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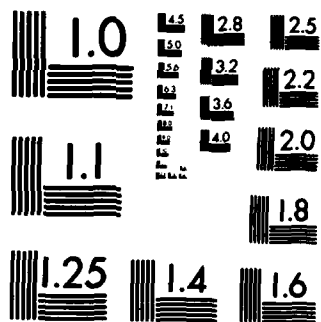
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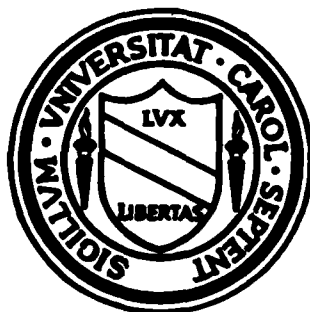




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A NON-COMMUTATIVE QUASI SUBADDITIVE ERGODIC THEOREM

by

Ryszard Jajte

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A NON-COMMUTATIVE QUASI SUBADDITIVE ERGODIC THEOREM

by

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Abstract: Let A be a von Neumann algebra with a faithful tracial normal state ϕ . Consider a sequence (x_n) of self-adjoint elements in $\mathcal{L}_1(A, \phi)$ for which there exists a ϕ -preserving $*$ -automorphism of A and (h_n) in $\mathcal{L}_1^+$ such that $x_{n+k} \leq x_n + \alpha^n x_k + \alpha^n h$ and $\inf_n n^{-1} \phi(x_n) > -\infty$. Then $n^{-1} x_n$ converges almost uniformly and in $\mathcal{L}_1$. This extends the result of Derrienic.

Keywords: von Neumann algebra, (quasi) subadditive ergodic theorem, trace, almost uniform convergence.

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0. Introduction.

→ This work is a contribution to the non-commutative pointwise ergodic theory which has been developed recently in a series of papers by ^{a few authors.} C. Lance [9], Y.G. Sinai and V. Anshelevich [13], F. Yeadon [16], B. Kümmerer [8], M.S. Goldstein [4], S. Watanabe [15], D. Petz [11], J.P. Conze and N. Dang Ngoc [1]. These authors extended to the von Neumann algebra context the classical Birkhoff's type theorems. ^{This document's} Our main goal is to prove a non-commutative version of a subadditive ergodic theorem, which generalizes our previous result [5] and contains as the special cases the 'commutative' results of Kingman [6,7] and Derrienic [2]. The last author very recently proved the following extension of Kingman's theorem.

Theorem 0. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Assume that T is a measurable measure preserving transformation of Ω . Put $Tf = f \circ T$ for $f \in L_1(\Omega, \mathcal{F}, \mu)$. Let (f_n) be a sequence of integrable functions on Ω satisfying the following conditions:

- 1) $f_{n+k} \leq f_n + T^n f_k + T^n h_k$ ($n, k = 1, 2, \dots$), where (h_k) is a sequence of positive functions such that $\sup_k \int_{\Omega} h_k d\mu < \infty$
- 2) $\inf_k k^{-1} \int_{\Omega} f_k d\mu > -\infty$.

Then $n^{-1} f_n$ converges in L_1 and almost everywhere.

We shall show that the result just formulated can be extended to the von Neumann algebra context in the case of finite trace. Let us begin with some notation and preliminaries. Let A be a von Neumann algebra (acting in a Hilbert space H) with a faithful normal tracial state ϕ . Let $\tilde{A}$ denote the $*$ -algebra of all operators in H affiliated

