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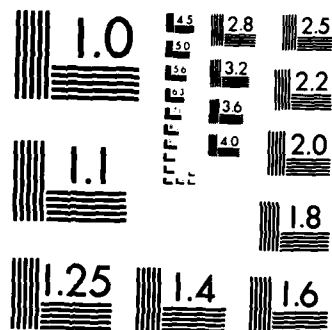
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MRC Technical Summary Report #2731

ON THE ASYMPTOTIC ANALYSIS OF
TRAVELLING SHOCKS AND PHASE
BOUNDARIES IN ELASTIC BARS

Thomas J. Pence

AD-A147 239

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ON THE ASYMPTOTIC ANALYSIS OF TRAVELLING SHOCKS
AND PHASE BOUNDARIES IN ELASTIC BARS

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ABSTRACT

This paper is concerned with the propagation of shocks and phase boundaries in elastic bars. We consider materials for which the one-dimensional stress response is piecewise linear and not monotonic. In the presence of an applied load the dynamical fields are described by a set of functional equations. These equations are treated asymptotically for a model problem involving a load which approaches a constant value. The dynamical fields approach the solution given by a corresponding Riemann problem at a rate t^{-n} where $n < -2$ is given in terms of the stress response.

t to the minus n power

original supplied keywords missing

AMS (MOS) Subject Classifications: 35M05, 35L67, 73D99, 35L65, 41A60, 39B30

Key Words: Phase Transitions, Elastic Solids, Functional Equations, *and*

Work Unit Number 2 (Physical Mathematics)

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SIGNIFICANCE AND EXPLANATION

When the stress-strain law governing the dynamics of elastic bars is nonlinear, certain loads give rise to travelling shock waves. If the stress-strain relation is not monotonic, then these shock waves may also describe phase boundaries. In the event the applied load approaches a constant value, the fields in a semi-infinite bar evolve in time toward a simple wave solution of an appropriate Riemann problem. The nature of the approach to this simple wave depends on the detailed loading program. For materials in which the stress-strain law is piecewise linear, the solution is governed by different linear wave equations in distinct regions of an (x,t) -plane. The curves separating these regions obey a set of functional-differential equations. These equations are derived and treated asymptotically in order to determine the large-time dependence of the solution on the complete loading history. The rate of approach to the simple wave solution is found to be independent of the early loading history.

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ON THE ASYMPTOTIC ANALYSIS OF TRAVELLING SHOCKS
AND PHASE BOUNDARIES IN ELASTIC BARS

Thomas J. Pence

I. INTRODUCTION. This investigation is concerned with the propagation of shocks and phase boundaries in elastic solids. Attention is restricted to one-dimensional motions; for simplicity, imagine a bar in which transverse displacements are absent. The problem is introduced in section II, where it is reviewed how change of phase phenomena can be modelled by means of a nonmonotonic stress-strain law. These laws have been studied previously in [1], [2], [3]; a relevant experimental study is [4]. This section also treats the simple wave that develops whenever a nonzero load σ_∞ is suddenly applied to the end of the bar. This simple wave would be expected to mirror in some fashion the ultimate state of affairs whenever the bar is gradually loaded at one end to the level σ_∞ provided waves are not subsequently reflected back from the opposite end. Issues involved in such an asymptotic study are discussed in the third section. Section IV addresses special considerations for materials in which the stress-strain law is piecewise linear. Then, in section V, we carry out an asymptotic analysis for an example problem involving such a material.

II. FORMULATION OF THE PROBLEM AND THE RIEMANN SOLUTION FOR IMPULSIVE LOADING. Consider a homogeneous, semi-infinite elastic bar which occupies $x \geq 0$ in a reference configuration. Pure longitudinal motion is governed by the momentum equation

$$(2.1) \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x},$$

where $u = u(x, t)$ is the longitudinal displacement of the bar and σ is the stress along the axis of the bar. The density in the reference state is taken to be one. Let

$$(2.2) \quad \epsilon = \frac{\partial u}{\partial x}, \quad v = \frac{\partial u}{\partial t}$$

denote respectively the strain and velocity in the bar. For elastic materials, the stress at time t at a location which was originally at position x is completely determined by the value of $\epsilon(x, t)$ by means of the constitutive relation $\sigma = \sigma(\epsilon)$. The sound speed of the material can be identified by expressing (2.1) in characteristic form and is found to be

$\sqrt{\sigma'(\epsilon)}$. Here the $'$ symbol is the usual shorthand notation for derivative. We shall focus attention on a hypothetical material for which the stress-strain law $\sigma(\epsilon)$ is given by the smooth curve in Fig. 1. Here

$\sigma(0) = 0$ and we shall restrict attention to positive values of σ and ϵ . This entails no loss in generality since compressive motions can be treated by considering a corresponding extensional problem for a material with a stress-strain curve found by reflecting the original curve through the origin. Unlike the state of affairs in gas dynamics, this technique gives correct results even in the presence of shocks [5].

