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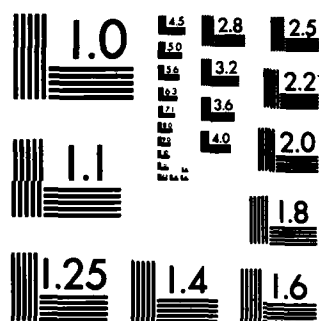
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OPTIMUM REPLACEMENT
OF A SYSTEM SUBJECT TO SHOCKS

by

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A system is subject to shocks. Each shock weakens the system and makes it more expensive to run. It is desirable to determine a replacement time for the system. Boland and Proschan [4] consider periodic replacement of the system and give sufficient conditions for the existence of an optimal finite period, assuming that the shock process is a non-homogeneous Poisson process and the cost structure does not depend on time. Block, Borges, and Savits [3] establish similar results assuming that cost structure is time dependent, still requiring that the shock process is nonhomogeneous Poisson process. We show via a sample path argument that the results of [3] and [4] hold for any counting process whose jump size is of one unit magnitude.			
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ABSTRACT

A system is subject to shocks. Each shock weakens the system and makes it more expensive to run. It is desirable to determine a replacement time for the system. Boland and Proschan (4) consider periodic replacement of the system and give sufficient conditions for the existence of an optimal finite period, assuming that the shock process is a non-homogeneous Poisson process and the cost structure does not depend on time. Block, Borges and Savits (3) establish similar results assuming that cost structure is time dependent, still requiring that the shock process is nonhomogeneous Poisson process. We show, via a sample path argument, that the results of (3) and (4) hold for any counting process whose jump size is of one unit magnitude.

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1. INTRODUCTION AND SUMMARY. A system is subject to shocks which cause the system to deteriorate. In references [1] - [2] life distribution properties of such systems are discussed for different types of deterioration processes. At best these processes are right continuous Markov processes. Our interest in this paper is to tackle a related but different problem: we assume that the normal cost of running the system is 'a' per unit of time and that each shock to the system increases the running cost by 'c' per unit of time. The cost of completely replacing the system is c_{sub} . The system is to be completely replaced at times $T, 2T, \dots$. Such replacement policies are known as periodic replacement policies. The value T is known as the period of the policy. In practice, reliability analysts are often asked to find the optimal value of the period, that is to say the value of T that minimizes some functional of the cost. Such functionals are normally taken to be the long-run average cost per unit of time or the discounted total cost. Boland and Proschan [4] treat the case where the shock process is a non-homogeneous Poisson process. Their proofs depend heavily on the fact that a non-homogeneous Poisson process can be transformed to a homogeneous Poisson process via a non-random time transformation and on the probabilistic properties of the Poisson process. Block, Borges and Savits [3] establish similar results when the cost structure is time dependent. They also assume that the shock process is a nonhomogeneous Poisson process. We show via a sample path argument that the results of Boland and Proschan, Block, Borges and Savits hold for any counting process whose jump size is a unit magnitude.

2. OPTIMAL PERIODIC REPLACEMENT FOR A SYSTEM SUBJECT TO REPEATED SHOCKS WITH TIME INDEPENDENT COST.

For $t \geq 0$ let

$N(t)$ = Number of shocks that the system is subject to during the

interval $[0, t]$ and let $N = (N(t), t \geq 0)$. Throughout we assume that the jumps of N are of one unit magnitude. Let (τ_n) be the sequence of the jump times of the process N .

Observe that the total cost of running the system per period for a given realization of the sequence (τ_n) is equal to

$$aT + c(\tau_2 - \tau_1) + \dots + c(N(T) - 1)(\tau_{N(T)} - \tau_{N(T)-1}) \\ + cN(T)(T - \tau_{N(T)}) + c_0$$

which can be written in the form

$$aT + c \int_0^T N(t) dt + c_0.$$

From Fubini's theorem it follows that the expected total cost of running the system per period is given by

$$(2.1) \quad aT + c \int_0^T M(t) dt + c_0$$

where $M(t)$ = expected number of shocks in $[0, t]$ and M is assumed to be a continuous function on $[0, \infty)$.