with (A, ϕ) in the sense of Segal [12]. In our case (of finite trace ϕ) the algebra $\tilde{A}$ coincides with the algebra of operators measurable in the sense of Nelson [10]. A sequence $(x_n) \subset \tilde{A}$ is said to be convergent bilaterally almost uniformly to an element $x \in \tilde{A}$ if, for each $\varepsilon > 0$, there is a projection $p \in A$ such that $(x_n - x)p \in A$ for n large enough, $\phi(1-p) < \varepsilon$ and $\|p(x_n - x)p\| \rightarrow 0$ as $n \rightarrow \infty$ (comp [12], [13], [10]). The symbol $\mathbb{L}_1(A, \phi)$ will stand for the Banach space of measurable operators which are integrable with respect to ϕ , with the norm $\|x\|_1 = \phi(|x|)$, where $|x| = (x^*x)^{1/2}$. In the sequel we shall identify $\mathbb{L}_1$ with the predual A_* of A [3]. The symbol $\mathbb{L}_1^{(s)}$ denotes the hermitian part of $\mathbb{L}_1$. For a self-adjoint operator x affiliated with A and a Borel set Z on the real line, the symbol $e_Z(x)$ will denote the spectral projection of x corresponding to the set Z . In particular, $x = \int_{\mathbb{R}} \lambda e(x) d\lambda$ (spectral representation).

We adopt the following definition

Definition 1: A sequence $(x_n; n = 1, 2, \dots)$ in $\mathbb{L}_1^{(s)}(A, \phi)$ is said to be quasi-subadditive if there is a ϕ -preserving $*$ -automorphism α of A and a sequence (h_k) of nonnegative elements of $\mathbb{L}_1^{(s)}$ with $C = \sup_k \phi(h_k) < \infty$ and such that

- (i) $x_{n+k} \leq x_n + \alpha^n x_k + \alpha^n h_k$ for $n, k = 1, 2, \dots$,
- (ii) $\gamma = \inf_k \frac{\phi(x_k)}{k} > -\infty$.

Of course, α in (i) should be treated as the (unique) extension of the automorphism α to $\mathbb{L}_1$.

Theorem I. Let (x_n) be a quasi-subadditive sequence in $\mathbb{L}_1(A, \phi)$. Then $n^{-1}x_n$ converges in $\mathbb{L}_1$ and bilaterally almost uniformly to an α -invariant element $\hat{x}$ of $\mathbb{L}_1$, and $\phi(\hat{x}) = \gamma$.

Before starting the proof of Theorem 1 we shall formulate a few lemmas. We shall follow the basic ideas of Kingman [6] and Derriencic [2]. However, in our context, we cannot perform some operations, for example, to pass to the $\liminf$

or $\limsup$ with the sequences of operators under considerations (as in [6] and [2] with sequences of real functions). To avoid such procedures we start with the following lemma.

Lemma 1. Let, for $n, m = 1, 2, \dots$, b_n, c_n and $a_n(m)$ be the elements of $\mathbb{L}_1^{(s)}(A, \phi)$ satisfying the inequalities $c_m \geq 0$, and

$$(1) \quad 0 \leq b_n \leq c_m + a_n(m) \text{ for } n > m.$$

Assume that $\phi(c_m) \rightarrow 0$ as $m \rightarrow \infty$ and $a_n(m) \rightarrow 0$ bilaterally almost uniformly as $n \rightarrow \infty$, for each $m = 1, 2, \dots$. Then $b_n \rightarrow 0$ bilaterally almost uniformly. This lemma is known [5]. For the sake of completeness we sketch the proof.

Proof. Let $\varepsilon > 0$. Choose a subsequence (m_s) of positive integers m in such a way that $\phi(c_{m_s}) < \varepsilon_{m_s}^2$ with $\sum_{s=1}^{\infty} \varepsilon_{m_s} < \varepsilon/2$. Putting $E = \bigwedge_{s=1}^{\infty} e_{[0, \varepsilon_{m_s}]}(c_{m_s})$, we have $\phi(1-E) < \varepsilon/2$. Let (P_m) be a sequence of projections from A such that $\phi(1-P_m) < \frac{\varepsilon}{2^{m+1}}$ and $\|P_m a_n(m) P_m\| \rightarrow 0$ as $n \rightarrow \infty$. Put $p = \bigwedge_{m=1}^{\infty} P_m$. Then, for each m , $\|p a_n(m) p\| \rightarrow 0$ as $n \rightarrow \infty$. In particular, $\|p a_n(m) p\| < \varepsilon_m$ for $n > N_m$. Putting $Q = E \wedge p$, we have $\phi(1-Q) < \varepsilon$ and $\|Q b_n Q\| \leq \|Q c_{m_s}\| + \|Q a_n(m) Q\| < 2\varepsilon_{m_s}$, for $n > \max(m_s, N_s)$, which means that $b_m \rightarrow 0$ bilaterally almost uniformly. $\square$

Lemma 2. Let N and S be two bounded linear functionals on A . Assume that N is normal (i.e. $N \in A_*$) and S is positive and singular in the sense of Takesaki ([14], p. 127). Then $N + S \geq 0$ implies $N \geq 0$.