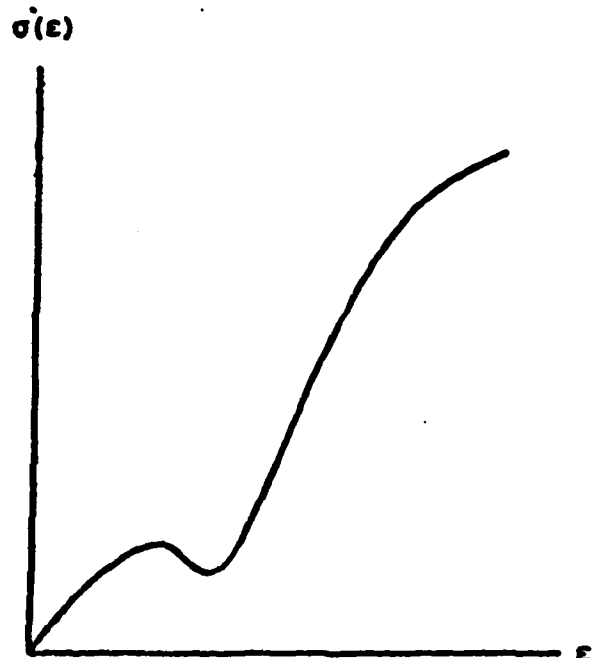


Fig. 1. Stress-strain curve

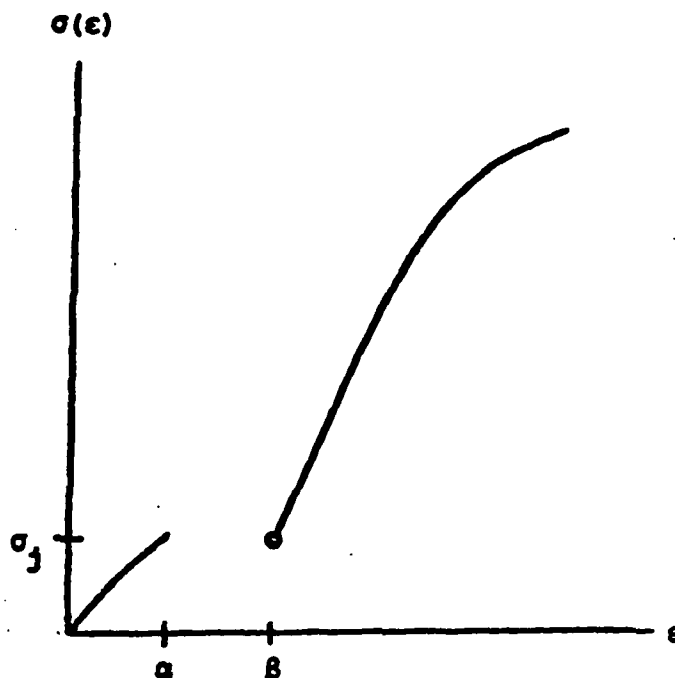


Fig. 2. Restricted stress-strain curve indicating the jump associated with a change of phase

The descending portion of the curve in Fig. 1 is its most conspicuous feature. In an equilibrium setting, strains associated with this portion are found to be unstable [1]. In the nonequilibrium case the sound speed is imaginary and the equation of motion is elliptic at these strains. These difficulties can be overcome by precluding these strains. This is accomplished in a natural fashion by considering an inverse to the $\sigma(\epsilon)$ relation, namely strain as a function of stress. Since the $\sigma(\epsilon)$ relation is not monotonic, there are innumerable such inverse functions. We take the particular inverse that leads to the clipped or restricted curve as shown in Fig. 2. The separated strain intervals are now associated with different material phases. Here the value of the transition stress σ_j is assumed to be a property of the material (see [4]). Several other methods for dealing with non-monotonic constitutive laws have also been studied (see [6], [7], [8]).

If a load $\sigma_0(t) > 0$ is applied to the end of the bar beginning at time $t = 0$, (2.1) is to be solved subject to

$$(2.3) \quad \sigma(\epsilon(0, t)) = \sigma_0(t), \quad t > 0,$$

$$(2.4) \quad \epsilon(x, 0) = 0, \quad v(x, 0) = 0, \quad x > 0.$$

The condition (2.4) indicates that the bar is taken to be initially undeformed and at rest. The condition (2.3) can also be written

$$(2.5) \quad \epsilon(0, t) = \epsilon_0(t), \quad t > 0,$$

where $\epsilon_0(t)$ is the strain associated with the stress $\sigma_0(t)$ by means of the curve in Fig. 2, thus

$$(2.6) \quad \sigma(\epsilon_0(t)) = \sigma_0(t).$$

In the event $\sigma_0(t)$ exceeds the value σ_j , $\epsilon_0(t)$ will be discontinuous and the strain field $\epsilon(x, t)$ will necessarily include a discontinuity front associated with a change of phase. In addition, other shock discontinuities of a more familiar kind may arise from the intersection of characteristics associated with different sound speeds of the nonlinear $\sigma(\epsilon)$ relation. Across any such discontinuity front, say $x = s(t)$, the jump in field quantities are to be restricted by the shock conditions

$$(2.7) \quad \frac{ds}{dt} [\epsilon] + [v] = 0, \quad \frac{ds}{dt} [v] + [\sigma] = 0.$$

In general the problem given by (2.1) - (2.7) does not admit simple analytical solutions.

An important practical problem associated with this system is that of impulsive loading. By this is meant the situation where $\sigma_0(t)$ is given by

$\sigma_0(t) \equiv \sigma_\infty$. In this case the problem, which is given on the quadrant $x > 0, t > 0$, is a variant of what is known as the Riemann Problem [2]. The solution follows from the absence of both length and time scales in the initial and boundary conditions. It is given by

$$(2.8) \quad \epsilon(x, t) = \tilde{\epsilon}(\lambda), \quad v(x, t) = \tilde{v}(\lambda)$$

where $\lambda = x/t$; $\tilde{\epsilon}(\lambda)$ can be found by a construction which is outlined in the following paragraph. Other treatments of similar problems may be found in [2].

Let $\hat{\sigma}(\epsilon)$ be the upper convex envelope on the interval $[0, \epsilon_\infty]$ of the clipped curve $\sigma(\epsilon)$ of Fig. 2. Here ϵ_∞ is the root of $\sigma(\epsilon_\infty) = \sigma_\infty$. The curve $\hat{\sigma}(\epsilon)$ will in general consist of a number of line segments connecting portions of the clipped curve. The curve $\hat{\sigma}(\epsilon)$ inherits the differentiability of the original restricted curve $\sigma(\epsilon)$ and so will be smooth at all values of strain in the interval $0 < \epsilon < \epsilon_\infty$ with the possible exception of $\epsilon = \alpha$. At the value $\epsilon = \alpha$, the derivative $\hat{\sigma}'(\epsilon)$ may - or may not - be discontinuous (see Fig. 3.) In the event $\hat{\sigma}'(\epsilon)$ is discontinuous at $\epsilon = \alpha$, $\hat{\sigma}'(\alpha)$ is to be regarded as given by the interval $[\hat{\sigma}'(\alpha+), \hat{\sigma}'(\alpha-)]$. In this way the graph of $\hat{\sigma}'(\epsilon)$ is a monotonically decreasing curve on $0 < \epsilon < \epsilon_\infty$. Hence the equation

