is represented by a spherical-harmonic expansion up to degree N , it may be appropriate to consider, for the isostatic reduction, only the effect for degrees $N > 180^\circ$, replacing T_{TI} by

$$(T_{TI})_{N>180^\circ} = T_{TI} - T_{TI}^N , \quad (82)$$

where T_{TI}^N represents a spherical-harmonic expansion for T_{TI} truncated at degree $N = 180$. This explains eq. (81).

The observations $\ell_i = (\bar{\xi} , \bar{\eta} , \Delta g)$, which represent linear functionals $L_i T$, are reduced in the same way, obtaining

$$\ell_i - L_i T_{TI} - L_i T_{EM}^N + L_i T_{TI}^N = L_i \delta T . \quad (83)$$

Adding the earth model reduction to the computational procedure outlined at the end of the preceding section, we thus have the following flow diagram:

 $(L_i T)$


reduction
TI, EM

observations referred to
Geodetic Reference System 1980

$$-L_i (T_{TI} + T_{EM}^N - T_{TI}^N)$$


 $(L_i \delta T)$


collocation


 δT


inverse reduction
TI, EM

$$+ T_{TI} + T_{EM}^N - T_{TI}^N$$


 T


$$N = (T/\gamma)_o, \quad \zeta = (T/\gamma)_h$$

Data. The topography in Austria is rather varied, with elevations up to 3800m . The density of astrogeodetic stations was 10 to 20 km ; the total number of deflections data used was 521 . No gravity anomalies were used in this first computation.

The topographic-isostatic reduction of the deflections of the vertical was made using a rather crude digital terrain model consisting of mean elevations for 20" x 20" rectangles. It has been obtained by digitizing a map 1:500000 . The standard error of this model is on the order of

$\pm 100\text{m}$. Investigations have shown that, in spite of its poor accuracy, the model is reasonably adequate for reduction of deflections of the vertical (it is totally inadequate for gravity!). In fact, the reduction error for ξ , η is approximately proportional to terrain inclination; it is thus very small if the deflection station is situated in an area of inclination zero. This is the case not only if the station lies in a horizontal plane, but also if it lies on the top of a mountain, as most deflection stations do.

For the collocation computation, the covariance function of Jordan and Heller (1978) was used for reasons of simplicity. The parameters of this function were determined from the data.

Results. It turned out that almost all of the signal (T , N , ζ) comes from the topographic-isostatic model and the $N = 180$ gravity model used. This part, $TI + EM$, lies between 41.5m and 47.5m . The contribution of collocation ($\gamma^{-1} T$) lies between $\pm 0.5\text{m}$, after removal of a pronounced trend on the order of 3m .

The efficiency of topographic-isostatic reduction can also be seen from the fact that topographic-isostatic has reduced the variance of the deflections of the vertical in Austria (the square of the average size of ξ and η) from 30 arcsec^2 to 5 arcsec^2 .

Of considerable interest is the effect of analytical continuation on the isostatically (+EM) reduced anomalous potential T . It is expressed by the difference $\gamma^{-1} T$ at the earth's surface minus $\gamma^{-1} T$ at sea level. This difference reaches a maximum of 13 cm in the Central Alps and is otherwise positive and negative.

Of even greater interest is the difference between height anomalies ($= \gamma^{-1}T$ at the earth's surface) and geoidal heights N ($= \gamma^{-1}T$ at sea level). The maximum of 35cm for $\zeta - N$ is reached at the Grossglockner mountain ($H = 3800m$). The results are in good agreement with the approximate formula (PG, p. 327)

$$\zeta - N = -(981gal)^{-1} \Delta g_R H ,$$

where Δg_R is the Bouguer anomaly in gal and H is the elevation in the same units as ζ and N .

A comprehensive information on the geoid in Austria, with regard to both observations and computations, can be found in (Lichtenegger, et al., 1983), of which the main results have also been presented in English at the XVIII General Assembly of IUGG/IAG in Hamburg, August 1983 (Bretterbauer, 1983; Erker, 1983; Grasegger, et al., 1983; Haitzmann, et al., 1983).

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