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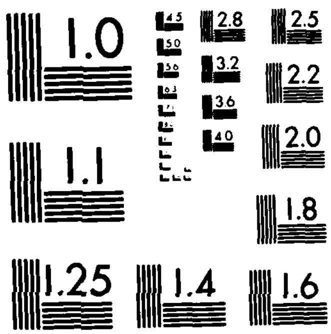

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STABLE MONOTONE VARIATIONAL INEQUALITIES

L. McLinden

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

Variational inequalities associated with monotone operators (possibly nonlinear and multivalued) and convex sets (possibly unbounded) are studied in reflexive Banach spaces. A variety of results are given which relate to a stability concept involving a natural parameter. These include characterizations useful as criteria for stable existence of solutions and also several characterizations of surjectivity. The monotone complementarity problem is covered as a special case, and the results are sharpened for linear monotone complementarity and for generalized linear programming.

AMS (MOS) Classification: 47H05, 49A29, 90C33, 90C05.

Key Words: monotone operators, variational inequalities, complementarity problems, generalized linear programming, stable solvability, surjectivity, reflexive Banach spaces, locally convex Hausdorff spaces.

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SIGNIFICANCE AND EXPLANATION

Variational inequalities (VI's), including complementarity problems, arise in many areas of applications, ranging from mathematical economics to structural mechanics to boundary value problems and mathematical physics. VI's are central also in mathematical optimization, which itself has an exceptionally wide range of application. One important general class of VI's involves monotone operators; this case already covers linear and convex constrained optimization and much more complex nonlinear phenomena as well.

One is usually interested in treating a given problem not in isolation but rather as one of a parametrized family of similar problems, with attention given to analyzing the nature of the dependency of the solution(s) on certain natural parameters involved in specifying the original problem. This paper introduces and studies a general notion of problem stability for VI's. Roughly speaking, a VI is called stable here if it is solvable and remains solvable for all small perturbations of the relevant parameters. Combined with monotonicity, such stability entails significant additional properties concerning the parametric solution behavior.

The heart of the paper develops a number of characterizations of, and sufficient conditions for, stability. The results are established for general monotone VI's in reflexive Banach spaces. When specialized to monotone complementarity problems, the results unify and considerably extend a large number of results from the finite-dimensional linear complementarity literature.

Responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STABLE MONOTONE VARIATIONAL INEQUALITIES

L. McLinden

1. Introduction. Let Z be a reflexive Banach space having dual W, with $\langle z,w\rangle$ denoting the value $w(z) \in \mathbb{R}$ of a continuous linear functional $w \in W$ at a point $z \in Z$. Let C be a given nonempty closed convex subset of Z, and let T be a given operator (possibly multivalued) from Z into W. The variational inequality $\mathcal{V}(a)$ associated with T, C and parameter $a \in W$ is to

$$\left.\begin{array}{l} \text{find } z \in C \cap \mathcal{D}(T) \text{ such that, for some } w \in T(z), \\ \langle z' - z, w - a\rangle \geq 0 \text{ for all } z' \in C \cap \mathcal{D}(T) . \end{array}\right\} \qquad \mathcal{V}(a)$$

Here, $\mathcal{D}(T) := \{z \in Z \mid T(z) \neq \emptyset\}$ is the effective domain of T. Also of use will be $\mathcal{R}(T) := \{w \in W \mid w \in T(z) \text{ for some } z \in \mathcal{D}(T)\}$, the range of T. A prominent instance of $\mathcal{V}(a)$ is the complementarity problem $\mathcal{C}(a)$, obtained by restricting C to be a cone. Writing Q instead of C in the cone case and putting $Q^* := \{w \in W \mid \langle z,w\rangle \geq 0 \text{ for all } z \in Q\}$, one can reformulate the problem into the more standard form

$$\left.\begin{array}{l} \text{find } z \in \mathcal{D}(T) \text{ such that, for some } w \in T(z), \\ \quad z \in Q,\ w-a \in Q^*,\ \langle z, w - a\rangle = 0 . \end{array}\right\} \qquad \mathcal{C}(a)$$

The case in which T is an anti-selfadjoint (i.e., skew-symmetric) linear operator is of particular interest, as it covers generalized linear programming problems. Problems of the above sort have long been recognized as being important to many different areas of application.

This paper treats problem $\mathcal{V}(a)$ under the assumption that T is a monotone operator, that is, satisfies

$$\langle z' - z, w' - w\rangle \geq 0 \quad \text{whenever } w \in T(z),\ w' \in T(z') .$$

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This monotone case is arguably the most basic. It encompasses not only constrained nonsmooth convex minimization and constrained nonsmooth convex-concave minimax problems, but also a wide range of other variational problems (e.g., involving differential and/or integral operators) which are not expressible in terms of such optimization. See Minty [39] and Stampacchia [56], respectively, for early, authoritative discussions of monotonicity and of variational inequalities. There is now a very large related literature having many different aspects. The reader might consult Brézis [5, 7], Browder [11], Kachurovskii [25], Ghizzetti [23], Auslender [4], Mosco [44], Kluge [30], Pascali-Sburlan [47], Cottle-Giannessi-Lions [14] and the references therein, in addition to the references cited below.

Several comments are in order concerning our general setting. The need to permit C to be unbounded is rather clear; for example, to cover the important complementarity problem $\mathcal{C}(a)$. Less evident, perhaps, is the value of permitting T to be multivalued. This capability is actually necessary for treating many applications; for example, $\mathcal{V}(a)$'s arising from nonsmooth convex or convex-concave optimization in which the problem's defining functionals fail to be continuously differentiable everywhere on relevant domains. Reflexive Banach spaces have been chosen because some of the key tools used in the majority of our results appear to be limited essentially to such spaces. Certain of our results, however, are valid in the locally convex Hausdorff setting; these will be indicated usually.

We generally consider $\mathcal{V}(0)$ to be the given problem $(a = 0$ being a harmless normalization here) and regard $\mathcal{V}(a)$, for a near the origin, as a perturbation of $\mathcal{V}(0)$. In optimization contexts such perturbations usually correspond to adding a linear functional to the problem's objective function and/or to varying the data vector which appears as the "right-hand-side" of the system of constraint inequalities.

The focus of the paper is provided by the following definition. We say $\mathcal{V}(0)$ is stably solvable (or just stable) if and only if $\Omega(a)$ is nonempty for all a in some neighborhood of the origin, where

$$\Omega(a) := \{z \in Z \mid z \text{ solves } \mathcal{V}(a)\} \quad \text{for all} \quad a \in W .$$

Our goal is twofold. First, briefly to draw attention to the abundance of powerful, general information already implicitly available for stable monotone problems by virtue of the relatively well developed theory of monotone operators. And second, as the heart of the paper, to derive a number of results relating to stability for the problems $\mathcal{V}(a)$. These include various characterizations of stability. Some of the corollaries are surjectivity results, in that they ensure $\mathcal{D}(\Omega) = W$, but without necessarily requiring coercivity of T on C. (Here, of course, $\mathcal{D}(\Omega) := \{a \in W \mid \Omega(a) \neq \emptyset\}$.)

The paper is organized as follows. In §2 we make precise how general theory for monotone operators converts into facts about stable monotone problems. Also, we introduce, and give various sufficiency criteria for, a certain blanket hypothesis of maximality which is needed for most of the results. In §3 we define the class of C's for which the strongest results hold. These are the sets C which have at least one "bounded nontrivial section". This notion is seen to be the proper generalization to general convex sets of the familiar notion of weakly compact base for cones.

Section 4 is the heart of the paper. Assuming $\mathcal{V}(0)$ has a "strictly feasible" point, Theorem 1 assures (without maximality) that the sets $\Omega(a)$ are uniformly bounded for a varying near the origin, and Corollary 1A gives a convenient a priori bound on $\Omega(0)$. Theorem 2 gives a sufficiency criterion for stability valid for single valued T's in general spaces. Theorem 3, on which many of the subsequent results depend, is a structure theorem for $\mathcal{B}(\Omega)$. Theorem 4 presents six characterizations of stability. Theorem 5 adds an extremely strong seventh characterization of stability valid whenever T enjoys some very weak form of strict monotonicity with respect to C.

In §5 we sharpen the earlier results for the generalized linear complementarity problem (abbreviated LCP), that is, problem $\mathcal{C}(a)$ with T linear. Theorem 6 gives further results for this situation. Theorem 7 gives still sharper results for the case in which T is anti-selfadjoint linear.

In §6 we apply the results to linear programming in reflexive Banach spaces.

Finally, a word on notation. We denote the <u>indicator function</u> of C and the <u>support function</u> of C by ψ_C and σ_C, respectively. Thus, $\psi_C\colon Z \to (-\infty,\infty]$ and $\sigma_C\colon W \to (-\infty,\infty]$ are the lower semicontinuous convex functions given by

$$\psi_C(z) = \begin{cases} 0 & \text{if } z \in C \\ \infty & \text{if } z \in Z\backslash C \end{cases}, \quad \sigma_C(w) = \sup_{z \in C} \langle z,w \rangle .$$

The <u>recession cone</u> of C and the <u>barrier cone</u> of C will be denoted by 0^+C and D, respectively. Thus,

$$0^+C = \{z \in Z \mid C + z \subset C\}\ , \quad D = \{w \in W \mid \sigma_C(w) < \infty\}\ .$$

The abbreviations

cl, int, ri, core, conv,

respectively, denote the operations of closure, interior, relative interior, algebraic interior, and convex hull applied to a set.

2. A maximality assumption and implications of stability. The designation $\Omega(a)$ for the solution set to $\mathcal{V}(a)$ effectively defines a multifunction (point-to-set function) Ω from W into Z via

$$\Omega(a) := \{z \in C \mid z \text{ solves } \mathcal{V}(a)\}\ .$$

We begin with a convenient representation of Ω. It involves N_C, the normality operator for C, which is the multifunction from Z into W defined by

$$N_C(z) := \begin{cases} \{w \in W \mid \langle z' - z, w\rangle \leq 0 \text{ for all } z' \in C\} & \text{if } z \in C \\ \emptyset \quad \text{if } z \in Z\backslash C\ . \end{cases}$$

Notice N_C is exactly $\partial\psi_C$, the subdifferential of ψ_C. The sum $T + N_C =: M$ of T and N_C is the multifunction from Z into W given by

$$M(z) := (T + N_C)(z) := T(z) + N_C(z)\ ,$$

with the convention that $S + \emptyset = \emptyset = \emptyset + S$ for any subset S of W. Writing M^{-1} for the multifunction inverse to M, which is defined by

$\mathcal{D}(M^{-1}) = \mathcal{R}(M)$, $\mathcal{R}(M^{-1}) = \mathcal{D}(M)$ and $z \in M^{-1}(w)$ if and only if $w \in M(z)$, one obtains easily that $\Omega(a) = M^{-1}(a)$ for all $a \in W$, that is,

$$\Omega = M^{-1} = (T + N_C)^{-1} . \tag{2.1}$$

Hence,

$$\mathcal{V}(0) \text{ is stable } \Longleftrightarrow 0 \in \operatorname{int} \mathcal{D}(\Omega) . \tag{2.2}$$

The following blanket assumption is in force for the remainder of the paper, unless otherwise stated:

$$M := T + N_C \text{ is } \underline{\text{maximal monotone}}. \tag{2.3}$$

This means M is monotone and, for each $(\hat{z}, \hat{w}) \in Z \times W$ such that $\hat{w} \notin M(\hat{z})$, there exist some $z \in \mathcal{D}(M)$ and $w \in M(z)$ for which $\langle \hat{z} - z, \hat{w} - w \rangle < 0$. In our discussion the sum operator M in (2.3) is automatically monotone, since T is monotone by hypothesis and N_C, being the subdifferential of the convex function ψ_C, is also monotone. We note that in fact here N_C is maximal monotone [51], since C is also nonempty and closed. The maximality of the sum in (2.3) is a subtle matter, but fortunately broad criteria are available to cover most cases of interest.

