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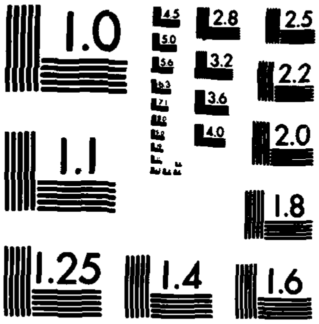
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INSTITUTE FOR PHYSICAL SCIENCE
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Laboratory for Numerical Analysis

Technical Note BN-1022

FEEDBACK, ADAPTIVITY AND A-POSTERIORI ESTIMATES IN FINITE ELEMENTS: AIMS, THEORY AND EXPERIENCE

by

I. Babuška

June 1984

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The paper deals with the principles of the adaptive procedures in the finite element computations. It characterizes the aims and formulates typical theorems related to the subject. One and two dimensional numerical examples illustrate the applicability of the theory for practical computations.

The paper reflects the invited lecture at the International Conference on Accuracy Estimates and Adaptive Refinement in Finite Element Computations, Lisbon, 1984.

FEEDBACK, ADAPTIVITY AND A-POSTERIORI ESTIMATES IN FINITE ELEMENTS: AIMS, THEORY AND EXPERIENCE

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Technical Note

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1. INTRODUCTION

Adaptive approaches and a-posteriori estimates of the accuracy of computed solutions have recently attracted great interest. Nevertheless the basic notions, aims and goals, principles for comparisons, etc. are not yet well established. We try in this paper to formulate some principle notions and approaches related to this new field of numerical mathematics and computational engineering and show their practical importance. We will introduce the notions of feedback, adaptive approaches, a-posteriori estimates, etc. and formulate illustrative theorems in a model setting. Although the available theoretical results cover a much larger area than formulated in this paper, the general theory still waits to be developed.

We will place some emphasis on the analysis of the one dimensional model problem treated by the h, p and h-p versions of the finitie element method. In this case we are relatively easily able to formulate the illustrative theorems and to show the general ideas.

The results mentioned here illustrate more general theory and its numerical implementation developed by a group at the University of Maryland. We mention in this connection especially contributions of W. Gui [1], B. Guo [2], Ch. Mesztenyi, A. Miller and M. Vogelius.

We refer also to [3], [4], [5], [6], [7] for some relevant approaches and ideas.

Our results relate mostly to linear elliptic problems. For nonlinear problems, problems of other type such as parabolic and hyperbolic, etc., some other aspects need to be emphasized; nevertheless we will not address them. For some of these aspects related to the solution of nonlinear problems, see [5], [8].

2. THE BASIC MODEL PROBLEM

2.1. The one dimensional problem

Let us be interested in the one dimensional problem

$$-(au')' + bu = f, \qquad x \in I = (0,1) \tag{2.1a}$$

$$u(0) = u(1) = 0 \tag{2.1b}$$

where

$$0 < \alpha_0 \leq a(x) \leq \alpha_1 < \infty, \quad 0 \leq b(x) \leq \alpha_1 < \infty. \tag{2.1c}$$

We will assume that f is such that the solution u_0 of the problem (2.1a,b) belongs to $\overset{0}{H}{}^1(I)$; i.e. $u_0 \in \overset{0}{H}{}^1(I)$ where

$$\overset{0}{H}{}^1(I) = \left\{u(x) \mid \|u\|_1^2 = \int_0^1 (u'^2+u^2)dx < \infty,\ u(0) = u(1) = 0\right\}. \tag{2.2}$$

The triple (a,b,f) will be referred to as <u>input data</u>.

We denote

$$\|u\|_{E(I)}^2 = \int_0^1 (au'^2+bu^2)dx. \tag{2.3}$$

The solution of (2.1) is formulated in the usual weak form: Find u_0 $\overset{0}{H}{}^1(I)$ so that

$$B(u_0,v) = \int_0^1 (au'v'+buv)dx = \int_0^1 f\ dx \tag{2.4}$$

holds for all $v \in \overset{0}{H}{}^1(I)$.

A special consideration will be given to the case when the solution u_0 $= w$ where

$$(2.5) \qquad w(x) = [x-\gamma]^{\alpha} + C_1 x + C_2,$$

$$[x-\gamma] = \begin{cases} x-\gamma, & x > \gamma, \\ 0, & x < \gamma, \end{cases}$$

$\alpha > 1/2$ not integral and C_i, $i = 1,2$ are chosen so that (2.1.b) holds. This solution (with $\gamma = 0$) is a one dimensional model of the solution with a singularity caused by the presence of cracks and corners in the two dimensional setting.

We are interested in the accuracy of the approximate solution measured in the energy norm (2.3):

$$\|u_0 - \tilde{u}\|_E < \tau \|u_0\|_E .$$

2.2. The two dimensional problems

Let us be interested in the elasticity problem (plane strain) of the cracked panel shown in Fig. 2.1. We assume that the Poisson ratio $\nu = .3$, Young's modul $E = 1$ and the tractions are such that the solution coincides with the solution for the slit half plane with stress intensity factors $k_I = k_{II} = 1$.

The energy norm (the analog to (2.3)) and the weak formulation (analogous to (2.4)) is the usual one.

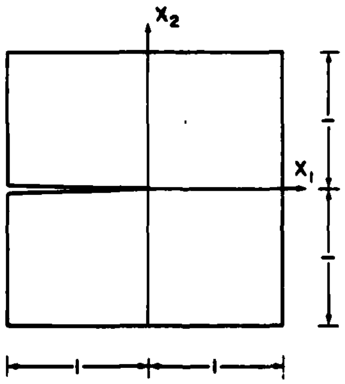


Figure 2.1. The scheme of cracked panel.

The second problem will be to solve on $S = \{x_1, x_2 \mid |x_1| < \frac{1}{2}, |x_2| < 1/2\}$ the boundary value problems

(2.6a)
$$\Delta u_i = 0 \quad \text{on} \quad S$$

(2.6b)
$$\frac{\partial u_i}{\partial n} = g_i, \quad \text{on} \quad \partial S, \quad i = 1,2$$

with

$$u_1 = \mathrm{Re}[(z - \tfrac{i}{2})\lg(z - \tfrac{i}{2})]$$

$$u_2 = \mathrm{Im}[(z - \tfrac{1}{2})\lg(z - \tfrac{1}{2})].$$

3. THE FINITE ELEMENT METHOD (FEM) FOR THE ONE DIMENSIONAL PROBLEM

Let

(3.1a)
$$\Delta =: 0 < x_0^\Delta < x_1^\Delta < \cdots < x_{M(\Delta)}^\Delta = 1,$$

