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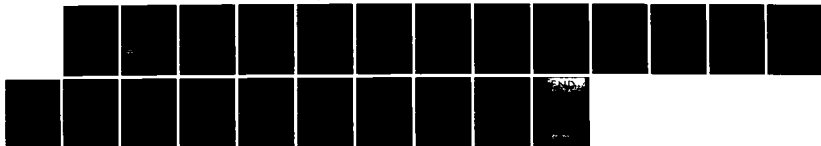
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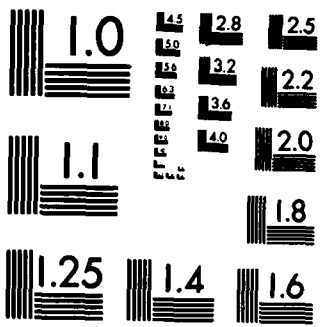
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OMNI TRANSFORMS: APPLICATIONS TO RENEWAL THEORY

Submitted to:

Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

Attention: Group Leader, Statistics and Probability
Mathematical and Physical Science

Submitted by:

Martin Krakowski
Research Associate

Carl Harris
Professor

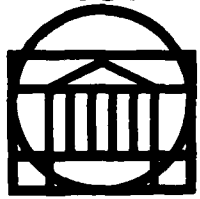
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SCHOOL OF ENGINEERING AND
APPLIED SCIENCE

DEPARTMENT OF SYSTEMS ENGINEERING

UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA 22901

A Technical Report

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OMNI-TRANSFORMS: APPLICATIONS TO RENEWAL THEORY

ABSTRACT

The theory of the renewal process has an important bearing on the single server queue, especially on the model $M/G/1$ (although some results are obtained for $G/G/1$ also). Since a renewal process can be viewed as a single-server saturated system (i.e., a system whose source keeps the server perpetually busy) this affinity is not surprising. Moreover, in deriving some results for the renewal process we gain insight into the use of the conservation method, based on the application of ergodic variables and of the so-called omni-transform, to the treatment of single-server queues.

The renewal process adds insight into the theory of mixtures of random variables and of omni-forms (i.e., linear functions of omni-transforms). It allows to convert, when possible in principle, an omni-equation involving derivatives into an omni-equation without derivatives, an equation which is a statement about a mixture of distributions.

1. The Omni-Transform

Let Y be a random variable and $\psi(Y)$ an arbitrary but well-behaved function of Y ; "well-behaved" means that the expectation of $\psi(Y)$ is finite, and that all formal operations which are applied are valid. This rough description will suffice, at least for the time being. In most of our applications Y will be a positive random variable.

Definition The expectation of $\psi(Y)$, an arbitrary (though well-behaved) function of Y , will be called the omni-transform of Y : $E[\psi(Y)]$.

This is a very wide definition. Thus, the Laplace transform of Y (i.e., the Laplace transform of the distribution of Y) is obtained by putting $\psi(Y) = e^{-sy}$; the generating function of the random variable N is obtained by putting $\psi(N) = z^N$. In a future report we will show how to derive omni-transforms from Laplace transforms and from generating functions.

The omni-transform owes its flexibility to the free choice of $\psi(Y)$; and its ease of operation to the fact that expectations, because of their additive property, are easy to handle. Some problems which require the inversion of a Laplace transform or the differentiation, at times repeated differentiation, of a transform are found with little effort from an omni-equation. The derivation of moments is especially simple. The inverse of an omni-transform is not even defined in any sense analogous to the inverse of a Laplace transform.

The applications of omni-transforms to the theory of runs and to Markov and semi-Markov chains will be described in a future report.

It is instructive to compare the equation

$$x = y \text{ (x and y are random variables)} \quad (1.1)$$

with the omni-equation

$$E[\psi(x)] = E[\psi(y)], \text{ for all } \psi. \quad (1.2)$$

Equation (1.1) states that x and y are identical, i.e., that " x " and " y " are different names for the same random variable. Equation (1.2) makes the weaker statement that x and y have the same distribution; nothing is said about x and y being identical, dependent, or independent.