$$(2.9) \quad \hat{\sigma}'(\tilde{\epsilon}) = \lambda^2$$

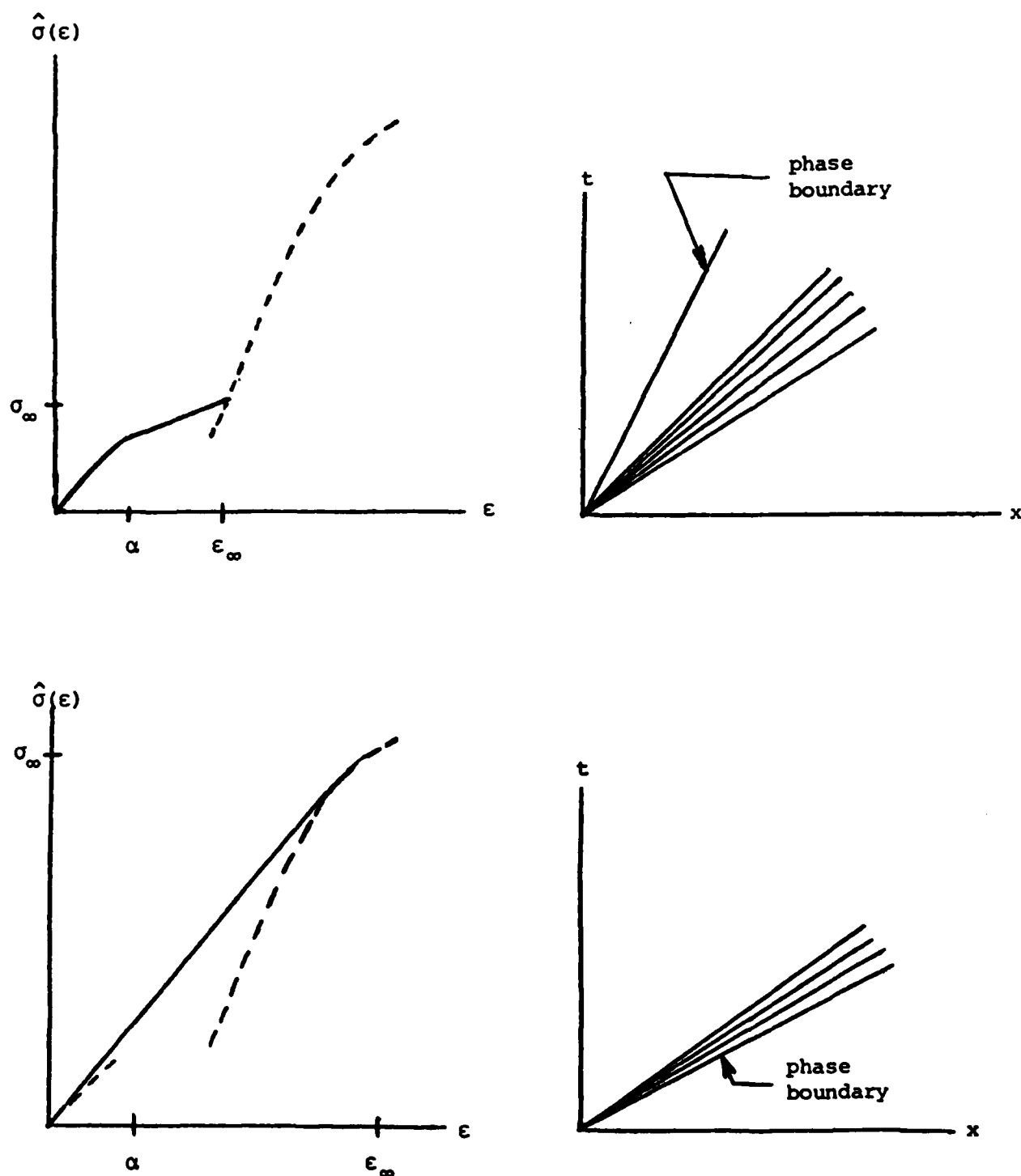


Fig. 3. Upper convex envelopes and corresponding simple waves for two values of σ_∞ . Here $\hat{\sigma}'(\epsilon)$ is discontinuous at $\epsilon = \alpha$ in the example on top; no such discontinuity in $\hat{\sigma}'(\epsilon)$ occurs in the bottom example.

has solutions $\tilde{\varepsilon}$ for each value of λ on the interval $[\sqrt{\hat{\sigma}'(\varepsilon_\infty)}, \sqrt{\hat{\sigma}'(0)}]$. Whenever $\hat{\sigma}(\varepsilon)$ contains line segments, (2.9) will have an interval of solutions in $\tilde{\varepsilon}$ for certain values of λ , say $\lambda = \lambda_1 \dots \lambda_n$. At all other values of λ , (2.9) uniquely determines a function $\tilde{\varepsilon}(\lambda)$. Next, $\tilde{\varepsilon}(\lambda)$ is extended to $0 < \lambda < \infty$, $\lambda \neq \lambda_1$, by defining $\varepsilon(\tilde{\lambda}) = 0$ for $\lambda > \sqrt{\hat{\sigma}'(0)} \equiv \lambda_0$ and $\tilde{\varepsilon}(\lambda) = \varepsilon_\infty$ for $\lambda < \sqrt{\hat{\sigma}'(\varepsilon_\infty)}$.