PROPOSITION 1. The blanket assumption (2.3) holds under any one of the following three conditions:

(a) $\mathcal{D}(T) \supset C$ and T is (monotone and) singlevalued and hemicontinuous (i.e., continuous with respect to the weak* topology of W) along each line segment in C;

(b) T is maximal monotone and $0 \in \mathrm{ri}(\mathrm{conv}\, \mathcal{D}(T) - C)$;

(c) T is maximal monotone and T is locally bounded (i.e., uniformly bounded in a neighborhood of) some point of $C \cap \mathrm{cl}\, \mathcal{D}(T)$.

Notice that the requirement $0 \in \mathrm{ri}(\mathrm{conv}\, \mathcal{D}(T) - C)$ in (b) is implied by

either $\emptyset \neq \mathcal{D}(T) \cap \mathrm{int}\, C$, or $\emptyset \neq C \cap \mathrm{int}\, \mathcal{D}(T)$,

or Z finite-dimensional and

$$\emptyset \neq \mathrm{ri}\, \mathcal{D}(T) \cap \mathrm{ri}\, C .$$

Criterion (b) of Proposition 1 follows from McLinden [37, Theo[illegible]]. The other criteria (including the special cases of (b) just mention[illegible] are consequences of Rockafellar [52, Theorems 1, 2 and 3].

In connection with applying criteria (b) - (c) above to verify (2.3), observe that T is known to be maximal monotone in each of the following general cases: (1) if T is the subdifferential of a lower semicontinuous proper convex function on Z (Rockafellar [51]); (2) if T is induced via a certain twist from the subdifferential of a closed proper convex-concave function on $Z = Z_1 \times Z_2$ (Rockafellar [54]); (3) if T is a closed, densely defined monotone linear transformation whose adjoint is also monotone (Brézis [6, Theorem 1]).

Further maximality criteria for both T and M, applicable when Z is a Hilbert space, can be found in Brézis [7].

Since maximal monotonicity (in the reflexive Banach setting) is preserved under passage to the inverse multifunction, the blanket hypothesis (2.3) is equivalent (cf. (2.1)) to the condition

$$\Omega \text{ is maximal monotone .} \tag{2.4}$$

Powerful facts now follow readily concerning the parametric solution multifunction Ω and, in particular (cf. (2.2)), stable $\mathcal{V}(0)$'s.

PROPOSITION 2. Assuming (2.4) holds (equivalently, (2.3)), each of the following holds:

(a) If $0 \in \operatorname{int} \mathcal{D}(\Omega)$, there exists a neighborhood U of the origin such that

$$\bigcup_{a \in U} \Omega(a) \quad \text{is bounded .}$$

(b) For each $a \in \operatorname{int} \mathcal{D}(\Omega)$ the set $\Omega(a)$ is nonempty closed convex and bounded, and Ω is upper semicontinuous on $\operatorname{int} \mathcal{D}(\Omega)$ from the norm topology to the weak* topology.

(c) The set

$$\{a \in \operatorname{int} \mathcal{D}(\Omega) \,|\, \Omega(a) \text{ is a singleton, say } z, \text{ and}$$
$$\|z_n - z\| \to 0 \text{ whenever } \|a_n - a\| \to 0,\ a_n \in \operatorname{int} \mathcal{D}(\Omega),\ z_n \in \Omega(a_n)\}$$

is a dense G_δ subset of $\operatorname{int} \mathcal{D}(\Omega)$.

(d) If Z is finite-dimensional, the set

$$\{a \in \operatorname{int} \mathcal{D}(\Omega) \,|\, \Omega \text{ fails to be differentiable at } a\}$$

has Lebesgue measure zero. In particular, Ω is singlevalued and Lipschitz continuous (Lebesgue-) almost everywhere on $\operatorname{int} \mathcal{D}(\Omega)$.

Parts (a) and (b) here follow from Rockafellar [50, Theorem 1]. Part (c) was established independently by Fitzpatrick [21] and Kenderov-Robert [29]. Part (d) was established by Mignot [38, Theorem 1.3].

Of course, any other properties established for maximal monotone operators also apply to Ω with the aid of (2.4), i.e., (2.3).

Finally, we wish to point out that general sufficiency criteria for

$$0 \in \mathcal{R}(\Omega) \quad \text{[solvability]} \tag{2.5}$$

and for

$$W = \mathcal{R}(\Omega) \quad \text{[global solvability; surjectivity of } M] , \tag{2.6}$$

respectively, have been provided by Rockafellar in [52, Theorem 5] and (partly codifying the singlevalued case obtained independently by Hartman-Stampacchia [24] and Browder [9]) in [52, Theorem 4]. We exhibit below several sufficiency criteria for (2.6) not requiring coercivity of the operator $T + N_C$. The main focus of the present paper, though, is on

$$0 \in \operatorname{int} \mathcal{R}(\Omega) \quad \text{[stable solvability]} , \tag{2.7}$$

the basic case intermediate between (2.5) and (2.6).

3. <u>Sets C which admit a bounded nontrivial section</u>. A prominent role is played below by certain closed convex sets C which, while possibly unbounded, yet have bounded "nontrivial" intersection with at least one closed halfspace. Let us make this precise. For any $w \in W$ and any $\epsilon \in [0,\infty)$, define

$$S(w,\epsilon) := \{z \in C \,|\, \langle z,w\rangle \geq -\epsilon + \sigma_C(w)\} \tag{3.1}$$

to be a <u>section</u> of C, where recall $\sigma_C(w) := \sup\{\langle z',w\rangle \,|\, z' \in C\}$. We say that $S(w,\epsilon)$ is a <u>nontrivial</u> section of C if and only if it is nonempty and $\epsilon > 0$. It is clear that any section of C is expressible as

$$S(w,\epsilon) = \partial_\epsilon \sigma_C(w) , \tag{3.2}$$

where $\partial_\epsilon \sigma_C$ designates the ϵ-approximate subdifferential of σ_C (e.g., [42, §10.h]), and that $S(w,\epsilon)$ is nontrivial if and only if $\epsilon > 0$ and $w \in D := \{w \in W | \sigma_C(w) < \infty\}$. A nontrivial section which is bounded constitutes the correct generalization, to general closed convex sets in reflexive Banach spaces, of the familiar notion of weakly compact base (cross-sectional truncation) of a cone.

These various sets are illustrated in $Z = \mathbb{R}^2$ by $C := \{z | z_2 \geq \exp z_1\}$. One gets $D = \{w | w_2 < 0 < w_1 \text{ or } w_2 \leq 0 = w_1\}$, $\mathcal{R}(N_C) = \{w | w_2 < 0 < w_1 \text{ or } w_2 = 0 = w_1\}$ and $0^+C = \{z | z_1 \leq 0 \leq z_2\}$, and both bounded and unbounded nontrivial sections of C exist.

Sets which admit a bounded nontrivial section have important properties. Among these is that

$$\left.\begin{array}{l}C \text{ has a bounded}\\ \text{nontrivial section}\end{array}\right\} \iff \emptyset \neq \text{int } D . \tag{3.3}$$

More specifically,

$$\left.\begin{array}{l}\text{there exists } \epsilon > 0 \text{ such}\\ \text{that } S(w,\epsilon) \text{ is a bounded}\\ \text{nontrivial section of } C\end{array}\right\} \iff w \in \text{int } D , \tag{3.4}$$

in which event

$$\left.\begin{array}{l}\text{for all } \lambda \in [0,\infty) \text{ the sets } S(w + a,\lambda) \text{ are}\\ \text{nonempty and uniformly bounded for all } a \text{ in}\\ \text{some neighborhood of the origin.}\end{array}\right\} \tag{3.5}$$

These facts follow from the next two results.

PROPOSITION 3. The support function σ_C of C is continuous on core D; in particular,

$$\text{core } D = \text{int } D\ . \tag{3.6}$$

Also,

$$0^+C = D^\circ := \{z \in Z \mid \langle z,w\rangle \leq 0 \text{ for all } w \in D\}\ . \tag{3.7}$$

This follows from Rockafellar [49, Corollaries 7C and 3C(d)] applied to the conjugate pair of functions σ_C and ψ_C. In the same manner the next result follows from independent work of Moreau [41, 42] and Rockafellar [49]; see also [3, Theorem 2 and ff.].

PROPOSITION 4. (a) For any $w \in$ core D and any $\mu \in [0,\infty)$, there exists a neighborhood U of the origin such that

$$\bigcup_{\substack{a \in U \\ 0 \leq \lambda \leq \mu}} S(w + a,\lambda) \quad \text{is bounded}$$

and each set $S(w + a,\lambda)$ appearing in this union is nonempty.

(b) On the other hand, for any $w \in W$, if $S(w,\epsilon)$ is nonempty and bounded for some $\epsilon > 0$, then $w \in$ core D and thus part (a) applies.

The property that C have a bounded nontrivial section is used in the rest of the paper (by virtue of (3.3), (4.24), (4.25)) as a convenient condition under which a number of implications involving stability criteria are strengthened to equivalences.

It is sometimes possible only to deal with a weaker notion than stability, one permitting $\mathcal{D}(\Omega)$ to have empty interior. Thus, we define $V(0)$ to be <u>quasistable</u> provided

$$0 \in \operatorname{ri} \mathcal{D}(\Omega), \tag{3.8}$$

"ri" denoting interior relative to the closed affine hull of $\mathcal{D}(\Omega)$. Concerning this notion, see (4.27) of Theorem 3 and Corollary 3C below. Also, in [36] can be found explicit quasistability results for parametric, noncoercive general convex minimization.