The equation

$$E[e^{-sx}] = E[e^{-sy}], \text{ for all } s \geq s_0, \quad (1.3)$$

which is a special case of (1.2) when $\psi(x) = e^{-sx}$, also leads to the conclusion that x and y have the same distribution since the family e^{-sx} is rich enough for this conclusion (indeed the validity of (1.3) for rational s is adequate). But omni-equations, of which (1.2) is the simplest specimen, are in many contexts much simpler to use than Laplace transforms.

2. The Renewal Process

The theory of the renewal process has, in our approach, an important bearing on the single server queue, especially on the model M/G/1 (although some results are obtained for G/G/1 also). Since a renewal process can be viewed as a single-server saturated system (i.e., a system whose source keeps the server perpetually busy) this affinity is not surprising. Moreover, in deriving some results for the renewal process we gain insight into the use of the conservation method, based on the use of ergodic variables, and into the use of omni-transforms.

Consider a service station (also called a renewal station) and an ample source of customers so that no gaps arise between services.

Any positive random variable can be interpreted as service time. We will usually denote the service time as x .

Note The service time x can be visualized in two physically different ways. One way is to follow the demographic or actuarial analogy and to treat the sojourn time x as a kind of lifetime subject to mortality law; the lifetime of an arrival is known only statistically. Another way is to assume that a customer arrives with a random load x whose precise value is revealed at the instant of arrival which is also the instant of entry into the station; and that the server works uniformly on the load x whose measure is the length of service. The second interpretation of x is usually more tractable analytically and has been commonly used in queuing theory in the context of "unfinished work." Unless otherwise specified, we follow the random-load interpretation.

Along with x we consider two associated random variables: "age" (or "seniority") and "residual life".

Definition of Age The incumbent's "age" (or "seniority") is the time, as of a given instant, which he has already spent in the station. The age can be observed continuously or in a Poisson manner; the two methods yield the same probability distribution, a fact used in Monte Carlo and in simulation models. When there is no ambiguity we can denote the age as g , a random variable.

Definition of "Residual Life" The incumbent's "residual life", or "residue" for terseness, in his remaining time to be spent in service as of a given instant; it will be designated by r when there is no ambiguity. The residue can be observed continuously or in a Poisson manner; the two methods yield the same probability distribution.

When x is interpreted as the working load which is numerically equal to the service time we have $dt = dg = -dr$, provided that the service is not completed during dt .

The notation r and g is satisfactory when one considers a service time which is not a function of other random variables and when no operations typified by "residue of a residue" enter into the problem. For such cases a more systematic notation is introduced.

Definition of the Operators R and G The operator R acting on the positive random variable x converts it into the residue of x : $Rx = \text{residue of } x$. Similarly, the operator G converts x into the age of x : $Gx = \text{age of } x$. Parentheses will be used in cases such as $R(x+y)$. An n -fold application of the operator R can be denoted by R^n . The operators are applied in order of their proximity to the argument. Thus, $G^2x = G(G(x))$ and $RGx = R(Gx)$.

In our approach to renewal and queuing models under steady-state conditions we will make much use of what we call ergodic processes.

Definition A process Y will be referred to as ergodic if during a random dt the expected change of $\psi(Y)$ is zero; here $\psi(Y)$ is an arbitrary function of

Y whose expectation is finite. This definition, ad hoc and not rigorous, is intended to capture the essence of conservation in stochastic models. Perhaps, in a future report we will define ergodic processes more rigorously. So far, a process has been declared to be ergodic on physical or economic grounds, e.g., the number of customers in a queue or the remaining load.

