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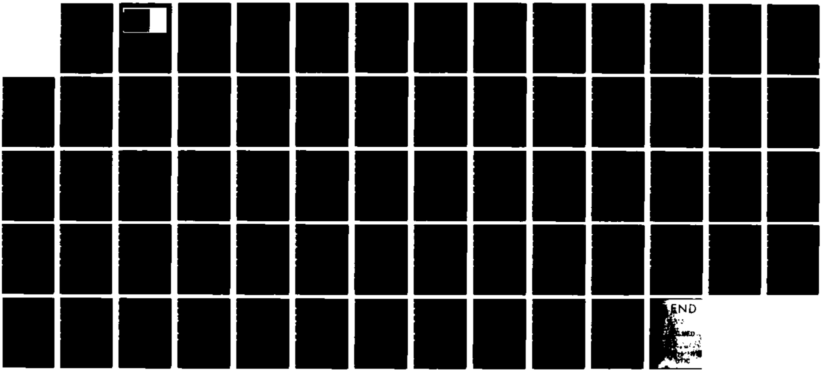

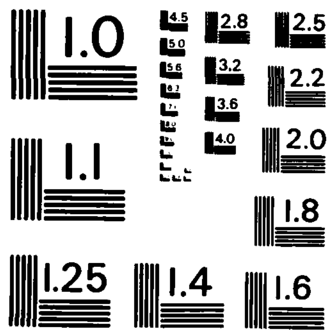

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MRC Technical Summary Report #2735

THE RESISTANCE AND MOBILITY FUNCTIONS OF TWO EQUAL SPHERES IN LOW REYNOLDS NUMBER FLOW

Sangtae Kim and Richard T. Mifflin

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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THE RESISTANCE AND MOBILITY FUNCTIONS OF TWO EQUAL SPHERES IN LOW REYNOLDS NUMBER FLOW

Sangtae Kim[*,1,2] and Richard T. Mifflin[**,1,3]

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ABSTRACT

The resistance and mobility functions which completely characterize the linear relations between the force, torque and stresslet and the translational and rotational velocities of two spheres in low-Reynolds-number flow have been calculated using a boundary collocation technique. The ambient velocity field is assumed to be a superposition of a uniform stream and a linear (vorticity and rate-of-strain) field. This is the first compilation of accurate expressions for the entire set of functions. Our calculations are in agreement with earlier results for all functions for which such results are available. Our technique is successful at all sphere-sphere separations except at the almost-touching (gaps of less than .005 diameter) configuration.

c square

New results for the stresslet functions have been used to determine Batchelor and Green's (1972) order c^2 coefficient in the bulk-stress (7.1 instead of their 7.6). The two-sphere functions have also been used to determine the motion of a rigid dumbbell in a linear field. We also show that certain functions have extrema. The source (FORTRAN) code is furnished in the appendix.

AMS (MOS) Subject Classifications: 76D05, 35Q10

Key Words: low Reynolds number, hydrodynamic interaction, resistance functions, mobility function, suspensions

Work Unit Number 2 (Physical Mathematics)

This report is also available as Rheology Research Center Report #94.

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*Department of Chemical Engineering and Mathematics Research Center, University of Wisconsin-Madison, Madison, WI 53705
**Department of Chemical Engineering, Engineering Quadrangle, Princeton University, Princeton, NJ 08544

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[3] The Fannie and John Hertz Foundation.

SIGNIFICANCE AND EXPLANATION

The calculation of hydrodynamic interactions between particles is needed for the understanding and control of many natural and manufacturing processes, for instance, those involving sedimentation, colloidal stability, suspension rheology, and cloud formation. A fundamental approach to these problems often requires detail information on the hydrodynamic interactions between two spheres, that is, the forces, torques and stress dipoles induced by the particle motions and the ambient velocity. Until now, the available information was incomplete.

This report furnishes the complete solution of the problem using a collocation approach. The results are in excellent agreement with all earlier solutions for special cases of the complete problem. The source (FORTRAN) code has been included.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the authors of this report.

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THE RESISTANCE AND MOBILITY FUNCTIONS OF TWO EQUAL SPHERES IN LOW REYNOLDS NUMBER FLOW

Sangtae Kim[*,1,2] and Richard T. Mifflin[**,1,3]

1. Introduction

In this work we have used the boundary collocation technique to calculate the set of functions which describe the hydrodynamic interaction between two rigid spheres in low-Reynolds-number flow. Such information is needed in theoretical investigations of the behavior of suspensions of small (sub-micron) particles as shown in the review articles by Batchelor (1974) and Jeffrey and Acrivos (1976). Specific applications are found in studies of sedimentation velocities (Batchelor 1972), rheological properties (Batchelor and Green 1972), Brownian diffusion (Batchelor 1976) and fixed-bed permeabilities (Howells 1974). In all cases the specific information that is required is the linear relation between the rigid-body motion of the spheres,

$$\underset{\sim}{U}_\alpha + \underset{\sim}{\Omega}_\alpha \times (\underset{\sim}{x} - \underset{\sim}{x}_\alpha),$$

in an ambient field, $\underset{\sim}{U}(\underset{\sim}{x}) = \underset{\sim}{U}^\infty + \underset{\sim}{\Omega} \times \underset{\sim}{x} + \underset{\approx}{E} \cdot \underset{\sim}{x}$ on one hand and on the other hand the force, torque and stresslet (moment) exerted by each sphere on the fluid. The notation is as follows: $\underset{\sim}{U}_\alpha$ and $\underset{\sim}{\Omega}_\alpha$ are the translational and rotational velocities of the sphere centered at $\underset{\sim}{x}_\alpha$, $\alpha = 1,2$; $\underset{\sim}{U}^\infty$, $\underset{\sim}{\Omega}$ and $\underset{\approx}{E}$ are the uniform stream, constant vorticity and rate-of-strain fields. The force, torque and stresslet on each sphere are given by the following integrals of the stress, $\underset{\approx}{\sigma}$, over the surface of sphere α:

$$\underset{\sim}{F}_\alpha = -\int_{S_p} \underset{\approx}{\sigma} \cdot \underset{\sim}{n} \, dA,$$

$$\underset{\sim}{T}_\alpha = -\int_{S_p} (\underset{\sim}{x} - \underset{\sim}{x}_\alpha) \times (\underset{\approx}{\sigma} \cdot \underset{\sim}{n}) \, dA,$$

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$$\underline{\underline{S}}_\alpha = -\frac{1}{2}\int_{S_p} [(\underline{x}-\underline{x}_\alpha)\underline{\sigma}\cdot\underline{n} + \underline{\sigma}\cdot\underline{n}(\underline{x}-\underline{x}_\alpha)]dA.$$

$\underline{n}$ is the outward normal vector for the surface.

Because of the linearity of the governing equations, the solution of our hydrodynamic interaction problem (with equal spheres) is completely specified by 19 independent scalar functions. We present here what we believe to be the first complete solution of this problem. We emphasize that our method can be reduced to a 60-line routine that, with the help of subprograms for the special functions (Legendre functions) calculates the entire collection of functions. Others have solved various subsets of this problem using bispherical coordinates (Stimson and Jeffery 1926; Goldman, Cox and Brenner 1966; Lin, Lee and Sather 1970), method of reflections (Happel and Brenner 1965) and lubrication theory (O'Neill and Majumdar 1970), as reviewed by Jeffrey and Onishi (1984). These authors also present a comprehensive solution of the important subset involving the force and torque in an ambient field composed of a uniform stream and vorticity field, using the twin-multipole variation of the method of reflections.

A more detailed discussion of hydrodynamic interaction is presented in following subsections where we review the resistance and mobility functions. In Section 2, we show how the boundary collocation technique of Gluckman et. al. (1971) and Lamb's velocity representation can be used to solve the boundary value problem associated with each resistance function. Formulae which relate the resistance functions to the coefficients in the velocity representation are found in Section 3. We state our principal results in Section 4 and illustrate sample applications in Section 5, including the correction of Batchelor and Green's (1972b) result for the coefficient of the c^2 term in the bulk stress (7.1 instead of their 7.6) and the hydrodynamic functions for the rotation of a rigid dumbbell in a linear field. We have placed our source code and sample calculations in the appendix.

1.1 The Resistance Problem

Following Brenner and O'Neill (1972), we define the resistance problem as that in which the force, torque and stresslet are to be determined for a specified particle motion in the ambient field. The linearity of the Stokes equations permits the expression of the forces, torques and stresslets in the following matrix form:

$$\begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{S}_1 \\ \mathbf{S}_2 \end{bmatrix} = \mu \begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} & \tilde{\mathbf{B}}^{11} & \tilde{\mathbf{B}}^{12} & \tilde{\mathbf{G}}^{11} & \tilde{\mathbf{G}}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} & \tilde{\mathbf{B}}^{21} & \tilde{\mathbf{B}}^{22} & \tilde{\mathbf{G}}^{21} & \tilde{\mathbf{G}}^{22} \\ \mathbf{B}^{11} & \mathbf{B}^{12} & \mathbf{C}^{11} & \mathbf{C}^{12} & \tilde{\mathbf{H}}^{11} & \tilde{\mathbf{H}}^{12} \\ \mathbf{B}^{21} & \mathbf{B}^{22} & \mathbf{C}^{21} & \mathbf{C}^{22} & \tilde{\mathbf{H}}^{21} & \tilde{\mathbf{H}}^{22} \\ \mathbf{G}^{11} & \mathbf{G}^{12} & \mathbf{H}^{11} & \mathbf{H}^{12} & \mathbf{M}^{11} & \mathbf{M}^{12} \\ \mathbf{G}^{21} & \mathbf{G}^{22} & \mathbf{H}^{21} & \mathbf{H}^{22} & \mathbf{M}^{21} & \mathbf{M}^{22} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 - \mathbf{U}(\mathbf{x}_1) \\ \mathbf{U}_2 - \mathbf{U}(\mathbf{x}_2) \\ \mathbf{\Omega}_1 - \mathbf{\Omega} \\ \mathbf{\Omega}_2 - \mathbf{\Omega} \\ -\mathbf{E} \\ -\mathbf{E} \end{bmatrix}$$

The 6X6 matrix of tensors has been named the grand resistance matrix by Rallison (1977).

We have followed the development and notation in Jeffrey and Onishi (1984) and Jeffrey (1984) throughout this section. $\mathbf{A}$, $\mathbf{B}$, $\tilde{\mathbf{B}}$ and $\mathbf{C}$ are second rank tensors, $\mathbf{G}$, $\tilde{\mathbf{G}}$, $\mathbf{H}$ and $\tilde{\mathbf{H}}$ are third rank tensors and the $\mathbf{M}$ tensors are fourth rank tensors. We shall see below that there are inter-relations between certain tensor pairs. These pairings are highlighted by using the same letters and the symbol, ~.

We first reduce the number of independent tensors by using properties that are independent of the particle geometry. The reciprocal theorem of Lorentz (1906) can be used to show that the resistance matrix is symmetric (Brenner and O'Neill 1972 and Hinch 1972), i.e.:

$$A^{\alpha\beta}_{ij} = A^{\beta\alpha}_{ji}, \qquad \tilde{B}^{\alpha\beta}_{ij} = B^{\beta\alpha}_{ji}, \qquad C^{\alpha\beta}_{ij} = C^{\beta\alpha}_{ji}, \qquad (1.1a,b,c)$$

$$\tilde{G}^{\alpha\beta}_{ijk} = G^{\beta\alpha}_{kij}, \qquad \tilde{H}^{\alpha\beta}_{ijk} = H^{\beta\alpha}_{kij}, \qquad M^{\alpha\beta}_{ijk\ell} = M^{\beta\alpha}_{k\ell ij}. \tag{1.1d,e,f}$$

We may impose additional relations because $\underline{\underline{S}}$ and $\underline{\underline{E}}$ are symmetric and traceless. The condition on $\underline{\underline{S}}$ permit us to set

$$G^{\alpha\beta}_{ijk} = G^{\alpha\beta}_{jik}, \qquad H^{\alpha\beta}_{ijk} = H^{\alpha\beta}_{jik}, \qquad M^{\alpha\beta}_{ijk\ell} = M^{\alpha\beta}_{jik\ell}, \tag{1.2a,b,c}$$

$$G^{\alpha\beta}_{iik} = 0, \qquad H^{\alpha\beta}_{iik} = 0, \qquad M^{\alpha\beta}_{iik\ell} = 0. \tag{1.2d,e,f}$$

while the conditions on $\underline{\underline{E}}$ require that

$$G^{\alpha\beta}_{ijk} = G^{\alpha\beta}_{ikj}, \qquad H^{\alpha\beta}_{ijk} = H^{\alpha\beta}_{ikj}, \qquad M^{\alpha\beta}_{ijk\ell} = M^{\alpha\beta}_{ij\ell k}, \tag{1.3a,b,c}$$

$$G^{\alpha\beta}_{ijj} = 0, \qquad H^{\alpha\beta}_{ijj} = 0, \qquad M^{\alpha\beta}_{ijkk} = 0. \tag{1.3d,e,f}$$

The symmetry of the two-sphere geometry implies that each tensor satisfies:

$$\underline{\underline{P}}^{\alpha\beta}(\underline{R}) = \underline{\underline{P}}^{(3-\alpha)(3-\beta)}(-\underline{R}), \tag{1.4}$$

where $\underline{R}=\underline{x}_2-\underline{x}_1$ is the center-to-center vector. Finally, the axisymmetry about the sphere-sphere axis implies that each tensor can be decomposed into expressions involving *no more* than three scalar functions (Brenner 1963, 1964). Jeffrey and Onishi (1984) designate these scalar functions as $X^P_{\alpha\beta}(R)$, $Y^P_{\alpha\beta}(R)$ and $Z^P_{\alpha\beta}(R)$ with P, α and β denoting the appropriate tensor $\underline{\underline{P}}^{\alpha\beta}$. They reserve the letter X for those funtions that arise from axisymmetric flows. (More specifically, we shall see later that the X, Y, and Z functions arise from boundary conditions involving spherical harmonics with the azimuthal number, m, equal to 0, 1 and 2 respectively). Thus, with $\underline{d} = \underline{R}/R$:

$$A^{\alpha\beta}_{ij} = X^A_{\alpha\beta}d_i d_j + Y^A_{\alpha\beta}(\delta_{ij}-d_i d_j), \tag{1.5a}$$

$$B^{\alpha\beta}_{ij} = \tilde{B}^{\beta\alpha}_{ji} = Y^B_{\alpha\beta}\varepsilon_{ijk}d_k, \tag{1.5b}$$

$$C_{ij}^{\alpha\beta} = X_{\alpha\beta}^{C} d_i d_j + Y_{\alpha\beta}^{C}(\delta_{ij} - d_i d_j), \tag{1.5c}$$

$$G_{ijk}^{\alpha\beta} = \tilde{G}_{kij}^{\beta\alpha} = X_{\alpha\beta}^{G}(d_i d_j - \tfrac{1}{3}\delta_{ij})d_k + Y_{\alpha\beta}^{G}(d_i\delta_{jk} + d_j\delta_{ik} - 2d_i d_j d_k), \tag{1.5d}$$

$$H_{ijk}^{\alpha\beta} = \tilde{H}_{kij}^{\beta\alpha} = Y_{\alpha\beta}^{H}(d_i\varepsilon_{jk\ell}d_\ell + d_j\varepsilon_{ik\ell}d_\ell), \tag{1.5e}$$

$$\begin{aligned} M_{ijk\ell}^{\alpha\beta} = {} & \tfrac{3}{2}X_{\alpha\beta}^{M}\,(d_i d_j - \tfrac{1}{3}\delta_{ij})(d_k d_\ell - \tfrac{1}{3}\delta_{k\ell}) \\ & + \tfrac{1}{2}Y_{\alpha\beta}^{M}(d_i\delta_{j\ell}d_k + d_j\delta_{i\ell}d_k + d_i\delta_{jk}d_\ell + d_j\delta_{ik}d_\ell - 4d_i d_j d_k d_\ell) \\ & + \tfrac{1}{2}Z_{\alpha\beta}^{M}(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \delta_{ij}\delta_{k\ell} + d_i d_j\delta_{k\ell} + \delta_{ij}d_k d_\ell \\ & \qquad - d_i\delta_{j\ell}d_k - d_j\delta_{i\ell}d_k - d_i\delta_{jk}d_\ell - d_j\delta_{ik}d_\ell + d_i d_j d_k d_\ell). \end{aligned} \tag{1.5f}$$

We now nondimensionalize these scalar functions so that they become functions only of the dimensionless separation parameter R/a. The dimensionless functions will be denoted with the symbol ^.

$$\underline{\underline{A}}^{\alpha\beta} = 6\pi a\hat{\underline{\underline{A}}}^{\alpha\beta}, \quad \underline{\underline{B}}^{\alpha\beta} = 4\pi a^2\hat{\underline{\underline{B}}}^{\alpha\beta}, \quad \underline{\underline{C}}^{\alpha\beta} = 8\pi a^3\hat{\underline{\underline{C}}}^{\alpha\beta}, \tag{1.6a-c}$$

$$\underline{\underline{G}}^{\alpha\beta} = 4\pi a^2\hat{\underline{\underline{G}}}^{\alpha\beta}, \quad \underline{\underline{H}}^{\alpha\beta} = 8\pi a^3\hat{\underline{\underline{H}}}^{\alpha\beta}, \quad \underline{\underline{M}}^{\alpha\beta} = \tfrac{20}{3}\pi a^3\hat{\underline{\underline{M}}}^{\alpha\beta}. \tag{1.6d-f}$$

The dimensionless functions for the tensors on the diagonal of the grand resistance matrix will approach unity for large R because the scales were chosen by considering the single-sphere result.