Proof. Let p be an arbitrary orthogonal projection from A . By Takesaki's theorem ([14], p. 134, Th. 3.8), a positive linear functional S on A is singular if and only if, for every non-zero projection $q \in A$, there exists a non-zero projection $r \in A$ such that $r \leq q$ and $S(r) = 0$. It follows that there is a sequence (p_k) of mutually orthogonal projections

in A such that $p = \sum_{k=1}^{\infty} p_k$ and $S(p_k) = 0$. Indeed, it is enough to take in

A a maximal family of mutually orthogonal non-zero projections (p_{α}) such that $p_{\alpha} \leq p$ and $S(p_{\alpha}) = 0$. Such family is at most countable (since ϕ is normal and faithful). Put $Q_n = \sum_{k=1}^n p_k$ ($n=1,2,\dots$). Then by the normality of N , we have that $N(p) = \lim_n N(Q_n) = \lim_n [N(Q_n) + S(Q_n)] \geq 0$.

Since p is arbitrary, by the spectral theorem we obtain $S(a) \geq 0$ for all $0 \leq a \in A$, which ends the proof. $\square$

Lemma 3. (comp. [2], [6]). Let (x_n) , (h_n) and α be as in Theorem 1.

Put

$$y_m = \frac{1}{m} \sum_{k=1}^m (x_k - \alpha x_{k-1}), \quad m \geq 1.$$

Then there exists a sequence (z_n) , $0 \leq z_n \in \mathbb{L}_1(A, \phi)$ such that, for every $m > n$, we have

$$(2) \quad \sum_{k=0}^{n-1} \alpha^k y_m \leq x_n + \alpha^n \left(\frac{1}{m} \sum_{k=1}^{m-1} h_k \right) + \frac{1}{m} z_n.$$

Moreover, $\sup_n \|y_n\|_1 < \infty$.

Proof. We have

$$m y_m = (I - \alpha) \left(\sum_{k=1}^{m-1} x_k \right) + x_m.$$

Consequently,

$$\begin{aligned} \sum_{i=0}^{n-1} \alpha^i y_m &= (I - \alpha^n) \left(\sum_{k=1}^{m-1} x_k \right) + \sum_{i=0}^{n-1} \alpha^i x_m = \\ &= \sum_{k=1}^n x_k + \sum_{k=1}^{m-n-1} (x_{k+n} - \alpha^n x_k) + \sum_{i=0}^{n-1} \alpha^i (x_m - \alpha^{n-i} x_{m-n+i}). \end{aligned}$$

By the quasi-subadditivity of (x_n) , we have that

$$x_{n+k} - \alpha^n x_k \leq x_n + \alpha^n h_k$$

and

$$x_m - \alpha^{n-i} x_{m-n+i} \leq x_{n-i} + \alpha^{n-i} h_{m-n+i},$$

hence

$$\sum_{i=0}^{n-1} \alpha^i y_m \leq mx_n + \alpha^n \left(\sum_{k=1}^{m-1} h_k \right) + \left[\sum_{k=1}^n x_k - (n+1)x_n + \sum_{i=0}^{n-1} \alpha^i x_{n-i} \right]$$

Putting

$$z_n = \sum_{k=1}^n x_k - (n+1)x_n + \sum_{i=0}^{n-1} \alpha^i x_{n-i},$$

we obtain (2). It remains to prove the boundness of (y_k) in Π_1 -norm.