Consider now the ergodic process $Y =$ "time spent in service by a customer of a renewal station." This process manifests itself as $x =$ "completed service at an instance of departure"; and as $g =$ "age of an incumbent in service." The frequency of departures (= frequency of entries) is $\mu = 1/E(x)$. The expected change of $\psi(Y)$ during a random dt is due to arrivals and departures and aging: $dt\mu E[\psi(0)]$ due to arrivals; $-dt\mu E[\psi(x)]$ due to departures; and $dtE[\psi'(g)]$ due to aging. The three contributions add up to zero:

$$+dt\mu E[\psi(0)] - dt\mu E[\psi(x)] + dtE[\psi'(g)] = 0.$$

Hence, replacing μ by $1/E(x)$, we have

$$E[\psi(x)] - \psi(0) = E(x)E[\psi'(g)]. \quad (2.1)$$

We will refer to equations valid for arbitrary $\psi()$ as omni-equations. Equation (2.1) is an important instance of a linear omni-equation, each coefficient of $\psi()$ and of its derivative being a constant. In a linear omni-equation there is never a term without $\psi()$ or without a derivative of $\psi()$.

In order to simplify the typography and to improve the esthetics of omni-equations we introduce the omni-convention.

The Omni-Convention: In a linear omni-equation it shall be understood that each $\psi()$ or $\psi'()$, etc. stands for $E[\psi()]$ or $E[\psi'()]$, etc. Thus, equation (2.1) can be written as

$$\psi(x) - \psi(0) = E(x) \psi'(g). \quad (2.2)$$

(Our omni-convention is analogous to the summation convention of tensor and matrix calculus, the so-called Einstein convention.)

Example If $\psi(x) = x^2$ then $\psi'(g) = 2g$; hence

$$E(g) = E(x^2)/2E(x). \quad (2.3)$$

Example If, more generally, $\psi(x) = x^{k+1}$ then $\psi'(g) = (k+1)g^k$ and (2.2)

yields

$$E(g^k) = E(x^{k+1})/(k+1)E(x). \quad (2.4)$$

Example If $\psi(x) = E(e^{-sx})$ then $\psi'(g) = -se^{-sg}$ and (2.2) yields

$$E(e^{-sg}) = [1 - E(e^{-sx})]/sE(x). \quad (2.5)$$

Equation (2.5) relates the Laplace transforms of x and $g = Gx$, that is, the Laplace transforms of their probability distributions.

We now define

$$F_x(t) = \text{Prob}[x \leq t] \quad \text{and} \quad F_g(t) = \text{Prob}[g \leq t], \quad (2.6)$$

and get all integrals extending from 0 to ∞ ,

$$E[\psi(x)] = \int \psi(t)d[1 - F_x(t)];$$

and since $F_x(t) \rightarrow 1$ as $t \rightarrow \infty$,

$$E[\psi(x)] = \psi(0) + \int [1 - F_x(t)]d\psi(t). \quad (2.7)$$

Furthermore,

$$E[\psi'(g)] = \int \psi'(t)dF_g(t) = \int F_g'(t)d\psi(t). \quad (2.8)$$

From (2.1), (2.7), and (2.8) we have

$$\int [1 - F_x(t)]d\psi(t) = E(x) \int F_g'(t)d\psi(t).$$

Since $\psi(t)$ is arbitrary the above equation implies that

$$F_g'(t) = [1 - F_x'(t)]/E(x). \quad (2.9)$$

(Note: Equation (2.9) can also be obtained by putting $\psi(x) = H(t-x)$ where $H(t) = 1$ for $t \geq 0$ and $H(t) = 0$ otherwise; and treating $H'(t)$ as the Dirac delta function.)

The reasoning which has led to the derivation of (2.1) can be repeated with $Z =$ "time to be spent in service by a customer of a renewal station" in place of Y ; and with $r =$ "residual time of an incumbent in service" in place of g . Note that the two random variables "completed service" and "entering load" have identical distributions.