This construction orders the sound speeds of the rays $x = \lambda t$ in such a manner that values of strain associated with higher sound speeds are located further down the bar compared to those strain values with lower sound speeds. In the process it naturally positions shock and phase boundary curves $x = \lambda_1 t$, as well as ensuring the correct values of ε on the x and t axes. By virtue of (2.1), (2.7), the function $\tilde{v}(\lambda)$ is found from $\tilde{\varepsilon}(\lambda)$ through

$$(2.10) \quad \tilde{v}(\lambda) = \begin{cases} 0 & \lambda > \lambda_0 \\ \tilde{v}(\lambda_1^-) - \int_{\varepsilon(\lambda_1^-)}^{\tilde{\varepsilon}(\lambda)} \sqrt{\hat{\sigma}'(s)} ds & \lambda_{i+1} < \lambda < \lambda_i, \\ & (i = 1 \dots n, \lambda_{n+1} \equiv 0) \end{cases}$$

where

$$v(\lambda_1^-) = v(\lambda_1^+) + \lambda_1 [\varepsilon(\lambda_1^+) - \varepsilon(\lambda_1^-)] .$$

The solution (2.8) of the impulsive load problem thus consists of a partitioning of the quadrant $x > 0$, $t > 0$ into sectors by rays through the origin. Across these rays the solution may be either smooth or discontinuous. In the sectors, the strain and velocity fields are either constant or arise from solutions of (2.9). In both cases it can be shown that at least one of the two Riemann invariants associated with (2.1) remains constant. This being the case, the solution is said to be a simple wave.

It is worth mentioning that if at some time $t_1 > 0$ the bar is subsequently impulsively unloaded back to $\sigma = 0$, the wave pattern which results (before any interactions with the loading waves which at t_1 are further down the bar) can be found by an analogous construction involving lower convex envelopes of $\sigma(\varepsilon)$.

III. NONIMPULSIVE LOADING. Whenever the load $\sigma_0(t)$ is not simply a positive constant, the solution of the system (2.1) - (2.7) is not a simple wave similarity solution like that of the previous section. Instead one must take account of both families of characteristics associated with (2.1). Suppose, however, that the applied load eventually attains or merely approaches the final value σ_∞ . In this event, one expects the corresponding simple wave solution with $\sigma_0(t) \equiv \sigma_\infty$ to give the ultimate number of shocks

and phase boundaries, and to yield the correct limiting order of the waves in the bar as t tends to infinity. The spacing of these waves will depend on the manner in which $\sigma_0(t) \rightarrow \sigma_\infty$, nevertheless the value of the dynamical fields in the limit $\lambda = x/t$ fixed, $t \rightarrow \infty$ is given by the simple wave solution.

The approach to the asymptotic state may be studied by decomposing the displacement u into the corresponding simple wave solution, and a correction to be denoted by $\hat{u}$. By virtue of (2.8), (2.2) the former is expressed $u_0(\lambda)t$; the latter $\hat{u}$ is assumed to be $o(t)$ as $t \rightarrow \infty$, λ fixed.

Thus

$$(3.1) \quad u = u_0(\lambda)t + \hat{u}(\lambda, t),$$

while the governing equation (2.1) becomes

$$(3.2) \quad \frac{2\lambda}{t^2} u_\lambda + \frac{\lambda^2}{t^2} u_{\lambda\lambda} - \frac{2\lambda}{t} u_{\lambda t} + u_{tt} - \sigma'(\frac{1}{t} u_\lambda) \frac{1}{t^2} u_{\lambda\lambda} = 0.$$

Here subscripts of λ and t denote partial differentiation. Entering (3.2) with (3.1) one obtains to leading and second order

$$(3.3) \quad 0 = \left\{ \frac{1}{t} [\lambda^2 - \sigma'(u'_0)] u''_0 \right\} + \left\{ \frac{1}{t^2} [\lambda^2 - \sigma'(u'_0)] \hat{u}_{\lambda\lambda} - \frac{2\lambda}{t} \hat{u}_{\lambda t} + \hat{u}_{tt} - \frac{1}{t^2} [2\lambda - \frac{d}{d\lambda} \sigma'(u'_0)] \right\} + \text{terms higher than second order.}$$

In arriving at the above result it is necessary to make use of the expansion

$$\sigma'(\frac{1}{t} u_\lambda) \sim \sigma'(u'_0) + \sigma''(u'_0) \frac{1}{t} \hat{u}_\lambda.$$

The expression in the last parenthesis of (3.3) dominates all succeeding expressions for large times and so must independently vanish. Thus one draws

$$(3.4) \quad u_0(\lambda) = c_1 \lambda + c_2$$

for constants c_1, c_2 , or

$$(3.5) \quad u_0(\lambda) = \int^\lambda \sigma'^{-1}(s^2) ds$$

where the superscript -1 denotes functional inverse. It is easily verified that (3.4) gives rise to the constant strain and velocity fields of the impulsive load problem, while (3.5) corresponds to solutions of (2.9). The Riemann solution discussed in the previous section is the unique way in which it is possible to assemble these solutions (3.4) and (3.5) in a manner which satisfies (2.4), (2.7), $\sigma_0(t) \equiv \sigma_\infty$ and also jumps over the specified interval (α, β) .

The nature of the approach to this simple wave is governed by the expression in the second parenthesis of (3.3). Requiring this expression to vanish leads to

$$(3.6) \quad -2\lambda \hat{u}_{\lambda t} + t \hat{u}_{tt} = 0$$

in the case of (3.5), while for (3.4) it is easiest to revert back to independent variables x and t whereupon one finds that the equation can be written

$$\hat{u}_{tt} = \sigma'(c_1) \hat{u}_{xx}.$$

The latter has the familiar solution

$$(3.7) \quad \hat{u} = A(x - \sqrt{\sigma'(c_1)} t) + B(x + \sqrt{\sigma'(c_1)} t),$$

for any functions $A(\cdot)$ and $B(\cdot)$. The former (3.7) is also easily solved

$$(3.8) \quad \hat{u} = \frac{1}{\sqrt{\lambda}} \bar{C}(\sqrt{\lambda} t) + \bar{D}(\lambda) = t C(xt) + D(x/t)$$

for any functions $C(\cdot)$ and $D(\cdot)$. Unfortunately the boundary condition (2.3) cannot be used directly for determining the free functions which have emerged from this treatment. Instead one anticipates appropriate conditions to arise from a more penetrating analysis of various short- and intermediate-time solutions to the problem, each of which is appropriate in a different region of the (x,t) -quadrant. In general one expects to continue the fields between these as yet unknown regions through a detailed matching layer analysis. Rather than pursuing this program, a method appropriate to certain special materials will be introduced.