This follows from the estimations

$$x_k \leq x_1 + (k-1) \leq x_1 + \alpha x_{k-1} + \alpha h_{k-1}.$$

and

$$y_m \leq x_1 + \alpha \left(\frac{1}{m} \sum_{k=1}^m h_{k-1} \right)$$

then we can write

$$\begin{aligned} \|y_m\|_1 &\leq \|x_1 + \alpha \left(\frac{1}{m} \sum_{k=1}^m h_{k-1} \right) - y_m\|_1 + \|x_1 + \alpha \left(\frac{1}{m} \sum_{k=1}^m h_{k-1} \right)\|_1 \leq \\ &\leq \phi(x_1) + \phi\left(\alpha \left(\frac{1}{m} \sum_{k=1}^m h_{k-1} \right)\right) - \frac{\phi(x_m)}{m} + \|x_1 + \alpha \left(\frac{1}{m} \sum_{k=1}^m h_{k-1} \right)\|_1 \leq \\ &\leq \phi(x_1) + \|x_1\| + 2 \sup_k \|h_k\|_1 - \inf_m \frac{\phi(x_m)}{m} = \text{const} < \infty. \quad \square \end{aligned}$$

Lemma 4. Let (x_n) , (h_n) and α be as in Theorem 1. Then there exist in $\Pi_1(A, \phi)$ two elements $\bar{x}$ and ω such that

$$(3) \quad \sum_{i=0}^{n-1} \alpha^i \bar{x} \leq x_n + \alpha^n(\omega) \quad \text{for } n = 1, 2, \dots$$

Moreover, $\phi(\bar{x}) = \gamma = \inf_n \frac{\phi(x_n)}{n}$

Proof. We follow here the idea of Kingman [6] of using the weak $*$ -

-compactness of the sequence (y_m) defined in Lemma 2. Identifying $\Pi_1(A, \phi)$ with the predual A_* of A and taking the natural injection of A_* into $A^* = (A_*)^{**}$, we can treat the operators y_m from Π_1 as the

continuous linear functionals on A . The images in $A^* = \mathbb{L}_1^{**}$ of elements $x, y, \dots$ of $\mathbb{L}_1$ we denote by $\tilde{x}, \tilde{y}, \dots$. In particular, $\tilde{y}_m(a) = \phi(y_m a)$ for $a \in A$. Moreover, we put $(\tilde{\alpha\sigma})(a) = \sigma(\alpha^{-1}a)$ for $\sigma \in A^*$ and $a \in A$. Then we have in particular $\widetilde{\alpha^{-1}y} = \tilde{\alpha^{-1}}\tilde{y}$ for $y \in \mathbb{L}_1$. Indeed, since ϕ is α -invariant, we have for $a \in A$, that

$$(\tilde{\alpha y})(a) = \phi(\alpha(y)a) = \phi(\alpha^{-1}(\alpha(y)a)) = \phi(y\alpha^{-1}(a)) = \tilde{y}(\alpha^{-1}a) = \tilde{\alpha y}(a).$$

The sequences (y_m) and $(\frac{1}{m} \sum_{k=1}^m h_k)$ are bounded in $\mathbb{L}_1$ -norm, so the sequences $(\tilde{y}_m)$ and $(\frac{1}{m} \sum_{k=1}^m \tilde{h}_k)$ are compact in A^* in the weak*-topology.

The formula (2) can be rewritten in the form

$$(4) \quad \sum_{k=1}^{n-1} \alpha^k \tilde{y}_m \leq \tilde{x}_n + \tilde{\alpha}^n \left(\frac{1}{m} \sum_{k=1}^m \tilde{h}_k \right) + \frac{1}{m} \tilde{z}_n.$$

Taking the suitable limit points for $(\tilde{y}_m)$ and $(\frac{1}{m} \sum_{k=1}^m \tilde{h}_k)$, say v_0 and δ , respectively, we can write

$$(5) \quad \sum_{i=0}^{n-1} \tilde{\alpha}^i v_0 \leq \tilde{x}_n + \tilde{\alpha}^n(\delta) \quad (n = 1, 2, \dots),$$

where $v \in A^*$ and $0 \leq \delta \in A^*$.