4. Main results: the general monotone variational inequality case. We begin with several results not requiring the maximality assumption (2.3). The first one provides a convenient criterion for uniform boundedness of the solution sets $\Omega(a)$ for a varying near the origin. It can be compared with Proposition 2(a). We write

$$T(C) := \bigcup \{T(z) \mid z \in C\} .$$

THEOREM 1. Even without assuming (2.3), if

$$0 \in \operatorname{conv} T(C) + \operatorname{int} D , \tag{4.1}$$

then there exists a neighborhood U of the origin such that

$$\bigcup_{a \in U} \Omega(a) \text{ is bounded} . \tag{4.2}$$

Proof. Hypothesis (4.1) gives

$$0 \in \tilde{w} + \operatorname{int} D , \quad \text{where} \quad \tilde{w} := \Sigma \lambda_k w_k , \tag{4.3}$$

for

$$w_k \in T(z_k) , \quad z_k \in C , \quad \lambda_k \geq 0 , \quad \Sigma \lambda_k = 1 \tag{4.4}$$

(summation over $k = 1,\ldots,m$). Choose any

$$\mu > \max\{0, \sigma_C(-\tilde{w}) + \gamma\} , \quad \text{where} \quad \gamma := \Sigma \lambda_k \langle z_k, w_k \rangle . \tag{4.5}$$

By (4.3) and Proposition 4(a), there exists a neighborhood U_o of the origin such that

$$\bigcup_{a \in U_o} S(a - \tilde{w}, \mu) \text{ is bounded} . \tag{4.6}$$

Consider any $a \in W$ and any $z \in \Omega(a)$. Then $z \in C$ and there exists $w \in T(z)$ such that $\langle z' - z, w - a \rangle \geq 0$ for all $z' \in C \cap \mathcal{D}(T)$. Consider any index k. Taking $z' = z_k$ gives

$$0 \geq \langle z_k - z, a - w \rangle ,$$

which together with monotonicity gives

$$\begin{aligned} \langle z_k, -w_k \rangle &\geq - \langle z_k, w_k \rangle + \langle z_k - z, w \rangle \\ &\geq - \langle z_k, w_k \rangle + \langle z_k - z, a \rangle . \end{aligned}$$

Multiplying through by λ_k and summing yields

$$\langle z, -\tilde{w} \rangle \geq - \gamma + \langle \tilde{z} - z, a \rangle ,$$

where $\tilde{z} := \Sigma \lambda_k z_k$, so that

$$\langle z, a - \tilde{w} \rangle \geq - \gamma + \langle \tilde{z}, a \rangle .$$

By the arbitrariness of z this implies that

$$\left.\begin{aligned} \Omega(a) &\subset \{z \in C \mid \langle z, a - \tilde{w} \rangle \geq - \gamma + \langle \tilde{z}, a \rangle\} \\ &\subset \{z \in C \mid \langle z, a - \tilde{w} \rangle \geq - \lambda + \sigma_C(a - \tilde{w})\} \\ &= S(a - \tilde{w}, \lambda) \end{aligned}\right\} \tag{4.7}$$

whenever

$$\lambda \geq \max\{0,\ \sigma_C(a - \tilde{w}) + \gamma - \langle \tilde{z},a \rangle\}\ . \tag{4.8}$$

For future use, observe that

$$\left.\begin{array}{l}\text{the second inclusion in (4.7) is an} \\ \text{equality if } \lambda = \sigma_C(a - \tilde{w}) + \gamma - \langle \tilde{z},a \rangle \geq 0\ .\end{array}\right\} \tag{4.9}$$

We now show that in (4.8) the choice $\lambda = \mu$ works for all a sufficiently small. Since $\sigma_C - \langle \tilde{z},\cdot \rangle =: \varphi$ is continuous at $-\tilde{w}$ by Proposition 3, for

$$\varepsilon := \mu - \max\{0,\ \sigma_C(-\tilde{w}) + \gamma\} \tag{4.10}$$

(cf. (4.5)) there exists a neighborhood U_1 of the origin such that

$$\varphi(a - \tilde{w}) \leq \varepsilon + \varphi(-\tilde{w}) \quad \text{for all} \quad a \in U_1\ .$$

By the choice (4.10), this yields

$$\sigma_C(a - \tilde{w}) + \max\{-\sigma_C(-\tilde{w}),\gamma\} - \langle \tilde{z},a \rangle \leq \mu \quad \text{for all} \quad a \in U_1\ . \tag{4.11}$$

Combining (4.11), (4.5), (4.8) and (4.7) yields

$$\Omega(a) \subset S(a - \tilde{w},\mu) \quad \text{for all} \quad a \in U_1\ . \tag{4.12}$$

Choosing $U := U_o \cap U_1$ and combining (4.12) with (4.6) establishes (4.2).

The explicitness of the preceding proof provides a potentially easily obtainable a priori estimate for $\Omega(a)$ which does not require knowing the neighborhood U in (4.2).

COROLLARY 1A. Even without assuming (2.3), for any

$$a \in \text{conv } T(C) + \text{int } D$$

one has

$$\Omega(a) \subset \{z \in C \,|\, \langle z,\tilde{w} - a\rangle \leq \gamma + \langle \tilde{z}, -a\rangle\} , \tag{4.13}$$

with the set on the right nonempty and bounded, whenever

$$(\tilde{z},\tilde{w}) := \Sigma\, \lambda_k(z_k,w_k) \quad \text{and} \quad \gamma := \Sigma\, \lambda_k\langle z_k,w_k\rangle \tag{4.14}$$

(summation over $k = 1,\ldots,m$) satisfy

$$w_k \in T(z_k)\ ,\quad z_k \in C,\quad \lambda_k \geq 0\ ,\quad \Sigma\, \lambda_k = 1\ , \tag{4.15}$$

$$a \in \tilde{w} + \text{int}\, D\ , \tag{4.16}$$

$$\gamma + \langle \tilde{z}, -a\rangle \geq \inf_{z' \in C} \langle z',\tilde{w} - a\rangle\ . \tag{4.17}$$

<u>Proof</u>. The argument given above for (4.7) establishes the inclusion (4.13) assuming only (4.15). Now suppose (4.16) and (4.17) also hold. By (4.17),

$$\lambda := \sigma_C(a - \tilde{w}) + \gamma - \langle \tilde{z},a\rangle \geq 0\ .$$

Hence, (4.9) implies the set on the right in (4.13) equals $S(a - \tilde{w},\lambda)$ for this λ, so by (3.2) it equals $\partial_\lambda\sigma_C(a - \tilde{w})$. Since (4.16) and Proposition 3 imply σ_C is continuous at $a - \tilde{w}$, this set is nonempty and bounded by Proposition 4(a).

The proofs just given for Theorem 1 and Corollary 1A apply equally well to any locally convex Hausdorff spaces Z and W paired in duality, with weak boundedness in Z and Mackey interior in W, if one assumes σ_C is finitely bounded above on some Mackey neighborhood in W (cf. [3, pages 453-455]).

In the other direction, if one specializes to $Z = \mathbb{R}^n$ and takes for C the usual nonnegative orthant, the estimate (4.13) refines (cf. [34]) to give the following ℓ_1-norm bound:

$$\Omega(a) \subset \{z \geq 0 \mid \sum_{i=1}^{n} z_i \leq \mu^{-1}(\gamma + \langle \tilde{z}, -a \rangle)\} , \tag{4.18}$$

where $\mu := \min\{\tilde{w}_i - a_i \mid i = 1,\ldots,n\}$, whenever a and the terms in (4.14) satisfy (4.15), $\tilde{w} - a > 0$ (coordinatewise) and $\gamma + \langle \tilde{z}, -a \rangle \geq 0$.

Most of our subsequent results use information about the structure of $\mathcal{B}(\Omega)$ (cf. (2.1)). Even without assuming (2.3) one has

$$\mathcal{B}(\Omega) \subset T(C) + \mathcal{R}(N_C) \subset T(C) + D \tag{4.19}$$

and

$$\text{int } D \subset \mathcal{R}(N_C) \subset D \ . \tag{4.20}$$

The first part of (4.20) follows from (3.2) and the nonemptiness assertion of Proposition 4(a). The first part of (4.19) is a simple rephrasing of the elementary inclusion

$$\mathcal{R}(T_1 + T_2) \subset \mathcal{R}(T_1) + \mathcal{R}(T_2) \ . \tag{4.21}$$

Employing the same technique as above, we can obtain a sufficiency criterion for stable solvability valid beyond the realm of reflexive Banach spaces.

THEOREM 2. Let Z and W be any locally convex Hausdorff spaces paired in duality, with Z (resp. W) assigned the weak (resp. Mackey) topology induced by the pairing. Assume that T is singlevalued and monotone on C, that T is continuous on the intersection of C with any finite-dimensional subspace of Z, and that σ_C

is finitely bounded above on some neighborhood in W. Then

$$T(C) + \operatorname{core} D \subset \operatorname{int} \mathcal{D}(\Omega) \tag{4.22}$$

(in addition to (4.19)).

<u>Proof</u>. Let $a \in T(C) + \operatorname{core} D$. Then there exists $\tilde{z} \in C$ such that $a - T\tilde{z} \in \operatorname{core} D$. By the argument given for (4.9) (begin just after (4.6)) we have

$$\Omega(a) \subset S(a - T\tilde{z}, \lambda) = \{z \in C \mid \langle z - \tilde{z}, T\tilde{z} - a\rangle \leq 0\} =: K ,$$

where $\lambda := \sigma_C(a - T\tilde{z}) - \langle\tilde{z}, a - T\tilde{z}\rangle$. (This corresponds to there being just one index k, and writing $(z_k, w_k) = (\tilde{z}, \tilde{w})$, $\gamma = \langle\tilde{z}, \tilde{w}\rangle$, $\tilde{w} = T\tilde{z}$.) Recall (3.2). By the theorem of Moreau [41], [42] (see also [3, Theorem 2]), $S(a - T\tilde{z}, \lambda)$ is compact and nonempty and $\operatorname{core} D = \operatorname{int} D$. For any $z \in C\backslash K$, monotonicity implies

$$\langle z - \tilde{z}, Tz - a\rangle \geq \langle z - \tilde{z}, T\tilde{z} - a\rangle > 0 .$$

Hence, an existence result of Brézis-Nirenberg-Stampacchia [8, page 297] applies to C and $T_a z := Tz - a$, yielding nonemptiness of $\Omega(a)$. This shows $T(C) + \operatorname{core} D \subset \mathcal{D}(\Omega)$. Since the set on the left is open (by $\operatorname{core} D = \operatorname{int} D$), (4.22) follows.

Theorem 2 strengthens an existence result of Allen [2, Theorem 3], which was based on earlier work of Ky Fan [19, Theorem 1], [20]. Allen requires that $z \to \langle z, Tz\rangle$ be (weakly) lower semicontinuuous on all of C and doesn't obtain the stability conclusion. (We note that [2, Theorem 3] ostensibly treats a general quasiconvex f in place of the indicator case $f = \psi_C$, but the proof provided requires f convex in order for the counterpart of the present set K to be convex, and this general convex case can be reduced to the indicator case by using Mosco's idea [43].)

From (4.19) and (2.2) follows immediately a weak necessary condition for problem $\mathcal{V}(0)$ to be stably solvable:

$$0 \in \operatorname{int}(\operatorname{conv} T(C) + D) . \tag{4.23}$$

We say "weak" not only because of the presence of the convex hull operation, but primarily because of the fact that, for two general monotone operators T_1 and T_2, the set $\mathcal{R}(T_1 + T_2)$ in (4.21) can be drastically smaller than $\mathcal{R}(T_1) + \mathcal{R}(T_2)$. For example, consider $T_1 = - T_2 = L$, where L is any nonzero anti-selfadjoint (i.e., skew-symmetric) bounded linear operator from Z into W. It is therefore noteworthy that such collapsing behavior cannot occur if T_2 is a normality operator, such as N_C, in a reflexive Banach setting and assumption (2.3) holds. This is established by the following structure theorem for $\mathcal{B}(\Omega)$. In particular, notice (4.25) shows that the weak necessary condition (4.23) is in fact also sufficient for stability of $\mathcal{V}(0)$. Note also that from (4.25), together with the elementary inclusion

$$\operatorname{conv} T(C) + \operatorname{int} D \subset \operatorname{int}(\operatorname{conv} T(C) + D) , \tag{4.24}$$

it follows that each set $\Omega(a)$ appearing in (4.2) is nonempty (since for U_o in (4.6) one has $-\tilde{w} + U_o \subset \operatorname{int} D$).

Henceforth, assumption (2.3) is in force.