The equation (2.2), has its analog therefore in

$$\psi(x) - \psi(0) = E(x)\psi'(r) \quad (2.10)$$

and (2.9) has its analog in

$$F_r'(t) = [1 - F_x(t)]/E(x), \quad (2.9a)$$

where $F_r(t) = \text{Prob}[r \leq t]$. In turn, Equations (2.3), (2.4), and (2.5) can be extended to

$$E(r) = E(g) = E(x^2)/2E(x) \quad (2.3a)$$

$$E(r^k) = E(g^k) = E(x^{k+1})/(k+1)E(x) \quad (2.4a)$$

$$E(e^{-sr}) = E(e^{-sg}) = E(1 - e^{-sx})/sE(x). \quad (2.5a)$$

Functions of Both r And g

Consider now the ergodic variable $Y = \psi(r,g)$ where $\psi(r,g)$ is an arbitrary function of r and g . The expected change in $\psi(r,g)$ during a random dt is made up of three components (if we omit contributions of order $(dt)^2$):

- (a) the continuous contribution while service is in progress equals

$$-D_1 \psi(r,g)dt + D_2 \psi(r,g)dt, \text{ since } dt = dg = -dr; \text{ here } D_j$$

is the partial differentiation operator for the j -th argument;

- (b) the contribution of arrivals equals $\psi(x,0)\mu dt$; and

- (c) the contribution of departures equals $-\psi(0,x)\mu dt$.

Therefore, with $\mu = 1/E(x)$,

$$\psi(x,0) - \psi(0,x) = E(x)[D_1\psi(r,g) - D_2\psi(r,g)]. \quad (2.11)$$

Example Let $\psi(r,g) = r^m g^n$ with $m \geq 1$ and $n \geq 1$. Then $\psi(x,0) = \psi(0,x) = 0$ and

$$mE[r^{m-1}g^n] = nE[r^m g^{n-1}]. \quad (2.12)$$

If $m = 2$ and $n = 1$ (cf. (2.4a)) we get

$$E(rg) = E(r^2/2) = E(g^2/2) = E(x^3)/6E(x). \quad (2.13)$$

Similarly, we can derive the expectations of higher mixed moments of r and g in terms of the moments of x . Repeated use of (2.12) yields

$$E[r^m g^{n-1}] = \frac{m!(n-1)!}{(m+n-1)!} E[g^{m+n-1}] \quad (2.14)$$

Replacing $n-1$ by n and applying (2.4a) we obtain

$$E[r^m g^n] = E[g^{m+n}] / \binom{m+n}{m} = E[x^{m+n+1}] / E(x) (m+n+1) \binom{m+n}{m} \quad (2.15)$$

which holds, as easily verified, for $m \geq 0$ and $n \geq 0$.

Example Let $\psi(r,g) = \phi(ar + bg)$. Note that this is a specification, and not an omni-equation, and the omni-convention does not apply; therefore we use two different Greek letters in the specification. Of course, $\phi(\)$ is an arbitrary function of its argument. Equation (2.11) implies the omni-equation (omni-convention applies)

$$\phi(ax) - \phi(bx) = E(x)(a-b)\phi'(ar + bg). \quad (2.16)$$

When $a = b$ (2.16) results in $0 = 0$. Let us therefore see what happens when $a \rightarrow b$. Let $a = b + \delta$; (2.16) becomes, upon dividing both sides by $a - b$

$$\frac{\phi(bx + \delta x) - \phi(bx)}{\delta} = E(x) \psi'(br + \delta r + bg).$$

When $\delta \rightarrow 0$ and $b = 1$ we have

$$E[x\phi'(x)] = E(x)E[\phi'(r+g)]. \quad (2.17)$$

(Note that in (2.17) no use is made of the omni-convention because the term $x\phi'(x)$ makes it nonlinear.)