In particular, we have for $n = 1$

$$(6) \quad \tilde{x}_1 - v_0 + \tilde{\alpha}(\delta) \geq 0$$

By Takesaki's theorem ([14], p. 127) we can write

$$(7) \quad \tilde{x}_1 - v_0 + \tilde{\alpha}(\delta) = \sigma_* + \sigma_s,$$

where σ_* and σ_s are normal and singular part of $\tilde{x}_1 - v_0 + \tilde{\alpha}(\delta)$, respectively.

Let $\delta = \delta_* + \delta_s$ be also the Takesaki's decomposition of δ , i.e. $\delta_* \in A_*$

and δ_s is a singular functional. Obviously, we have $\sigma_s \geq 0$ and $\delta_s \geq 0$.

Let us remark that $\tilde{\alpha}(\delta_s)$ is also a singular functional. Indeed, supposing

that there is a positive normal functional $b \leq \tilde{\alpha}(\delta_s)$, we would have

$\tilde{\alpha}^{-1}b \leq \delta_s$ and, consequently, $\tilde{\alpha}^{-1}b = 0$, so $b = 0$.

Summing up, the formula (7) gives us, for some $z \notin A_*$,

$$(8) \quad v_0 = z + \tilde{\alpha}(\delta_s) - \sigma_s$$

with positive singular functionals σ_s and δ_s . Thus, by formula (5),

we have

$$(9) \quad \sum_{i=0}^{n-1} \tilde{\alpha}^i(z) - \sum_{i=0}^{n-1} \tilde{\alpha}^i(\sigma_s) + \sum_{i=0}^{n-1} \tilde{\alpha}^{i+1}(\delta_s) \leq \tilde{x}_n + \tilde{\alpha}^n(\delta_s) + \tilde{\alpha}^n(\delta_*)$$

Since $\sum_{i=0}^{n-1} \tilde{\alpha}^{i+1}(\delta_s) \geq 0$, we obtain

$$(10) \quad \sum_{i=0}^{n-1} \tilde{\alpha}^i(z) \leq \tilde{x}_n + \tilde{\alpha}^n(\delta_*) + \tilde{\alpha}^n(\delta_s) + \sum_{i=0}^{n-1} \tilde{\alpha}^i(\sigma_s),$$

or, equivalently,

$$(11) \quad N + S \geq 0,$$

where $N = \tilde{x}_n + \tilde{\alpha}^n(\delta_*) - \sum_{i=0}^{n-1} \tilde{\alpha}^i(z)$

and $S = \tilde{\alpha}^n(\delta_s) + \sum_{i=0}^{n-1} \tilde{\alpha}^i(\sigma_s)$.

By Lemma 2, $N \geq 0$, i.e. we have

$$(12) \quad \sum_{i=0}^{n-1} \tilde{\alpha}^i(z) \leq \tilde{x}_n + \tilde{\alpha}^n(\delta_*) \quad (n = 1, 2, \dots)$$

The last formula can be easily translated into

$$(13) \quad \sum_{i=0}^{n-1} \alpha^i(\bar{x}) \leq x_n + \alpha^n(\omega),$$

for some $\bar{x}, \omega \in \mathbb{L}_1(A, \phi)$.

To end the proof it remains to show that $\phi(\bar{x}) = \gamma$, or, what is equivalent, that $z(1) = \gamma$. In order to prove this we shall follow the general idea of [2] and use the uniqueness of the Takesaki's decomposition of

$f \in A^*$ into its normal and singular parts.

Let us notice that formula (12) gives

$$(14) \quad z(1) \leq \inf_n \frac{x_n(1)}{n} = \inf_n \frac{\phi(x_n)}{n} = \gamma$$

Moreover,

$$v_0(1) = \lim_s m_s^{-1} \sum_{k=1}^m (\tilde{x}_k - \tilde{\alpha} \tilde{x}_{k-1})(1) = \lim_s \frac{\phi(x_{m_s})}{m_s} = \gamma.$$

Thus, since $z = v_0 + \sigma_s - \tilde{\alpha}(\delta_s)$, we obtain

$$\gamma \geq z(1) = v_0(1) - \tilde{\alpha}(\delta_s)(1) + \delta_s(1) \geq \gamma - \tilde{\alpha}(\delta_s)(1) + \sigma_s(1).$$

Consequently, since $\tilde{\alpha}(\delta_s)(1) = \delta_s(1) \leq \delta(1) = \|\delta\| \leq \sup_k \|h_k\|_1 = C$,

we have $\gamma \geq z(1) \geq \gamma - C$.