1.2 The Mobility Problem

Following Batchelor (1976), we define mobility problems as those in which the particle forces and torques are prescribed in the ambient field and the particle motion and stresslet are the unknowns. The formulation of the mobility problem is rather awkward from a mathematical perspective since the boundary conditions involve the unknowns, but in many problems the forces and torques are the prescribed physical quantities and the particles must move accordingly. In later sections, we shall first solve the resistance problem and then use the relations between the mobility and resistance tensors to solve the mobility problem. Again, the linearity of the Stokes equation allows us to write:

$$\begin{bmatrix} \mathbf{U}_1-\mathbf{U}(\mathbf{x}_1) \\ \mathbf{U}_2-\mathbf{U}(\mathbf{x}_2) \\ \boldsymbol{\Omega}_1-\boldsymbol{\Omega} \\ \boldsymbol{\Omega}_2-\boldsymbol{\Omega} \\ \mu^{-1}\mathbf{S}_1 \\ \mu^{-1}\mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{11} & \mathbf{a}^{12} & \tilde{\mathbf{b}}^{11} & \tilde{\mathbf{b}}^{12} & \tilde{\mathbf{g}}^{1} \\ \mathbf{a}^{21} & \mathbf{a}^{22} & \tilde{\mathbf{b}}^{21} & \tilde{\mathbf{b}}^{22} & \tilde{\mathbf{g}}^{2} \\ \mathbf{b}^{11} & \mathbf{b}^{12} & \mathbf{c}^{11} & \mathbf{c}^{12} & \tilde{\mathbf{h}}^{1} \\ \mathbf{b}^{21} & \mathbf{b}^{22} & \mathbf{c}^{21} & \mathbf{c}^{22} & \tilde{\mathbf{h}}^{2} \\ \mathbf{g}^{11} & \mathbf{g}^{12} & \mathbf{h}^{11} & \mathbf{h}^{12} & \mathbf{m}^{1} \\ \mathbf{g}^{21} & \mathbf{g}^{22} & \mathbf{h}^{21} & \mathbf{h}^{22} & \mathbf{m}^{2} \end{bmatrix} \begin{bmatrix} \mu^{-1}\mathbf{F}_1 \\ \mu^{-1}\mathbf{F}_2 \\ \mu^{-1}\mathbf{T}_1 \\ \mu^{-1}\mathbf{T}_2 \\ \mathbf{E} \end{bmatrix}$$

As in the resistance problem, the number of unknowns can be reduced by applying the reciprocal theorem, and the consequences of $\mathbf{S}$ and $\mathbf{E}$ being symmetric and traceless. *Thus:*

$$a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}, \qquad \tilde{b}_{ij}^{\alpha\beta} = b_{ji}^{\beta\alpha}, \qquad c_{ij}^{\alpha\beta} = c_{ji}^{\beta\alpha}, \tag{1.7a,b,c}$$

$$\tilde{g}_{ijk}^{\alpha} = g_{jki}^{1\alpha} + g_{jki}^{2\alpha}, \qquad \tilde{h}_{ijk}^{\alpha} = h_{jki}^{1\alpha} + h_{jki}^{2\alpha}, \tag{1.7d,e}$$

$$m_{ijk\ell}^{1} + m_{ijk\ell}^{2} = m_{k\ell ij}^{1} + m_{k\ell ij}^{2} . \tag{1.7f}$$

and equations which are analagous to (1.2) and (1.3).

As before, the two-sphere symmetry allows the following decompositions:

$$a_{ij}^{\alpha\beta} = x_{\alpha\beta}^{a} d_i d_j + y_{\alpha\beta}^{a}(\delta_{ij} - d_i d_j), \tag{1.8a}$$

$$b_{ij}^{\alpha\beta} = y_{\alpha\beta}^{b} \varepsilon_{ijk} d_k, \tag{1.8b}$$

$$c_{ij}^{\alpha\beta} = x_{\alpha\beta}^{c} d_i d_j + y_{\alpha\beta}^{c}(\delta_{ij} - d_i d_j), \tag{1.8c}$$

$$g_{ijk}^{\alpha\beta} = x_{\alpha\beta}^{g}(d_i d_j - \tfrac{1}{3}\delta_{ij}) d_k + y_{\alpha\beta}^{g}(d_i \delta_{jk} + d_j \delta_{ik} - 2 d_i d_j d_k), \tag{1.8d}$$

$$h^{\alpha\beta}_{ijk} = y^h_{\alpha\beta}(d_i\epsilon_{jk\ell}d_\ell + d_j\epsilon_{ik\ell}d_\ell), \tag{1.8e}$$

$$\begin{aligned} m^{\alpha\beta}_{ijk\ell} = {} & \frac{3}{2}x^m_{\alpha\beta}\,(d_id_j - \frac{1}{3}\delta_{ij})(d_kd_\ell - \frac{1}{3}\delta_{k\ell}) \\ & + \frac{1}{2}y^m_{\alpha\beta}(d_i\delta_{j\ell}d_k + d_j\delta_{i\ell}d_k + d_i\delta_{jk}d_\ell + d_j\delta_{ik}d_\ell - 4d_id_jd_kd_\ell) \\ & + \frac{1}{2}z^m_{\alpha\beta}(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \delta_{ij}\delta_{k\ell} + d_id_j\delta_{k\ell} + \delta_{ij}d_kd_\ell \\ & \qquad - d_i\delta_{j\ell}d_k - d_j\delta_{i\ell}d_k - d_i\delta_{jk}d_\ell - d_j\delta_{ik}d_\ell + d_id_jd_kd_\ell). \end{aligned} \tag{1.8f}$$

The nondimensionalizations of these functions are as follows:

$$\hat{\underline{\underline{a}}}^{\alpha\beta} = 6\pi a\underline{\underline{a}}^{\alpha\beta}, \qquad \hat{\underline{\underline{b}}}^{\alpha\beta} = 4\pi a^2\underline{\underline{b}}^{\alpha\beta}, \qquad \hat{\underline{\underline{c}}}^{\alpha\beta} = 8\pi a^3\underline{\underline{c}}^{\alpha\beta}, \tag{1.9a-c}$$

$$\hat{\underline{\underline{g}}}^{\alpha\beta} = \underline{\underline{g}}^{\alpha\beta}/(2a), \qquad \hat{\underline{\underline{h}}}^{\alpha\beta} = \underline{\underline{h}}^{\alpha\beta}, \qquad \underline{\underline{m}}^{\alpha\beta} = \frac{20}{3}\pi a^3\hat{\underline{\underline{m}}}^{\alpha\beta}. \tag{1.9d-f}$$

1.3 <u>Relations between the Resistance and Mobility Functions</u>

Our numerical technique solves the resistance problem. We obtain the mobility functions by using the following relations between the (dimensional) mobility and resistance functions. In matrix form, we have:

$$\begin{pmatrix} \underline{\underline{a}}^{11} & \underline{\underline{a}}^{12} & \tilde{\underline{\underline{b}}}^{11} & \tilde{\underline{\underline{b}}}^{12} \\ \underline{\underline{a}}^{21} & \underline{\underline{a}}^{22} & \tilde{\underline{\underline{b}}}^{21} & \tilde{\underline{\underline{b}}}^{22} \\ \underline{\underline{b}}^{11} & \underline{\underline{b}}^{12} & \underline{\underline{c}}^{11} & \underline{\underline{c}}^{12} \\ \underline{\underline{b}}^{21} & \underline{\underline{b}}^{22} & \underline{\underline{c}}^{21} & \underline{\underline{c}}^{22} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}}^{11} & \underline{\underline{A}}^{12} & \tilde{\underline{\underline{B}}}^{11} & \tilde{\underline{\underline{B}}}^{12} \\ \underline{\underline{A}}^{21} & \underline{\underline{A}}^{22} & \tilde{\underline{\underline{B}}}^{21} & \tilde{\underline{\underline{B}}}^{22} \\ \underline{\underline{B}}^{11} & \underline{\underline{B}}^{12} & \underline{\underline{C}}^{11} & \underline{\underline{C}}^{12} \\ \underline{\underline{B}}^{21} & \underline{\underline{B}}^{22} & \underline{\underline{C}}^{21} & \underline{\underline{C}}^{22} \end{pmatrix}^{-1}$$

$$\begin{bmatrix} \underline{\underline{g}}^{11} & \underline{\underline{g}}^{12} & \underline{\underline{h}}^{11} & \underline{\underline{h}}^{12} & \underline{\underline{m}}^{1} + \underline{\underline{M}}^{11} + \underline{\underline{M}}^{12} \\ \underline{\underline{g}}^{21} & \underline{\underline{g}}^{22} & \underline{\underline{h}}^{21} & \underline{\underline{h}}^{22} & \underline{\underline{m}}^{2} + \underline{\underline{M}}^{21} + \underline{\underline{M}}^{22} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\underline{G}}^{11} & \underline{\underline{G}}^{12} & \underline{\underline{H}}^{11} & \underline{\underline{H}}^{12} \\ \underline{\underline{G}}^{21} & \underline{\underline{G}}^{22} & \underline{\underline{H}}^{21} & \underline{\underline{H}}^{22} \end{bmatrix} \begin{bmatrix} \underline{\underline{a}}^{11} & \underline{\underline{a}}^{12} & \tilde{\underline{\underline{b}}}^{11} & \tilde{\underline{\underline{b}}}^{12} & \tilde{\underline{\underline{g}}}^{1} \\ \underline{\underline{a}}^{21} & \underline{\underline{a}}^{22} & \tilde{\underline{\underline{b}}}^{21} & \tilde{\underline{\underline{b}}}^{22} & \tilde{\underline{\underline{g}}}^{2} \\ \underline{\underline{b}}^{11} & \underline{\underline{b}}^{12} & \underline{\underline{c}}^{11} & \underline{\underline{c}}^{12} & \tilde{\underline{\underline{h}}}^{1} \\ \underline{\underline{b}}^{21} & \underline{\underline{b}}^{22} & \underline{\underline{c}}^{21} & \underline{\underline{c}}^{22} & \tilde{\underline{\underline{h}}}^{2} \end{bmatrix}$$

The relations for the functions follow as:

$$\begin{bmatrix} x^a_{11} & x^a_{12} \\ x^a_{21} & x^a_{22} \end{bmatrix} = \begin{bmatrix} X^A_{11} & X^A_{12} \\ X^A_{21} & X^A_{22} \end{bmatrix}^{-1} \tag{1.10}$$

$$\begin{bmatrix} x^c_{11} & x^c_{12} \\ x^c_{21} & x^c_{22} \end{bmatrix} = \begin{bmatrix} X^C_{11} & X^C_{12} \\ X^C_{21} & X^C_{22} \end{bmatrix}^{-1} \tag{1.11}$$

$$\begin{bmatrix} y^a_{11} & y^a_{12} & y^b_{11} & y^b_{21} \\ y^a_{21} & y^a_{22} & y^b_{12} & y^b_{22} \\ y^b_{11} & y^b_{12} & y^c_{11} & y^c_{12} \\ y^b_{21} & y^b_{22} & y^c_{21} & y^c_{22} \end{bmatrix} = \begin{bmatrix} Y^A_{11} & Y^A_{12} & Y^B_{11} & Y^B_{21} \\ Y^A_{21} & Y^A_{22} & Y^B_{12} & Y^B_{22} \\ Y^B_{11} & Y^B_{12} & Y^C_{11} & Y^C_{12} \\ Y^B_{21} & Y^B_{22} & Y^C_{21} & Y^C_{22} \end{bmatrix}^{-1} \tag{1.12}$$

$$\begin{bmatrix} x^g_{11} & x^g_{12} \\ x^g_{21} & x^g_{22} \end{bmatrix} = \begin{bmatrix} X^G_{11} & X^G_{12} \\ X^G_{21} & X^G_{22} \end{bmatrix} \begin{bmatrix} x^a_{11} & x^a_{12} \\ x^a_{21} & x^a_{22} \end{bmatrix} \tag{1.13}$$

$$\begin{bmatrix} y^g_{11} & y^g_{12} & -y^h_{11} & -y^h_{12} \\ y^g_{21} & y^g_{22} & -y^h_{21} & -y^h_{22} \end{bmatrix}$$

$$= \begin{bmatrix} Y^G_{11} & Y^G_{12} & -Y^H_{11} & -Y^H_{12} \\ Y^G_{21} & Y^G_{22} & -Y^H_{21} & -Y^H_{22} \end{bmatrix} \begin{bmatrix} y^a_{11} & y^a_{12} & y^b_{11} & y^b_{21} \\ y^a_{21} & y^a_{22} & y^b_{12} & y^b_{22} \\ y^b_{11} & y^b_{12} & y^c_{11} & y^c_{12} \\ y^b_{21} & y^b_{22} & y^c_{21} & y^c_{22} \end{bmatrix}. \tag{1.14}$$

$$x^m_\alpha = -(X^M_{\alpha 1} + X^M_{\alpha 2}) + X^G_{\alpha 1}(x^g_{11} + x^g_{21}) + X^G_{\alpha 2}(x^g_{12} + x^g_{22}), \tag{1.15a-c}$$

$$y^m_\alpha = -(Y^M_{\alpha 1} + Y^M_{\alpha 2}) + Y^G_{\alpha 1}(y^g_{11} + y^g_{21}) + Y^G_{\alpha 2}(y^g_{12} + y^g_{22})$$

$$+ Y^H_{\alpha 1}(y^h_{11} + y^h_{21}) + Y^H_{\alpha 2}(y^h_{12} + y^h_{22}),$$

$$z^m_\alpha = -(Z^M_{\alpha 1} + Z^M_{\alpha 2}), \qquad \text{for } \alpha = 1,2.$$

The above equations hold for two unequal spheres. If we limit the analysis to the case of two equal spheres, then the symmetry relation, equation (1.4), implies that subscripts "22" and "21" may be replaced everywhere by "11" and "12" respectively. For functions associated with $\underline{\underline{B}}$, $\underline{\underline{b}}$, $\underline{\underline{G}}$ and $\underline{\underline{g}}$, this substitution requires a change in sign. From here on, without loss of generality, we shall restrict our attention to the "11" and "12" functions.

2. Boundary Collocation

The boundary collocation technique developed by Gluckman, Pfeffer and Weinbaum (1971) has been used to solve a wide variety of low-Reynolds-number problems where the system boundaries do not conform to a single orthogonal co-ordinate system. A related technique was developed by O'Brien (1968) to calculate the flow past a slightly deformed sphere. The earliest applications of the technique were limited to axisymmetric problems which were solved using the stream function. Since then, the technique has been applied directly to the Stokes equation and three dimensional problems including the sedimentation of three spheres with centers in a vertical plane (Ganatos, Pfeffer and Weinbaum 1978), the motion of a sphere between two parallel infinite plates (Ganatos, Pfeffer and Weinbaum 1980) and the sedimentation of a sphere in an inclined channel (Ganatos, Weinbaum and Pfeffer 1982). For our two-sphere problem, a suitable co-ordinate system exists. Nevertheless we use the collocation technique because accurate numerical results are obtained with minimal human computation.

The essential idea behind the collocation technique is as follows. The velocity field can be represented by an expansion in terms of basis functions, each of which satisfies the equations of motion. In general, the number of elements in the basis set is not finite because of the interactions between the spheres. However, the higher order elements are usually unimportant. Consequently, the series can be truncated at N terms and the coefficients of the retained basis functions determined by setting the boundary condition at N collocation points (hence the name boundary collocation). It is found empirically that the lower order coefficients converge rapidly as N is increased. This is important since, as shown later, the force, torque and stresslet depend only on the first and second order coefficients.

2.1 The Velocity Representation

The velocity field, $\underline{v}$, satisfies the Stokes equation:

$$-\nabla p + \mu\nabla^2\underline{v} = \underline{0} \tag{2.1}$$

and the equation of continuity:

$$\nabla\cdot\underline{v} = 0. \tag{2.2}$$

The boundary conditions are those associated with each resistance problem.

Our goal in this section is to construct general forms of the velocity representations and boundary conditions which together encompass the complete set of resistance problems. Then, any resistance problem of interest can be obtained by selecting the appropriate set of parameters. This structure is readily passed on to the computer codes and the result is a versatile, yet short subroutine (less than 60 lines of code) which calculates the entire collection of functions.

As shown in Happel and Brenner (1965), the disturbance velocity field can be represented using Lamb's general solution:

$$\underline{v}(\underline{x}) - \underline{v}^{\infty}(\underline{x}) = \sum_{n=1}^{\infty} \Big\{ \nabla\Phi_{-n-1} + \nabla\times(\underline{x}\chi_{-n-1}) + \frac{(n+1)}{n(2n-1)}\, \underline{x}\, p_{-n-1} - \frac{(n-2)}{2n(2n-1)} r^2\nabla p_{-n-1} \Big\}, \tag{2.3}$$

where p_{-n-1}, Φ_{-n-1} and χ_{-n-1} are exterior spherical harmonics.