(3.1b)
$$I_{j+1}^\Delta = (x_j^\Delta, x_{j+1}^\Delta), \quad h_j^\Delta = x_{j+1}^\Delta - x_j^\Delta,$$
$$\hat{h}(\Delta) = \max_j h_j^\Delta, \qquad \check{h}(\Delta) = \min h_j^\Delta,$$

(3.1c)
$$p^\Delta = (p_1^\Delta, \ldots, p_{M(\Delta)}^\Delta), \quad p_j^\Delta > 1, \quad \text{integral.}$$

Δ is called mesh, x_j^Δ the nodal points, I_j^Δ the elements and p_j^Δ the degrees. We denote by $S(\Delta, p^\Delta) \subset \overset{\circ}{H}{}^1(I)$ the set of all functions which restricted to I_j^Δ are polynomials of degree p_j^Δ. For $\kappa(\Delta) = (\Delta, p^\Delta)$ we define

$$(3.1d) \qquad N(\kappa(\Delta)) = N(\Delta,p^{\Delta}) = \sum_{j=1}^{M(\Delta)} (p_j^{\Delta}+1) - (M(\Delta)+1).$$

$N(\kappa(\Delta))$ is the dimension of $S(\Delta,p^{\Delta})$ (i.e. the number of degrees of freedom).

The finite element solution

$$\tilde{u} = \tilde{u}(\kappa(\Delta)) = \tilde{u}(\Delta,p^{\Delta}) \in S(\Delta,p^{\Delta})$$

is defined in the usual way:

$$(3.2) \qquad B(\tilde{u},v) = \int_0^1 fv\,dx \quad \text{for every } v \in S(\Delta,p^{\Delta}).$$

The formulation of the finite element solution is quite general and includes <u>all three versions of the method</u> namely <u>the h, p and h-p versions</u> of the finite element method.

In two dimensions the formulation is analogous but more complex.

4. THE FEEDBACK FINITE ELEMENT METHOD. THE ACCEPTANCE CRITERION

By the <u>feedback</u> finite element method we will understand a <u>specific construction</u> of a sequence $\underline{\kappa} = (\kappa_1,\ldots)$, $\kappa_i = (\Delta_i,p^{\Delta_i})$, and the sequence of associated finite element solutions $\tilde{u}_i = \tilde{u}(\kappa_i)$ $i = 1,2,\ldots$.

The sequence $\underline{\kappa}$ is usually constructed recursively and is called a <u>trajectory</u>. In general κ_i depends on all κ_ℓ, $\tilde{u}_\ell$, $\ell = 1,2,\ldots i-1$ and on the input data (a,b,f).

We will write

$$(4.1) \qquad \kappa_i = A(\kappa_1,\ldots,\kappa_{i-1}, \tilde{u}_1,\ldots,\tilde{u}_{i-1}, a,b,f)$$

and call A a <u>transition operator</u>. In addition we will assume that

$\underline{\kappa} \in Q$, i.e. κ_i is restricted to a special distributions of nodal points and degrees. The set Q is given <u>a-priori</u> and the transition operator which leads to a trajectory $\underline{\kappa} \in Q$ will be called a <u>Q-transition operator</u>.

To every $(\kappa_i,\tilde{u}_i)$ an <u>error estimator</u>

$$E_i = E_i(\kappa_i,\tilde{u}_i,a,b,f) \tag{4.2}$$

is defined and the solution $\tilde{u}_i$ is accepted if

$$E_i(\kappa_i,\tilde{u}_i,a,b,f) \leq \tau\|\tilde{u}_i\|_{E(I)} \tag{4.3}$$

where τ is an a-priori given tolerance.

5. PRINCIPLES OF THE SELECTION OF A TRANSITION OPERATOR

Obviously there are many possible transition operators which could be considered in practice. An optimal selection depends on many factors, the class of problems, computer technology, etc. Nevertheless, it is of major interest to introduce notions which will allow one to characterize basic properties of feedback procedures, to make comparisons and to develop a systematic mathematical theory.

5.1. <u>Adaptivity with respect to the convergence measure</u>

Given the transition operator A, the problem is to describe the set R_1 of the input data (a,b,f) and the initial meshes κ_1 so that

$$\|u_0-\tilde{u}(\kappa_i)\|_{E(I)} \to 0 \quad \text{as} \quad i \to \infty. \tag{5.1}$$

Such an operator A (and the feedback finite element method) will be called <u>adaptive relative to R_1</u> with respect to <u>the convergence measure</u>. Obviously, in comparing two transition operators from this point of view, we prefer the one whose set $\mathcal{R}_1$ is larger or covers the set of input data which are more relevant in applications. Any feedback finite element method used in practice <u>has to be adaptive</u> with respect to the convergence measure for a large set $\mathcal{R}_1$. Therefore, a mathematical characterization of $\mathcal{R}_1$ is of major importance.

5.2. <u>Adaptivity with respect to the convergence rate measure</u>

Define

$$\begin{aligned} \Phi(N) &= \Phi(N,Q,a,b,f) \\ &= \inf_{\substack{\kappa\in Q \\ N(\kappa)=N}} \|u_0-\tilde{u}(\kappa)\|_{E(I)} \end{aligned} \tag{5.2}$$

and call $\Phi(N)$ the <u>rate function</u>. Obviously $\Phi(N)$ is the smallest possible error obtainable by the mesh of any trajectory of [illegible] with the degree of freedom N. (We remark that $\Phi(N)$ has not necessarily been defined for all N.)

Given the Q-transition operator $\mathbf{A}$ and a number $\rho > 1$, the problem is to describe the set $\mathcal{R}_2(\rho)$ of the input data (a,b,f) and the initial meshes κ_1 so that

$$\sup_{i=1,2,\ldots} \frac{\|\tilde{u}(\kappa_i)-u_0\|_{E(I)}}{\Phi(N(\kappa_i))} < \rho \tag{5.3}$$

or the set

$$\tilde{\mathcal{R}}_2 \equiv \bigcup_{\rho>1} \mathcal{R}_2(\rho). \tag{5.4}$$

The analysis of the rate function $\Phi(N)$ for various sets of input data and the admissible set of trajectories $\underline{\kappa} \in Q$ is of major importance. It gives a guide for the design of an effective transition operator.

The Q-transition operator A (resp. the feedback finite element method) satisfying (5.3) (resp. (5.4)) is called *ρ-adaptive* (resp. *adaptive*) *relative to $R_2(\rho)$* (resp. $\tilde{R}_2$) with respect to *the convergence rate measure*.

Obviously the definition is strongly tied to (5.3), where the error of the ideal mesh is in the denominator. We can weaken (5.3) in order to increase $R_2(\rho)$ and still *practically* require that the performance of $\{\kappa_i\}$ to be nearly ideal. For example, we can ask instead (5.3) that

$$\sup_{i=1,2,\dots} \frac{\|\tilde{u}(\kappa_i)-u_0\|_E}{[\Phi(N(\kappa_i))]^{\beta}} < \rho \tag{5.5}$$

with $\beta = 1 - \varepsilon$ and $\varepsilon > 0$ small.