From standard renewal theory argument it follows that the long-run average cost per unit of time is given by

$$A(T) = [aT + c \int_0^T M(t) dt + c_0] / T.$$

2.2 REMARK. Boland and Proschan [4] in their lemma 1.1 and Theorem 1.2 exploit the probabilistic behavior of the non-homogeneous Poisson process to prove (2.1) when the shock process N is a non-homogeneous Poisson process.

Observe that $A(T)$ is a differentiable function of T and that the first order derivative of A at T is given by

$$A'(T) = [c \int_0^T (M(T) - M(t)) dt - c_0] / T^2.$$

Moreover

$$\int_0^T (M(T) - M(t)) dt$$

is positive and increasing,

$$\lim_{T \rightarrow 0} A'(T) = -\infty, \text{ and } \lim_{T \rightarrow \infty} A(T) = \lim_{T \rightarrow \infty} M(T).$$

We seek to find the value of the periodic replacement time that minimizes the long-run average cost of running the system per unit of time and we refer to such a value by the optimal periodic replacement time. The proof of the following theorem follows from the above observations.

2.3 THEOREM. The optimal value of the periodic replacement time always exists and is equal to the unique solution of the integral equation

$$\int_0^T [M(t) - M(t)] dt = co/c.$$

Moreover it is finite if and only if

$$\lim_{T \rightarrow \infty} \int_0^T [M(t) - M(t)] dt > co/c.$$

2.4 EXAMPLE. (Nonstationary Pure Birth Shock Models) Assume that shocks occur according to a nonstationary pure birth process as follows: Shocks occur according to a Markov process; given that k shocks occurred in $[0, t]$, the probability of a shock occurring in $[t, t + \Delta)$ is equal to $\lambda_k(t) = o(\Delta)$, while the probability of more than one shock occurring in $[t, t + \Delta) = o(\Delta)$. Observe that in this case the shock process N is a nonstationary Markov process. However, the pair $N^*(t) = (N(t), t)$ form a stationary Markov process. Throughout we will assume without loss of generality that $N(0) \equiv 1$. For any point (k, t) in the state

space $E = \{1, 2, \dots\} \times \mathbb{R}_+$ the symbol $E^{k,t}$ stands for the expectation when the process at time t is in state k . For any bounded function defined on E the infinitesimal generator is defined by

$$Af(k,t) = \lim_{s \rightarrow t} \{[E^{k,t} f(N_s, s) - f(k,t)]/(s-t)\}$$

It is well known that

$$E^{1,0} f(N_t, t) - f(1,0) = \int_0^t E^{1,0} f(N_s, s) ds.$$

If f is such that for any (k, t) in E , $f(k, t) = f(k)$ for some f defined on $\{1, 2, \dots\}$, then it is easy to see that

$$Af(k,t) = \lambda_k(t) [f(k+1) - f(k)].$$

In particular if $f(k, t) = k$ for each k in $\{1, 2, \dots\}$, then

$$Af(k, t) = \lambda_k(t)$$

and therefore

$$E^{1,0} N(t) - 1 = \int_0^t E^{1,0} (\lambda_{N(s)}(s)) ds$$

i.e.,

$$M(t) - 1 = \int_0^t E^{1,0} (\lambda_{N(s)}(s)) ds.$$

Thus

$$\begin{aligned} \int_0^T [M(t) - M(t)] dt &= \int_0^T \int_t^T E^{1,0} (\lambda_{N(s)}(s)) ds dt \\ &= \int_0^T s E^{1,0} (\lambda_{N(s)}(s)) ds. \end{aligned}$$

If $E(\lambda_{N(s)}(s)) = O(s^{-2})$ as $s \rightarrow \infty$, then from Theorem 2.3 it follows that the optimal value of the periodic replacement time exists and is finite.