THEOREM 3. One always has

$$\operatorname{int} \mathcal{D}(\Omega) = \operatorname{int}(T(C) + D) = \operatorname{int}(\operatorname{conv} T(C) + D) \tag{4.25}$$

and

$$\operatorname{cl} \mathcal{D}(\Omega) = \operatorname{cl}(T(C) + D) = \operatorname{cl}(\operatorname{conv} T(C) + D) \ . \tag{4.26}$$

If $\phi \neq \operatorname{ri}(\operatorname{conv} T(C) + D)$, then

$$\operatorname{ri} \mathcal{D}(\Omega) = \operatorname{ri}(\operatorname{conv} T(C) + D) \ . \tag{4.27}$$

Formula (4.25), which sharpens the inclusion (4.22), will itself be sharpened below in Corollaries 4A and 5A.

<u>Proof</u>. By the positive homogeneity of N_C, the operator $M := T + N_C$ satisfies $M + N_C = M$. By this and (2.3), $M + N_C$ is maximal monotone. Since $N_C = \partial\sigma_C$ and (using $M + N_C = M$) $\mathcal{D}(M) = \mathcal{D}(M) \cap \mathcal{D}(N_C) \subset \mathcal{D}(N_C)$, it follows from McLinden [37, Theorem] that

$$\mathcal{R}(M + N_C) \simeq \mathcal{R}(M) + D \simeq \operatorname{conv}(\mathcal{R}(M) + D) \ , \tag{4.28}$$

where $\simeq$ designates that the sets on either side have the same interior and the same closure, and also, provided $\phi \neq \operatorname{ri} \operatorname{conv}(\mathcal{R}(M) + D)$, that

$$\operatorname{ri} \mathcal{R}(M + N_C) = \operatorname{ri} \operatorname{conv}(\mathcal{R}(M) + D) \ . \tag{4.29}$$

Observe that

$$\begin{aligned} \mathcal{R}(M) + D &= \bigcup_{z \in C \cap \mathcal{D}(T)} (T(z) + N_C(z) + D) \\ &= \bigcup_{z \in C} (T(z) + D) = T(C) + D \ , \end{aligned} \tag{4.30}$$

where the middle equality uses $\mathcal{R}(N_C) \subset D$, the fact D is a cone, and $0 \in N_C(z)$ for $z \in C$. Therefore, using convexity of D,

$$\mathrm{conv}(\mathcal{R}(M) + D) = \mathrm{conv}\, T(C) + D . \qquad (4.31)$$

Combining (4.30) - (4.31) with (4.28) - (4.29) yields (4.25) - (4.27), since $\mathcal{R}(M + N_C) = \mathcal{R}(M) = \mathcal{D}(\Omega)$.

The power of Theorem 3 is suggested by the following result, which recovers quite efficiently certain results of Minty [38a] and Rockafellar [50, Theorem 1], [53, Theorem 2] proved originally by different methods.

COROLLARY 3A. Let Z be a reflexive Banach space with dual W, and let T be a maximal monotone operator (possibly multivalued) from Z into W. Then

$$\mathrm{int}\, \mathcal{R}(T) = \mathrm{int\, conv}\, \mathcal{R}(T)$$

and

$$\mathrm{cl}\, \mathcal{R}(T) = \mathrm{cl\, conv}\, \mathcal{R}(T) .$$

In particular, both $\mathrm{int}\, \mathcal{R}(T)$ and $\mathrm{cl}\, \mathcal{R}(T)$ are convex, $\mathcal{R}(T)$ is dense (resp. all of W) exactly when $\mathrm{conv}\, \mathcal{R}(T)$ is dense (resp. all of W), and $\emptyset \neq \mathrm{int\, conv}\, \mathcal{R}(T)$ implies

$$\mathrm{cl\, int}\, \mathcal{R}(T) = \mathrm{cl}\, \mathcal{R}(T) , \quad \mathrm{int\, cl}\, \mathcal{R}(T) = \mathrm{int}\, \mathcal{R}(T) .$$

If $\phi \neq \mathrm{ri\ conv}\ \mathcal{R}(T)$, then

$$\mathrm{ri}\ \mathcal{R}(T) = \mathrm{ri\ conv}\ \mathcal{R}(T) .$$

All of the above facts hold also with $\mathcal{R}$ replaced everywhere by $\mathcal{D}$.

<u>Proof</u>. Choose $C := Z$. One has $N_C(z) = \{0\}$ for all z, $D = \{0\}$ and $T^{-1} = \Omega$. The assumed maximality of T ensures (2.3) holds for this C. Thus, Theorem 3 yields the first and third assertions. The second follows easily. Since T^{-1} satisfies the same assumptions as T, the part already established applies to it, yielding the last assertion.

The surjectivity (i.e., global existence) condition

$$W = \mathcal{D}(\Omega) \tag{4.32}$$

is known to hold if $C \cap \mathcal{D}(T)$ is bounded or, more generally, if $T + N_C$ is coercive. (See the comment surrounding (2.6).). Part (a) of the following corollary characterizes such surjectivity (and incidentially gives a necessary condition for coercivity of $T + N_C$). Additional, more specialized sufficient conditions for (4.32) not requiring coercivity of $T + N_C$ appear below in Corollary 5A (see also Theorem 5') and Theorem 6.

COROLLARY 3B. (a) One has

$$W = \mathcal{D}(\Omega) \iff W = \mathrm{conv}\ T(C) + D .$$

(b) In particular, (4.32) holds if C is bounded, which occurs if and only if $D = W$. If $0^+C = \{0\}$, then C is bounded if it admits a bounded nontrivial section (cf. §3) or if it is weakly locally compact.

Proof. (a) is immediate from (4.25). For (b), by Rockafellar [49, Theorem 5B] C bounded is equivalent to $D = W$. Now assume $0^+C = \{0\}$. If C is weakly locally compact, Köthe [31, page 343] implies C bounded. Finally, assume C has a bounded nontrivial section. If C were unbounded, Rockafellar [49, Theorem 5B] would imply D is a proper dense subset of W. On the other hand, (3.3) implies D has nonempty interior, which (since D is convex) implies $\mathrm{int}(\mathrm{cl}\, D) = \mathrm{int}\, D$. Since the latter would be contradicted by proper denseness, C must be bounded.

Part (b) of Corollary 3B, given mainly for completeness, deals once and for all with the relatively uninteresting (in the stability context of this paper) case in which C has a bounded nontrivial section and $0^+C = \{0\}$. For all subsequent results the reader could assume that $0^+C \neq \{0\}$.

For general monotone variational inequality problems $\mathcal{V}(a)$ it is helpful to make the following definitions. The terms used are consistent with traditional terminology in the optimization literature, and their use here is justified by Theorem 3. For any $a \in W$, define $\mathcal{V}(a)$ to be <u>strictly feasible</u>, <u>strongly feasible</u>, <u>feasible</u>, or <u>weakly feasible</u>, respectively, according to whether the parameter a belongs to the set

$$T(C) + \mathrm{int}\, D,\ \mathrm{ri}(\mathrm{conv}\, T(C) + D),\ T(C) + D,\ \text{or}\ \mathrm{conv}\, T(C) + \mathrm{cl}\, D\ .$$

The following is immediate from (4.27) of Theorem 3 and the fact that a finite-dimensional convex set has nonempty relative interior.

COROLLARY 3C. One always has

$$\text{strong feasibility} \Rightarrow \text{quasistability}$$

(recall (3.8)), with the converse also true when Z is finite-dimensional.

The following theorem provides a number of characterizations of stability. More will be added in Theorems 5 and 6.

THEOREM 4. Consider the following seven conditions:

(I) $\Omega(0)$ is nonempty and bounded [compact existence].

(II) $\Omega(a)$ is nonempty and uniformly bounded for all a in some neighborhood of the origin.

(III) $0 \in \mathcal{B}(\Omega)$ and Ω is locally bounded at the origin (i.e., $\Omega(a)$ is uniformly bounded for all a in some neighborhood of the origin).

(IV) $0 \in \operatorname{int} \mathcal{B}(\Omega)$ [stability].

(V) $0 \in \operatorname{int}(\operatorname{conv} T(C) + D)$.

(VI) $0 \in \operatorname{conv} T(C) + \operatorname{int} D$.

(VII) $0 \in T(C) + \operatorname{int} D$ [strict feasibility].

One always has

$$(I) \Leftarrow (II) \Leftrightarrow (III) \Leftrightarrow (IV) \Leftrightarrow (V) \Leftarrow (VI) \Leftarrow (VII) .$$

If C admits a bounded nontrivial section (see §3), then all seven conditions (I) - (VII) are pairwise equivalent, and furthermore,

$$0 \in \mathcal{B}(\Omega)\backslash \operatorname{int} \mathcal{B}(\Omega) \Rightarrow \Omega(0) \text{ contains a halfline .} \tag{4.33}$$

<u>Proof</u>. Clearly, (VII) $\Rightarrow$ (VI) $\Rightarrow$ (V) and (II) $\Rightarrow$ (I) and (II) $\Rightarrow$ (III). By Rockafellar [50, Theorem 1], (III) $\Rightarrow$ (IV) $\Rightarrow$ (II). By Theorem 3, (IV) $\Leftrightarrow$ (V). We conclude the proof by establishing (4.33) and (I) $\Rightarrow$ (IV) $\Rightarrow$ (VII) under the assumption C admits a bounded nontrivial section, that is (cf. (3.3)), assuming

$$0 \neq \operatorname{int} D . \tag{4.34}$$

Observe that Theorem 3 with (4.34) and (4.24) imply

$$0 \neq \operatorname{int} \mathcal{B}(\Omega) ,$$

Corollary 3A and (2.3) imply

$$\text{int}\,\mathcal{D}(\Omega) = \text{int conv}\,\mathcal{D}(\Omega) ,$$

and also

$$\text{int conv}\,\mathcal{D}(\Omega) = \text{int cl conv}\,\mathcal{D}(\Omega)$$

(since conv $\mathcal{D}(\Omega)$ is convex with nonempty interior). Therefore,

$$\phi \neq \text{int}\,\mathcal{D}(\Omega) = \text{int cl conv}\,\mathcal{D}(\Omega) . \tag{4.35}$$

In view of (4.35), if $0 \in \mathcal{D}(\Omega)\backslash\text{int}\,\mathcal{D}(\Omega)$, then Rockafellar [50, Lemma 3] implies $\Omega(0)$ contains at least one halfline. This establishes (4.33), from which follows (I) $\Rightarrow$ (IV). Finally, let (IV) hold. By (4.34) there exists some $b \in -\,\text{int}\,D$. Then (IV) implies there exists $\varepsilon > 0$ sufficiently small that

$$a := \varepsilon b \in \text{int}\,\mathcal{D}(\Omega) \subset \mathcal{D}(\Omega) = \mathcal{R}(T + N_C)$$

and $a \in -\,\text{int}\,D$. Hence, there exists (z,w) such that $z \in C$, $w \in T(z)$ and $a \in w + N_C(z)$. Therefore,

$$-\,w \in N_C(z) - a \subset D + \text{int}\,D \subset \text{int}(D + D) = \text{int}\,D$$

(since D is a convex cone), and so

$$0 \in w + \text{int}\,D \subset T(C) + \text{int}\,D .$$

Part of Theorem 4 relates to previous work. The existence part of (VII) $\Rightarrow$ (I) generalizes results obtained for the case $Z = \mathbb{R}^n$ and C a cone by Moré [40, Theorem 3.2] (see also Saigal [55]), Karamardian [28, Theorem 4.1] and McLinden [35, Theorem 1]. Also, (VI) $\Rightarrow$ (I) generalizes in the same way a result of Mangasarian-McLinden [34, Theorem 1.4].