Following Ganatos et. al. (1978), the velocity is written as a superposition of two expansions, one centered at sphere 1 and the other at sphere 2.

$$\underline{v}(\underline{x}) - \underline{v}^{\infty}(\underline{x}) = \sum_{n=1}^{\infty} \{ \nabla\phi_{-n-1}^{(1)} + \nabla\times(\underline{r}_1 \chi_{-n-1}^{(1)}) + \frac{(n+1)}{n(2n-1)} \underline{r}_1\, p_{-n-1}^{(1)} - \frac{(n-2)}{2n(2n-1)} r_1^2 \nabla p_{-n-1}^{(1)} + \nabla\phi_{-n-1}^{(2)} + \nabla\times(\underline{r}_2 \chi_{-n-1}^{(2)}) + \frac{(n+1)}{n(2n-1)} \underline{r}_2\, p_{-n-1}^{(2)} - \frac{(n-2)}{2n(2n-1)} r_2^2 \nabla p_{-n-1}^{(2)} \}, \tag{2.4}$$

with $\underline{r}_\alpha = \underline{x}-\underline{x}_\alpha$, $r_\alpha = |\underline{r}_\alpha|$ for $\alpha = 1,2.$

The spherical harmonics are expanded as:

$$p_{-n-1}^{(\alpha)} = \sum_{m=0}^{n} r^{-n-1} P_n^m(\cos\theta_\alpha)\, [a_{0n}^{(\alpha)}\delta_{0m} + a_{mn}^{(\alpha)}\sin m\phi]\ ,$$

$$\phi_{-n-1}^{(\alpha)} = \sum_{m=0}^{n} r^{-n-1} P_n^m(\cos\theta_\alpha)\, [b_{0n}^{(\alpha)}\delta_{0m} + b_{mn}^{(\alpha)}\sin m\phi]\ ,$$

$$\chi_{-n-1}^{(\alpha)} = \sum_{m=0}^{n} r^{-n-1} P_n^m(\cos\theta_\alpha)\, c_{mn}^{(\alpha)}\cos m\phi\ .$$

For each resistance function, we will actually require only one particular m, the value which appears in the surface velocity, and the ϕ-dependence will factor from our problem as shown in the next subsection. Thus for two-sphere problems, a one-dimensional collocation (in θ) is possible, even if the flow is three-dimensional.

2.2 Application of the Boundary Conditions

The disturbance field must decay far away from both particles. At the sphere surface, the disturbance velocity must equal a surface velocity, $\underline{v}_s$, which is the difference between the particle's rigid-body motion and the ambient velocity. Thus all relevant cases are included in the following expression of the boundary condition at sphere 1:

$$\underline{v}_s = \sum_{\ell=1}^{2} \sum_{m=0}^{\ell} [\ \nabla\{r_1^\ell P_\ell^m(\cos\theta)[A_{\ell 0}\delta_{0m} + A_{\ell m}\sin m\phi]\ \} + \nabla\times\{\underline{r}_1\ r_1^\ell P_\ell^m(\cos\theta) B_{\ell m}\cos m\phi]\ \}\ ]. \tag{2.5}$$

In (2.5) the cosmϕ terms are omitted for m≥1, since those resistance problems are equivalent to those obtained from the sinmϕ terms (with a rotational co-ordinate transformation about the sphere-sphere axis). The following table shows the required $\underset{\sim}{v}_s$ for each resistance function.

Table 1.

Non-zero Coefficient	$\underset{\sim}{v}_s$	Resistance Function(s)
1) $A_{10} = 1$	Translation along sphere-sphere axis.	$X_A^{\alpha\beta}, X_G^{\alpha\beta}$
2) $A_{11} = 1$	Translation perpendicular to axis.	$Y_A^{\alpha\beta}$, $Y_B^{\alpha\beta}$, $Y_G^{\alpha\beta}$
3) $A_{20} = 1$	Axisymmetric straining.	$X_M^{11}+X_M^{12}$
4) $A_{21} = 1$	Rate-of-strain as in ZX shear flow.	$Y_M^{11}+Y_M^{12}$
5) $A_{22} = 1$	Hyperbolic straining in XY plane.	$Z_M^{11}+Z_M^{12}$
6) $B_{10} = 1$	Rotation about sphere-sphere axis.	$X_C^{\alpha\beta}$
7) $B_{11} = 1$	Rotation with axis perpendicular to sphere-sphere axis.	$Y_C^{\alpha\beta}$, $Y_H^{\alpha\beta}$

We now examine the form taken by Lamb's representation at the sphere boundaries. We use the cylindrical coordinate system (z,R,ϕ) as shown in Figure 1. The z, R and ϕ velocity components in equation (2.4) are equated to the corresponding components of the surface velocity in (2.5). Dependence on ϕ occurs for problems with m≥1, but factors out as follows: a factor of sinmϕ in the z-component and R-component equations and a factor of cosmϕ in the ϕ-component equation. Thus the boundary conditions on sphere 1 (i.e. $r_1 = 1$) requires that:

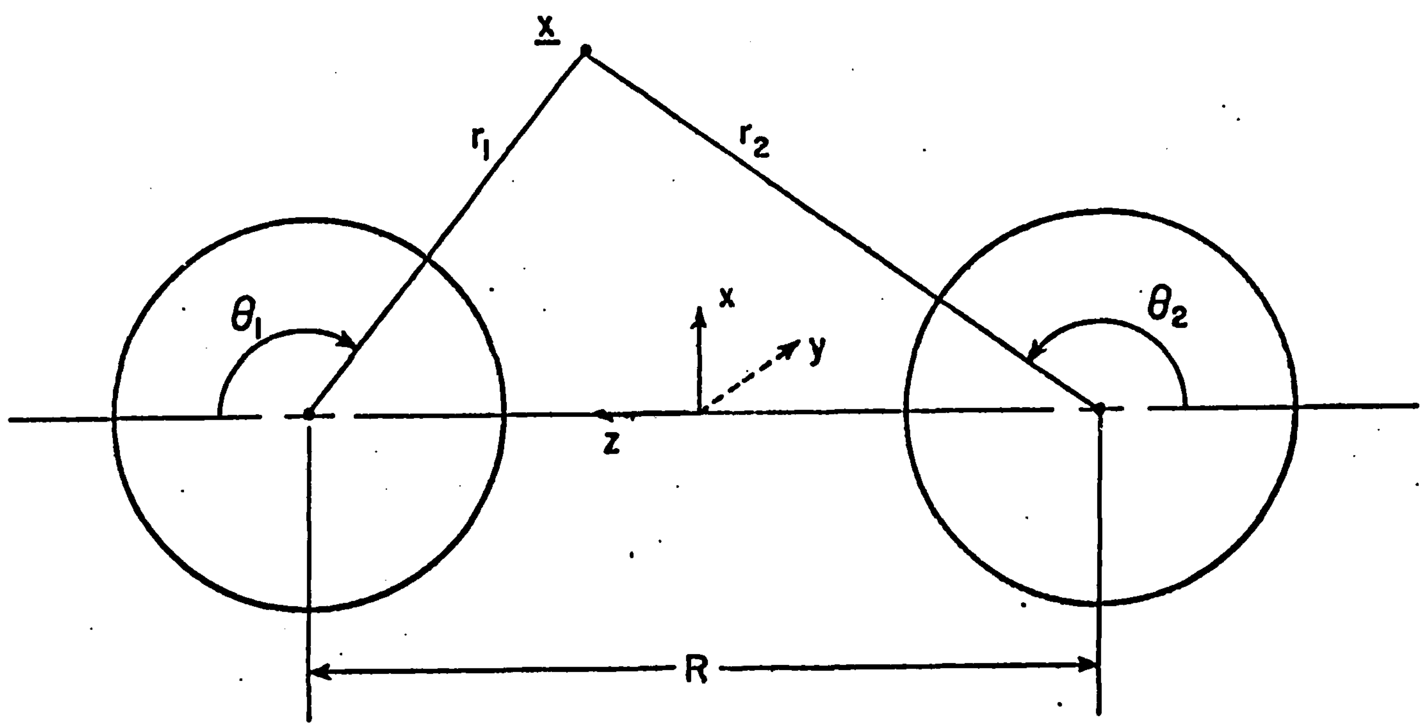


Figure 1. The two-sphere geometry

$$\sum_{\alpha=1}^{2} (-1)^{\alpha-1} \sum_{n=\ell}^{\infty} \{ a_{mn}^{(\alpha)} [\frac{(n+1)}{(4n-2)} r_\alpha^{-n} \xi_\alpha P_n^m(\xi_\alpha) - \frac{(n-2)}{n(4n-2)} r_\alpha^{-n} (1-\xi_\alpha^2) P_n^{m\prime}(\xi_\alpha)]$$

$$+ b_{mn}^{(\alpha)} [-(n+1) r_\alpha^{-n-2} \xi_\alpha P_n^m(\xi_\alpha) + r_\alpha^{-n-2} (1-\xi_\alpha^2) P_n^{m\prime}(\xi_\alpha)]$$

$$+ m c_{mn}^{(\alpha)} [r_\alpha^{-n-1} P_n^m(\xi_\alpha)] \}$$

$$= A_{\ell m} [\ell \xi_1 P_\ell^m(\xi_1) + (1-\xi_1^2) P_\ell^{m\prime}(\xi_1)] + B_{\ell m} m P_\ell^m(\xi_1) \tag{2.6a}$$

$$\sum_{\alpha=1}^{2} \sum_{n=\ell}^{\infty} \{ a_{mn}^{(\alpha)} [\frac{(n+1)}{(4n-2)} r_\alpha^{-n} \sin\theta_\alpha P_n^m(\xi_\alpha)$$

$$+ \frac{(n-2)}{n(4n-2)} r_\alpha^{-n} [\xi_\alpha P_n^{m+1}(\xi_\alpha) + m \sin\theta_\alpha P_n^m(\xi_\alpha)]]$$

$$+ b_{mn}^{(\alpha)} r_\alpha^{-n-2} [-(n+1) \sin\theta_\alpha P_n^m(\xi_\alpha) - [\xi_\alpha P_n^{m+1}(\xi_\alpha) + m \sin\theta_\alpha P_n^m(\xi_\alpha)]]$$

$$- c_{mn}^{(\alpha)} r_\alpha^{-n-1} P_n^{m+1}(\xi_\alpha) \} \tag{2.6b}$$

$$= A_{\ell m} [\ell \sin\theta_1 P_\ell^m(\xi_1) - [\xi_1 P_\ell^{m+1}(\xi_1) + m \sin\theta_1 P_\ell^m(\xi_1)]] - B_{\ell m} P_\ell^{m+1}(\xi_1)$$

$$\sum_{\alpha=1}^{2} \sum_{n=\ell}^{\infty} \{ -m a_{mn}^{(\alpha)} [\frac{(n-2)}{n(4n-2)} r_\alpha^{-n} P_n^m(\xi_\alpha)/\sin\theta_\alpha]$$

$$+ m b_{mn}^{(\alpha)} [r_\alpha^{-n-2} P_n^m(\xi_\alpha)/\sin\theta_\alpha] + c_{mn}^{(\alpha)} [r_\alpha^{-n-1} \sin\theta_\alpha P_n^{m\prime}(\xi_\alpha)] \}$$

$$= A_{\ell m} m P_\ell^m(\xi_1)/\sin\theta_1 + B_{\ell m} \sin\theta_1 P_\ell^{m\prime}(\xi_1) , \tag{2.6c}$$

where $\xi_\alpha = \cos\theta_\alpha$. There is an analogous set of equations from sphere 2. Equations (2.6a) and (2.6c) follow directly from the z-component and ϕ-component equations. Equation (2.6b) is obtained by subtracting the ϕ-component equation from the R-component equation. This last maneuver allows us to remove the singularities from the poles. Further details are given in Section 2.5.

The series is now truncated at N terms. The 6N unknown coefficients, a^1_{mn}, b^1_{mn}, c^1_{mn}, a^2_{mn}, b^2_{mn} and c^2_{mn}, are determined by applying the truncated version of (2.6a-c) at 2N collocation points on the surfaces and solving the resulting 6N×6N system. We remind the reader that the parameters m and ℓ are specified by the resistance problem. In the next section, the computations are simplified by exploiting the mirror symmetry with respect to the XY plane.

2.3 Mirror Symmetry about the XY Plane

For the general problem of two unequal spheres, it has been shown that the larger sphere requires more points (Liao and Krueger 1980). For two equal spheres, the points are distributed in equal numbers between the two, at equidistant spacings (Gluckman, Weinbaum and Pfeffer 1971). Furthermore, we decompose each resistance problem into subproblems that exploit the fore-aft mirror symmetry with respect to the XY plane. Symmetry dictates that the coefficients in the series centered at sphere 1 either equal or are negatives of the corresponding coefficients in the other series. This also holds for the truncated expansion as long as the collocation points on sphere 2 are placed at the mirror images of the points chosen for sphere 1.

An examination of the resistance problems reveals that they either posess one of the following two types of symmetry or may be decomposed into two subproblems, with a subproblem of each symmetry type. A velocity field with mirror symmetry with respect to the XY plane satisfies:

$$v_x(x,y,z) = v_x(x,y,-z),$$
$$v_y(x,y,z) = v_y(x,y,-z),$$
$$v_z(x,y,z) = -v_z(x,y,-z),$$

(The flow vectors in the half-spaces are mirror images of each other). A field with mirror anti-symmetry satisfies:

$$v_x(x,y,z) = -v_x(x,y,-z),$$
$$v_y(x,y,z) = -v_y(x,y,-z),$$
$$v_z(x,y,z) = v_z(x,y,-z).$$

For problems with these symmetries, the coefficients for the terms centered at sphere 2 are given by:

$$a_{mn}^{(2)} = Sa_{mn}^{(1)}$$

$$b_{mn}^{(2)} = Sb_{mn}^{(1)}$$

$$c_{mn}^{(2)} = Sc_{mn}^{(1)}$$

where the symmetry parameter, S, is defined by:

$$S = \begin{cases} 1 & \text{for problems with mirror symmetry,} \\ -1 & \text{for problems with mirror anti-symmetry,} \end{cases}$$

and the collocation equation from a point on the surface of sphere 2 becomes identical to that from the image point on sphere 1. Thus equations (2.6a-c) and their counterparts from sphere 2 reduce to:

$$\begin{aligned}\sum_{n=\ell}^{\infty} \Big\{ & a_{mn}^{(1)} \sum_{\alpha=1}^{2} (-S)^{\alpha-1} \Big[\frac{(n+1)}{(4n-2)} r_\alpha^{-n} \xi_\alpha P_n^m(\xi_\alpha) - \frac{(n-2)}{n(4n-2)} r_\alpha^{-n} (1-\xi_\alpha^2) P_n^{m\prime}(\xi_\alpha) \Big] \\ & + b_{mn}^{(1)} \sum_{\alpha=1}^{2} (-S)^{\alpha-1} \Big[-(n+1) r_\alpha^{-n-2} \xi_\alpha P_n^m(\xi_\alpha) + r_\alpha^{-n-2} (1-\xi_\alpha^2) P_n^{m\prime}(\xi_\alpha) \Big] \\ & + m c_{mn}^{(1)} \sum_{\alpha=1}^{2} (-S)^{\alpha-1} \Big[r_\alpha^{-n-1} P_n^m(\xi_\alpha) \Big] \Big\} \\ = {} & A_{\ell m}\big[\ell \xi_1 P_\ell^m(\xi_1) + (1-\xi_1^2) P_\ell^{m\prime}(\xi_1)\big] + B_{\ell m} m P_\ell^m(\xi_1) \end{aligned} \tag{2.7a}$$

$$\sum_{n=\ell}^{\infty} \{ a_{mn}^{(1)} \sum_{\alpha=1}^{2} S^{\alpha-1} [\frac{(n+1)}{(4n-2)} r_\alpha^{-n} \sin\theta_\alpha P_n^m(\xi_\alpha)$$

$$+ \frac{(n-2)}{n(4n-2)} r_\alpha^{-n} [\xi_\alpha P_n^{m+1}(\xi_\alpha) + m\sin\theta_\alpha P_n^m(\xi_\alpha)]]$$

$$+ b_{mn}^{(1)} \sum_{\alpha=1}^{2} S^{\alpha-1} r_\alpha^{-n-2} [-(n+1) \sin\theta_\alpha P_n^m(\xi_\alpha) - [\xi_\alpha P_n^{m+1}(\xi_\alpha) + m\sin\theta_\alpha P_n^m(\xi_\alpha)]]$$

$$- c_{mn}^{(1)} \sum_{\alpha=1}^{2} S^{\alpha-1} r_\alpha^{-n-1} P_n^{m+1}(\xi_\alpha) \} \qquad (2.7b)$$

$$= A_{\ell m}[\ell\sin\theta_1 P_\ell^m(\xi_1) - [\xi_1 P_\ell^{m+1}(\xi_1) + m\sin\theta_1 P_\ell^m(\xi_1)]] - B_{\ell m} P_\ell^{m+1}(\xi_1)$$

$$\sum_{n=\ell}^{\infty} \{ -ma_{mn}^{(1)} \sum_{\alpha=1}^{2} S^{\alpha-1} [\frac{(n-2)}{n(4n-2)} r_\alpha^{-n} P_n^m(\xi_\alpha)/\sin\theta_\alpha]$$

$$+ mb_{mn}^{(1)} \sum_{\alpha=1}^{2} S^{\alpha-1} [r_\alpha^{-n-2} P_n^m(\xi_\alpha)/\sin\theta_\alpha]$$

$$+ c_{mn}^{(1)} \sum_{\alpha=1}^{2} S^{\alpha-1} [r_\alpha^{-n-1} \sin\theta_\alpha P_n^{m\prime}(\xi_\alpha)] \}$$

$$= A_{\ell m} m P_\ell^m(\xi_1)/\sin\theta_1 + B_{\ell m} \sin\theta_1 P_\ell^{m\prime}(\xi_1) \; . \qquad (2.7c)$$

Thus each resistance function is calculated by solving an appropriate set of problems of the form given by (2.7a-c). We shall see that each resistance function is a linear combination of a small number of the lower order coefficients.