Let us take $x'_k = x_{2k}$ for $k = 1, 2, \dots$. The sequence (x'_k) is obviously quasi-subadditive with respect to α^2 (and with the same constant $C = \sup_k \|h_k\|_1$). We can now repeat the same reasoning as for (x_k) . In

particular, we put

$$y'_m = \frac{1}{m} \sum_{k=1}^m (x_{2k} - \alpha^2 x_{2k-2})$$

and obtain the formula analogue to (8)

$$(15) \quad z' = v'_0 + \sigma'_s - \tilde{\alpha}^2(\delta'_s),$$

where v'_0 is a weak* limit point of y'_m , z' - the corresponding element of A_* , σ'_s and δ'_s the corresponding singular functionals in Takesaki's decompositions of suitable α' and δ' . Since $n^{-1}\phi(x_{2n}) \rightarrow 2\gamma$, we obtain

$$(16) \quad 2\gamma \geq z'(1) \geq 2\gamma - C$$

Let us notice now that

$$(17) \quad y'_m = \frac{1}{m} \sum_{k=1}^m (x_{2k} - \alpha x_{2k-1}) + \alpha \left(\frac{1}{m} \sum_{k=1}^m (x_{2k-1} - \alpha x_{2k-2}) \right).$$

The sequences

$$(18) \quad \frac{1}{m} \sum_{k=1}^m (x_{2k} - \alpha x_{2k-1}) \text{ and } \frac{1}{m} \sum_{k=1}^m (x_{2k-1} - \alpha x_{2k-2})$$

are bounded in $\mathbb{L}_1$ -norm (since $x_r - \alpha x_{r-1} \leq x_1 + \alpha h_{r-1}$; compare the proof of Lemma 3). Taking the suitable weak*-limit points v_1 and v_2 of these sequences, we obtain

$$(19) \quad v_0 = \frac{1}{2} (v_1 + v_2) \text{ and } v'_0 = v_1 + \alpha v_2.$$

Denote by z_i the normal parts of v_i ($i = 1, 2$). The uniqueness of Takesaki's decomposition gives immediately

$$(20) \quad z = \frac{1}{2} (z_1 + z_2) \text{ and } z' = z_1 + \alpha z_2$$

Consequently, we have $z(1) = \frac{1}{2} (z_1(1) + z_2(1)) = \frac{1}{2} z'(1)$, and (16) gives

$$(21) \quad \gamma \geq z(1) \geq \gamma - \frac{C}{2}.$$

The repetition of the procedure just described will give us

$$(22) \quad \gamma \geq z(1) \geq \gamma - \frac{C}{2^n} \text{ for } n = 1, 2, \dots$$

which means that $\gamma = z(1)$ and ends the proof of lemma 4. $\square$

Proof of Theorem 1.

Let (x_n) satisfy the conditions (i) and (ii) of Definition 1. By lemma 4 there exist in A two elements $\bar{x}$ and ω such that

$$(23) \quad u_n = x_n + \alpha^n(\omega) - \sum_{i=0}^{n-1} \alpha^i \bar{x} \geq 0, \text{ with } \phi(\bar{x}) = \gamma.$$

It is easily seen that (u_n) is a nonnegative quasi-subadditive sequence in $\mathbb{L}_1$ satisfying the condition

$$(24) \quad \inf_n \frac{\phi(u_n)}{n} = \inf_n \frac{\phi(x_n)}{n} - \gamma = 0.$$

Moreover, by Yeadon's theorem [16], the averages $n^{-1} \sum_{i=0}^{n-1} \alpha^i x$ converge

bilaterally almost uniformly and in $\mathbb{L}_1$, so it is clear now that it suffices to prove our theorem for the quasi-subadditive sequences (x_n) which are nonnegative and satisfy the condition

$$(25) \quad \inf_n \frac{\phi(x_n)}{n} = 0.$$

In this case the convergence in $\mathbb{L}_1$ (to zero) of $\frac{x_n}{n}$ is obvious, so it remains to show the bilateral almost uniform convergence of $\frac{x_n}{n}$ to zero.