The following corollary points out, among other things, that strict feasibility and the superficially weaker condition

$$0 \in \operatorname{conv} T(C) + \operatorname{int} D ,$$

which we call distributed strict feasibility, are actually equivalent whenever they make sense, i.e., whenever $\emptyset \neq \operatorname{int} D$.

COROLLARY 4A. Assume C admits a bounded nontrivial section. Then the four conditions

compact existence, stability, strict
feasibility, distributed strict feasibility

are pairwise equivalent. Moreover,

$$\operatorname{int} \mathcal{D}(\Omega) = \operatorname{conv} T(C) + \operatorname{int} D = T(C) + \operatorname{int} D \tag{4.36}$$

(in addition to (4.25) of Theorem 3).

Proof. The first assertion simply restates the equivalence among (I), (IV), (VII), (VI). For the second, consider any fixed $a \in W$. Since the multifunctions T_a and M_a defined by

$$T_a(z) := T(z) - a \quad , \quad M_a(z) := M(z) - a$$

satisfy

$$\mathcal{G}(T_a) = \mathcal{G}(T) - (0,a) \ , \quad \mathcal{G}(M_a) = \mathcal{G}(M) - (0,a) \ ,$$

where $\mathcal{G}(\cdot)$ denotes the graph of a multifunction considered as a subset of $Z \times W$, one has T_a monotone (resp., M_a maximal monotone) if and only if T is monotone (resp., M is maximal monotone). Also, the multifunction $\Omega_a := (M_a)^{-1}$ satisfies $\mathcal{D}(\Omega_a) = \mathcal{D}(\Omega) - a$. Now apply to T_a and Ω_a the equivalence among (IV), (VI), (VII).

COROLLARY 4B. (a) In order for $\Omega(a)$ to be nonempty and uniformly bounded for all a in some neighborhood of the origin, it is sufficient that, for some $\tilde{w} \in \mathrm{conv}\, T(C)$ and $\tilde{\varepsilon} > 0$, the set

$$\{z \in C \mid \langle z,\tilde{w}\rangle \leq \tilde{\varepsilon} + \inf_{z' \in C} \langle z',\tilde{w}\rangle\}$$

be bounded.

(b) When C admits a bounded nontrivial section, in order for $\Omega(0)$ to be nonempty and bounded, it is necessary that some $\tilde{w} \in T(C)$ be such that for all $\lambda \in [0,\infty)$ the sets

$$\{z \in C \mid \langle z,\tilde{w} - a\rangle \leq \lambda + \inf_{z' \in C} \langle z',\tilde{w} - a\rangle\}$$

be nonempty and bounded for all a in some neighborhood of the origin.

<u>Proof</u>. Part (a) restates (VI) $\Rightarrow$ (II) with the aid of (3.4). Part (b) restates (I) $\Rightarrow$ (VII) with the aid of both (3.4) and (3.5).

COROLLARY 4C. If C admits a bounded nontrivial section, then $\Omega(a)$ is unbounded if and only if $a \in \mathcal{D}(\Omega)\backslash \mathrm{int}\, \mathcal{D}(\Omega)$, in which event $\Omega(a)$ contains at least one halfline.

<u>Proof</u>. For given $a \in W$ apply Theorem 4 to the operator $T_a := T - a$, using the equivalence (I) $\Leftrightarrow$ (IV) and also (4.33).

Corollary 4C overlaps a result of Robinson [48, Theorem 2], who treats the case of $Z = \mathbb{R}^n$, C polyhedral and T a positive semidefinite matrix, and obtains for it additional, detailed information.

The next theorem and corollary augment significantly Theorem 4 and Corollary 4A. We give two formulations. For the simpler one, recall that <u>T is strictly monotone on C</u> provided

$\langle z' - z, w' - w \rangle > 0$ whenever

$$w \in T(z),\ w' \in T(z'),\ z \in C,\ z' \in C,\ z \neq z' .$$

THEOREM 5. Along with conditions (I) through (VII) of Theorem 4, consider the following condition:

(VIII) $0 \in \text{conv}\ T(C) + \text{cl}\ D$ [weak feasibility].

The eight conditions (I) - (VIII) are pairwise equivalent whenever the following assumptions hold:

(a) C admits a bounded nontrivial section and $0^+C \neq \{0\}$;

(b) T is strictly monotone on C;

(c) either (c_1) T is singlevalued monotone and hemicontinuous along each line segment in $C \subset \mathcal{D}(T)$, or else (c_2) T is maximal monotone with $C \subset \text{int conv}\ \mathcal{D}(T)$.

The proof is deferred; more specifically, a stronger result will be proved below.

COROLLARY 5A. Let assumptions (a), (b) and (c) of Theorem 5 hold. Then the five conditions

compact existence, stability, strict feasibility,
distributed strict feasibility, weak feasibility

are pairwise equivalent. Moreover,

$$\text{int}\ \mathcal{D}(\Omega) = \mathcal{D}(\Omega) = \text{conv}\ T(C) + \text{cl}\ D \qquad (4.37)$$

(in addition to (4.36) of Corollary 4A and (4.25) of Theorem 3). In particular,

$$\text{conv}\ T(C) + \text{cl}\ D \text{ closed} \Rightarrow W = \mathcal{D}(\Omega) .$$

Proof. This follows just as Corollary 4A does from Theorem 4. (Use (4.19) also.)

The conclusions of Theorem 5 and Corollary 5A remain valid under considerable relaxations of hypotheses (b) and (c_2). For this, we introduce two definitions. Let us say T is asymptotically strictly monotone on C provided for each $z \in C$ there exists $\zeta \geq 0$ such that

$$\left.\begin{array}{l} w \in T(z)\ ,\ w' \in T(z')\ , \\ z' - z \in (0^+C)\backslash \zeta B\ , \\ w' - w \in -(0^+C)^{\circ}\ , \end{array}\right\} \Rightarrow \langle z' - z, w' - w\rangle > 0\ ,$$

where $B := \{z \in Z \,|\, \|z\| \leq 1\}$. This condition is satisfied trivially by any T strictly monotone on C (take $\zeta = 0$). Incidentally, it is easy to see that $\Omega(a)$ consists of at most one point when T is strictly monotone on C. The second definition is motivated by the need to have a certain auxiliary operator, which occurs in the proof of Theorem 5' below, be maximal monotone. We say that T and C are in good position provided for each $z \in C \cap \mathcal{D}(T)$ there exists $d \in \text{int } D$ such that

for every $\delta_1 < 0$ there exists $\delta_2 < \delta_1$ for which $0 \in \text{ri}(\text{conv } \mathcal{D}(T) - C_z)$,

where

$$C_z := z + \{\tilde{z} \in 0^+C \,|\, \delta_2 \leq \langle \tilde{z}, d\rangle \leq \delta_1\}\ . \qquad (4.38)$$

This condition is satisfied in a trivial way if $C \subset \text{int conv } \mathcal{D}(T)$ and hypothesis (a) of Theorem 5 holds (use $C_z \subset C$ and Lemma 2 below). More interesting is the following general sufficient condition.

LEMMA 1. Assume C admits a bounded nontrivial section and $0^+C \neq \{0\}$. Then T and C are in good position if for each $z \in C \cap \mathcal{D}(T)$ there exists $\zeta \geq 0$ such that either

$$z + ((0^+C)\setminus\zeta B) \subset \text{int conv } \mathcal{D}(T) \tag{4.39}$$

or else

$$\phi \neq \text{int } 0^+C \quad \text{and} \quad z + ((\text{int } 0^+C)\setminus\zeta B) \subset \text{conv } \mathcal{D}(T) \ . \tag{4.40}$$

For proving this as well as Theorem 5' below, the following is useful.

LEMMA 2. Assume C admits a bounded nontrivial section and $0^+C \neq \{0\}$. Then for all $d \in \text{int } D$ (recall (3.3)) and al $\delta < 0$ the set $\{z \in 0^+C \mid \langle z,d \rangle = \delta\}$ generates a base for 0^+C.

<u>Proof</u>. Let $d \in \text{int } D$. If any nonzero $\tilde{z} \in 0^+C$ satisfied $0 \leq \langle \tilde{z},d \rangle$ then, for any $\bar{z} \in C$,

$$\bar{z} + \{\lambda\tilde{z} \mid 0 \leq \lambda < \infty\} \subset \{z \in C \mid \langle z,d \rangle \geq \langle \bar{z},d \rangle\}$$

$$= S(d,\epsilon) \quad \text{for} \quad \epsilon := \sigma_C(d) - \langle \bar{z},d \rangle \in [0,\infty)$$

would imply $S(d,\epsilon)$ unbounded, in violation of (3.4) - (3.5). Therefore $\langle z,d \rangle < 0$ for all nonzero $z \in 0^+C$. The lemma follows from this.

<u>Proof of Lemma 1</u>. Let $z \in C \cap \mathcal{D}(T)$. Pick any $d \in \text{int } D$, and let $\delta_1 < 0$. By hypothesis there exists $\zeta \geq 0$ such that either (4.39) or (4.40) holds. Pick $\delta_o < -\zeta\|d\|$. Then

$$\langle \tilde{z},d \rangle \leq \delta_o \Rightarrow \tilde{z} \notin \zeta B \ . \tag{4.41}$$

Put $\mu := \min\{\delta_0, \delta_1\}$ and pick $\delta_2 \leq \mu - \varepsilon$ for some (any) $\varepsilon > 0$. Then $\delta_2 < \delta_1$. Define C_z as in (4.38). By $\delta_2 < \mu < 0$ and Lemma 2,

$$\text{there exists } \tilde{z} \in 0^+C \text{ with } \delta_2 < \langle \tilde{z}, d \rangle < \mu . \tag{4.42}$$

By choice of μ and (4.41), such a $\tilde{z}$ satisfies

$$\tilde{z} \notin \zeta B \tag{4.43}$$

as well as $\langle \tilde{z}, d \rangle < \delta_1$. So $z' := z + \tilde{z} \in C_z$. If (4.39) holds, (4.43) yields $z' \in \text{int conv } \mathcal{B}(T)$ as well, so that

$$0 = z' - z' \in \text{int conv } \mathcal{B}(T) - C_z \subset \text{int}(\text{conv } \mathcal{B}(T) - C_z) .$$

Now suppose (4.40) holds. Then

$$\text{int } C_z = z + \{\tilde{z} \in \text{int } 0^+C \mid \delta_2 < \langle \tilde{z}, d \rangle < \delta_1\} ,$$

and the $\tilde{z}$ in (4.42) can be chosen from $\text{int } 0^+C$. Then $z' := z + \tilde{z} \in \text{int } C_z$, and by (4.43) and (4.40) also $z' \in \text{conv } \mathcal{B}(T)$. Hence

$$0 = z' - z' \in \text{conv } \mathcal{B}(T) - \text{int } C_z \subset \text{int}(\text{conv } \mathcal{B}(T) - C_z)$$

in this case also.