IV. PIECEWISE LINEAR STRESS RESPONSE We turn now to a class of model materials in which the stress-strain curve consists of a number of linear segments. Such models are often associated with the theory of plasticity, nevertheless here we shall assume that loading and unloading follow the same stress-strain curve.

The fields arising from an impulsive load can be found by the procedure discussed in section 2. As before, each line segment in the upper convex envelope $\hat{\sigma}(\epsilon)$ of $\sigma(\epsilon)$ is associated with a front across which the strain suffers a discontinuity. For the materials now under consideration, $\hat{\sigma}(\epsilon)$ will consist of nothing but line segments. It is convenient to distinguish among three types of discontinuity fronts by drawing a distinction between the line segments comprising $\hat{\sigma}(\epsilon)$. We shall say

- a phase boundary is a front which is associated with a line segment which spans a clipped portion of the original curve $\sigma(\epsilon)$,
- a contact discontinuity is a front which is associated with a line segment which coincides with the original curve $\sigma(\epsilon)$, and

a shock

is a front which is associated with a line segment which neither coincides with the original curve, nor spans a clipped portion of the curve.

The solution of the impulsive load problem will consist of partitioning of the (x,t) -quadrant into sectors of constant strain and velocity. These sectors are separated by either shocks, phase boundaries or contact discontinuities.

We now inquire as to what is the relation between this simple wave solution and the fields which would be produced if the bar were gradually loaded to the level σ_∞ ? As before, the simple wave solution will yield the limiting order of waves in the bar as t tends to infinity. It will also produce the correct final number and ordering of shocks and phase boundaries. The contact discontinuities, however, are not associated with true curves of discontinuity in the field variables. Instead they indicate nondispersive wave packets across which the dynamical fields gradually change.

The approach to this simple wave can be examined by exploiting the linearity of the material in the different strain intervals. The difficulty lies in determining the regions in the (x,t) -plane in which the strain takes values in the individual intervals. The boundaries of these curves must be either curves of constant strain or curves across which the strain jumps between values from different intervals. In either case these conditions lead to functional equations when the strains are expressed in terms of D'Alembert's solution to the linear wave equation.

In order to illustrate this, we consider the material of Fig. 5. The particular form of this stress-strain curve is motivated by its similarity to the material previously introduced in Fig. 1. We shall suppose that

$$\lim_{t \rightarrow \infty} \sigma_0(t) = \sigma_\infty \text{ where } \sigma_\infty \text{ is as indicated in Fig. 5. The corresponding}$$

impulsive load solution is also given in Fig. 5. This solution consists of a phase boundary and a contact discontinuity separating regions in which the dynamical fields are constant. Suppose further that

$$\sigma_0(0) = 0, \quad \sigma_0'(t) > 0.$$

We shall let $x = s(t)$ denote the phase boundary, while $x = q(t)$ shall denote the curve upon which the strain has value μ , thus

$$(4.1) \quad \epsilon(q(t), t) = \mu.$$

Let A_0, A_1, A_2 denote the regions in the (x,t) -plane in which the strain lies in the respective intervals $(0, \alpha), (\beta, \mu), (\mu, \infty)$. In each

region A_1 the strain obeys the equation $\frac{\partial^2 \epsilon}{\partial t^2} = c_1^2 \frac{\partial^2 \epsilon}{\partial x^2}$ which implies that