To this end let us fix a positive integer m and, for $n > m$, write $n = mk + r$ with $0 \leq r \leq m-1$. Then we have

$$0 \leq x_n \leq x_{mk} + \alpha^{mk} x_r + \alpha^{mk} h_r \leq \sum_{i=0}^{k-1} \alpha^{im} x_m + \alpha^{mk} x_r + \alpha^{mk} h_r \leq \sum_{i=0}^{k-1} \alpha^{im} x_m + \alpha^{mk} z_m,$$

where $z_m = \sum_{k=1}^{m-1} (x_k + h_k)$.

By Yeadon's theorem, the averages

$$(26) \quad \frac{1}{k} \sum_{i=0}^{k-1} \alpha^{im} x_m$$

converge bilaterally almost uniformly to some α^m -invariant element

$$\hat{x}_m \in \mathbb{L}_1(A, \phi), \text{ such that } \phi(\hat{x}_m) = \phi(x_m).$$

Putting

$$(27) \quad a_n(m) = \frac{1}{k} \sum_{i=0}^{k-1} \alpha^{im} \frac{x_m - \hat{x}_m}{m} + \frac{1}{k} \alpha^{mk} z_m,$$

where $k = k(m)$ is defined by the equality $n = mk + r$ ($0 \leq r \leq m-1$), we get

$$(28) \quad 0 \leq \frac{x_n}{n} \leq \frac{\hat{x}_m}{m} + a_n(m) \quad (n > m)$$

We shall reduce the proof of the theorem to Lemma 1. We shall show that

$a_n(m) \rightarrow -$ bilaterally almost uniformly as $n \rightarrow \infty$, for every $m = 1, 2, \dots$

By Yeadon's theorem, the first term on the right hand side of (27) tends to zero so it remains to show that $\frac{1}{k} \alpha^{mk} \rightarrow 0$ almost uniformly as $k \rightarrow \infty$ (for a fixed m). Put $\beta = \alpha^m$ and let

$$z_m = \int_0^\infty \lambda e(d\lambda)$$

be the spectral representation of z_m . Then $\int_0^\infty \lambda \phi(e(dx)) < \infty$ since $z \in \mathbb{L}_1(A, \phi)$, and $\beta^k(e(d\lambda))$ is the spectral measure of $\beta^k z_m$. Moreover, $\phi(e(d\lambda)) = \phi(\beta^k e(d\lambda))$ for all k . Thus, having taken $0 < \lambda_n \rightarrow 0$, we can write

$$\sum_{k=1}^\infty \phi(e_{(\lambda_n, \infty)} \{ \frac{1}{k} \beta^k z_m \}) = \sum_{k=1}^\infty \phi(e_{(k\lambda_n, \infty)} \{ z_m \}) < \infty$$

(since $z_m \in \mathbb{L}_1$).

Let $\varepsilon > 0$, and choose (k_n) in such a way that

$$\sum_{k=k_n}^\infty \phi(e_{(k\lambda_n, \infty)} \{ z \}) < 2^{-n} \varepsilon$$

holds for $n = 1, 2, \dots$

Putting

$$Q_{k,n} = e_{[0, \lambda_n]} \{ \frac{1}{k} \beta^k z_m \}$$

and $Q = \bigwedge_{n=1}^\infty \bigwedge_{k=k_n}^\infty Q_{k,n}$, we have

$$\phi(1 - Q) < \sum_n 2^{-n} \varepsilon = \varepsilon.$$

Moreover, $||\frac{1}{k} \beta^k z_m Q|| < \lambda_n$ for $k > k_n$ which means that $\frac{1}{k} \beta^k z_m \rightarrow 0$ almost uniformly as $k \rightarrow \infty$, for $m = 1, 2, \dots$