A simple illustration of the conditions in Lemma 1 is provided in $Z = \mathbb{R}^2$ with C the nonnegative quadrant and any T satisfying

$$\mathcal{B}(T) = C \cup \{z \mid z_1 \leq 0,\ z_2 + 1 \geq \exp(-z_1)\} .$$

Here, $0^+C = C = -D$. For any z on the nonegative z_1-axis, (4.39) fails and (4.40) holds. For any other z in $C \cap \mathcal{B}(T)$, (4.39) and (4.40) both hold.

Equipped with the two definitions given above, we can present a fancier version of Theorem 5.

THEOREM 5'. The conclusions of Theorem 5 and Corollary 5A are still valid if the assumptions (b) and (c_2) are relaxed, respectively, to

(b') T is asymptotically strictly monotone on C, and

(c_2') T is maximal monotone, and T and C are in good position.

Theorem 5' clearly subsumes Theorem 5. By Theorem 4, in order to prove Theorem 5' it suffices to establish the following.

LEMMA 3. One has (VIII) $\Rightarrow$ (VI) if hypotheses (a), (b') and either (c_1) or (c_2') hold.

This lemma was inspired by a certain result of Karamardian for the problem $\mathcal{C}(0)$ in $\mathbb{R}^n$ [28, Theorem 4.2 and Corollary 4.1] (which built on his earlier result [26, Theorem 3(i)]).

<u>Proof</u>. Assume that (VIII), (a), (b') and either (c_1) or (c_2') hold. Hypothesis (VIII) gives

$$0 \in \hat{w} + \mathrm{cl}\, D\ , \quad \text{where} \quad \hat{w} = \Sigma\, \lambda_k w_k\ ,$$

for $w_k \in T(z_k)$, $z_k \in C$, $\lambda_k \geq 0$, $\Sigma\, \lambda_k = 1$ (summation over $k = 1,\ldots,m$). Consider z_k for fixed k. Let $\zeta \geq 0$ be as guaranteed for this z_k by (b'). Consider first the case of (c_2'). Let $d \in \mathrm{int}\, D$ be as guaranteed for this z_k by (c_2'). Pick $\delta_1 < -\zeta\|d\|$. Then

$$\langle \tilde{z}, d \rangle \leq \delta_1 \Rightarrow \tilde{z} \notin \zeta B\ . \tag{4.44}$$

By (c_2') there exists $\delta_2 < \delta_1$ for which

$$0 \in \operatorname{ri}(\operatorname{conv} \mathcal{D}(T) - C_k) \ , \tag{4.45}$$

where we write

$$C_k := z_k + A_k \ , \ A_k := \{\tilde{z} \in 0^+C \,|\, \delta_2 \leq \langle \tilde{z}, d \rangle \leq \delta_1\} \ . \tag{4.46}$$

By (4.45) and Proposition 1(b), the operator $T + N_{C_k}$ is maximal monotone. Also, (3.4) - (3.5) imply its effective domain is bounded, since

$$\mathcal{D}(T + N_{C_k}) = \mathcal{D}(T) \cap C_k \subset C_k \subset S(d,\epsilon) \tag{4.47}$$

for $\epsilon := \sigma(d) - \langle z_k, d \rangle - \delta_2$. Therefore, Rockafellar [52, Theorem 4] implies $W = \mathcal{R}(T + N_{C_k})$, so

$$w_k \in \mathcal{R}(T + N_{C_k}) \ . \tag{4.48}$$

Now consider the case of (c_1). For any $d \in \operatorname{int} D$ and any $\delta_2 < \delta_1 < -\zeta\|d\|$, define C_k and A_k again via (4.46). Then C_k is again bounded, by (4.47). Therefore (c_1) and the Browder/Hartman-Stampacchia theorem (e.g., [24, Theorem 4]) imply $W = \mathcal{R}(T + N_{C_k})$. Thus, (4.48) holds in either case, (c_1) or (c_2'). Hence, there exists (z_k', w_k') with

$$w_k' \in T(z_k') \ , \ z_k' \in C_k \ , \ w_k - w_k' \in N_{C_k}(z_k') \ .$$

Then

$$0 \geq \langle z - z_k', w_k - w_k' \rangle \ \text{ for all } \ z \in C_k \ ,$$

so

$$\langle z - z_k, w_k' - w_k \rangle \geq \langle z_k' - z_k, w_k' - w_k \rangle =: \gamma \ \text{ for all } \ z \in C_k. \tag{4.49}$$

Since $\gamma \geq 0$ by monotonicity of T,

$$\langle z, w_k' - w_k \rangle \geq 0 \quad \text{for all} \quad z \in A_k . \tag{4.50}$$

Since A_k generates a base for 0^+C by Lemma 2, (4.50) implies $w_k' - w_k \in -(0^+C)^o$. Also, $z_k' \in C_k$ implies $z_k' - z_k \in 0^+C\setminus\{0\}$ and, using (4.44), $z_k' - z_k \notin \zeta B$. Therefore, (b') implies

$$\gamma > 0 . \tag{4.51}$$

Since A_k is bounded (being a translate of C_k), it is equicontinuous. Hence (using (4.51)), there exists a neighborhood U of the origin such that

$$\langle z, w \rangle \geq -\gamma \quad \text{for all} \quad w \in U \quad \text{and all} \quad z \in A_k . \tag{4.52}$$

Adding (4.52) and (4.49) yields

$$\langle z, w_k' - w_k + w \rangle \geq 0 \quad \text{for all} \quad z \in A_k \quad \text{and all} \quad w \in U .$$

By Lemma 2 this implies

$$w_k' - w_k + U \subset -(0^+C)^o$$

and thus, using (3.7),

$$w_k - w_k' \in \operatorname{int}(0^+C)^o = \operatorname{int} \operatorname{cl} D .$$

Since this holds for all indices k,

$$\Sigma \lambda_k (w_k - w_k') \in \operatorname{int} \operatorname{cl} D .$$

Hence, for $\tilde{w} := \Sigma \lambda_k w_k'$ we obtain

$$-\tilde{w} \in -\hat{w} + \operatorname{int} \operatorname{cl} D \subset \operatorname{cl} D + \operatorname{int} \operatorname{cl} D$$

$$\subset \operatorname{int}(\operatorname{cl} D + \operatorname{cl} D) = \operatorname{int} \operatorname{cl} D = \operatorname{int} D ,$$

and thus

$$0 \in \tilde{w} + \text{int } D . \tag{4.53}$$

Since $z_k' \in C_k \subset C$ and $w_k' \in T(z_k')$, also

$$\tilde{w} \in \text{conv } T(C) . \tag{4.54}$$

Combining (4.53) and (4.54) establishes (VI).

5. Refinements: the monotone LCP case. The general results of §4 all apply, of course, to the linear complementarity problem (abbreviated LCP) case, in which T is linear and C is a cone. Here we derive for this situation two additional theorems (Theorems 6 and 7 below) and consequences. Also, here most of the earlier results will be recast for convenience, and/or sharpened, for this important case.

From now on we denote by L the operator T, assumed linear, and by Q the set C, assumed to be a cone. By linearity of L we mean either that L is an operator which is singlevalued monotone and linear on $Q \subset \mathcal{D}(L)$ or else that L is a closed, densely defined linear transformation which is monotone, whose adjoint is also monotone, and which satisfies $0 \in \text{int}(\mathcal{D}(L) - Q)$. Throughout this section we assume L falls into one of these two cases. Either case ensures that the blanket hypothesis (cf. (2.3))

$$M := L + N_Q \text{ is maximal monotone} \tag{5.1}$$

is met (by parts (a) and (b) of Proposition 1 plus the result of Brézis cited following Proposition 1).

Since now $D = Q^o$, by putting $Q^* := -Q^o$ we can rewrite the parametric solution multifunction Ω as

$$\Omega(p) = \{z \in \mathcal{D}(L) \mid z \in Q,\ Lz - p \in Q^*,\ \langle z, Lz - p\rangle = 0\} \quad (5.2)$$

for all $p \in W$. Note that, in conformity with the general scheme of §4, we write $Lz - p \in Q^*$ in (5.2). Keeping this in mind, there should be no confusion in comparing the present results with those in the finite-dimensional LCP literature, where the corresponding constraint is now often written as $Mz + q \geqq 0$.

COROLLARY 1A. Even without (5.1), for any $p \in L(Q) - \operatorname{int} Q^*$ the set in (5.2) satisfies

$$\Omega(p) \subset \{z \in Q \mid \langle z, L\tilde{z} - p\rangle \leq \gamma + \langle \tilde{z}, -p\rangle\} ,$$

with the set on the right nonempty and bounded, whenever

$$\tilde{z} := \Sigma\, \lambda_k z_k \quad \text{and} \quad \gamma := \Sigma\, \lambda_k \langle z_k, Lz_k\rangle$$

(summation over $k = 1,\ldots,m$) satisfy

$$z_k \in Q \cap \mathcal{D}(L) \ , \ \lambda_k \geq 0 \ , \ \Sigma\, \lambda_k = 1 \ ,$$

$$p \in L\tilde{z} - \operatorname{int} Q^* \ , \ \gamma + \langle \tilde{z}, -p\rangle \geq 0 \ .$$

THEOREM 2. Let Z and W be any locally convex Hausdorff spaces paired in duality, with Z (resp. W) assigned the weak (resp. Mackey) topology induced by the pairing. Assume L is singlevalued monotone and linear on $Q \subset \mathcal{D}(L)$. Then

$$L(Q) - \text{int } Q^* \subset \text{int } \mathcal{B}(\Omega) \quad . \tag{5.3}$$

Of course, (5.3) is in addition to the general inclusions (cf. (4.19))

$$\mathcal{B}(\Omega) \subset L(Q) + \mathcal{R}(N_Q) \subset L(Q) - Q^* \ , \tag{5.4}$$

$$- \text{ int } Q^* \subset \mathcal{R}(N_Q) \subset - Q^* \ . \tag{5.5}$$

In regard to (5.4) a well known result of Cottle [12, page 241] shows that

$$\mathcal{B}(\Omega) = L(Q) - Q^* \tag{5.6}$$

when $Z = \mathbb{R}^n$, $Q = \mathbb{R}^n_+ := \{z | z_i \geq 0 \text{ for } i = 1,\ldots,n\}$ and L is a positive semidefinite matrix. This result was extended to successively larger classes of matrices by Lemke [32, Theorem 4], Eaves [18], Garcia [22, Theorem 3.4], and Doverspike [16, Theorem 3.2].

Lemke in fact gave a celebrated pivoting-type algorithm for finding an element of $\Omega(p)$, without assuming such exists, which as a byproduct *establishes (5.6) for* L *a copositive plus matrix.* We wish to point out that a slight refinement of Lemke's proof of [32, Theorem 4] shows that (5.6) holds (in $Z = \mathbb{R}^n$ with $Q = \mathbb{R}^n_+$) with his copositive plus assumption on L weakened to:

$$z \in Q \Rightarrow \langle z, Lz \rangle \geq 0 \ ,$$

and

$$[z \in Q, \ Lz \in Q^*, \ 0 = \langle z, Lz \rangle] \Rightarrow Lz + L^* z \in Q^* \ .$$

THEOREM 3. One always has

$$\mathrm{int}\ \mathcal{B}(\Omega) = \mathrm{int}(L(Q) - Q^*) \ , \tag{5.7}$$

$$\mathrm{cl}\ \mathcal{B}(\Omega) = \mathrm{cl}(L(Q) - Q^*) \ , \tag{5.8}$$

and, if $\emptyset \neq \mathrm{ri}(L(Q) - Q^*)$, also

$$\mathrm{ri}\ \mathcal{B}(\Omega) = \mathrm{ri}(L(Q) - Q^*) \ . \tag{5.8a}$$

COROLLARY 3B. One has

$$W = \mathcal{B}(\Omega) \iff W = L(Q) - Q^* \ .$$

THEOREM 4. For any fixed $p \in W$, consider the following six conditions:

(I) $\Omega(p)$ is nonempty and bounded [compact existence].