$$(4.2) \quad \epsilon(x, t) = f_1(x + c_1 t) + g_1(-x + c_1 t)$$

for as yet undetermined functions f and g . The velocity in A_1 must then

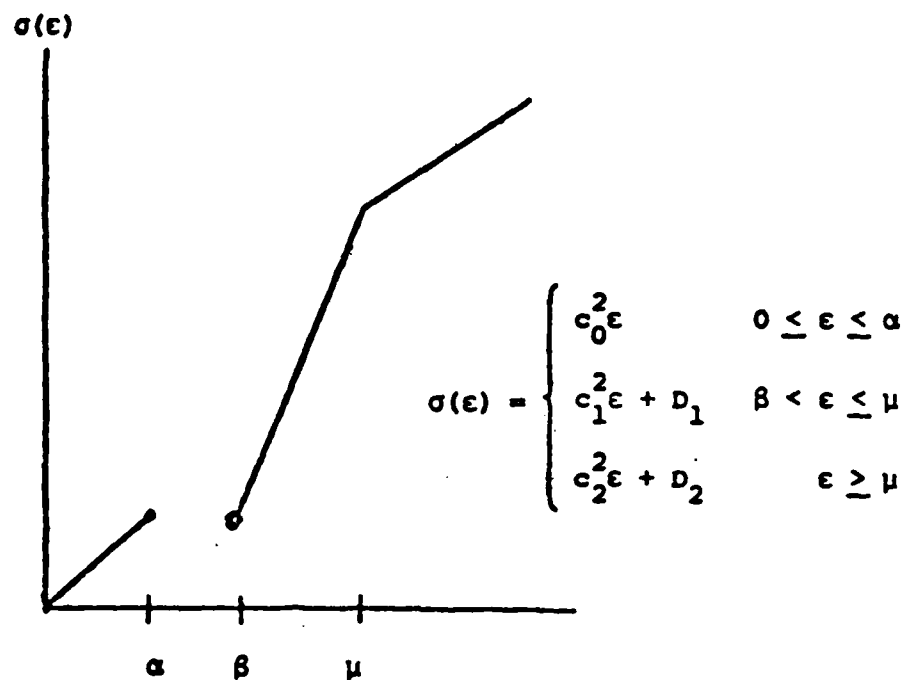


Fig. 4. Model material with piecewise linear stress response

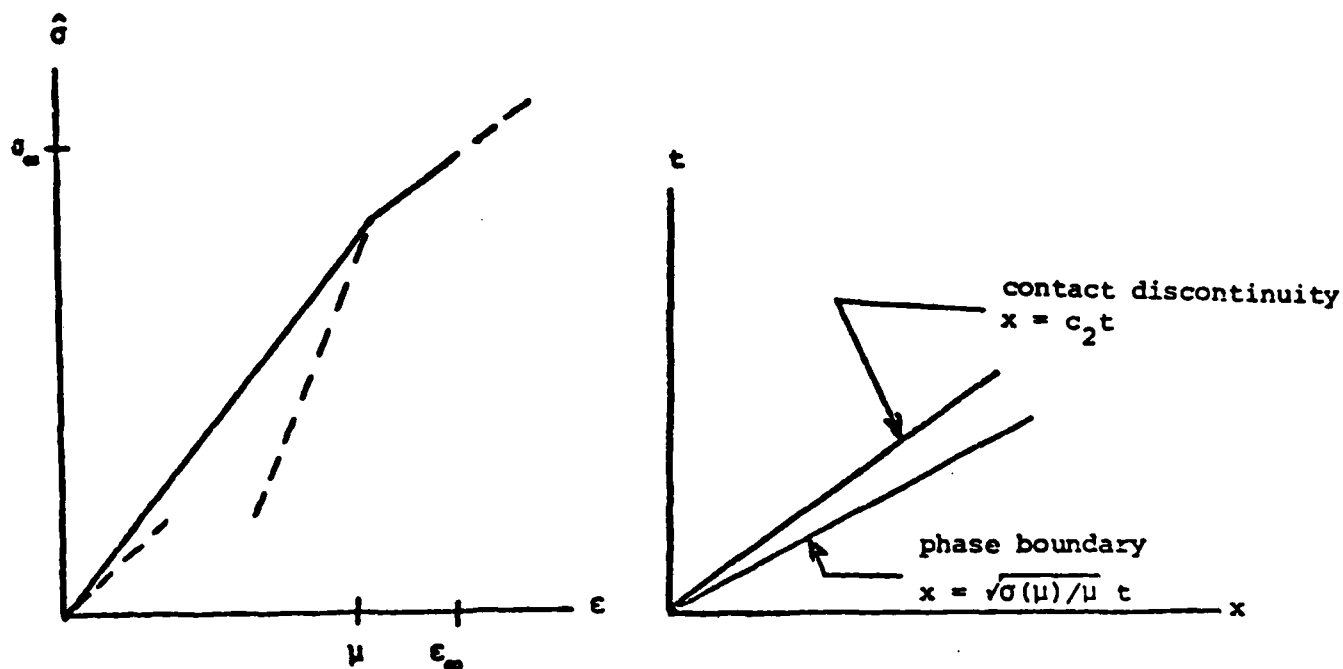


Fig. 5. Upper convex envelope $\hat{\sigma}(\epsilon)$ and corresponding simple wave for the material in Fig. 4 when $\epsilon_\infty > \mu$

be $v(x,t) = c_1 f_1(x + c_1 t) - c_1 g_1(-x + c_1 t)$. The f_1 and g_1 , along with $s(t)$ and $q(t)$, are eight unknown functions, each of a single argument. They must satisfy eight conditions given by: the two initial conditions (2.4), the boundary condition (2.3), the two shock conditions (2.7), two conditions stemming from (4.1), and a condition that the velocity is continuous across $x = q(t)$. The last three conditions in the above list give rise to functional equations since $q(t)$ appears as an argument of f_1, f_2, g_1, g_2 . Moreover, since $\dot{s}(t)$ appears in (2.7), the condition on $x = s(t)$ furnishes a pair of functional differential equations. In the expressions which follow, we shall employ parenthesis solely to indicate the argument of a function.

From the initial conditions (2.4) it can be shown that the Riemann invariant associated with the characteristics $\frac{dx}{dt} = -c_0$ is identically zero on A_0 . Thus $f_0(z) \equiv 0$ which, by virtue of (4.2), (2.5), yields

$$g_0(z) = \begin{cases} \epsilon_0(z/c_0) & z > 0, \\ 0 & z < 0. \end{cases}$$

The other six unknown functions do not admit simple solution representations. In what follows we shall restrict attention to large times. The analysis will be shortened considerably by taking advantage of the simple wave solution depicted in Fig. 5. This is not necessary; the asymptotic simple wave could be deduced from the analysis. Such a program, however, requires the consideration - too lengthy to be included here - of numerous possible cases.

V. LARGE-TIME DYNAMICAL FIELDS. The boundary condition (2.5) can be incorporated directly into the functions f_2, g_2 . In place of the functions f_1, g_1, f_2, g_2 we shall instead employ f, g, h through the expressions

$$(5.1) \quad \epsilon(x,t) = \begin{cases} f(x + c_1 t) + g(-x + c_1 t) & \text{in } A_1 \\ h(x + c_2 t) - h(-x + c_2 t) + \epsilon_0(t - x/c_2) & \text{in } A_2 \end{cases}$$

$$(5.2) \quad v(x,t) = \begin{cases} c_1 f(x + c_1 t) - c_1 g(-x + c_1 t) & \text{in } A_1 \\ c_2 h(x + c_2 t) + c_2 h(-x + c_2 t) - c_2 \epsilon_0(t - x/c_2) & \text{in } A_2 \end{cases}$$

The five unknowns f, g, h, s, q are to be determined from: two conditions holding on $x = q(t)$ which stem from (4.1), a condition expressing the continuity of velocity on $x = q(t)$, and the two shock conditions (2.7) which hold on $x = s(t)$. The first two of the conditions holding on $x = q(t)$ become

$$(5.3) \quad f(q(t) + c_1 t) + g(-q(t) + c_1 t) = \mu,$$

$$(5.4) \quad h(q(t) + c_2 t) - h(-q(t) + c_2 t) + \epsilon_0(t - \frac{1}{c_2} q(t)) = \mu.$$

With the aid of these two equations, the continuity of velocity condition may be written as

$$(5.5) \quad f(q(t) + c_1 t) = \frac{c_2}{c_1} h(q(t) + c_2 t) + \frac{1}{2} \left(1 - \frac{c_2}{c_1}\right) \mu.$$

The two shock conditions can be manipulated into the form

$$(5.6) \quad [c_1 + \dot{s}(t)] f(s(t) + c_1 t) + \frac{D_1}{2c_1} = 0, \quad t \text{ sufficiently large,}$$

$$(5.7) \quad [c_1 - \dot{s}(t)] g(-s(t) + c_1 t) + \frac{D_1}{2c_1} = 0, \quad t \text{ sufficiently large.}$$

The phrase "t sufficiently large" indicates that these are the equations which hold once the phase boundary $x = s(t)$ has become the leading disturbance. This eventuality follows from the simple wave solution, moreover the simple wave solution also indicates that $\dot{s}(t) \rightarrow \sqrt{\sigma(\mu)/\mu}$ and $\dot{q}(t) \rightarrow c_2$.