Applying Lemma 1 to inequality (28) and using the fact that

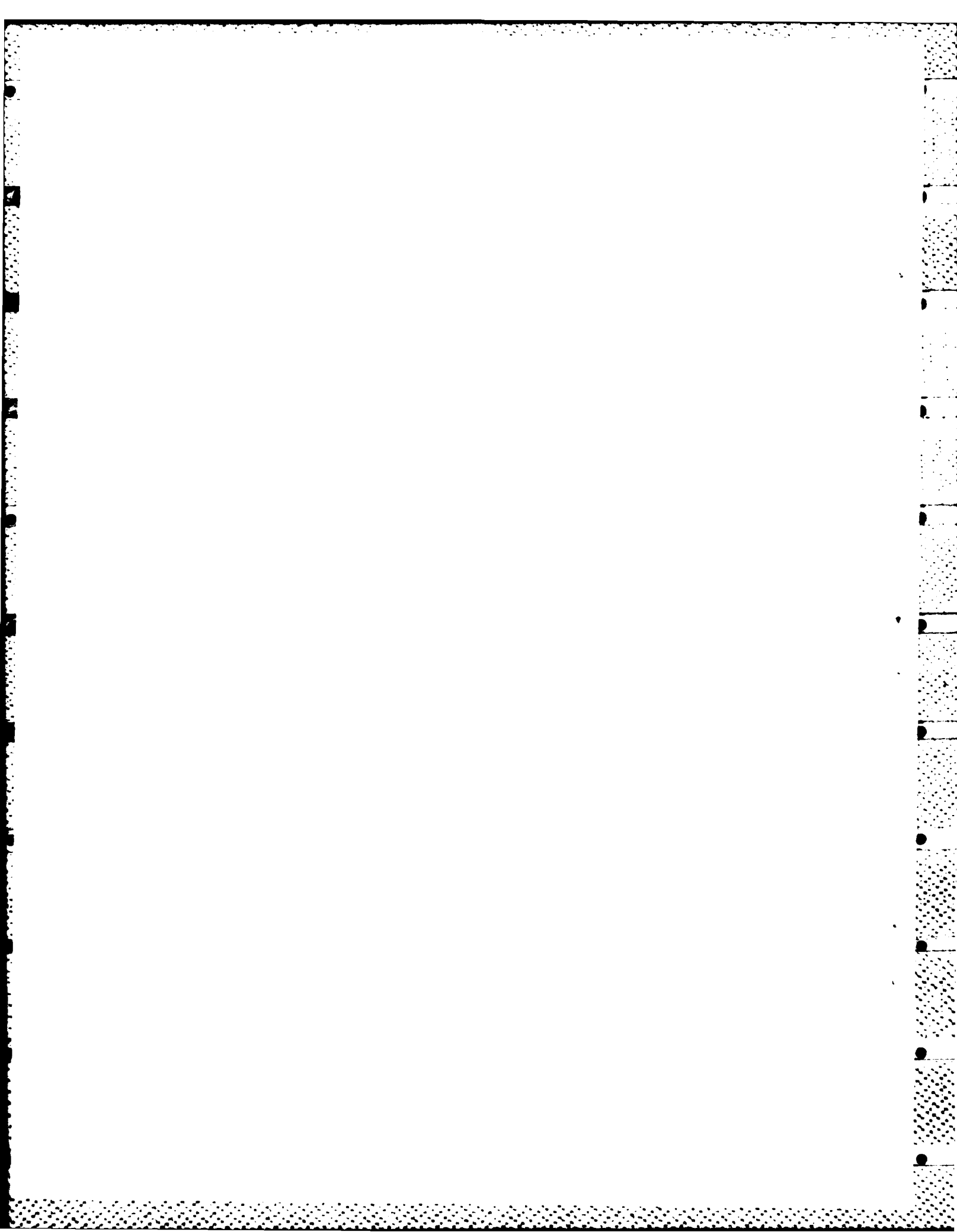
$$\frac{\phi(\hat{x}_m)}{m} = \frac{\phi(x_m)}{m} \rightarrow 0, \text{ we get } \frac{x_n}{n} \rightarrow 0 \text{ bilaterally almost uniformly as } n \rightarrow \infty.$$

The proof of Theorem 1 is completed. $\square$

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