(II) $\Omega(p + a)$ is nonempty and uniformly bounded for all a in some neighborhood of the origin.

(III) $p \in \mathcal{B}(\Omega)$ and Ω is uniformly bounded for all a in some neighborhood of p.

(IV) $p \in \mathrm{int}\ \mathcal{B}(\Omega)$ [stability].

(V) $p \in \mathrm{int}(L(Q) - Q^*)$.

(VI) $p \in L(Q) - \mathrm{int}\ Q^*$ [strict feasibility].

One always has

$$(I) \Leftarrow (II) \iff (III) \iff (IV) \iff (V) \Leftarrow (VI) \ .$$

If Q has a bounded base (i.e., nontrivial bounded section), then all six conditions (I) - (VI) are pairwise equivalent, and furthermore,

$$p \in \mathcal{B}(\Omega)\backslash \mathrm{int}\ \mathcal{B}(\Omega) \Rightarrow \Omega(p) \text{ contains a halfline} \ . \tag{5.9}$$

Proof. Apply the general Theorem 4 to $C := Q$ and $T_p(z) := Lz - p$, obtaining conclusions in terms of the associated multifunction $\Omega_p := (T_p + N_C)^{-1}$. Since $\Omega_p(a) = \Omega(p + a)$ for Ω as in (5.2), the conclusions can be rewritten as indicated.

Parts of Theorem 4 (LCP case) relate to previous work on the LCP for the case $Z = \mathbb{R}^n$ and $Q = \mathbb{R}^n_+$. For this case, Mangasarian [33, Theorem 2] has given a number of characterizations of our condition (I), assuming L is a copositive plus matrix. Also for this case, work of Doverspike [16, Theorem 3.3] sharpens the implication (I) $\Rightarrow$ (IV) for L belonging to a class of matrices more general than copositive plus.

COROLLARY 4A. Assume Q has a bounded base. Then the three conditions

compact existence, stability, strict feasibility,

are pairwise equivalent. Moreover,

$$\operatorname{int} \mathcal{D}(\Omega) = L(Q) - \operatorname{int} Q^* \tag{5.10}$$

(in addition to (5.7)).

COROLLARY 4B. Let any $p \in W$ be fixed. (a) In order for $\Omega(p + a)$ to be nonempty and uniformly bounded for all a in some neighborhood of the origin, it is sufficient that, for some $\tilde{\varepsilon} > 0$ and $\tilde{z} \in Q \cap \mathcal{D}(L)$ satisfying $L\tilde{z} - p \in Q^*$ the set

$$\{z \in Q \mid \langle z, L\tilde{z} - p\rangle \le \tilde{\varepsilon}\}$$

be bounded. (b) When Q has a bounded base, in order for $\Omega(p)$ to be nonempty and bounded, it is necessary that some $\tilde{z} \in Q \cap \mathcal{D}(L)$ satisfy

$L\tilde{z} - p \in \text{int } Q^*$ and be such that for all $\lambda \in [0,\infty)$ the sets

$$\{z \in Q \mid \langle z, L\tilde{z} - p - a\rangle \leq \lambda\}$$

are bounded for all a in some neighborhood of the origin.

COROLLARY 4C. If Q has a bounded base, then $\Omega(p)$ is unbounded if and only if $p \in \mathcal{D}(\Omega)\setminus\text{int } \mathcal{D}(\Omega)$, in which event $\Omega(p)$ contains at least one halfline.

This overlaps a result of Cottle [13, Theorem 3.1], who treats the case $Z = \mathbb{R}^n$ and $Q = \mathbb{R}^n_+$ with L a copositive plus matrix.

The following theorem sharpens Theorem 4 (LCP case) for the pivotal parameter choice $p = 0$.

THEOREM 6. Let (I_o) through (VI_o) denote the conditions of Theorem 4 (LCP case) corresponding to $p = 0$. Consider also the following four conditions:

(I_o') Only $z = 0$ solves the system

$$z \in Q \cap \mathcal{D}(L) \ , \quad Lz \in Q^* \ , \quad \langle z, Lz\rangle = 0 \ ;$$

(I_o'') L is asymptotically strictly monotone on Q (defined in §4 following Corollary 5A);

(IV_o') $W = \mathcal{D}(\Omega)$;

(V_0') $W = L(Q) - Q^*$.

One always has

$$(I_o) \Leftrightarrow (I_o') \Leftrightarrow (I_o'') \Leftarrow (II_o) \Leftrightarrow (III_o) \Leftrightarrow$$

$$\Leftrightarrow (IV_o) \Leftrightarrow (IV_o') \Leftrightarrow (V_0') \Leftrightarrow (V_o) \Leftarrow (VI_o) \ .$$

If Q has a bounded base, then all ten of these conditions are pairwise equivalent.

Proof. In view of Theorem 4 (LCP case) and Corollary 3B (LCP case) it will suffice to show that $(I_o) \Rightarrow (I_o') \Leftrightarrow (I_o'')$ and $(IV_o) \Rightarrow (IV_o')$. The latter follows from (5.7), since $L(Q) - Q^*$ is a cone. To see $(I_o) \Rightarrow (I_o')$, observe that if some nonzero $z \in \mathcal{D}(L)$ satisfied $z \in Q$, $Lz \in Q^*$ and $\langle z, Lz\rangle = 0$, then by all the positive homogeneity it would follow that $\Omega(0) \supset \{\lambda z \,|\, 0 \leq \lambda < \infty\}$, showing $\Omega(0)$ unbounded. Finally, (I_o') is equivalent to the condition

$$\langle z, Lz\rangle > 0 \quad \text{whenever} \quad 0 \neq z \in Q \cap \mathcal{D}(L) \quad \text{and} \quad Lz \in Q^* .$$

This implies (take $\zeta = 0$) and is implied by (use homogeneity) the condition

$$\begin{aligned} &\text{there exists} \quad \zeta \geq 0 \quad \text{such that} \quad \langle z, Lz\rangle > 0 \\ &\text{whenever} \quad z \in (Q \cap \mathcal{D}(L))\backslash \zeta B \quad \text{and} \quad Lz \in Q^* . \end{aligned}$$

This is implied by (take $z'' = z$, $z' = 0$) the condition that for every $z' \in Q \cap \mathcal{D}(L)$ there exists $\zeta \geq 0$ such that

$$\left.\begin{aligned} &\langle z'' - z', L(z'' - z')\rangle > 0 \quad \text{whenever} \\ &z'', z' \in \mathcal{D}(L),\ z'' - z' \in Q\backslash \zeta B,\ L(z'' - z') \in Q^* . \end{aligned}\right\} \tag{5.11}$$

Since (5.11) depends not on z' but only on $z = z'' - z'$ the converse implication also holds. By linearity of L, this last condition is equivalent to (I_o'').

In view of Theorem 6, nothing new is contributed to the LCP case by Theorem 5'. This is because Lemma 3, when applied to the present LCP case, amounts to the implication $(I_o'') \Rightarrow (VI_o)$ under hypotheses slightly stronger than required by Theorem 6.

Parts of Theorem 6 relate to previous work on the LCP for the case $Z = \mathbb{R}^n$ and $Q = \mathbb{R}^n_+$. In this case the equivalence among the conditions (IV_o'), (V_0'), (VI_o) has been shown by Mangasarian [33, Corollary 3] for L a copositive plus matrix. The equivalence between (I_o') and (IV_o') overlaps, again for the case $Z = \mathbb{R}^n$ and $Q = \mathbb{R}^n_+$, results of Aganagic-Cottle [1] and Pang [46], both of which depend on a result of Karamardian [27, Theorem 4.1].

The following is a theorem of the alternative which of course covers positive semidefinite matrices.

COROLLARY 6A. If Q has a bounded base, then exactly one of the following alternatives holds:

(i) there exists z such that $0 \neq z \in Q \cap \mathcal{D}(L)$, $Lz \in Q^*$, $\langle z, Lz \rangle = 0$.

(ii) there exists z such that $z \in Q \cap \mathcal{D}(L)$, $Lz \in \operatorname{int} Q^*$.

<u>Proof</u>. This expresses $(I_o') \Leftrightarrow (VI_o)$.

Let us say L is <u>skew-symmetric</u> (i.e., anti-selfadjoint) if $\langle Lz_1, z_2 \rangle = \langle z_1, -Lz_2 \rangle$ for all $z_1, z_2 \in \mathcal{D}(L)$. This implies that $\langle z, Lz \rangle = 0$ for all $z \in \mathcal{D}(L)$.

COROLLARY 6B. If $Q \neq \{0\}$ has a bounded base and L is skew-symmetric, then there exists a nonzero $z \in Q \cap \mathcal{D}(L)$ such that $Lz \in Q^*$.

<u>Proof</u>. Suppose $0 \in L(Q) - \operatorname{int} Q^*$, that is, $L\tilde{z} \in \operatorname{int} Q^*$ for some $\tilde{z} \in Q \cap \mathcal{D}(L)$. Necessarily $\tilde{z} \neq 0$, since otherwise $0 = L\tilde{z} \in \operatorname{int} Q^*$,

implying $Q^* = W$, in violation of $Q \neq \{0\}$. Then the proof of Lemma 2 yields $\langle \tilde{z}, L\tilde{z} \rangle > 0$, violating skew-symmetry. Therefore $0 \notin L(Q) - \text{int } Q^*$. Now apply Corollary 6A.