In particular if for $k = 1, 2, \dots$

$$\lambda_k(t) = k \lambda(t)$$

then we have that

$$M(t) - 1 = \int_0^t \lambda(s) M(s) ds.$$

The above equation has the solution

$$M(t) = e^{\int_0^t \lambda(s) ds}$$

and

$$E(\lambda_{N(t)}(t)) = \lambda(t) \exp\left(\int_0^t \lambda(s) ds\right)$$

which is clearly of order s^{-2} as $s \rightarrow \infty$ assuming that λ is an increasing function in s . In this case the optimal value of the periodic replacement policy exists and is finite. If $\lambda(s) \equiv 1$, then the optimal value of the periodic replacement time is the unique finite solution of the integral equation

$$e^T(T-1) = (c_0 - c) / c$$

Moreover if $\lambda(t) = t$ then the optimal value of the periodic replacement policy exists and is equal to $[\ln((c_0 + c)/c)]^{1/2}$

2.5 REMARK. The non-homogeneous Poisson process case discussed in [4] is a special case of the model discussed in Example 2.4 with

$\lambda_k(t) \equiv \lambda(t)$ for each $k = 1, 2, \dots$ for some λ defined on R_+ .

2.6 EXAMPLE. Assume that the shock process is a renewal process with a renewal function $M(t)$. Suppose that the common distribution function of the interarrival times is absolutely continuous with respect to the Lebesgue measure and has a derivative function $m(t) = \frac{d}{dt} M(t)$. It follows that if $m(t) = O(t^{-2})$ as $t \rightarrow \infty$ then the optimal value of the periodic replacement policy exists and is finite. For example if the interarrival times have finite mean, then $m(t) = O(1)$ as $t \rightarrow \infty$ and hence the optimal periodic replacement policy exists and is finite.

3. OPTIMAL PERIODIC REPLACEMENT FOR A SYSTEM SUBJECT TO REPEATED

SHOCKS WITH TIME DEPENDENT COST STRUCTURE. In this section we

discuss a model similar to the one discussed in §2 but with general costs that depend on the number of shocks and the time at which shocks occur. For $t \geq 0$ let $N(t)$ = Number of shocks the system is subject to during $[0, t]$. The shock process $N = (N(t), t \geq 0)$ is assumed to have jumps of size one almost everywhere; let (τ_n) be the sequence describing the jump times of the shock process N . The cost of operating the system per unit of time in $[\tau_i, \tau_{i+1})$ is $c_i(u)$, $i = 0, 1, \dots$, and u in R_+ , $\tau_0 \equiv 0$. The normal cost of running the system is a per unit of time and the cost of completely replacing the system is c_0 . The system is to be completely replaced at times $T, 2T, \dots$ and the shock process resets at zero at each of these replacements. By the same argument discussed in §2 the expected total cost of running the system per period is given by

$$aT + \int_0^T E c_{N(t)}(t) dt + c_0$$

and the long run average cost of running the system per a unit of time is equal to $[aT + \int_0^T E c_{N(t)}(t) dt + c_0]/T$. Define $h(t) = E(c_{N(t)}(t))$.

The following theorem is the analogue of Theroem 2.3 for this type of problem.

3.1 THEOREM. If h is continuous, increasing, then the optimal value of the periodic replacemnt time exists and is the unique solution of the integral equation

$$\int_0^T [h(T) - h(t)] dt = c_0/c$$

Moreover it is finite if and only if

$$\lim_{T \rightarrow \infty} \int_0^T [h(T) - h(t)] dt > c_0/c.$$

Above we only considered the problem of determining the optimal periodic replacement time over an infinite time horizon. Now we consider the finite horizon problem. We would like to find a replacement time T that minimizes the total expected cost over the interval $[0, t]$. We assume the same cost structure discussed above. Let $H(u) = \int_0^u h(s) ds$. For a period T the total expected cost in $[0, t]$ is equal to

$$K_t(T) = \begin{cases} at + m [H(T) + c_0] + H(s - mT) & \text{if } mT < t < (m+1)T, \\ at + m H(T) + (m-1) c_0 & \text{if } mT = t. \end{cases}$$

3.2 THEOREM. If $H(u)$ is a continuous function on $[0, t]$, then $K_t(\cdot)$ is continuous on $[0, t]$ except possibly at the points $t, t/2, t/3, \dots$ and is right continuous at these points.

3.3 THEOREM. Assume that $h(u)$ is a continuous increasing function on $[0, t]$. Then $K_t(T)$ is minimized on $[0, t]$ at one of the points $t, t/2, t/3, \dots$.

3.4 REMARK. Theorems 3.2 and 3.3 are established in [3] when the shock process is a non-homogeneous Poisson process.

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