5.3. *Adaptivity with respect to the work measure*

Let us define the computational effort of determining the finite element solution $\tilde{u}(\kappa_i)$ by an expression $W(\kappa_i)$. We can, for example, take $W(\kappa_i) = [N(\kappa_i)]^{\theta}$ where θ is a properly chosen number.

Computation of $\tilde{u}(\kappa_i)$ assumes that $\tilde{u}(\kappa_j)$, $j = 1,\dots,i-1$ were already determined. Therefore, the total work $\overline{W}(\kappa_i)$ needed in determining $\tilde{u}(\kappa_i)$ is:

$$\overline{W}(\kappa_i) = \sum_{j=1}^{i} W(\kappa_j). \tag{5.6}$$

We introduce now the *proportionality factor* $\chi(\kappa_i)$:

(5.7) $$\chi(\kappa_i) = \frac{\overline{W}(\kappa_i)}{W(\kappa_i)}.$$

The problem now is to describe the set $R_3(\rho)$ of input data and the initial meshes κ_1 so that

(5.8) $$\sup_{i=1,2,\ldots} \chi(\kappa_i) < \rho.$$

The transition operator **A** resp. the feedback finite element method satisfying (5.8) is called ρ- adaptive with respect to the work measure relative to $R_3(\rho)$.

5.4. The adaptivity with respect to the effectivity index of the error estimator

Let

(5.9) $$\Theta(\kappa_i) = \frac{E_i(\kappa_i,\tilde{u}_i,a,b,f)}{\|u_0-\tilde{u}(\kappa_i)\|_E}.$$

$\Theta(\kappa_i)$ is called the effectivity index of the estimator. Given a transition operator **A**, the problem now is to describe the set $R_4(\rho)$ of the input data and the initial meshes κ_1 such that

(5.10) $$\frac{1}{\rho} < \Theta(\kappa_i) < \rho$$

and the set $\tilde{R}_4$ such that

(5.11] $$|\Theta(\kappa_i)-1| \to 0 \quad \text{as} \quad i \to \infty.$$

If in addition $\Theta(\kappa_i) > 1$ then we will say that R_i is an upper estimator. The problem now is to describe the set $R_4(\rho)$ (resp. $\tilde{R}_4$). The pair $(\Delta,\mathcal{E})$ will be called *adaptive with respect to the effectivity index relative to* $R_{\mathcal{A}}(\rho)$ *or* $\tilde{R}_{\mathcal{A}}$ when (5.10) resp. (or (5.11)) holds.

5.5. Adaptivity of the FEM for two dimiensional problems

The notions of adaptivity for two dimensional problems are defined in analogy to those for one dimension.

The notions of feedback and adaptivity similar to as introduced above were first suggested in [7].

5.6. The selection of the feedback finite element method

We have introduced some basic notions which allow one to compare various feedback FEMs and error estimates. Of course other important comparison could be considered. Nevertheless, we believe the goals of the feedback procedures always have to be exactly defined and theoretically and experimentally investigated.

6. RATE FUNCTION Φ AND ITS PRACTICAL MEANING

The rate function Φ introduced in Section 5.2 is of principal importance because it describes the performance of an "ideal" feedback procedure. It gives also guidelines for the design of concrete feedback FEM by allowing one to judge consequences of programming and other considerations (influencing e.g. the set Q).

6.1. One dimensional model problem

We now mention here some typical illustrative results in the case when $u_0 = w$ with w given by (2.5).

Theorem 6.1. Let $\gamma < 1$ in (2.5) and a, b satisfy (2.1c). Let Q_1 be the set of all meshes such that $\hat{h}(\Delta)/\check{h}(\Delta) < \sigma < \infty$ and $p^{\Delta} = (p,\ldots,p)$, $p > 1$

(i.e. consider the classical h-version with quasiuniform mesh and elements of degree p). Then

(6.1) $$C_1(\alpha,\gamma,p,\sigma)N^{-\mu} \le \Phi(N,Q_1) \le C_2(\alpha,\gamma,p,\sigma)N^{-\mu}$$

where

a) for $0 \le \gamma < 1$:

(6.2) $$\mu = \min\left(p,\alpha - \frac{1}{2}\right),$$

b) for $\gamma < 0$:

(6.3) $$\mu = p.$$

<u>Theorem 6.2</u>. Let $0 \le \gamma < 1$ and a, b, satisfy (2.1c). Let Q_2 be the set of all meshes and $p^\Delta = (p,\ldots,p)$, $p \ge 1$ (i.e. we consider the classical h-version with arbitrary meshes and elements of degree p). Then

(6.4) $$C_1(\alpha,\gamma,p)N^{-p} \le \Phi(N,Q_2) \le C_2(\alpha,\gamma,p)N^{-p}.$$

We see an essential downgrading of the performance of the h version if we restrict ourselves to the quasiuniform meshes and the solution is not smooth. In the case of arbitrary meshes the performance is governed here <u>only</u> by the degree p and not the smoothness of the solution (characterized by α).

(6.4) shows also that use of higher degrees is advantageous. (We will see this further in connection with the p-version and by the numerical examples in Section 7.3.) Nevertheless, we have to assume here that the achieved accuracy is sufficiently good, i.e. we are in an asymptotic range. This could be practically impossible if α is close to $1/2$ (e.g. $\alpha = .55$ and we are restricted by double precision to implement proper meshes without round-off problems as will be seen in Section 7.3).

Theorem 6.3. Let $\gamma = 0$ and a, b satisfy (2.1c). Let Q_3 be the set of all meshes and all distribution of p (i.e. we consider the general h-p version). Then

$$\frac{C_1(\alpha)}{N^t} r^{\sqrt{(\alpha - 1/2)N}} < \Phi(N,Q_3) < C_2(\alpha) r^{\sqrt{(\alpha - 1/2)N}} \tag{6.5}$$

where

$$r = (\sqrt{2} - 1)^2, \quad 0 < t < \frac{1}{2}\alpha. \tag{6.6}$$

We see that the h-p version leads to an exponential rate with respect to $\sqrt{N}$.

Let us address now the p-version. To show essential featuraes we will restrict ourself to the case when only one element is used, i.e. $M(\Delta) = 1$.