Since $\phi'()$ is an arbitrary function we can replace it by another arbitrary function designated by $\psi()$, thus reverting to the more customary designation, and (2.17) becomes, equivalently,

$$E[\psi(r+g)] = E[x\psi(x)]/E(x). \quad (2.18)$$

Example Let $\psi(r,g) = \phi(r-g)$. From (2.16) with $a = 1$ and $b = -1$ we get

$$\phi(x) - \phi(-x) = 2E(x)\phi'(r-g). \quad (2.19)$$

Let $\phi(r-g) = (r-g)^k$. We have from (2.19)

$$E[(r-g)^k] = E[x^{k+1} - (-x)^{k+1}]/2(k+1)E(x). \quad (2.20)$$

Hence, with the aid of (2.4a) we have

$$\text{even } k \rightarrow E[(r-g)^k] = E(x^{k+1})/(k+1)E(x) = E(g^k) = E(r^k) \quad (2.20a)$$

$$\text{odd } k \rightarrow E[(r-g)^k] = 0. \quad (2.20b)$$

Let us now derive the joint probability density $p(t,u)$ of the residue $r = Rx$ and the age $g = Gx$. This $p(t,u)$ is the probability density perceived by either a perpetual or a poissonian observer of the processes x , r , and g . [Note: r and g can also be interpreted in terms of a stationary population in a demographic model.]

Theorem The joint probability density $p(t,u)$ of $r = Rx$ and $g = Gx$ satisfies

$$p(t,u) = f_x(t+u)/E(x) \quad (2.21)$$

where $f_x(t)$ is the p.d. of the lifetime x .

Proof This proof is based on (2.11). All integrals in the proof are from 0 to ∞ .

$$E[D_1\psi(t,u)] = \int \int p(t,u)D_1\psi(t,u)dtdu.$$

Now,

$$\int p(t,u)D_1\psi(t,u)dt = p(t,u)\psi(t,u)\Big|_{t=0}^{\infty} - \int \psi(t,u)D_1p(t,u)dt$$

and, since $p(t,u)$ tends to 0 as u tends to ∞ ,

$$\int p(t,u)D_1\psi(t,u)dt = -p(0,u)\psi(0,u) - \int p(t,u)D_1p(t,u)dt$$

and

$$\int \int p(t,u)D_1(t,u)dtdu = - \int p(0,u)\psi(0,u)du - \int \int p(t,u)D_1p(t,u)dtdu$$

Proceeding similarly for $E[D_2\psi(r,g)]$ we find that the right-hand side of (2.11) is

$$E(x) E[D_1\psi(r,g) - D_2\psi(r,g)] = E(x)\{- \int p(0,u)\psi(0,u)du + \int p(t,u)\psi(t,0)dt - \int \int [D_1p(t,u) - D_2p(t,u)]\psi(t,u)dtdu\} \quad (2.22)$$

The left-hand side of (2.11) is

$$E[\psi(x,0) - \psi(0,x)] = \int f_x(t)\psi(t,0)dt - \int f_x(t)\psi(0,t)dt. \quad (2.23)$$

Comparing (2.22) and (2.23) we see that the two sides of (2.11) are equal if, and only if, the following three conditions are satisfied:

- (a) $D_1p(t,u) = D_2p(t,u)$,
- (b) $f_x(t) = E(x)p(0,t)$, and
- (c) $f_x(t) = E(x)p(t,0)$.

It follows from (a) that

$$p(t,u) = h(t+u) \text{ for some function } h(\cdot). \quad (2.24)$$

Therefore, (b) and (c) each becomes

$$f_x(t) = E(x)h(t). \quad (2.25)$$

Replacing t by $t+u$ in (2.25) results in $f_x(t+u) = E(x)h(t+u)$; replacing $h(t+u)$ by $p(t,u)$ as allowed by (2.24) results in

$$p(t,u) = f_x(t+u)/E(x) \quad (2.21)$$

which completes the proof.