The remainder of Section 2 contains information relating to the code development from (2.7a-c). Readers who are more interested in the end applications may prefer to skip to Section 3.

2.4 Axisymmetric Problems

In axisymmetric problems certain velocity components vanish identically because of symmetry. The 3N×3N collocation system reduces to either an N×N or 2N×2N system. Axisymmetric problems with $v_\phi=0$ have m=0 so that equation

(2.7c) and c_{mn} vanish identically leaving a 2N×2N system. On the other hand, axisymmetric swirl problems with v_ϕ as the only nonvanishing component reduce to an N×N system since a_{mn}, b_{mn}, (2.7a) and (2.7b) vanish identically. The reduced system can be obtained from the $c_{mn}^{(1)}$ terms of (2.7b) (i.e. the 2-3 block of the larger system).

2.5 Stability of the Collocation System of Equations

In this section, we review earlier reports on the stability of the collocation system of equations and present new findings.

Problems occur in the collocation scheme if points are placed at the equator, $\theta_1 = \pi/2$, and the poles. If a point is placed at the equator, the terms from sphere 1, which are normally greater than those from sphere 2 vanish (since $\cos\theta_1$ vanishes at the equator) thus destabilizing the system. This problem was circumvented in the original work by Gluckman et. al. (1971) by using twin points at 89° and 91°, an approach which they justified by examining the convergence behavior, in the limit of small ε, for twin points at $90-\varepsilon$ and $90+\varepsilon$ degrees.

At the poles, the system is indeterminate because equations from the R and ϕ components become identical. Gluckman et. al. (1971) avoided this problem by not placing any points at the poles.

We have avoided the problem at the equator by using an even number of points spaced at equidistant intervals. The indeterminacy at the poles was removed by taking the difference of the R-component and ϕ-component equations to arrive at (2.6b) and (2.7b). The source of the indeterminacy is then apparent. Equation (2.7a-c) have zeros of multiplicity m, m+1 and m-1 at the poles. They may be removed by factoring $\sin^m\theta_1$, $\sin^{m+1}\theta_1$ and $\sin^{m-1}\theta_1$ respectively.

When the spheres are far apart, accurate solutions are obtained with as few as four points. Tests show that our collocation scheme compares favorably with that of Gluckman et. al. (1971) for R/a between 2.1 and 10.0. Furthermore, in the strong interaction region, our scheme converges faster because the error profile in the gap region is reduced by the "Hermite interpolation" nature of the approximant. Although we do not place a point at the equator, the large number of points used in the near-field, e.g. N=60, ensures the presence of collocation points near the equator.

The code development required a system solver and a routine for the spherical harmonics. We used the LINPAK routines DGECO and DGESL to invert the system. However, essentially identical results (15 significant figures) were obtained with other (slower) routines. A recursion scheme was developed for the spherical harmonics because our application required the harmonics divided by factors of $\sin^m\theta$. The stability of these routines was spot checked by comparison with the tables in Abramowitz and Stegun (1964).

3. Calculation of Resistance Functions

In this section, we extract the resistance functions from the information contained in the collocation solution. There are several methods for obtaining the force, torque and stresslet on a particle from the velocity solution (see Chapter 3 of Happel and Brenner, 1965). We present here a simple but powerful method.

The velocity field which was previously represented by Lamb's general solution can also be represented by the twin multipole expansion (Jeffrey, 1974):

$$\underline{v} = \sum_{\alpha=1}^{2} \{ \underline{F}_\alpha - (\underline{\underline{S}}_\alpha + \underline{\underline{T}}_\alpha)\cdot\nabla + \cdots \}\cdot\underline{\underline{J}}(\underline{x}-\underline{x}_\alpha)/(8\pi\mu),$$

with the Oseen tensor, $\underline{\underline{J}}$ defined by

$$\underline{\underline{J}}(\underline{x}) = \frac{1}{|\underline{x}|}\underline{\underline{\delta}} + \frac{1}{|\underline{x}|^3}\underline{x}\underline{x} .$$

This representation is useful because it reveals that the force on particle α appears as the coefficient in the term that decays as $|\underline{x}-\underline{x}_\alpha|^{-1}$ while the dipole moments, $\underline{\underline{S}}_\alpha$ and $\underline{\underline{T}}_\alpha = -\frac{1}{2}\underline{\underline{\varepsilon}}\cdot\underline{T}_\alpha$ appear as the coefficients in the terms that decay as $|\underline{x}-\underline{x}_\alpha|^{-2}$. More explicitly, in the notation of Chwang and Wu (1975), the force, torque and stresslet appear as the coefficient of the Stokeslet, Rotlet and Stresslet. Therefore, we rearrange the terms in Lamb's representation to form the Stokeslet, Rotlet and Stresslet and obtain the relation between the force and dipole moments and the coefficients (in Lamb's representation).

The above ideas are put into practice for each resistance problem in the following two-step procedure.

1) In the first step, we take the arbitrary but convenient convention of setting the appropriate $A_{\ell m}$ or $B_{\ell m}$ equal to one (and set all others equal to zero).

2) The force, torque and stresslet in the multipole expansion are expressed in terms of the resistance functions as given by (1.5a-f). This expansion has the same form as the (rearranged) Lamb's representation.

After the above prescribed algebra, we arrive at formulae for the resistance functions in terms of the coefficients in Lamb's solution - $a_{mn}(\ell,m,S)$, $b_{mn}(\ell,m,S)$ and $c_{mn}(\ell,m,S)$. The arguments, ℓ, m and S indicate which coefficient is retained in the surface velocity (see Table 1 on page 13) and the type of mirror symmetry. As discussed earlier, each function requires a superposition of a mirror symmetric and mirror anti-symmetric problem, with the exception of the scalar functions from $\underline{\underline{M}}$ which already possess the symmetry. As shown below the mirror-symmetric solutions are summed to give the 1-1 functions and their difference is taken to give the 1-2 functions.

The functions associated with translational motions and rate-of-strain fields, with the argument (ℓ,m,S) denoting which $A_{\ell m}$ is set equal to one, are given by:

$$\hat{X}^A_{11} = \frac{1}{3}[a_{01}(1,0,-1) + a_{01}(1,0,1)] \tag{3.3a}$$

$$\hat{X}^A_{12} = \frac{1}{3}[a_{01}(1,0,-1) - a_{01}(1,0,1)] \tag{3.3b}$$

$$\hat{Y}^A_{11} = \frac{1}{3}[a_{11}(1,1,-1) + a_{11}(1,1,1)] \tag{3.3c}$$

$$\hat{Y}^A_{12} = -\frac{1}{3}[a_{11}(1,1,-1) - a_{11}(1,1,1)] \tag{3.3d}$$

$$\hat{Y}^B_{11} = -[c_{11}(1,1,-1) + c_{11}(1,1,1)] \tag{3.3e}$$

$$\hat{Y}^B_{12} = c_{11}(1,1,-1) - c_{11}(1,1,1) \tag{3.3f}$$

$$\hat{X}^G_{11} = -\frac{1}{4}[a_{02}(1,0,-1) + a_{02}(1,0,1)] \tag{3.3g}$$

$$\hat{X}^G_{12} = -\frac{1}{4}[a_{02}(1,0,-1) - a_{02}(1,0,1)] \tag{3.3h}$$

$$\hat{Y}^G_{11} = -\frac{1}{4}[a_{12}(1,1,-1) + a_{12}(1,1,1)] \tag{3.3i}$$

$$\hat{Y}^G_{12} = \frac{1}{4}[a_{12}(1,1,-1) - a_{12}(1,1,1)] \tag{3.3j}$$

$$\hat{X}^M_{11} + \hat{X}^M_{12} = \frac{1}{10}a_{02}(2,0,1) \tag{3.3k}$$

$$\hat{Y}^M_{11} + \hat{Y}^M_{12} = \frac{1}{10}a_{12}(2,1,-1) \tag{3.3ℓ}$$

$$\hat{Z}^M_{11} + \hat{Z}^M_{12} = \frac{1}{10}a_{22}(2,2,1). \tag{3.3m}$$

The functions associated with rotational motions, with the argument (ℓ,m,S) denoting which $B_{\ell m}$ is set equal to one, are given by:

$$\hat{X}^C_{11} = \frac{1}{2}[c_{01}(1,0,-1) + c_{01}(1,0,1)] \tag{3.3n}$$

$$\hat{X}^C_{12} = -\frac{1}{2}[c_{01}(1,0,-1) - c_{01}(1,0,1)] \tag{3.3o}$$

$$\hat{Y}^C_{11} = \frac{1}{2}[c_{11}(1,1,-1) + c_{11}(1,1,1)] \tag{3.3p}$$

$$\hat{Y}^C_{12} = \frac{1}{2}[c_{11}(1,1,-1) - c_{11}(1,1,1)] \tag{3.3q}$$

$$\hat{Y}^H_{11} = -\frac{1}{8}[a_{12}(1,1,-1) + a_{12}(1,1,1)] \tag{3.3r}$$

$$\hat{Y}^H_{12} = -\frac{1}{8}[a_{12}(1,1,-1) - a_{12}(1,1,1)]. \tag{3.3s}$$

This completes the calculation of the resistance functions. The mobility functions are calculated from the resistance functions as prescribed by equations (1.10) to 1.15).

4. Results and Conclusions

In this section, we examine the results for the resistance and mobility functions and compare them, where available, with those obtained by other means. The resistance and mobility functions are plotted vs. the reduced sphere-sphere separation, R/a, in Figures 2a-ℓ and 3a-ℓ respectively. These plots were generated using 12 collocation points for $R \geq 3$ and 24 points for $R<3$. At these levels, convergence has been obtained far beyond the resolution of the plotter. We note that Y^H_{11}, $Y^M_{11}+Y^M_{12}$, y^c_{12} and y^m_1 are not monotonic functions but have extrema. This phenomenon is consistent with the requirement that two neutrally-buoyant spheres move as a rigid body when in contact.

For comparative purposes, an extensive table (not presented here but available upon request from SK) was created for all functions. In the construction of the table, the number of collocation points was increased until convergence was obtained to five significant figures. This table was used as "the collocation results" in the comparison. Except for almost touching spheres, the number of collocation points and therefore the order of the system of equations was well within the memory limitations of a VAX 11/780. At sphere-sphere gaps of 0.01a the most difficult function, X^A_{11}, converged to three significant figures at 60 collocation points. The program in its present form allows up to 100 points.

In the far-field, asymptotic solutions are either available or readily obtained by the method of reflections. The collocation results matched these far-field solutions for all functions. In fact, the far-field solutions were the primary defense against program bugs.

A more stringent test was available for resistance and mobility functions from $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$, $\underline{\underline{a}}$, $\underline{\underline{b}}$ and $\underline{\underline{c}}$ because of the recent twin-multipole expansion

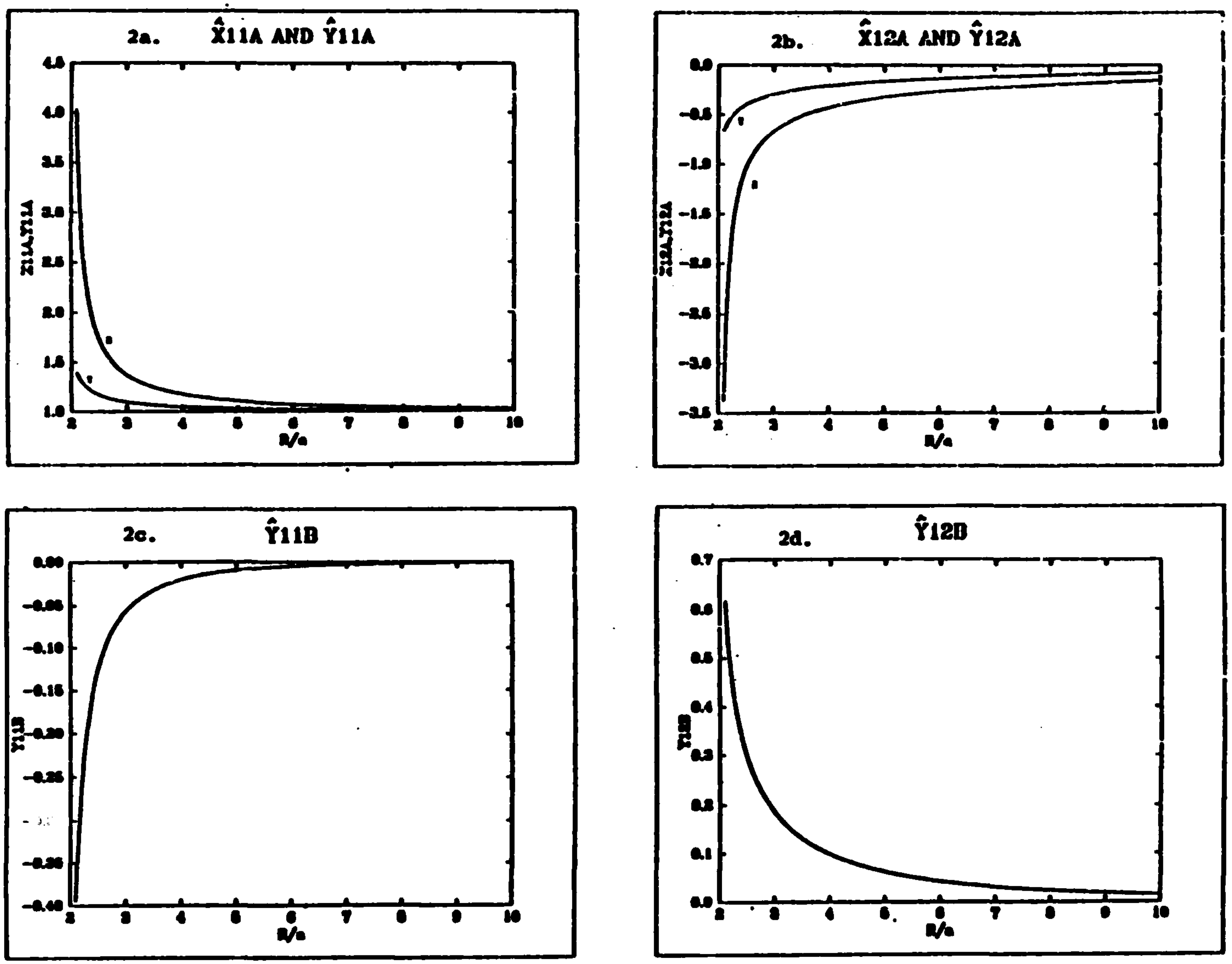
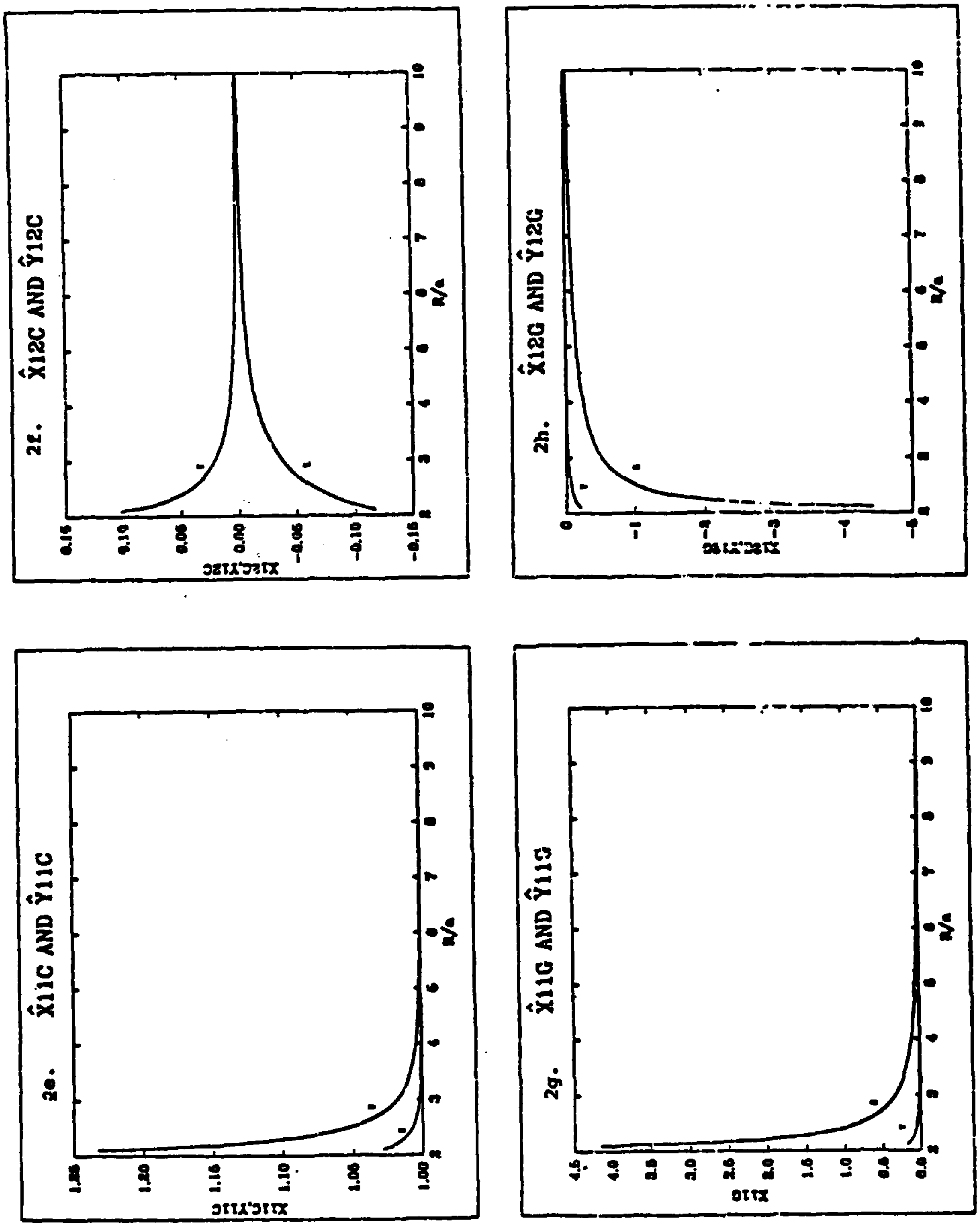
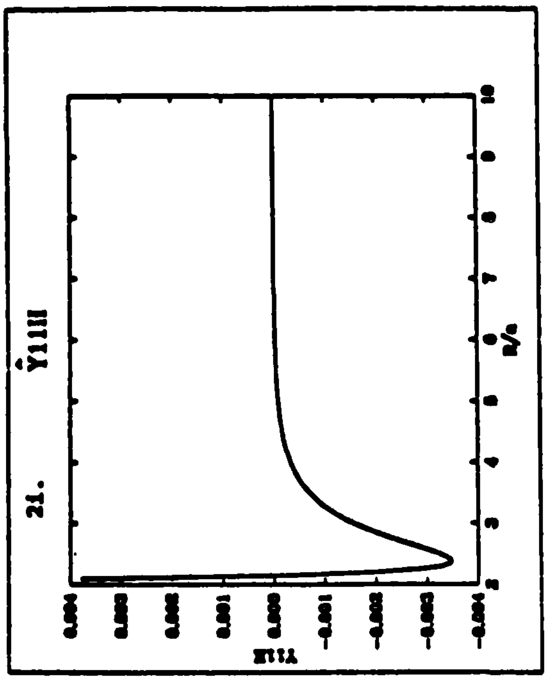
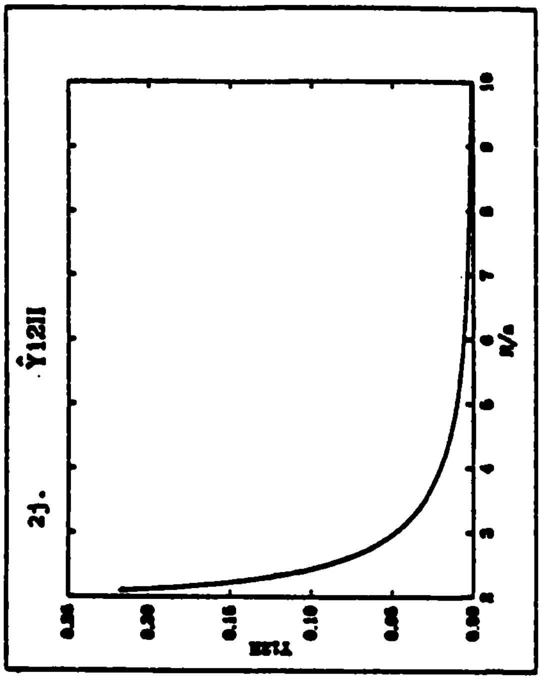
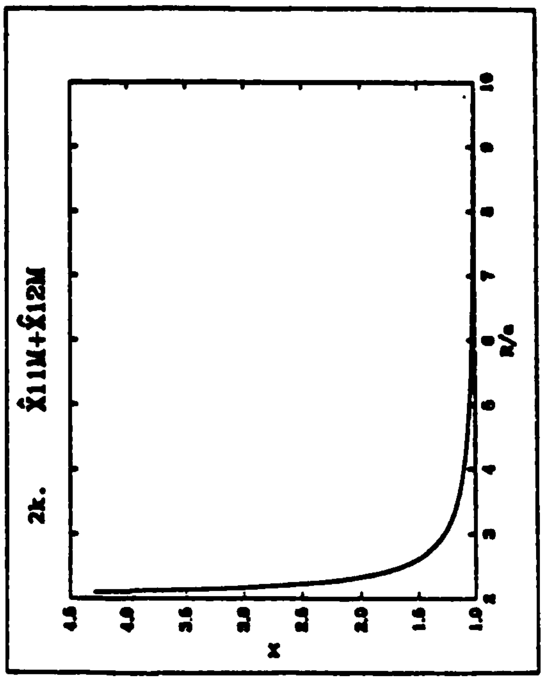