Theorem 6.4. Let $a = 1$, $b = 0$ and Q_4 be the set of all meshes with one element. Then

a) for $\gamma = 0$:

$$\Phi(N,Q_4) = \frac{\Gamma(\alpha)^2}{\sqrt{2\alpha-1}} \frac{|\sin \pi(\alpha-1)|}{\pi} \frac{1}{N^{2\alpha-1}} (1+o(1)) \tag{6.7}$$

b) for $\gamma < 0$:

Let

$$q = \frac{\sqrt{1-\gamma} - \sqrt{-\gamma}}{\sqrt{1-\gamma} + \sqrt{-\gamma}},$$

then

b1) for $0 < q^2 < 1 - \frac{1}{N}$:

$$(6.8)\qquad \frac{C_1}{\sqrt{1-q^2}}\,\frac{q^{N-\alpha}}{N^{\alpha}}\left\{\frac{1}{N^{\alpha-1/2}}+(1-q^2)^{\alpha-1/2}\right\} < \Phi(N,Q_4)$$

$$< C_2\Big(1+\frac{1}{\sqrt{1-q^2}}\Big)\,\frac{q^{N-\alpha}}{N^{\alpha}}\left\{\frac{1}{N^{\alpha-1/2}}+(1-q^2)^{\alpha-1/2}\right\}$$

b2) for $1-\frac{1}{N} < q^2 < 1$:

$$(6.9)\qquad C_1\,\frac{q^{N-\alpha}}{N^{\alpha-1/2}}\left\{\frac{1}{N^{\alpha-1/2}}+(1-q^2)^{\alpha-1/2}\right\} < \Phi(N,Q_4)$$

$$< C_2\,\frac{q^{N-\alpha}}{N^{\alpha-1/2}}\left\{\frac{1}{N^{\alpha-1/2}}+(1-q^2)^{\alpha-1/2}\right\}$$

where C_1 and C_2 are positive and independent of N and q

c) for $0 < \gamma < 1$:

$$(6.10)\qquad C(\alpha,\gamma)\,\frac{1}{N^{\alpha-1/2}} < \Phi(N,Q_4)$$

$$< C(\alpha,\gamma)\begin{cases}\dfrac{1}{N^{2\alpha-1}} & \text{for } \theta < \delta < \dfrac{1}{N}\\[2ex] \left(\dfrac{\delta}{N}\right)^{\alpha-1/2} & \text{for } \dfrac{1}{N} < \delta < \dfrac{\pi}{2}\end{cases}$$

where $\delta = \min(\theta,\pi-\theta)$, $\theta = \arccos(2\gamma-1)$. For $\gamma < 0$ the rate is exponential, for $\gamma = 0$ the rate is twice than that for $0 < \gamma < 1$. Comparing $\Phi(N,Q_4)$ and $\Phi(N,Q_1)$ we see that the performance of the p-version is never worse than the h-version with arbitrary p for the uniform mesh but it can be twice ($\gamma = 0$) or more ($\gamma < 0$) as good. This is true under much more general assumptions. If the mesh is optimal then the h-version with sufficiently high p outperforms for $\gamma > 0$ the p-version. The h-p version outperforms both as can be seen through comparison of Theorems 6.2, 6.3 and 6.4.

Theorems 6.1-6.4 describe the performance of "ideal" finite element versions. Although we considered only the special case with $u_0 = w$, similar behaviours and results are valid more generally.

So far we considered meshes without any restrictions on the locations of nodal points. Now we will consider the case when these positions are restricted and analyze the effects of this restriction. This will allow us for example to assess the effectivity of the procedure which is related to the search for positions of the nodal points.

We will consider sequence of meshes $\{\kappa_i\}$ (i.e. trajectories) which are recursive and such that Δ_{i+1} consists of some bisected elements of Δ_i. The set of trajectories of this type will be denoted by Q_i^* in contrast to Q. The meaning of Q_i^* (in comparison with Q_i) is obvious. Let us now mention some typical theorems.

Theorem 6.5. Let the assumptions of Theorem 6.2 hold. Then

$$4^{-p} < \frac{\Phi(N,Q_2)}{\Phi(N,Q_2^*)} < 1. \tag{6.11}$$

We remark that the left hand side of (6.11) is conservative and likely can be improved.

Theorem 6.6. Let the assumption of Theorem 6.3 hold. Then

$$\Phi(N,Q_3^*) < \tilde{C}_2(\alpha) r_*^{\sqrt{(\alpha - 1/2)N}} \tag{6.12}$$

$$r_* = r^{\sqrt{\lg \frac{1}{2} / \lg(\sqrt{2}-1)}} \qquad (6.13)$$

where r is given by (6.6).

Theorems of the mentioned type are important for the design of feedback procedures. For example, although the estimate (6.11) is conservative, we see that if the optimal positions of the nodal points are not obtained by at most one or two iterations, it is more effective to increase the number of nodal points with the use of recursive meshes than to further seek the optimal position.

6.2. The two dimensional model problem

The situation in two dimensions is similar. As illustration we will mention theorems which deal with our first model of elasticity problem.

Theorem 6.7. Consider the h-version with quasi-uniform mesh. Then

$$C_1(p,\sigma)N^{-1/4} < \Phi(N) < C_2(p,\sigma)N^{-1/4}. \qquad (6.14)$$

Theorem 6.8. Consider the h-version with arbitrary meshes. Then

$$C_1(p)N^{-p/2} < \Phi(N) < C_2(p)N^{-p/2}. \qquad (6.15)$$

We see that Theorems 6.7 and 6.8 are analogous to 6.1 and 6.2 where N is replaced by $N^{1/2}$.

As the analog of Theorem 6.3 we have

Theorem 6.9. Consider the h-p version, Then

$$\Phi(N) < C\,e^{-\beta\sqrt[3]{N}}, \qquad \beta > 0. \qquad (6.16)$$

There is no sharp estimate for β available, neither is it known whether the cube root in (6.16) is the best possible.

In contrast to the one dimensional case, a restriction can be placed here not only on the meshes but also on the structure of polynomials of the elements. It can be shown that β in (6.16) can be increased if the polynomial elements <u>do not</u> necessarily consist of <u>all</u> shape functions of the same degree (i.e. are chosen selectively) in comparison with the case when all shape functions of the same degree are used. This fact could be important for the design of a feedback procedure. See e.g. [9].

7. SOME SPECIFIC FEEDBACK PROCEDURES

We will formulate a specific feedback finite element method and analyze its properties in the light of Section 6.