It is noteworthy that we have derived the mixed moments of r and g , given by (2.15), before we derived the probability density of r and g given by (2.21). Likewise, we derived the moments of g given by (2.4) before deriving the probability density of g given by (2.9). Such freedom in the order of derivation is common when one is working with the omni-transform. Indeed, it is easier to find the mixed moments from (2.15) than from the joint probability density of r and g given by (2.21).

Example Consider a function $\alpha(r+g)$. Its expectation is, using (2.21),

$$E[\alpha(r+g)] = \int_{t=0}^{\infty} \int_{u=0}^{\infty} \alpha(t+u) f_x(t+u) dt du / E(x)$$

With $y = t+u$ and $z = t$ we get (noting that the Jacobian of the transformation equals 1)

$$E[\alpha(r+g)] = \int_{y=0}^{\infty} \int_{z=0}^y \alpha(y) f_x(y) dz dy / E(x) = \int_{y=0}^{\infty} y \alpha(y) f_x(y) dy / E(x)$$

Therefore,

$$E[\alpha(r+g)] = E[x\alpha(x)] / E(x). \quad (2.26)$$

When $\alpha(r+g) = (r+g)^k$ equation (2.26) becomes

$$E[(r+g)^k] = E(x^{k+1}) / E(x), \quad (2.27)$$

an equation known from the theory of stationary populations.

3. The Shifted Renewal Equation

The renewal equation in (x,r) form is

$$\psi(x) = \psi(0) + E(x)\psi'(r). \quad (3.1)$$

Let $\psi(x) = \phi(x+b)$ where $\phi(\)$ is another arbitrary function, and b is, for the time being, a constant. Then (3.1) becomes

$$\phi(x+b) = \phi(b) + E(x)\phi'(r+b). \quad (3.2)$$

The omni-convention applies, of course, to the $\phi(\)$ as well as to the $\psi(0)$. We can think, if we wish, of b in (3.2) as an outcome of a random variable y which is independent of the variables x and r . Thus,

$$\phi(x+y) = \phi(y) + E(x)\phi'(r+y). \quad (3.3)$$

In (3.3) the expectation of $\phi(x+y)$ is taken with respect to x and y ; of $\phi(y)$ with respect to y ; and of $\phi'(r+y)$ with respect to r and y . We will refer to (3.3) as the "shifted renewal equation."

We will refer to y as the "shift" and to x as the "core". In (3.3), the core x must be positive while the shift y need not be positive.

When the r.v. y is positive we can choose to treat it as the core in $\phi(x+y)$ and to treat x as the shift. Using now the operator R (recall that $Rx =$ residue of x) we write

$$\phi(x+y) = \phi(x) + E(y)\phi'(Ry+x) \quad (3.4)$$

while (3.3) can be written as

$$\phi(x+y) = \phi(y) + E(x)\phi'(Rx+y). \quad (3.3a)$$

The shifted renewal equation plays an important role in our treatment of the M/G/1 model, and some role in the G/G/1 model. This topic is left to a future report. The use of the shifted renewal equation in the integration of omni-equations is described in Section 4 of this report.

4. Mixtures of Random Variables

Definition We say that the random variable Y is a mixture of the random variables x_i with weights p_i if

$$\text{Prob}(Y \leq t) = p_1 \text{Prob}(x_1 \leq t) + \dots + \text{Prob}(x_k \leq t) \quad (4.1)$$

where $p_i \geq 0$ and $\sum p_i = 1$. We can similarly define a mixture of random processes.

Equation (4.1) can be stated in an equivalent form using omni-transforms:

$$\psi(Y) = p_1 \psi(x_1) + \dots + p_k \psi(x_k) \quad (4.2)$$

In this report we consider only finite mixtures, i.e., with $k < \infty$. [For a continuous mixture we can write

$$E[\psi(Y)] = \int p(t) \psi(x, t) dt, \text{ with } p(t) \geq 0 \text{ and } \int p(t) dt = 1 \quad (4.3)]$$

A mixture of random variables can be visualized as a two-stage experiment in which first one of the x_i is drawn (with probability p_i); and then a value of x_i is drawn (according to the distribution of x_i).