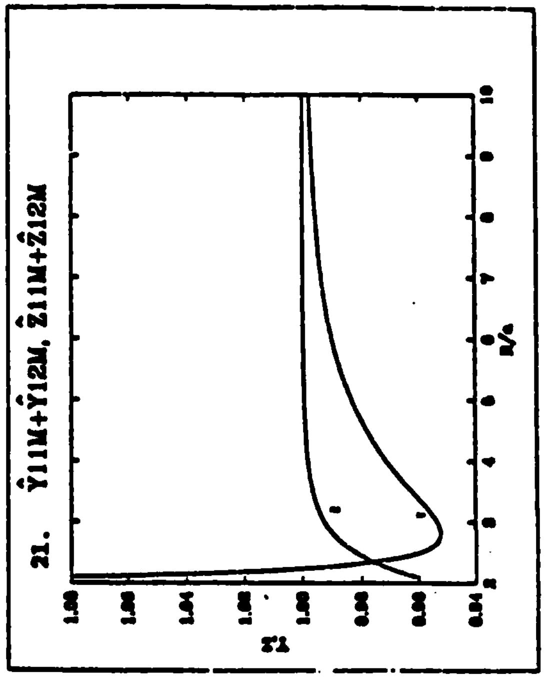
Figure 2. The resistance functions vs. sphere-sphere separation.


2f. $\hat{X}$12C AND $\hat{Y}$12C
2h. $\hat{X}$12G AND $\hat{Y}$12G
2e. $\hat{X}$11C AND $\hat{Y}$11C
2g. $\hat{X}$11G AND $\hat{Y}$11G
R/a


2i.
$\hat{Y}$11M
Y11M
R/a


2j.
$\hat{Y}$12M
Y12M
R/a


2k.
$\hat{X}$11M+$\hat{X}$12M
X
R/a


2l.
$\hat{Y}$11M+$\hat{Y}$12M, $\hat{Z}$11M+$\hat{Z}$12M
YZ
R/a

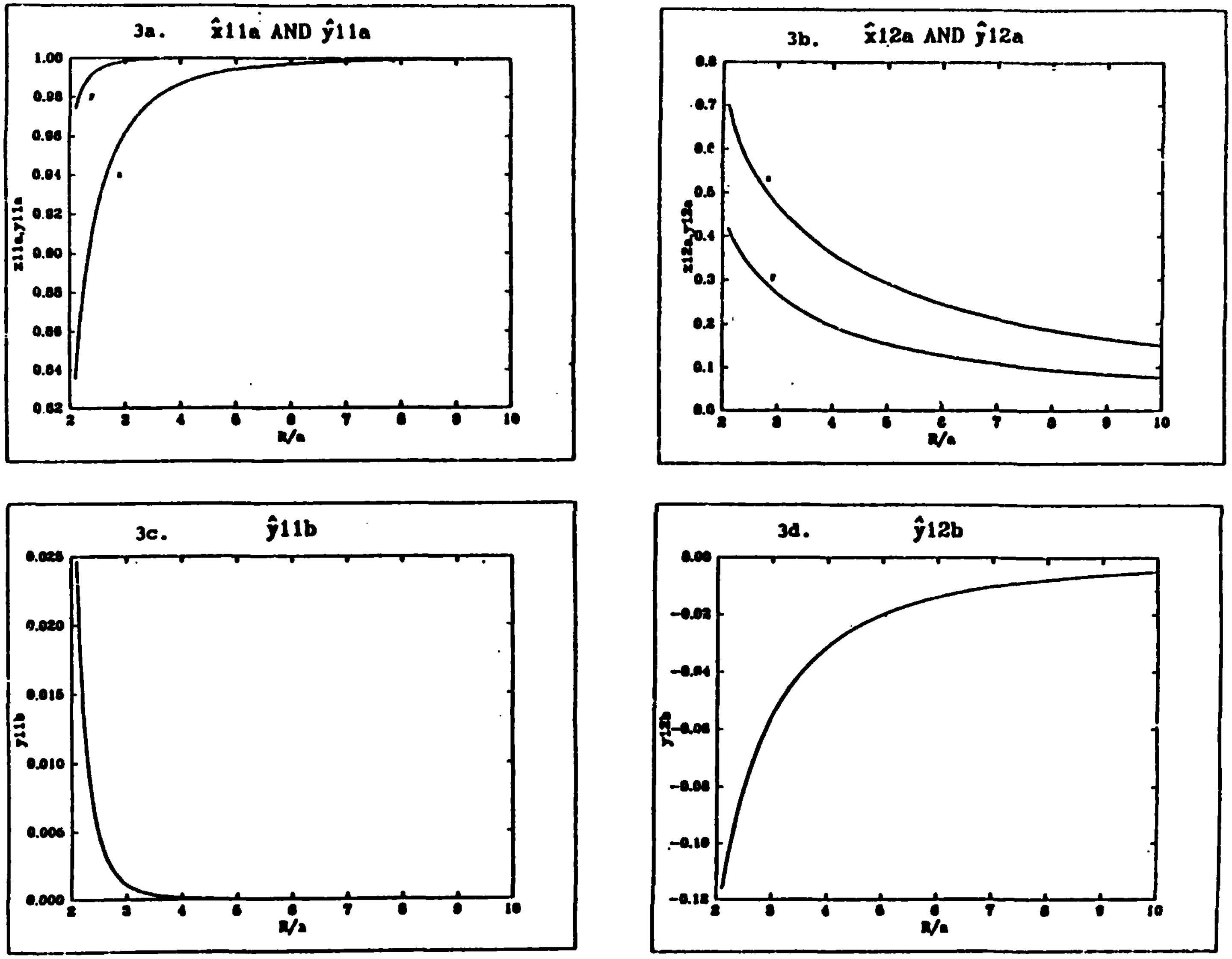
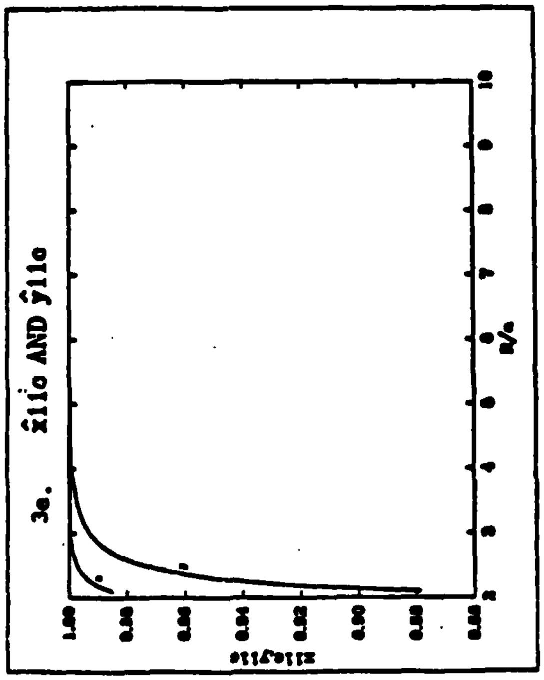
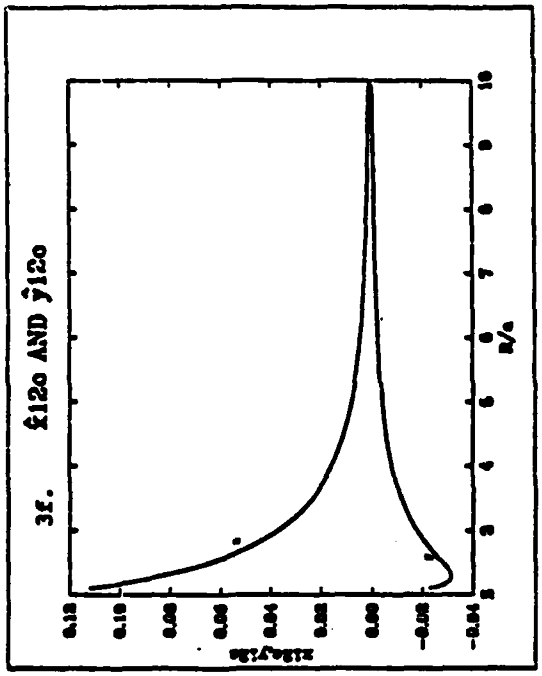
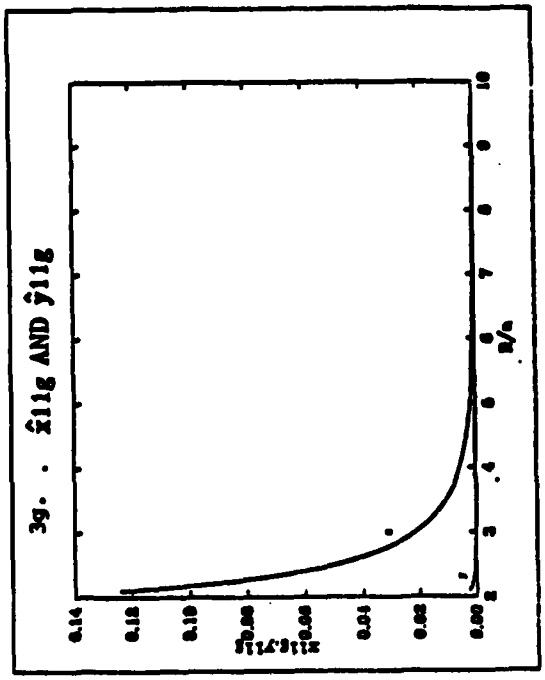
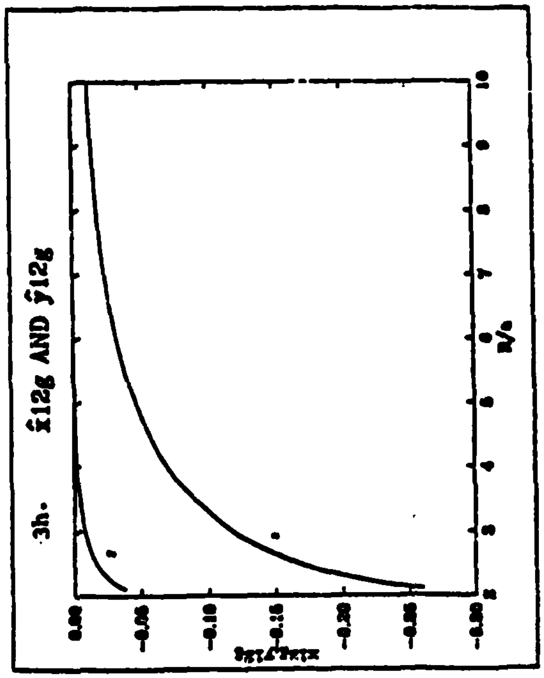


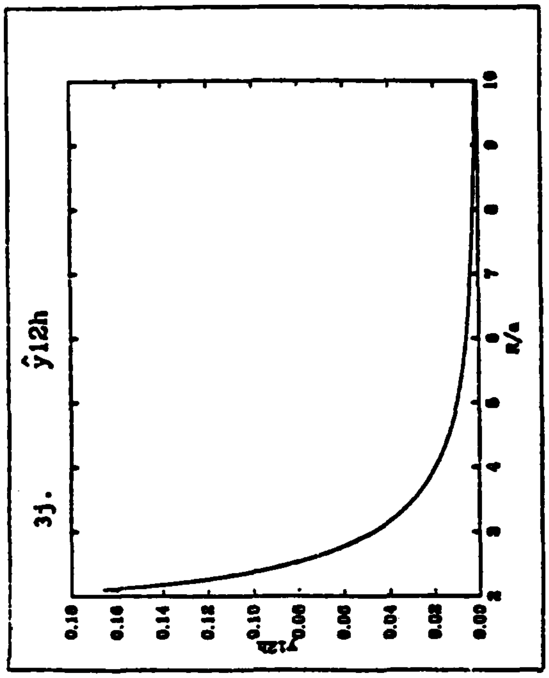
Figure 3. The mobility functions vs. sphere-sphere separation.