7.1. The feedback procedure for the h-version

Let us consider the problem (2.1a,b,c) and for simplicity use only $p = 1$. Let

$$(7.1) \qquad (\tilde{\eta}_j^\Delta)^2 = \int_{x_{j-1}^\Delta}^{x_j^\Delta} (az'^2 + bz^2)dx$$

where $z(x)$ is such that

$$(7.2) \qquad -(az')' + bz = f - [-(a\tilde{u}')' + b\tilde{u}] = r(x)$$

$$(7.2a) \qquad z(x_{j-1}^\Delta) = z(x_j^\Delta) = 0.$$

$z(x)$ is the solution of the original problem on I_j^Δ when $f(x)$ is replaced by $r(x)$. We will assume that the number $\tilde{\eta}_j^\Delta$ can be well approximated by η_j^Δ which is easily computable. For example we define

(7.3) $$(\eta_j^\Delta)^2 = \frac{h_j^2}{12\ a(x^\Delta_{j-1/2})} \int_{x^\Delta_{j-1}}^{x^\Delta_j} r^2(x)dx$$

$$x^\Delta_{j-1/2} = \frac{x^\Delta_{j-1}+x^\Delta_j}{2}.$$

The number η_j^Δ is called an <u>error indicator</u>. The error estimator $\mathcal{E}$ is now defined as follows:

(7.4) $$\mathcal{E}^2 = \sum_{j=1}^{M(\Delta)} (\eta_j^\Delta)^2.$$

We will assume that

$$|\eta_j^\Delta - \tilde{\eta}_j^\Delta| < \phi(h_j^\Delta)|\tilde{\eta}_j^\Delta|$$

where $\phi(h_j^\Delta) \to 0$ as $h_j^\Delta \to 0$ which allows us consider $\tilde{\eta}_j^\Delta$ and η_j^Δ as identical.

Assume now that Δ_i, $\tilde{u}_i$, $\eta_j^{\Delta_i}$ have been computed. Then Δ_{i+1} is construced by bisecting all elements of Δ_i for which $\eta_j^{\Delta_i} > \lambda \max_\ell \eta_\ell^{\Delta_i}$ where $0 < \lambda < 1$ is an a-priori given parameter which steers the process.

We will now state basic theorems related to this feedback procedure.

<u>Theorem 7.1</u>. The set $\mathcal{R}_1$ (see Section 5) consists of all input data satisfying (2.1c), all solutions of $\mathring{H}^1(I)$ and an arbitrary initial mesh κ_1.

<u>Theorem 7.2</u>. For any $0 < \lambda < 1$, $\tilde{\mathcal{R}}_2 \subset \mathcal{R}_1$, $\tilde{\mathcal{R}}_2 \neq \mathcal{R}_1$.

Theorem 7.2 shows that to get adaptivity with respect to the convergence rate, stringent restriction on $\tilde{\mathcal{R}}_2$ has to be made. In [4] we formulated

sufficient conditions so that $(a,b,f) \in \tilde{R}_2$.

As an illustration we state

<u>Theorem 7.3</u>. Let (2.abc) be satisfied. Assume in addition that $a(x)$ satisfies Hölder condition with exponent $\theta > \frac{1}{2}$. Then if $0 < \lambda < 1$, the input data (a,b,f) such that $u_0 = w$ and arbitrary initial mesh Δ_1 belong to $\tilde{R}_2$.

Combining Theorems 7.2, 7.3 and 6.2 we see that

$$\|u_0-\tilde{u}_1\|_E < C(\gamma,\alpha)N^{-1} .$$

Under what condition the input data belong to $R_2(\rho)$ with an a-priori given ρ is not known.

Theorem 6.1 shows immediately that if $\lambda = 0$, $\gamma > 0$ and $\alpha < 3/2$ then $u_0 = w$ does not belong to $\tilde{R}_2$. Nevertheless, if $\gamma < 0$ and $0 < \lambda < 1$, $u_0 = w$ belongs to $\tilde{R}_2$.

<u>Theorem 7.4</u>. Let $\lambda = 1$. If $u_0 = w$ and $0 < \gamma < 1$, then the input data do not belong to $R_3(\rho)$ for any ρ. If $0 < \lambda < \lambda_0 < 1$ and a, b satisfy the assumptions of Theorem 7.3, then $u_0 = w$ belongs to $R_3(\rho)$ with $\rho < \rho_0(\alpha,\gamma)$.

<u>Theorem 7.5</u>. Let conditions (2.abc) hold, a', b be continuous and f $L_2(I)$. Then

$$|\theta(\kappa)-1| < C\hat{h}^2(\Delta)$$

with C dependent on the input data but independent of the mesh.

Any design of a feedback approach has to involve a compromise among various requirements. Theorems of the type mentioned above (in larger generality) proven theoretically or experimentally (as conjectures) allow one to make a rational choice and an assessment of the design.

7.2. The feedback procedure for the h-p version

Analogously, but in a much more complex way a feedback procedure for the h-p version can be designed. The main difficulty is to decide when to refine the mesh and when to increase the degree. We cannot go into details here, but remark that we have designed a feedback approach and analyzed its properties. The results will be shown in numerical examples.

7.3. Numerical examples for the one dimensional problem

In this section we will show numerical data and analyze them from the theoretical point of view. We will also mention in this connection additional theoretical results pertinent to the analysis.

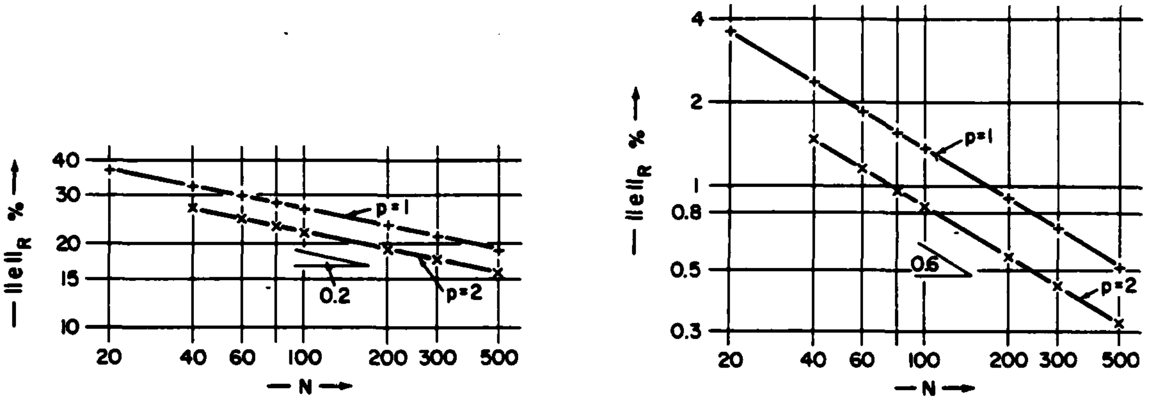
Let $e = u_0 - \hat{u}$ and

$$\|e\|_R = \frac{\|e\|_E}{\|u_0\|_E}$$

and consider the performance of the finite element method when $u_0 = w$ and w be defined by (2.5). We consider $a = 1$, $b = 0$ in (2.1).