Whereas sums of random variables, especially of sums of independent random variables (and their convolutions and limits) have been dealt with extensively in the literature, mixtures have not been dealt with often. The use of omni-transforms in the theory of mixtures appears to be new.

From the renewal equation we get

$$\psi(Y) = \psi(0) + E(Y) \psi'(RY) \quad (4.4)$$

and

$$\psi(x_i) = \psi(0) + E(x_i) \psi'(Rx_i) \quad (4.5)$$

where R is the residue operator defined by $Rx = \text{residue of } x$.

It follows from (4.2), (4.4), and (4.5) that

$$E(Y) \psi(RY) = p_1 E(x_1) \psi(Rx_1) + \dots + p_k E(x_k) \psi(Rx_k) \quad (4.6)$$

Thus, RY is a mixture of the Rx_i with weights $p_i E(x_i) / E(Y)$. (Note that these weights sum to one since $E(Y) = \sum p_i E(x_i)$.)

We now derive an expression for the residue of the sum of two independent positive random variables, x_1 and x_2 . From (3.3) we get

$$\psi(x_1 + x_2) = \psi(x_2) + E(x_1)\psi'(Rx_1 + x_2) \quad (4.7)$$

Now, we treat Rx_1 in (4.7) as the shift and x_2 as the core.

Therefore,

$$\psi'(Rx_1 + x_2) = \psi'(Rx_1) + E(x_2)\psi''(Rx_1 + Rx_2) \quad (4.8)$$

Since,

$$\psi(x_2) = \psi(0) + E(x_2)\psi'(Rx_2)$$

we get from (4.7) and (4.8)

$$\begin{aligned} \psi(x_1+x_2) &= \psi(0) + E(x_1)\psi'(Rx_1) + E(x_2)\psi'(Rx_2) \\ &\quad + E(x_1x_2)\psi''(Rx_1+Rx_2) \end{aligned} \quad (4.9)$$

But we also have, treating $x_1 + x_2$ as the core (with shift zero),

$$\psi(x_1+x_2) = \psi(0) + E(x_1 + x_2)\psi'(R(x_1 + x_2)).$$

From above and (4.9) we have

$$\begin{aligned} E(x_1+x_2)\psi'(R(x_1+x_2)) &= E(x_1)\psi'(Rx_1) + E(x_2)\psi'(Rx_2) \\ &\quad + E(x_1x_2)\psi''(Rx_1 + Rx_2). \end{aligned} \quad (4.10)$$

Define now $\phi(\cdot) = \psi'(\cdot)$. Then $\phi'(\cdot) = \psi''(\cdot)$ and (4.10) can be written as

$$\begin{aligned} E(x_1+x_2)\phi(R(x_1+x_2)) &= E(x_1)\phi(Rx_1) + E(x_2)\phi(Rx_2) \\ &\quad + E(x_1x_2)\phi'(Rx_1 + Rx_2). \end{aligned} \quad (4.11)$$

However, since $\phi(\cdot)$ is a general function of its argument just as $\psi(\cdot)$ is, we can replace each ϕ by ψ , thus obtaining

$$\begin{aligned} E(x_1+x_2)\psi(R(x_1+x_2)) &= E(x_1)\psi(x_1) + E(x_2)\psi(Rx_2) \\ &\quad + E(x_1x_2)\psi'(Rx_1+Rx_2) \end{aligned} \quad (4.12)$$

Thus, we obtain (4.12) from (4.10), where each term is of order one or higher, by "integrating" each term. Conversely, if we start with (4.12) we can "differentiate" each term. When "integrating" or "differentiating" a linear omni-equation we obtain again a valid linear omni-equation; when "integrating"

an omni-equation there is no need to worry about a constant term: none is needed. Of course, we "integrate" only omni-equations whose each term is of order at least one.