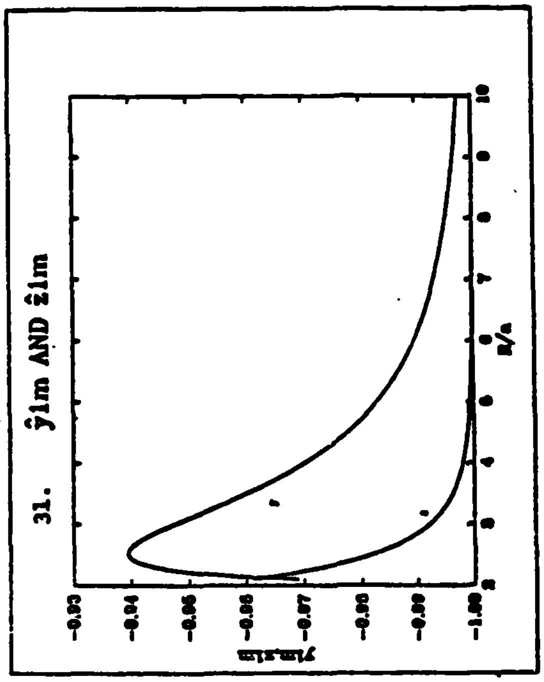

3e. $\hat{x}11o$ AND $\hat{y}11o$
R/a

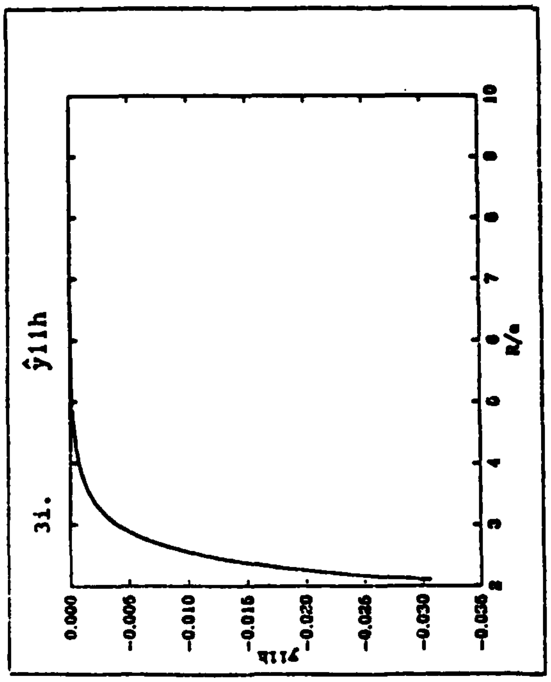

3f. $\hat{x}12o$ AND $\hat{y}12o$
R/a

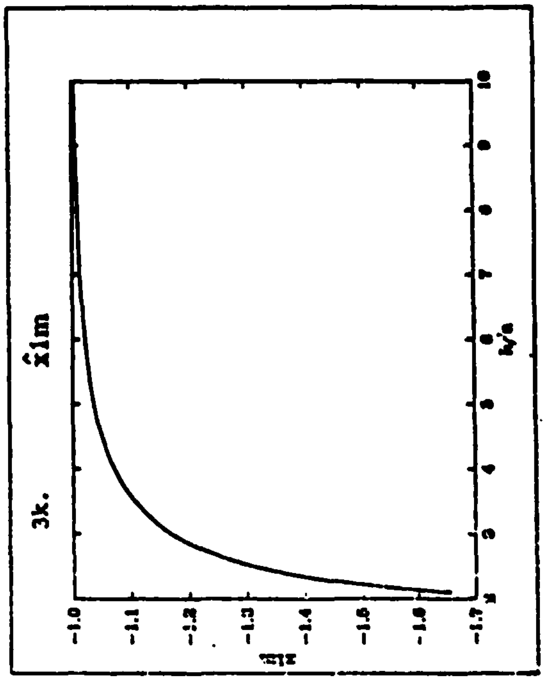

3g. $\hat{x}11g$ AND $\hat{y}11g$
R/a


3h. $\hat{x}12g$ AND $\hat{y}12g$
R/a


3j.
ŷ12h
R/a


3l.
ŷ1m AND ẑ1m
R/a


3l.
ŷ11h
R/a

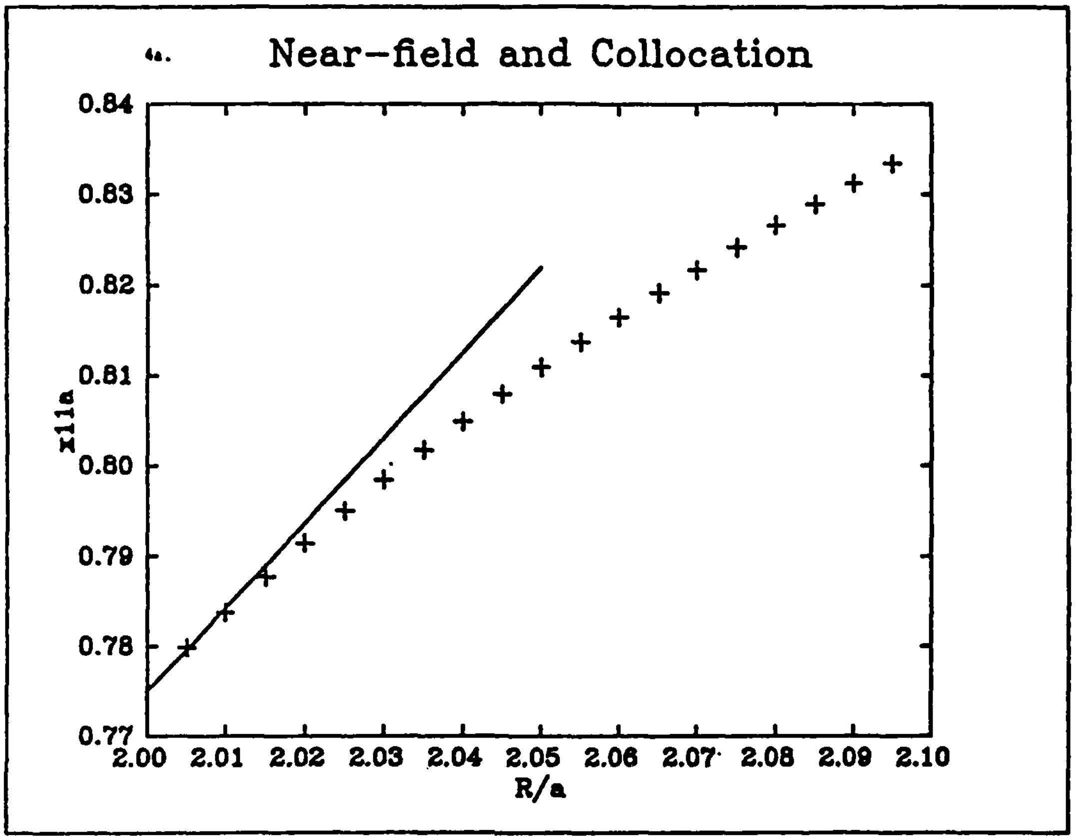

3k.
x̂1m
R/a

results of Jeffrey and Onishi (1984). The collocation results matched those obtained from a fifteen-term form of their expansion solution. The agreement was exact (in the sense that it was limited only by machine roundoff errors e.g. over 12 significant figures) except at the small separations mentioned above, where both techniques require special modifications.

Accurate results are also available for certain combinations of the $\underline{g}$ functions. The relative velocity between two spheres in a shear field which was obtained by Lin, Lee and Sather (1970) using bispherical coordinates furnishes a test for $x^g_{11}-x^g_{12}$ and $y^g_{11}-y^g_{12}$. Here again, agreement was obtained to all significant figures presented in the earlier work.

In the near-field, the collocation results for $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{a}$, $\underline{b}$, and $\underline{c}$ were compared with the near-field lubrication solutions. The latter are collected from the literature and presented in Jeffrey and Onishi (1984). Plots for the mobility functions presented in Figures 4a-j show that the two solutions match in an overlap region, but also show that the collocation technique, at least in its current form, cannot "turn the sharp corner" in the y functions.

In conclusion, we believe that we have successfully calculated, using the boundary collocation technique, the complete set of resistance and mobility functions required for determining the force, torque, stresslet on and motions of two equal sized spheres in an ambient velocity composed of a uniform stream and linear field. The results are accurate over all separations except at almost-touching ($R/a < 2.01$). In particular, this report provides an accurate algorithm for the computation of the stresslet functions over a wide range of sphere-sphere separations.



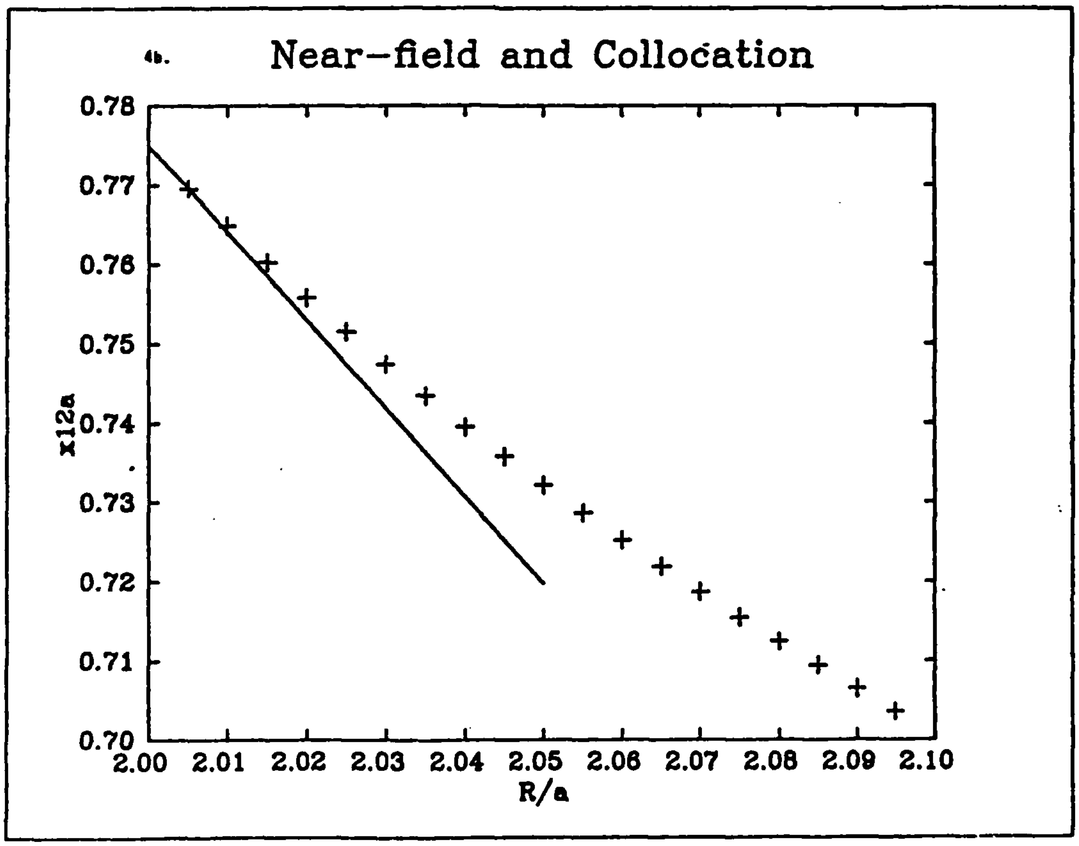
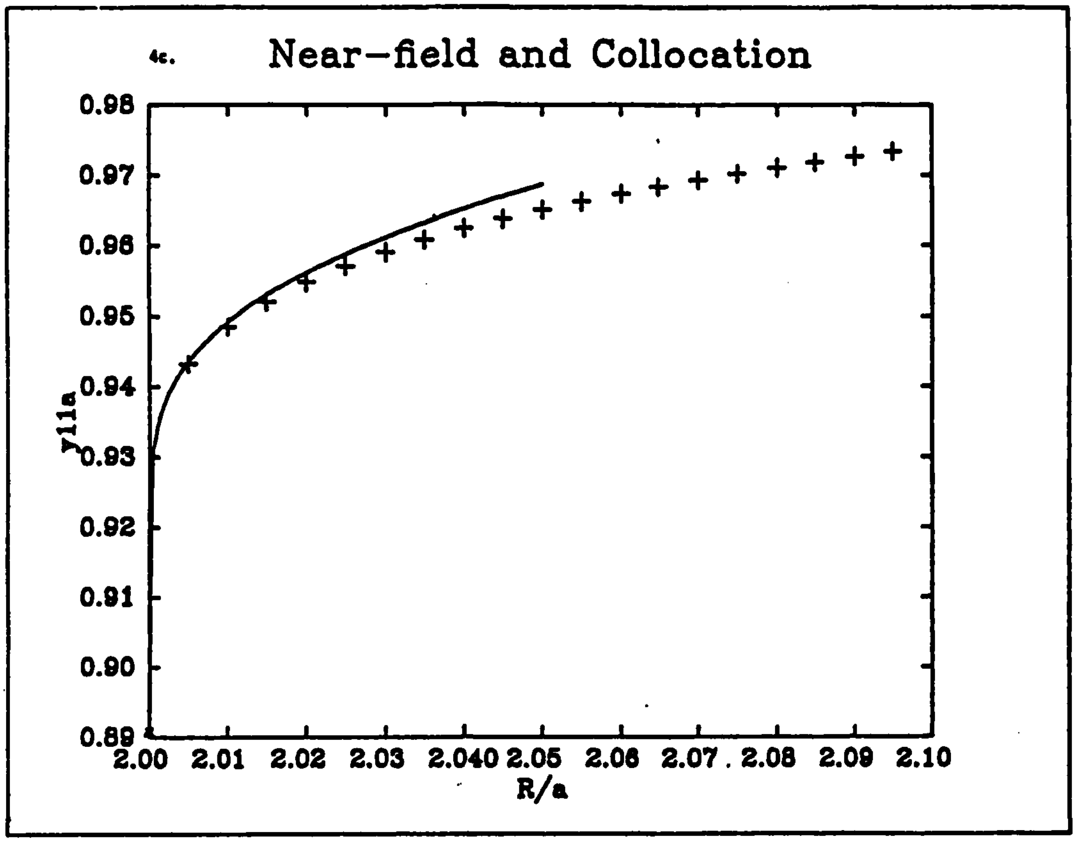
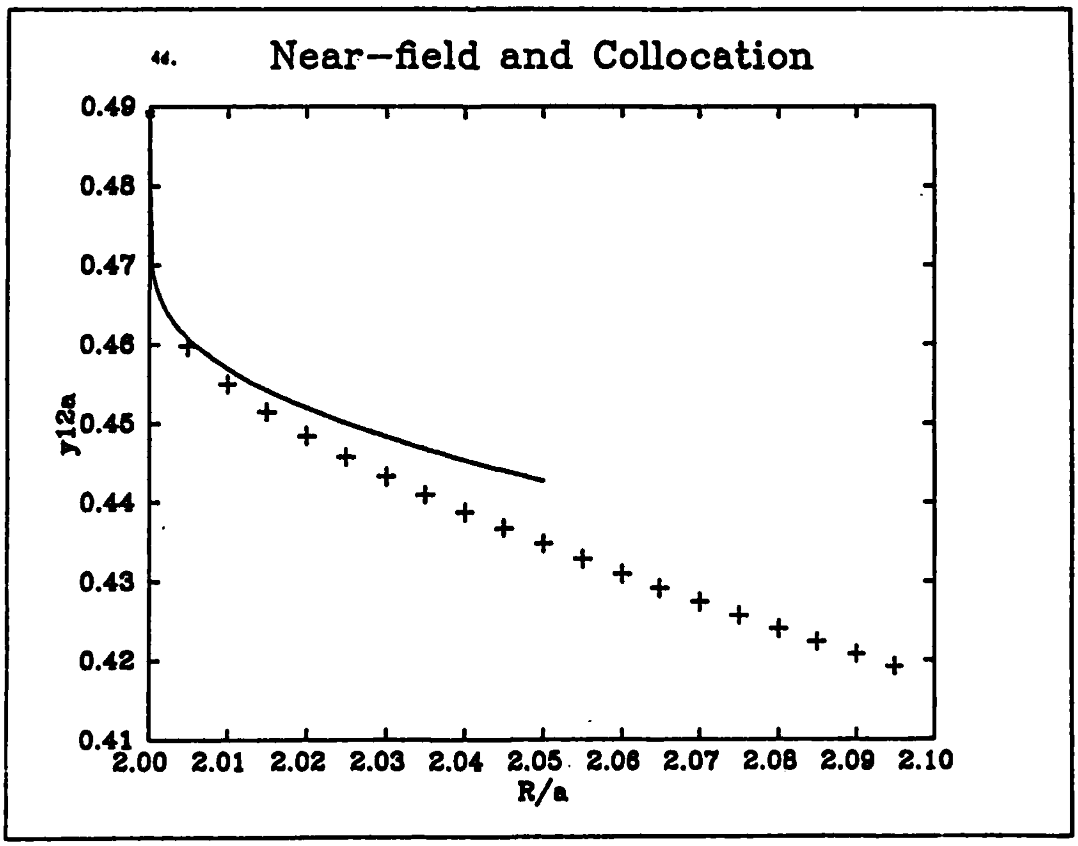


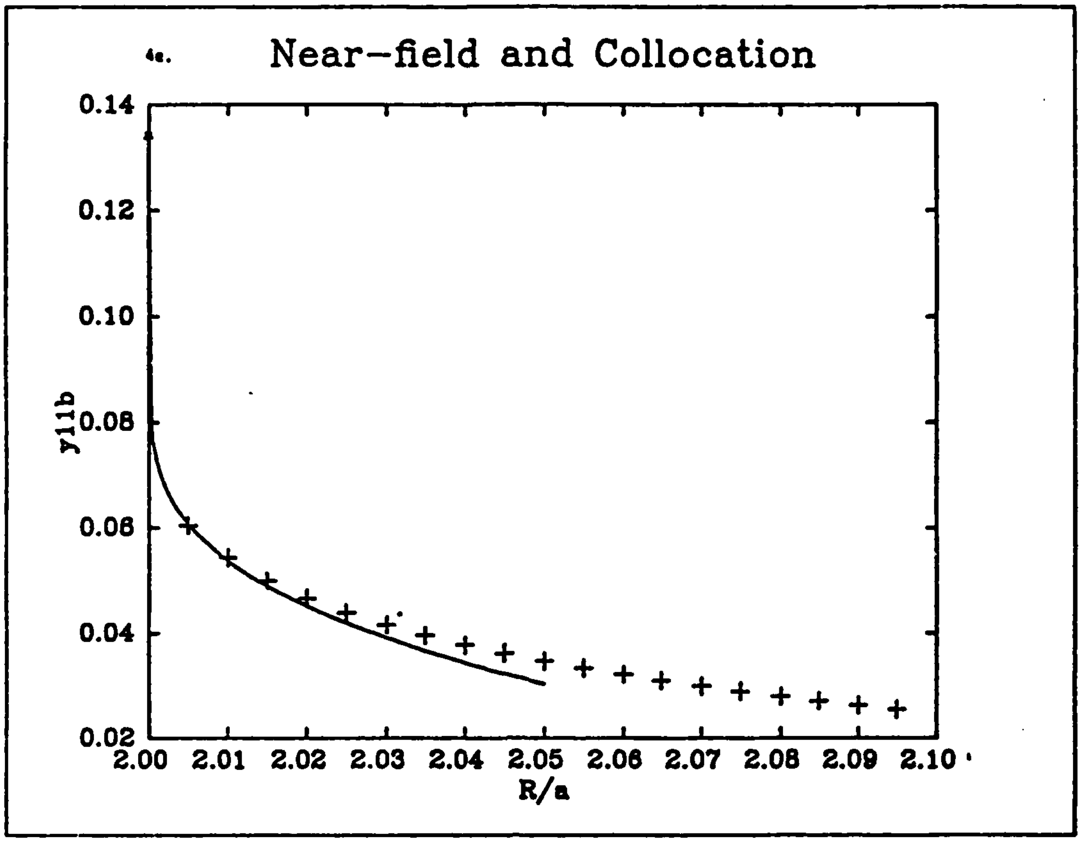
Figure 4. Comparison of the collocation and near-field solutions.