Figs. 7.1 ab show the performance of the h-version with <u>uniform</u> mesh for $\gamma = .3$ and $\alpha = .7$ and 1.1 (in log log scale). The shown slope is the theoretical one based on Theorem 6.1 ($\mu = .2$ and $\mu = .6$).

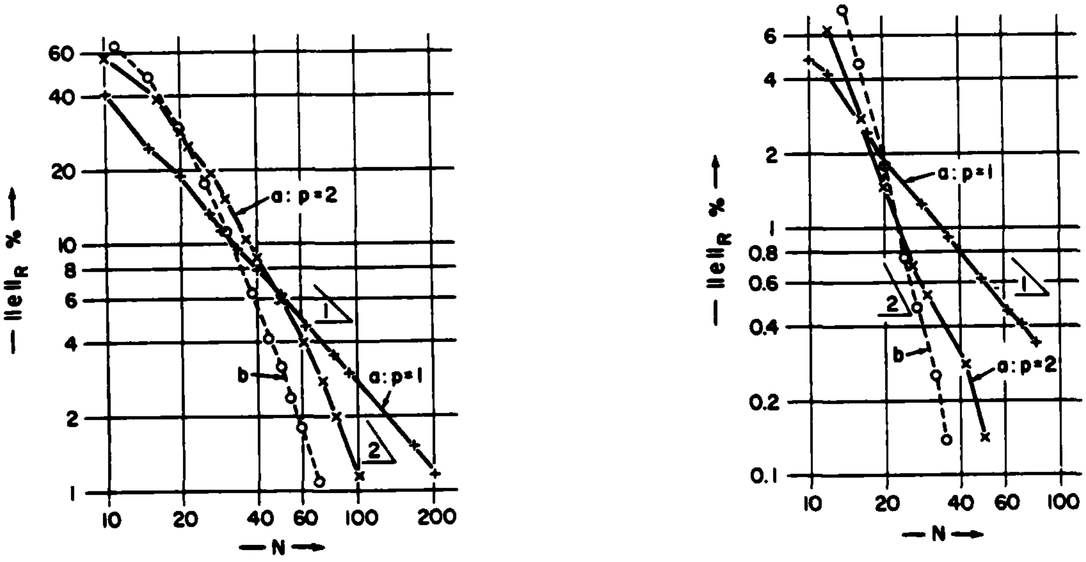
Figures 7.2 ab depict the performance of the feedback h-version (for $p = 1,2$) and the feedback h-p version.



γ = .3   α = .7

γ = .3   α = 1.1

Figure 7.1. The performance of the h-version with uniform mesh.



γ = .3   γ = .7

γ - .3   γ = 1.1

Figure 7.2. The performanace of the feedback h-version and the feedback h-p version. a: h version, b: h-p version.

We see that the rate of convergence of the feedback h-version is $O(N^{-1})$, and $O(N^{-2})$ for $p = 1$ and $p = 2$ independently of α. Recalling Theorems 6.2 and 7.3 we see that the method is adaptive with respect to the convergence rate measure. The feedback h-p version shows exponential rate.

Table 7.1 shows the coordinates of the nodal points constructed by the feedback h-version ($p = 1$) and $M = 80$. We see that the points are located very close to the singularity point (.3) which prevents to get higher accuracy due to limited computer accuracy.

TABLE 7.1

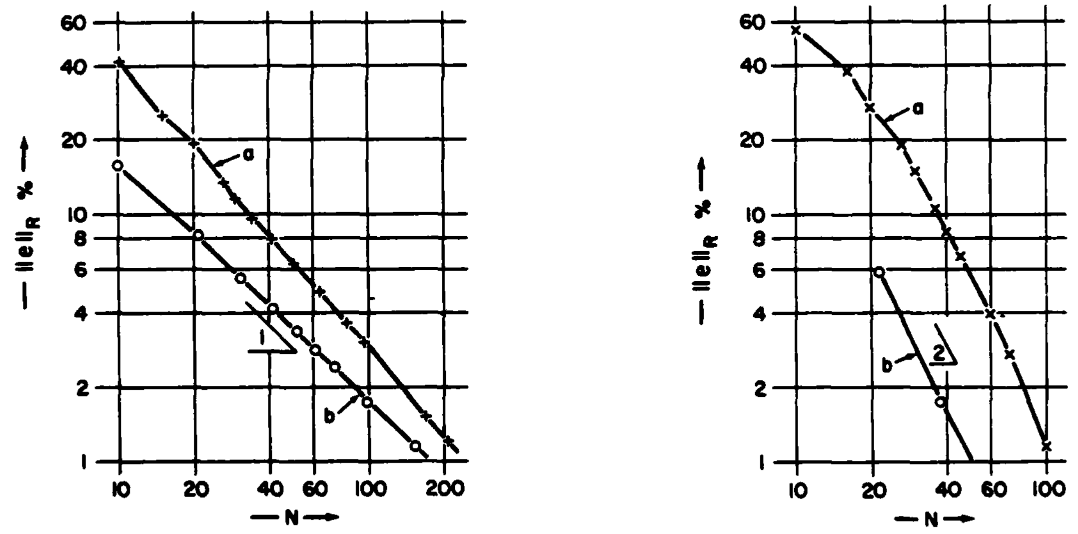
The locations of the nodal points for the feedback h-version
($p = 1$, $M = 80$, $\gamma = .3$, $\alpha = .7$)

n	χ_n^Δ	n	χ_n^Δ
0	0.00	36	.3000183105468750
1	.25	.	. . . .
2	.28125	46	.301025390625
3	.286875	47	.3046875
4	.298828125	.	. . . .
5	.2998046875	66	.390625
6	.2999267578125	.	. . . .
7	.2999877929875	71	.50
8	.29999542236283	72	.53125
9	.2999992370605468	.	. . . .
.	. . . .	80	.93750
19	.2999999999956344	81	1.00
20	.3000000000029104		
.	. . . .		

Figs. 7.3 ab are the illustration of Theorem 6.5; the graph b is the error for the asymptotic optimal mesh and a of the feedback based on bisecting. We see that the bisecting restriction increases the error roughly by the factor 4^p.

Fig. 7.4 shows in semilogarithmic scale ($\log\|e\|_R$ and $\sqrt{N}$) the performance of the h-p feedback FEM for $\alpha = .7$ and $\alpha = 1.1$. Using Theorem 6.3 (in slightly generalized form) the graphs in the asymptotic range should be linear. The slopes b,d and ac shown in the figure are based on (6.5) and (6.12). We see very good agreement from which we conclude that the feedback is adaptive with respect to the convergence rate.