In Corollary 6B it can happen that the z in question satisfies $Lz = 0$ (e.g., if $Q \subset L^{-1}(0)$). Also, recall that Tucker's theorem [57, Theorem 5], which addresses the case $Z = \mathbb{R}^n$ and $Q = \mathbb{R}^n_+$ with L a skew-symmetric matrix, assures that the z in Corollary 6B can be taken to satisfy $z + Lz \in \text{int } Q^*$ also. We note that this additional property may fail for other choices of Q, as is illustrated in $\mathbb{R}^2$ by $L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and

$$Q = \{z \mid \alpha z_2 \geq z_1 \text{ and } \alpha z_2 \geq -z_1\} \text{ for any } 0 < \alpha \leq \sqrt{2} - 1 .$$

It follows from the proof given below for (5.13) that under the hypothe.'s of Corollary 6B there exists $z_0 \in Z$ such that

$$0 \neq z_0 \text{ and } \langle z_0, a \rangle \leq 0 \text{ for all } a \in \mathcal{D}(\Omega) . \tag{5.12}$$

THEOREM 7. Assume $Q \neq \{0\}$ has a bounded base and that L is skew-symmetric, $\mathcal{D}(L) = Z$ and L is continuous. Then for each $p \in \mathcal{D}(\Omega) \setminus \text{int } \mathcal{D}(\Omega)$ there exists $z_p \in Z$ such that

$$0 \neq z_p \text{ and } \langle z_p, a \rangle \leq \langle z_p, p \rangle \text{ for all } a \in \mathcal{D}(\Omega) , \tag{5.13}$$

or equivalently,

$$0 \neq z_p \in Q , \quad Lz_p \in Q^* , \quad \langle z_p, p \rangle = 0 , \tag{5.14}$$

or equivalently,

$$0 \neq z_p \in 0^+\Omega(p) . \tag{5.15}$$

In particular, there exists $z_0 \in Z$ such that

$$0 \neq z_0 \in 0^+\Omega(0) = Q \cap L^{-1}(Q^*) = 0^+\Phi(p) \quad \text{for all} \quad p \in L(Q) - Q^* , \tag{5.16}$$

where

$$\Phi(p) := \{z \in \mathcal{D}(L) \mid z \in Q,\ Lz - p \in Q^*\} . \tag{5.17}$$

Proof. Assume Q has a bounded base. Then $\operatorname{int} \mathcal{D}(\Omega)$ is nonempty, by (3.3) and (5.7), and convex, by Corollary 3A applied to $\Omega = (L + N_Q)^{-1}$ via (5.1). Hence, by the separation theorem, for each $p \in \mathcal{D}(\Omega)\setminus\operatorname{int} \mathcal{D}(\Omega)$ there exists z_p such that (5.13) holds. In view of (5.7), it is routine to show (5.13) equivalent to (5.14) (use $-L^* = L$ and $Q^{oo} = Q$). Now assume $\mathcal{D}(L) = Z$ and that L is continuous. Since for any $p \in L(Q) - Q^*$ the nonempty set $\Phi(p)$ given by (5.17) can be written

$$\Phi(p) = Q \cap L^{-1}(Q^* + p) ,$$

it follows from Rockafellar [49, Theorem 2A(b)] and continuity of L that

$$0^+\Phi(p) = Q \cap 0^+L^{-1}(Q^* + p) .$$

From the definitions, $0^+L^{-1}(Q^* + p) \supset L^{-1}0^+(Q^* + p)$. On the other hand, suppose $\tilde{z} \in 0^+L^{-1}(Q^* + p)$. Pick any $z \in L^{-1}(Q^* + p)$. By [49, Theorem 2A(b)], $z + \lambda\tilde{z} \in L^{-1}(Q^* + p)$ for all $\lambda \geq 0$, so $Lz + \lambda L\tilde{z} \in Q^* + p$ for all $\lambda \geq 0$. Hence $L\tilde{z} \in 0^+(Q^* + p)$, using [49, Theorem 2A(b)] once more. Thus,

$$0^+L^{-1}(Q^* + p) = L^{-1}0^+(Q^* + p) .$$

Since clearly $0^+(Q^* + p) = Q^*$, we have shown

$$0^+\Phi(p) = Q \cap L^{-1}(Q^*) \quad \text{for all} \quad p \in L(Q) - Q^* . \tag{5.18}$$

Now assume L is skew-symmetric. Then

$$\Omega(p) = \Phi(p) \cap \{z \in Z \mid \langle z,p \rangle = 0\} ,$$

so (using [49, Theorem 2A(b)] again)

$$0^+\Omega(p) = 0^+\Phi(p) \cap \{z \in Z \mid \langle z,p \rangle = 0\} \quad \text{for all} \quad p \in \mathcal{B}(\Omega) . \tag{5.19}$$

The equivalence between (5.15) and (5.14) now follows by combining (5.19) with (5.18). The last assertion follows from what has already been established, since $0 \in \mathcal{B}(\Omega) \backslash \text{int } \mathcal{B}(\Omega)$ when L is skew-symmetric and $Q \neq \{0\}$ (by the proof of Corollary 6B).

Conclusion (5.16) can be compared with a result obtained by Lemke [32, Theorem 5] for the case $Z = \mathbb{R}^n$ and $Q = \mathbb{R}^n_+$ with L a positive semidefinite matrix. He showed that for each $p \in L(Q) - Q^*$ for which $\Phi(p)$ is nondegenerate, $\Phi(p)$ must contain at least n rays (some of which might conceivably be parallel). Notice that $\Phi(p)$ nondegenerate implies $\Omega(p)$ has at most one element, hence by Cottle's result (5.6) exactly one element, and thus by (5.15), (5.9) and (5.10) requires that $p \in L(Q) - \text{int } Q^*$.

A toy illustration of Theorem 7 (yet one adequate for §6) is provided in $\mathbb{R}^2$ with $Q = \mathbb{R}^2_+$ and $L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. One has $L(Q) - Q^* = \{w \mid w_1 \in \mathbb{R}, w_2 \leq 0\} = \mathcal{B}(\Omega)$. For $p \in L(Q) - Q^*$, $\Phi(p)$ is given by:

$$\{z \mid 0 \le z_1 \le -p_2,\ p_1 \le z_2 < \infty\} \quad \text{if} \quad p_1 \ge 0 \quad \text{and} \quad p_2 \le 0 ,$$

$$\{z \mid 0 \le z_1 \le -p_2,\ 0 \le z_2 < \infty\} \quad \text{if} \quad p_1 \le 0 \quad \text{and} \quad p_2 \le 0 .$$

For $p \in \mathcal{D}(\Omega)$, $\Omega(p)$ is given by:

$$\{z \mid z_1 = 0,\ p_1 \le z_2 < \infty\} \quad \text{if} \quad p_1 \ge 0 \quad \text{and} \quad p_2 = 0 ,$$

$$\{(-p_2, p_1)\} \quad \text{if} \quad p_1 > 0 \quad \text{and} \quad p_2 < 0 ,$$

$$\{z \mid 0 \le z_1 \le -p_2,\ 0 = z_2\} \quad \text{if} \quad p_1 = 0 \quad \text{and} \quad p_2 < 0 ,$$

$$\{(0,0)\} \quad \text{if} \quad p_1 < 0 \quad \text{and} \quad p_2 < 0 ,$$

$$\{z \mid z_1 = 0,\ 0 \le z_2 < \infty\} \quad \text{if} \quad p_1 < 0 \quad \text{and} \quad p_2 = 0 .$$

6. Application to linear programming in reflexive Banach spaces.

All the results of §§4-5 apply to the fundamental problems called linear programs, as follows. Let X (resp. Y) be a reflexive Banach space with dual V (resp. U), and let Q_1 (resp. Q_2) be a nonempty closed convex cone in X (resp. Y). Write $Q_i^* = -Q_i^o$ for $i = 1,2$. Let A be a closed, densely defined linear transformation from X into U. Recall that then the adjoint A^* of A, mapping Y into V, is also a closed, densely defined linear transformation, and if $\mathcal{D}(A) = X$ with A continuous, then $\mathcal{D}(A^*) = Y$ with A^* continuous. For each fixed parameter pair $(c,b) \in V \times U$, the optimization problems

$$\inf\{\langle x,c\rangle \mid x \in Q_1 \cap \mathcal{D}(A),\ Ax - b \in Q_2^*\} , \qquad P(c,b)$$

$$\sup\{\langle b,y\rangle \mid y \in Q_2 \cap \mathcal{D}(A^*),\ -A^*y + c \in Q_1^*\} \qquad D(c,b)$$

correspond to the classical primal-dual pair of linear programming problems. For a comprehensive treatment of the basic, finite-dimensional case, see Dantzig [15]. Extension of the theory to infinite dimensions was initiated by Duffin [17]; see Nakamura-Yamasaki [45] and the references therein for further infinite-dimensional results.

To apply the present results to the dual pair $P(c,b)$, $D(c,b)$ of linear programs, one verifies first without difficulty that a pair (x,y) is such that x solves $P(c,b)$ and y solves $D(c,b)$ if and only if it satisfies the conditions

$$x \in Q_1 \cap \mathcal{D}(A) \ , \ Ax - b \in Q_2^* \ , \tag{6.1}$$

$$y \in Q_2 \cap \mathcal{D}(A^*) \ , \ -A^*y + c \in Q_1^* \ , \tag{6.2}$$

$$\langle x,c \rangle = \langle b,y \rangle \ . \tag{6.3}$$

Next, introduce

$$Z := X \times Y \ , \ W := V \times U \ , \ \langle z,w \rangle := \langle x,v \rangle + \langle u,y \rangle \ ,$$

$$Q := Q_1 \times Q_2 \ , \ p := (-c,b) \ , \ a := (-v,u) \ ,$$

and define a closed, densely defined linear transformation L from Z into W via

$$L(z) = L(x,y) := (-A^*y, Ax) \text{ on } \mathcal{D}(L) := \mathcal{D}(A) \times \mathcal{D}(A^*) \ . \tag{6.4}$$

This leads to the adjoint given by

$$L^* = -L \text{ on } \mathcal{D}(L^*) = \mathcal{D}(L) \ , \tag{6.5}$$

so L is skew-symmetric. It can be shown that

$$L \text{ is maximal monotone} \tag{6.6}$$

(for such an A) without using the reflexive Banach space assumption, that is, in the general setting of X and V (resp. Y and U) locally convex Hausdorff spaces paired in duality.

To ensure the blanket hypothesis (5.1), we assume that

$$\text{either} \qquad Q_1 \subset \mathcal{D}(A) \quad \text{and} \quad Q_2 \subset \mathcal{D}(A^*) \tag{6.7a}$$

$$\text{or} \qquad 0 \in \operatorname{int}(\mathcal{D}(A) - Q_1) \quad \text{and} \quad 0 \in \operatorname{int}(\mathcal{D}(A^*) - Q_2) \ . \tag{6.7b}$$

This is of course satisfied trivially if $\mathcal{D}(A) = X$ with A continuous.

One can check easily that, for $p = (-c,b)$, the solution set $\Omega(p)$ in (5.2) and the feasible set $\Phi(p)$ in (5.17) here take the form

$$\begin{aligned} \Omega(p) &= \{(x,y) \mid (x,y) \text{ satisfies } (6.1), (6.2), (6.3)\} \\ &= \Phi(p) \cap \{(x,y) \mid \langle x,c \rangle = \langle b,y \rangle\} \\ &= \{x \mid x \text{ solves } P(c,b)\} \times \{y \mid y \text{ solves } D(c,b)\} \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} \Phi(p) &= \{(x,y) \mid (x,y) \text{ satisfies } (6.1), (6.2)\} \\ &= \{x \mid x \text{ feasible for } P(c,b)\} \times \{y \mid y \text{ feasible for } D(c,b)\} \ . \end{aligned} \tag{6.9}$$

The condition frequently invoked in §§4-5 that Q have a bounded base (i.e., bounded nontrivial section) becomes here the condition

$$\text{both } Q_1 \text{ and } Q_2 \text{ have a bounded base ,} \tag{6.10}$$

which by (3.3) is equivalent to the condition

$$\emptyset \neq \operatorname{int} Q_1^* \quad \text{and} \quad \emptyset \neq \operatorname{int} Q_2^* \ . \tag{6.11}$$

With the above identifications, all the results of §§4-5 apply to P(c,b) and D(c,b) with (c,b) treated as the parameter. Various new facts are obtained about linear programming.

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Variational inequalities associated with monotone operators (possibly nonlinear and multivalued and convex sets (possibly unbounded are studied in reflexive Banach spaces. A variety of results are given which relate to a stability concept involving a natural parameter. These include characterizations useful as criteria for stable existence of solutions and also several characterizations of surjectivity. The monotone complementarity problem is covered as a special case, and the results are sharpened for linear monotone complementarity and for generalized linear programming.

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