Note now that neither (4.10) nor (4.11) nor (4.12) is, explicitly at least, a statement about a mixture because not all terms are of the same order. Some omni-equations with terms of different order can be "integrated" into an equation of order zero which is a statement about a mixture. Thus, consider the omni-equation with positive random variables x and w

$$\lambda[\psi(w+x) - \psi(w)] = \psi'(w) - (1-\rho)\psi'(0) \quad (4.13)$$

which comes up in the M/G/1 model. (The interpretation of (4.13) is not now important.) We luckily recognize that (cf. (3.2))

$$\psi(w+x) - \psi(w) = E(x)\psi'(Rx+w) \quad (4.14)$$

which along with (4.13) results in

$$\lambda E(x)\psi'(w+Rx) = \psi'(w) - (1-\rho)\psi'(0). \quad (4.15)$$

We can now replace each ψ' by ψ , thus obtaining

$$\lambda E(x)\psi(w+Rx) = \psi(w) - (1-\rho)\psi(0). \quad (4.16)$$

Let us now put $\psi(\cdot) = 1$. Then (4.16) becomes

$$\lambda E(x) = 1 - (1-\rho) \text{ and } \lambda E(x) = \rho.$$

Therefore, (4.16) can be written as

$$\psi(w) = (1-\rho)\psi(0) + \rho\psi(w+Rx). \quad (4.17)$$

Thus, using the renewal equation we have succeeded in genuinely integrating the omni-equation (4.13), that is converting (4.13) into an equivalent equation whose all terms are of order zero. (The "integration" of (4.10) to (4.12) is essentially a typographical operation.) Such an omni-equation is equivalent to a statement about a mixture. In particular, (4.17) is equivalent to

$$\text{Prob}[w \leq t] = (1-\rho)\text{Prob}[0 \leq t] + \rho\text{Prob}[w+Rx \leq t] \quad (4.18)$$

i.e.,

$$\text{Prob}[w \leq t] = (1-\rho) + \rho \text{Prob}[w+Rx \leq t]. \quad (4.19)$$

The coefficients on the right-hand side of (4.18) are positive and add up to one. This makes the distribution of w a mixture of 0 with weight $1-\rho$, and of the distribution of $w+x$ with weight ρ .

When x is known we can obtain from (4.17) the moments of w ; and from (4.19) an integral equation for the distribution of w . We shall pursue this problem in connection with the discussion of the model M/G/1.

What about the renewal equation itself,

$$\psi(x) = \psi(0) + E(x)\psi'(Rx),$$

can this be integrated into a mixture? No, it cannot and neither can a shifted renewal equation. (It is a case of a wonder physician who can cure everyone else but not himself.) Equations (4.7) and (4.12) are, in essence, already equivalent to a shifted renewal equation and an attempt to convert them into a mixture will result in $0 = 0$. It is for this reason that we selected (4.13) to convert into a mixture. We will derive (4.13) in a future report.

Note Equation (4.9) can be extended to the case of $\psi(x_1 + \dots + x_k)$. We shall write down, without proof, the analog of (4.9) for $k = 3$. The general pattern does emerge from $k = 2$ and $k = 3$.

$$\begin{aligned} \psi(x_1 + x_2 + x_3) &= \psi(0) + E(x_1)\psi'(Rx_1) + E(x_2)\psi'(Rx_2) \\ &+ E(x_3)\psi'(Rx_3) + E(x_1x_2)\psi''(Rx_1 + Rx_2) \\ &+ E(x_2x_3)\psi''(Rx_2 + Rx_3) + E(x_3x_1)\psi''(Rx_3 + Rx_1) \\ &+ E(x_1x_2x_3)\psi'''(Rx_1 + Rx_2 + Rx_3). \end{aligned} \quad (4.20)$$

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