Fig. 7.5 is illustration of Theorem 6.4. It depicts the performance of the p-version (a) with one element only for various γ. We see large sensitivity to γ of the accuracy for higher p. The figure also shows the performance of the h resp h-p feedback FEM (graph b resp. graph c)



$\gamma = .3$, $\alpha = .7$, $p = 1$

$\gamma = .3$, $\alpha = .7$, $p = 2$

Figure 7.3. The performance of the h-verson with the optimal mesh and the feedback h-version. a: the feedback h-version, b: the h-version with optimal mesh.

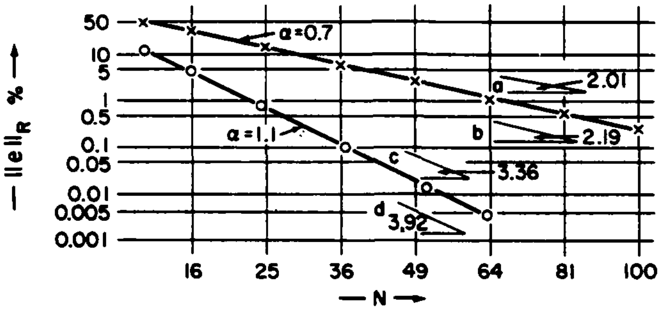


Figure 7.4. The performance of the h-p feedback, b,d: slopes based on (6.5), a,c: slopes based on (6.12).

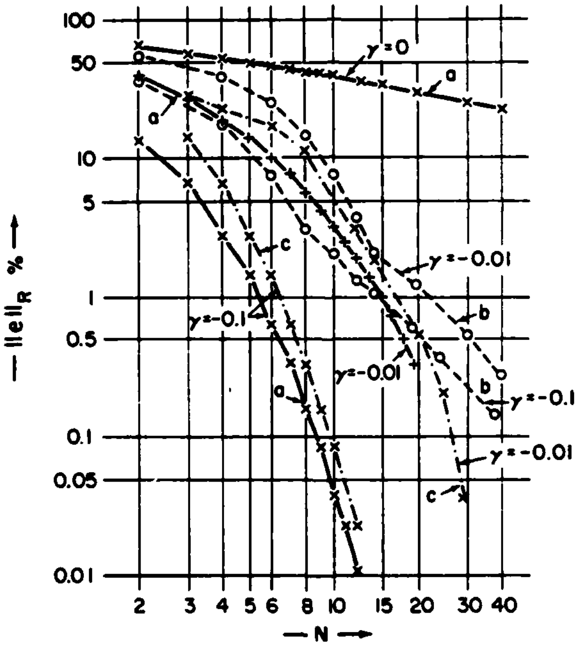


Figure 7.5. The performance of the p and feedback h and h-p versions for $\gamma < 0$, a: p-version, b: feedback h-version, c: feedback h-p version.

7.4. Feedback procedure for the h-version in two dimensions

During the last few years a feedback program FEARS (with $p = 1$) was developed at the University of Maryland. For more about FEARS, see e.g. [10].

FEARS has, among others, the following properties:

a) It has been proven that it is adaptive with respect to the convergence measure for a wide class of input data.

b) It is conjectured that it is adaptive with respect to the convergence rate measure for a large class of input data of practical importance. Although there is much experimental evidence to support this conjecture, the proof is not available.

c) Experience shows that FEARS is adaptive with respect to the work measure with $\rho < 2$ or 3 in all practical cases. The theoretical proof is not available.

d) It has been proven that FEARS is adaptive with respect to the effectivity index of the error estimator for a $\rho > 1$ with ρ independent of the mesh and domain, and large class of input data.

e) It has been proven that $\Theta(\kappa_1) \to 1$ under certain relatively restrictive assumptions. There is a conjecture that these assumptions are not necessary.

f) It is conjectured that the estimator is an upper one for a large class of input data provided that the achieved accuracy is sufficiently good. The mathematical proof is not available, but experimental results nevertheless support this conjecture.

7.5. Numerical illustration

We solved by Fears the cracked panel elasticity problem introduced in Section 2.2. Table 7.2 shows the sequence of meshes, constructed by FEARS, the true (relative) error and the effectivity index. Fig. 7.6 shows the mesh with $N = 617$.

TABLE 7.2

The performance of FEARS computation.

MESH	DOF	$\frac{\|e\|_E}{\|u_0\|_E}$ %	Θ
1	101	26.38	.885
2	143	21.35	.991
3	221	16.79	1.058
4	301	13.61	1.116
5	617	9.63	1.088

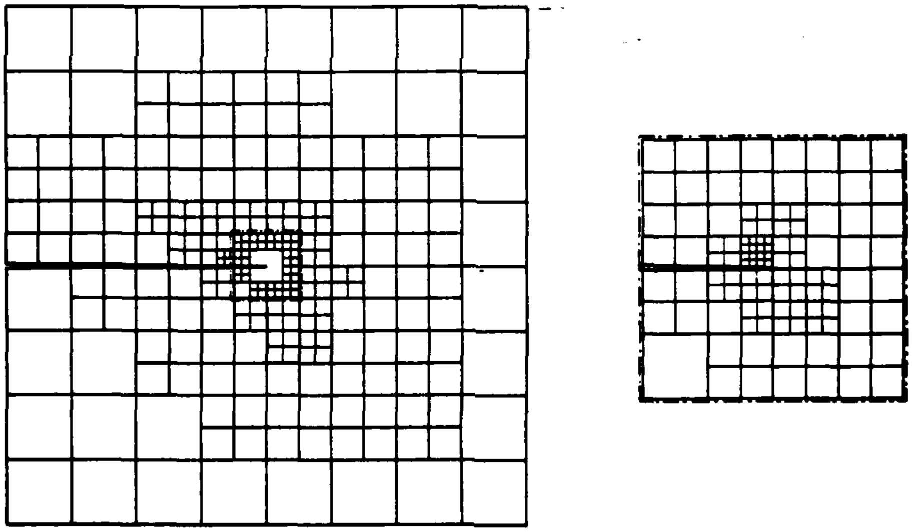


Figure 7.6. The mesh constructed by FEARS for $N = 617$.

Table 7.2 shows:

a) The rate of convergence is $O(N^{-1/2})$ and supports the conjecturae that FEARS is adaptive with respect to the convergence rate measure.

b) The error estimator is an upper one when the error is sufficiently small.

c) FEARS is adaptive with respect to the effectivity index of the estimator.

d) Because $N_{i+1}/N_i = 1.3$ to 2, and it has been seen that the work (machine time) is proportional to $N^{1.8}$, FEARS is adaptive with respect to the work measure with $\rho = 2$ in our particular case.