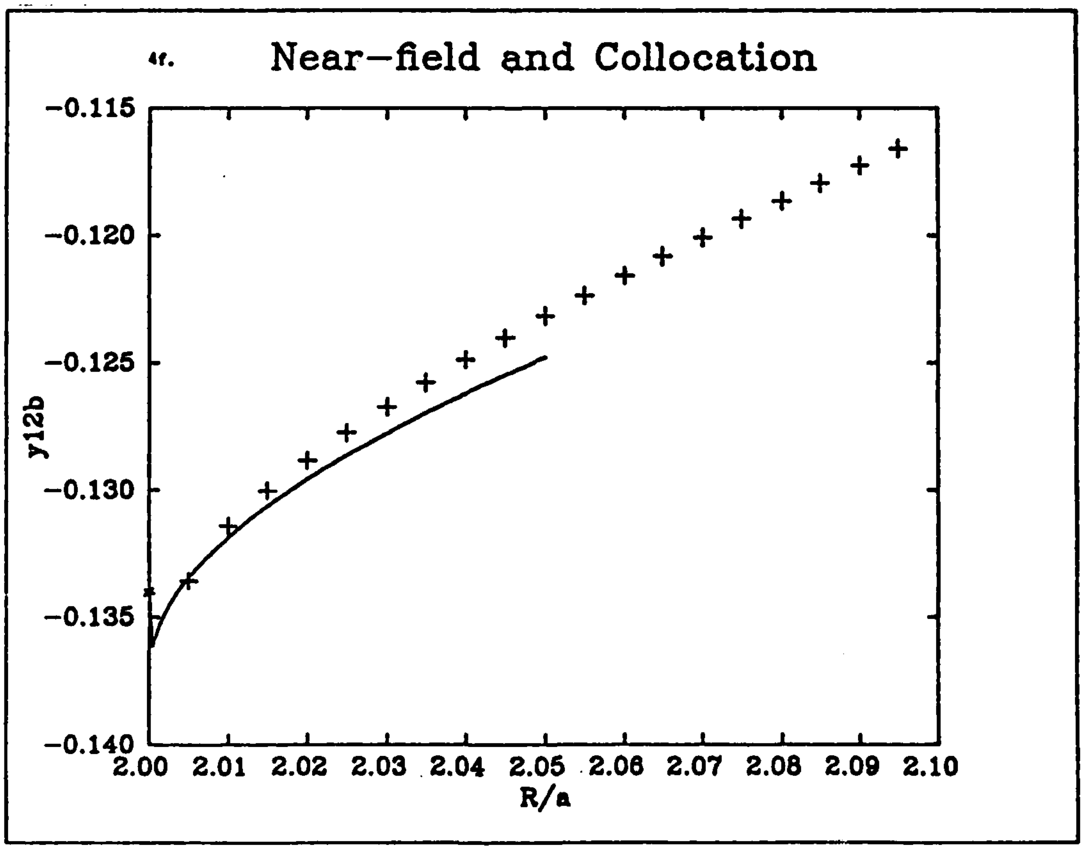


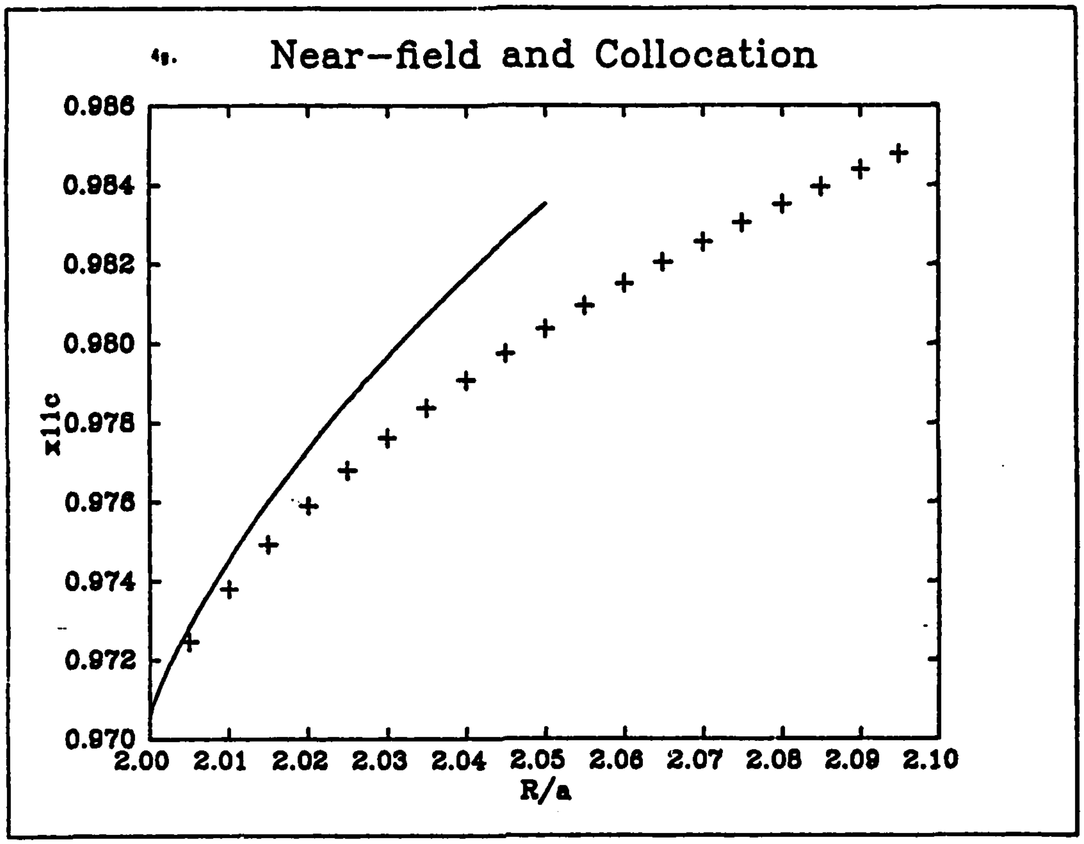
4c.
Near-field and Collocation
y11a
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
0.90
0.89
2.00
2.01
2.02
2.03
2.040
2.05
2.06
2.07
2.08
2.09
2.10
R/a

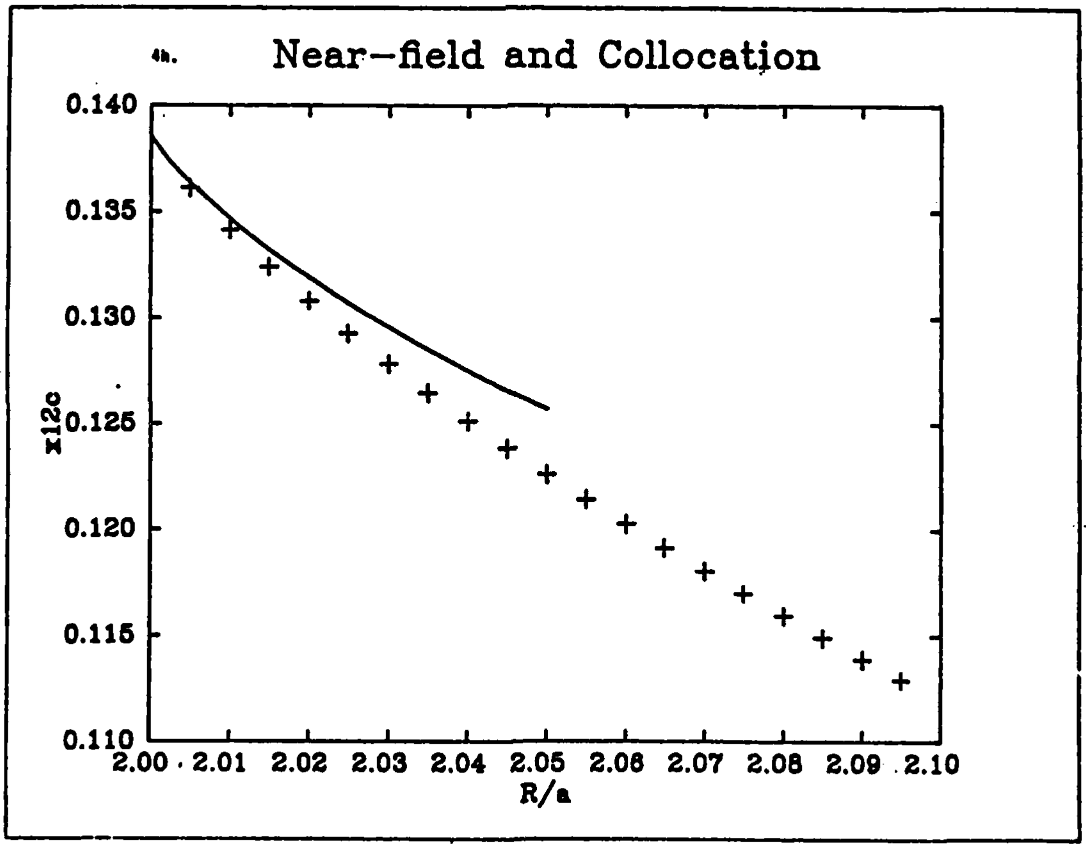


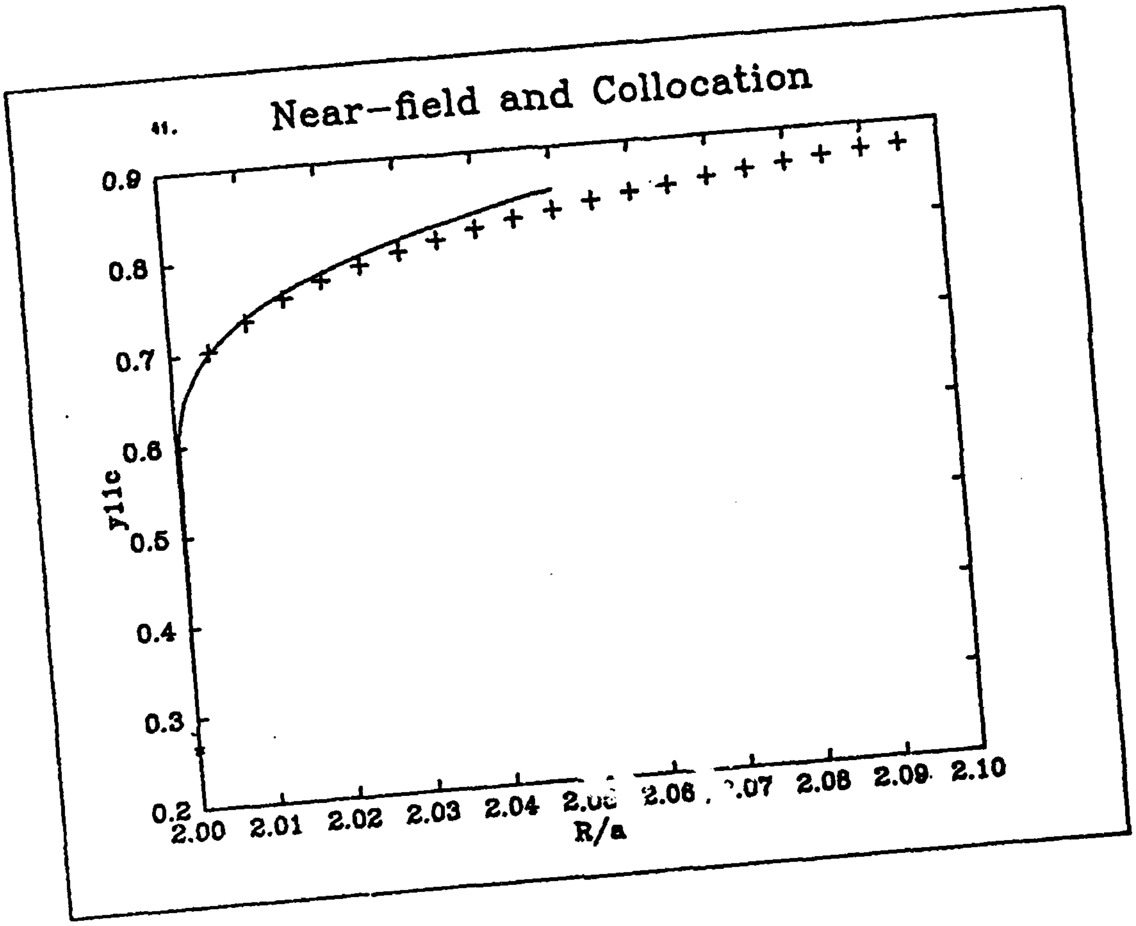
4d.
Near-field and Collocation
y12a
0.49
0.48
0.47
0.46
0.45
0.44
0.43
0.42
0.41
2.00
2.01
2.02
2.03
2.04
2.05
2.06
2.07
2.08
2.09
2.10
R/a

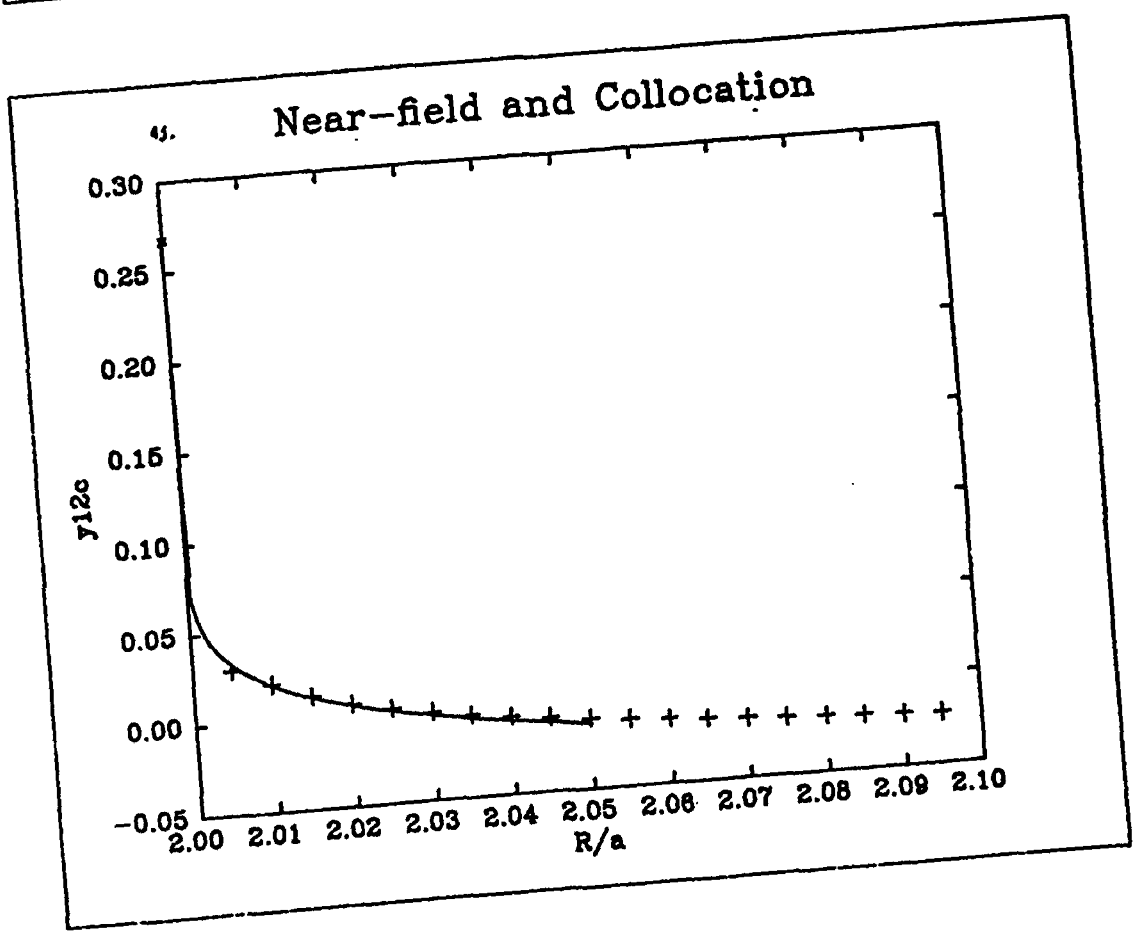

4e.
Near-field and Collocation
y11b
0.14
0.12
0.10
0.08
0.06
0.04
0.02
2.00
2.01
2.02
2.03
2.04
2.05
2.06
2.07
2.08
2.09
2.10
R/a