Assume that the wave and front speeds obey the ordering

$$(5.8) \quad c_1 > \dot{s}(t) > c_2 > \dot{q}(t) > 0$$

for t sufficiently large. One may conclude directly from (5.3) - (5.7) that

$$f(z) \rightarrow f_\infty, \quad g(z) \rightarrow g_\infty, \quad h(z) \rightarrow h_\infty \quad \text{as } z \rightarrow \infty,$$

$$\dot{s}(t) \rightarrow \alpha, \quad \dot{q}(t) \rightarrow \beta \quad \text{as } t \rightarrow \infty.$$

The values for f_∞ , g_∞ , and α are obtained from a consideration of the equations (5.3), (5.6), (5.7) for large times. As $t \rightarrow \infty$ these equations become respectively

$$f_\infty + g_\infty = \mu, \quad [c_1 + \alpha] f_\infty + \frac{D_1}{2c_1} = 0, \quad [c_1 - \alpha] g_\infty + \frac{D_1}{2c_1} = 0,$$

which give

$$(5.9) \quad \alpha = \sqrt{\frac{2}{c_1} + \frac{D_1}{\mu}} = \sqrt{\frac{\sigma(\mu)}{\mu}}, \quad f_\infty = \frac{-D_1}{2c_1[c_1 + \alpha]}, \quad g_\infty = \frac{-D_1}{2c_1[c_1 - \alpha]}.$$

The values

$$(5.10) \quad h_\infty = \frac{c_1}{c_2} f_\infty - \frac{1}{2} \left[\frac{c_1}{c_2} - 1 \right] \mu = \frac{1}{2} \mu \left[1 - \frac{\alpha}{c_2} \right], \quad \beta = c_2,$$

follow from (5.4), (5.5). Since $q(t) \rightarrow c_2 t$ and $s(t) \rightarrow \frac{\sqrt{\sigma(\mu)}}{\mu} t$ as $t \rightarrow \infty$, it is immediate that A_0, A_1, A_2 tend toward the sectors of the simple wave solution displayed in Fig. 5. Upon entering the values of $f_\infty, g_\infty, h_\infty$ into (5.1), (5.2), the corresponding sector values for strain and velocity are recovered.

In order to study the approach to these values let

$$\begin{aligned}
 (5.11) \quad s(t) &= \frac{\sqrt{\sigma(\mu)}}{\mu} t + s_1(t) \\
 q(t) &= c_2 t + q_1(t) \\
 f(z) &= f_\infty + f_1(z) \\
 g(z) &= g_\infty + g_1(z) \\
 h(z) &= h_\infty + h_1(z)
 \end{aligned}$$

for new functions s_1, q_1, f_1, g_1, h_1 . Upon substituting from these expressions into (5.3) - (5.7) and considering a balance of the second order terms, one obtains asymptotic expressions for these functions. This procedure applied to (5.6) yields

$$\begin{aligned}
 0 &= [c_1 + \alpha + \dot{s}_1(t)] \{f_\infty + f_1([c_1 + \alpha]t + s_1(t))\} + \frac{D_1}{2c_1} \\
 &= f_\infty \dot{s}_1(t) + [c_1 + \alpha + \dot{s}_1(t)] f_1([c_1 + \alpha]t + s_1(t)) \\
 &\sim \dot{s}_1(t) f_\infty + [c_1 + \alpha] f_1([c_1 + \alpha]t) ,
 \end{aligned}$$

which in turn implies

$$(5.12) \quad \dot{s}_1(t) \sim - \frac{[c_1 + \alpha]}{f_\infty} f_1([c_1 + \alpha]t) .$$

Similarly (5.7) leads to

$$(5.13) \quad \dot{s}_1(t) \sim \frac{[c_1 - \alpha]}{g_\infty} g_1([c_1 - \alpha]t) ,$$

while (5.3) leads to

$$(5.14) \quad f_1([c_1 + c_2]t) \sim - g_1([c_1 - c_2]t) .$$

Eliminating g_1 and s_1 between (5.12) (5.13) (5.14) reveals that f_1 obeys

$$f_1\left(\left[\frac{c_1 + c_2}{c_1 - c_2}\right]z\right) \sim \left[\frac{c_1 + \alpha}{c_1 - \alpha}\right]^2 f_1\left(\left[\frac{c_1 + \alpha}{c_1 - \alpha}\right]z\right), \quad \text{as } z \rightarrow \infty.$$

This condition will be written

$$(5.15) \quad f_1(k_2 z) \sim k_1 f_1(z) \quad \text{as } z \rightarrow \infty$$

where

$$(5.16) \quad k_1 = \left[\frac{c_1 - \alpha}{c_1 + \alpha}\right]^2, \quad k_2 = \frac{[c_1 + \alpha][c_1 - c_2]}{[c_1 - \alpha][c_1 + c_2]}.$$