8. THE MULTIPLE LOAD PROBLEM

So far we addressed only the problem of adaptive procedures for <u>one</u> load only. In practice we have often consider few loads (boundary conditions) and for obvious computational reasons, we wish to use one mesh for all these cases. Theoretically (and practically) the multiple load problem does not bring any difficulties. We will show it in a second example in Section 2.2. We (theoretically) solve a (noncoupled) system

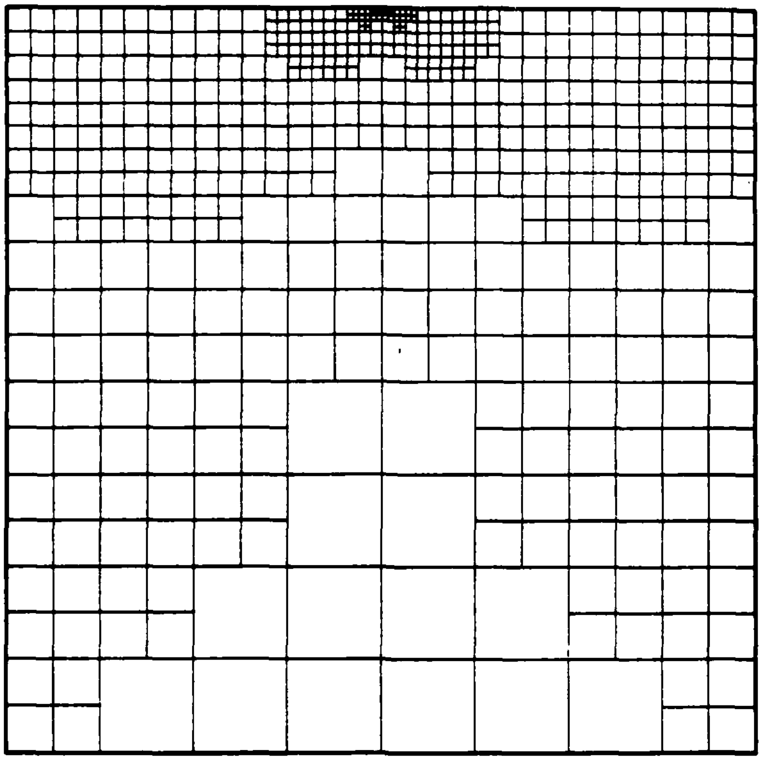
$$\Delta u_1 = 0, \qquad \frac{\partial u_1}{\partial n} = g_1$$

$$\Delta u_2 = 0, \qquad \frac{\partial u_2}{\partial n} = g_2$$

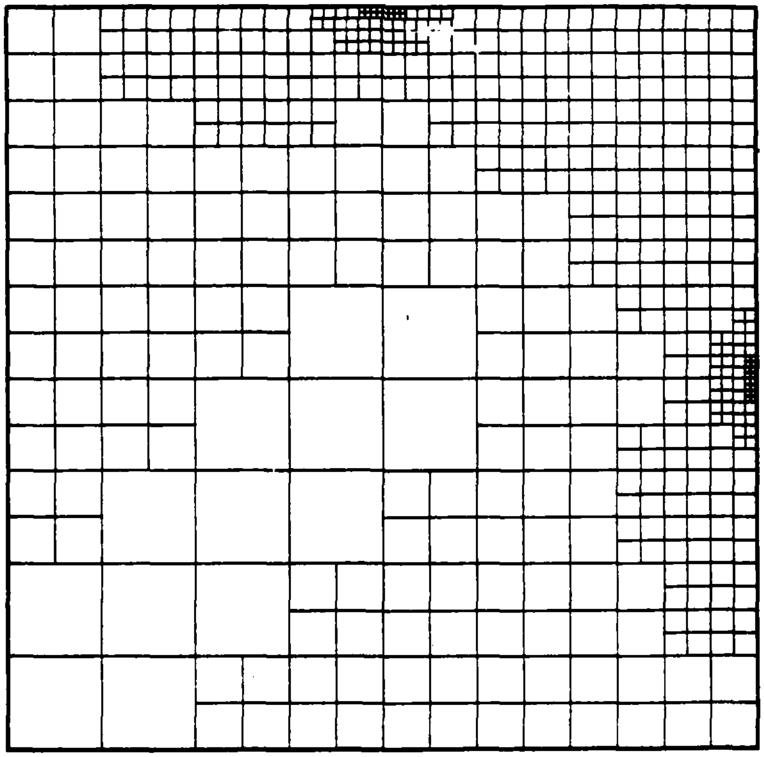
and define (for example)

$$\|R\|_E^2 = \|e_1\|_E^2 + \|e_2\|_E^2 .$$

(Of course we determine u_1, u_2 by using one stiffnes matrix only). Figure 8.1a shows the mesh (constructed by FEARS) for u_1 $(N = 538)$ (the mesh for u_2 is the same but rotates by 90^o). Figure 8.1b shows the mesh for multiple loads g_1, g_2, $N = 531$. Figure 8.2 shows $\|e_i\|_E$ for the mesh constructed for u_i as simple load case $i = 1,2$ and for the mesh which is used simultaneously for both cases.



a.



b.

Fig. 8.1. The meshes constructed by FEARS for simple and multiple load approach. a: The mesh for simple load g_1 (N = 538). b: The mesh for multiple load g_1, g_2 (N = 531).

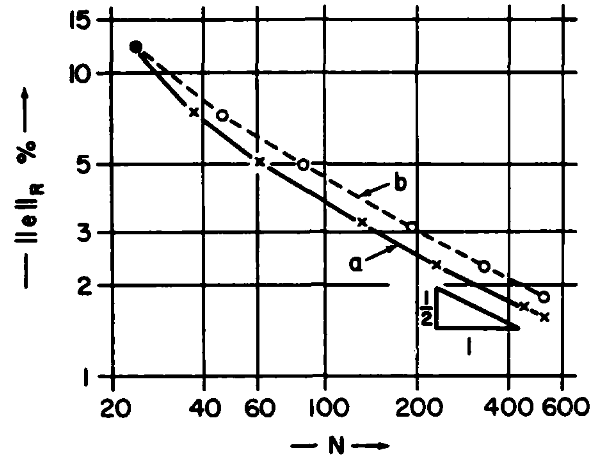


Fig. 8.2. The error $\|e_1\|_E$ using single load and multiple load meshes
a: single load; b: multiple load.

9. FEEDBACK PROCEDURES WITH RESPECT TO OTHER ACCURACY REQUIREMENTS

In the previous section we concentrated exclusively on the accuracy measured in the energy norm. Nevertheless, in structural mechanics we are often interested in the computation of some critical values (functionals) as for example, stresses in some points, stress intensity factors, etc.

There are various ways to extract these value from the finite element solution. These extractions are implemented by the <u>postprocessing technique</u>. The adaptivity and the a-posteriori estimates for the postprocessing approach have some special features. We cannot go here more in details and we refer only to [3] [11].

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