It follows from (5.8) that

$$(5.17) \quad k_2 > 1 > k_1 > 0.$$

The asymptotic expression (5.15) indicates that

$$(5.18) \quad f_1(z) \sim Az^n, \quad n = \ln k_1 / \ln k_2,$$

where A is undetermined. In light of (5.17), it follows that $n < 0$ so that $f_1(z)$ is indeed dominated by f_∞ whenever $z \rightarrow \infty$, as was assumed in the development leading to (5.15). Moreover the following argument demonstrates that

$$(5.19) \quad n < -2.$$

Proof: (5.18) implies $(k_1^{1/2} k_2)^n = k_1^{1 + 1/2 n}$, which, in conjunction with

$$0 < k_1^{1/2} k_2 = \frac{c_1 - c_2}{c_1 + c_2} < 1 \quad \text{and} \quad n < 0 \quad \text{yields} \quad k_1^{1 + 1/2 n} > 1. \quad \text{This last}$$

result, along with (5.17), yields $1 + 1/2 n < 0$, which is (5.19). /

One finds from (5.18), (5.12), (5.13) that

$$(5.20) \quad g_1(z) \sim k_1^{-\left(1 + \frac{n}{2}\right)} Az^n, \quad \dot{s}_1(t) \sim - \frac{[c_1 + \alpha]^{n+1}}{f_\infty} At^n.$$

On account of (5.20) and (5.19) one has

$$(5.21) \quad s_1(t) \sim s_0 - \frac{[c_1 + \alpha]^{n+1}}{(n+1)f_\infty} At^{n+1},$$

where s_0 is undetermined.

The asymptotic behavior of the corrections $h_1(z)$ and $q_1(t)$ are found from the remaining equations (5.4), (5.5). Entering (5.5) with (5.11), (5.10) one obtains

$$\frac{c_2}{c_1} h_1(2c_2 t + q_1(t)) - f_1([c_1 + c_2]t + q_1(t)) = 0$$

from which it follows that

$$(5.22) \quad h_1(z) \sim \frac{c_1}{c_2} \left[\frac{c_1 + c_2}{2c_2} \right]^n A z^n.$$

Equation (5.4) is the most delicate in the analysis. The exact equation yields

$$(5.23) \quad \mu = h_\infty + h_1(2c_2 t + q_1(t)) - h(-q_1(t)) + \varepsilon_0(-q_1(t)/c_2).$$

In this equation $h_1(2c_2 t + q_1(t)) = o(1)$ as $t \rightarrow \infty$. It is, however, not appropriate to expand $h(-q_1(t))$ by means of (5.11) unless $q_1(t) \rightarrow -\infty$. Assume for the moment that $q_1(t) \rightarrow -\infty$; then $\varepsilon_0(-q_1(t)/c_2) \rightarrow \varepsilon_\infty$,

$h(-q_1(t)) \rightarrow h_\infty$ so that (5.23) gives up $\varepsilon_\infty = \mu$. Since $\varepsilon_\infty > \mu$ the assumption is false. Instead (5.23) demands that $q_1(t) \rightarrow q_0 > -\infty$ with $\mu = h_\infty - h(-q_0) + \varepsilon_0(-q_0/c_2)$. In the event that $h(z)$ admits a Taylor expansion at $z = -q_0$, then (5.22), (5.23) yields

$$\{h(-q_0) + \frac{1}{c_2} \varepsilon_0(-q_0/c_2)\} [-q(t) + q_0] \sim h_1(2c_2 t) \sim [c_1/c_2] [c_1 + c_2]^n A t^n.$$

This in turn implies that $q(t) \sim q_0 + O(t^n)$ whenever the coefficient in parenthesis is not zero. Collecting the results from this section we have

$$\begin{aligned} f(z) &= f_\infty + A z^n + o(z^n) \\ g(z) &= g_\infty + A k_1 z^{-(1 + \frac{n}{2})} + o(z^n) \\ (5.24) \quad s(t) &= at + s_0 - \frac{[c_1 + a]^{n+1}}{(n+1)f_\infty} A t^{n+1} + o(t^{n+1}) \\ h(z) &= h_\infty + A \frac{c_1 [c_1 + c_2]^n}{2^n c_2^{n+1}} z^n + o(z^n) \\ q(t) &= c_2 t + q_0 + O(t^n). \end{aligned}$$

The constants $f_\infty, h_\infty, g_\infty, \alpha, k_1, n$ are given in (5.9), (5.10), (5.16), (5.18). The constants A, q_0, s_0 are as yet unspecified. Indeed, in order to assign values to A, q_0, s_0 one may surmise that it is necessary to take account of the complete loading history $\sigma_0(t)$. Consider, for example, the constant s_0 . The phase boundary $x = s(t)$ asymptotically approaches the line $x = \alpha t + s_0$. This line issues from the t axis at time $t = -s_0/\alpha$, so that s_0 defines a time scale for the problem. The only source of such a time scale lies in the applied load $\sigma_0(t)$.

The dynamical fields are obtained by substituting from (5.24) into (5.1), (5.2). This yields

$$\epsilon(x,t) \sim \epsilon_\infty + \frac{Ac_1[c_1 + c_2]^n}{2^n c_2^{n+1}} \left[\left[c_2 + \frac{x}{t} \right]^n - \left[c_2 - \frac{x}{t} \right]^n \right] t^n,$$

$$v(x,t) \sim -\alpha\mu - c_2[\epsilon_\infty - \mu] + \frac{Ac_1[c_1 + c_2]}{2^n c_2^{n+1}} \left[\left[c_2 + \frac{x}{t} \right]^n - \left[c_2 - \frac{x}{t} \right]^n \right] t^n$$

in region A_2 . In region A_1 one obtains

$$\epsilon(x,t) \sim \mu + A \left[\left[c_1 + \frac{x}{t} \right]^n + \left[\frac{c_1 + \alpha}{c_1 - \alpha} \right]^{n+2} \left[c_1 - \frac{x}{t} \right]^n \right] t^n,$$

$$v(x,t) \sim -\alpha\mu + Ac_1 \left[\left[c_1 + \frac{x}{t} \right]^n - \left[\frac{c_1 + \alpha}{c_1 - \alpha} \right]^{n+2} \left[c_1 - \frac{x}{t} \right]^n \right] t^n.$$

Although the value of A is not found from this analysis, the exponent n governing the rate of approach to the simple wave is determined from both the stress-strain behavior of the material and the ultimate level of the applied load by means of (5.18).

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