4f.
Near-field and Collocation
y12b
−0.115
−0.120
−0.125
−0.130
−0.135
−0.140
2.00
2.01
2.02
2.03
2.04
2.05
2.06
2.07
2.08
2.09
2.10
R/a


4g.
Near-field and Collocation
0.986
0.984
0.982
0.980
0.978
0.976
0.974
0.972
0.970
x11c
2.00 2.01 2.02 2.03 2.04 2.05 2.06 2.07 2.08 2.09 2.10
R/a


4h.
Near-field and Collocation
0.140
0.135
0.130
0.125
0.120
0.115
0.110
x12c
2.00 2.01 2.02 2.03 2.04 2.05 2.06 2.07 2.08 2.09 2.10
R/a


Near-field and Collocation
y11c
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
2.00 2.01 2.02 2.03 2.04 2.06 2.08 2.09 2.10
R/a


Near-field and Collocation
y12c
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
2.00 2.01 2.02 2.03 2.04 2.05 2.06 2.07 2.08 2.09 2.10
R/a

5. Sample Applications

5.1 The Bulk Stress in a Suspension of Spheres to Order c^2

Batchelor and Green (1972b) have derived the following rigorous expression for the viscosity of a suspension of identical rigid spheres in a steady pure straining motion.

$$\mu_{eff}/\mu = 1 + \frac{5}{2}c + c^2\left\{\frac{5}{2} + \frac{15}{2}\int_2^\infty J(\zeta)q(\zeta)\zeta^2 d\zeta\right\}, \tag{5.1}$$

where $\zeta = R/a$ and $J(\zeta)$ and $q(\zeta)$ are determined from the mobility functions as

$$J(\zeta) = -1 - \frac{1}{5}\left\{\hat{x}_1^m + 2\hat{y}_1^m + 2\hat{z}_1^m\right\}, \tag{5.2}$$

$$q(\zeta) = [1 - A(\zeta)]^{-1} \exp\left\{\int_\zeta^\infty \frac{3(B-A)}{x(1-A)}dx\right\}. \tag{5.3}$$

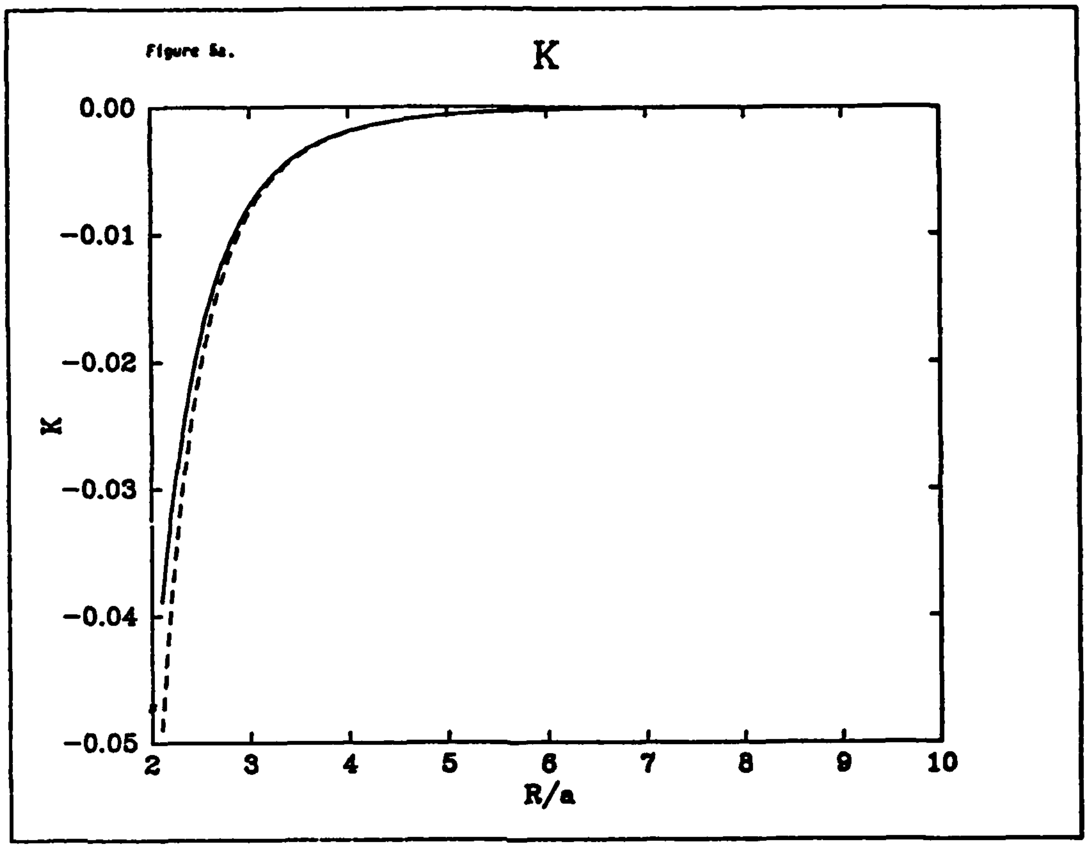
$A(\zeta)$ and $B(\zeta)$ are mobility functions that arise in the expression for the relative velocity between the two spheres and are as follows:

$$A(\zeta) = 4(\hat{x}_{11}^g - \hat{x}_{12}^g)/\zeta, \tag{5.4}$$

$$B(\zeta) = 8(\hat{y}_{11}^g - \hat{y}_{12}^g)/\zeta. \tag{5.5}$$

At the time of their work, information on J was limited to the far-field region (obtainable by the method of reflections) and the value at touching. As mentioned in their paper, the interpolation of the J curve in the region $2.0025<\zeta<3$ was the primary source of uncertainty in the final numerical result.

Figures 5a-5d show plots of J and also their stresslet functions K, L and M. The dashed line is the result obtained using the method of reflections. The "x" at R/a = 2 indicates the values for touching spheres. As is the case



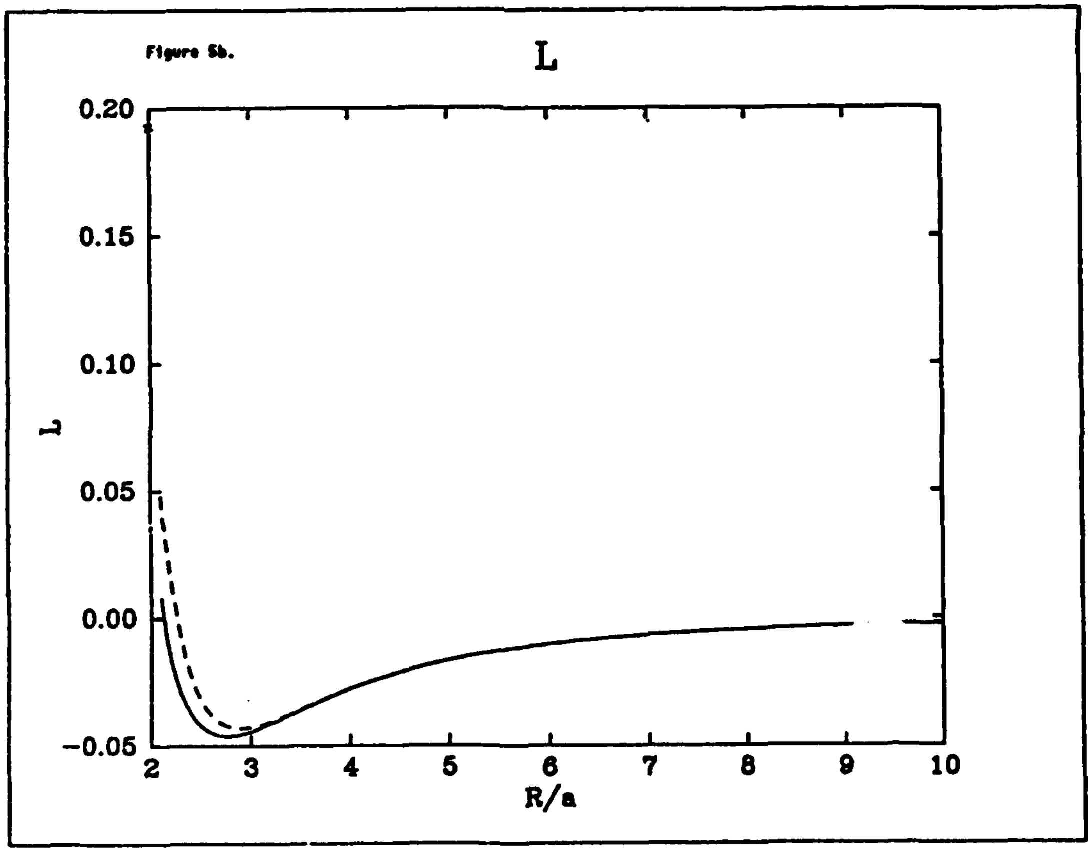


Figure 5. Comparison of the collocation and method of reflection solutions - for the stresslet functions of Batchelor and Green.

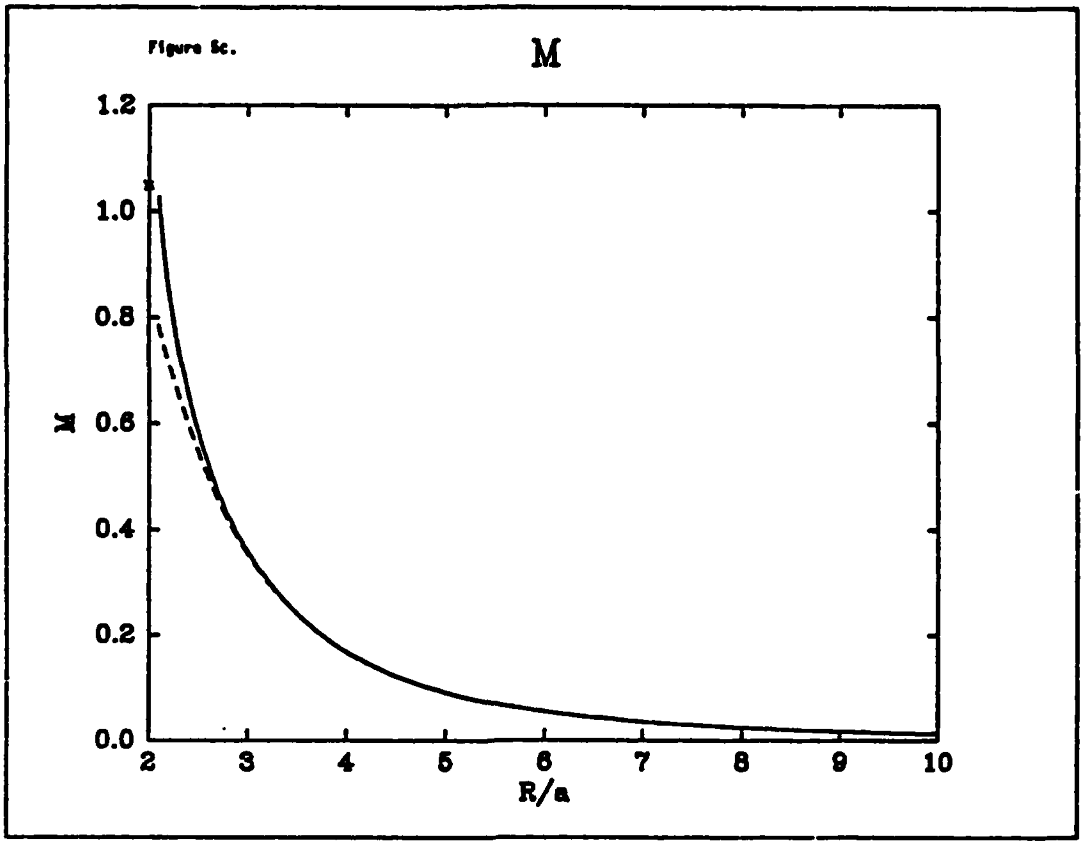


Figure 5c.
M
1.2
1.0
0.8
M
0.6
0.4
0.2
0.0
2
3
4
5
6
7
8
9
10
R/a

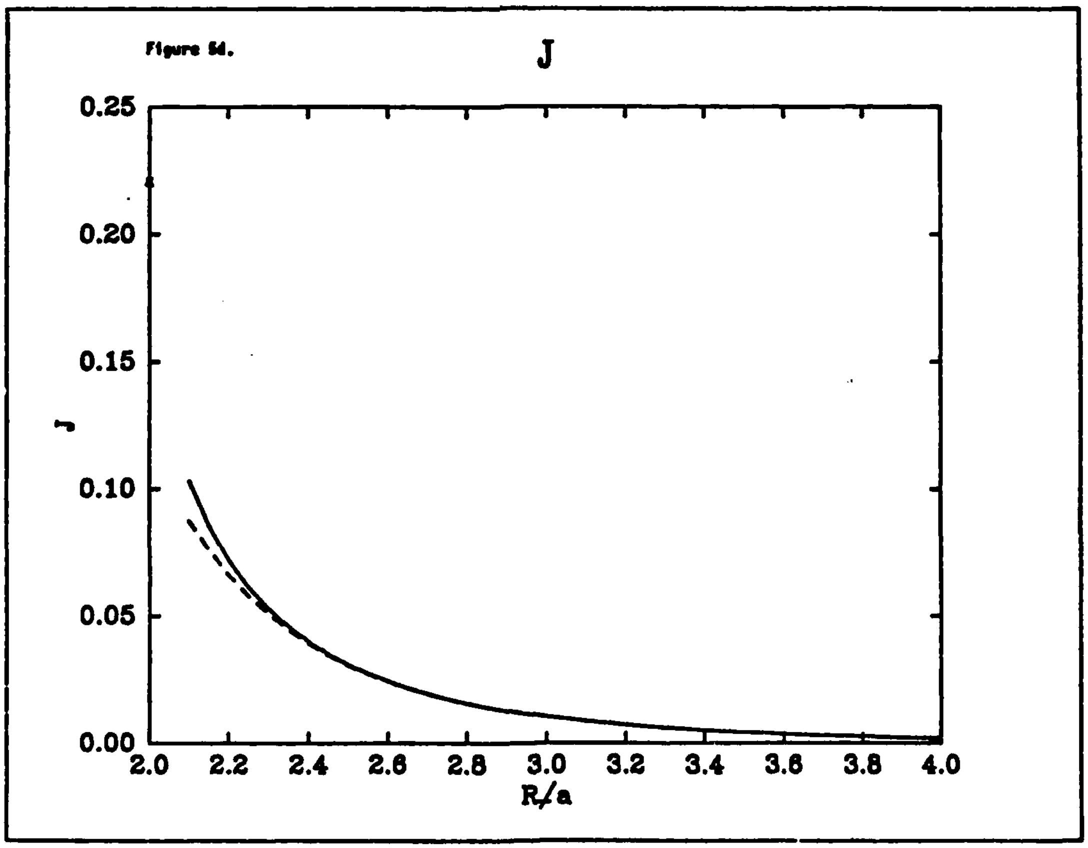


Figure 5d.
J
0.25
0.20
0.15
J
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
R/a

with many of the mobility problems, we see that the method of reflections result is accurate to quite small separations. The explanation is that in mobility problems, the leading terms in the higher order reflections are from the relatively weak dipole-dipole interactions. We note that L and M are not monotonic.

When (5.1) is evaluated with the collocation result, we find that the integral is less than the earlier estimate. Specifically, the integral from 2.0025 to 3.0 is found to be 0.384 instead of 0.449. Thus the coefficient of the c^2 term is 7.1 instead of 7.6, i.e.,

$$\mu_{eff}/\mu = 1 + \frac{5}{2}c + 7.1c^2.$$

The new result is within the ±10% uncertainty bound stated by Batchelor and Green.

5.2 The motion of a rigid dumbbell in a shear-field

Rigid dumbbells have been used to model suspensions of polymers with stiff backbones (see Bird et. al. 1977). In their models the hydrodynamic problem is simplified by assuming that the dumbbell consists of two point forces connected by a rigid rod. (The rod has no hydrodynamic resistance). In this section, we show that if one replaces the point forces with spheres, the necessary hydrodynamic functions can be extracted from our two-sphere functions.

The motion of a neutrally buoyant, axisymmetric particle in a shear field was completely solved by Bretherton (1962) who showed that the motion of almost all axisymmetric particles was the same as that of some ellipsoid of revolution. Thus, for our dumbbells, we use our two-sphere functions to find the "effective spheroid".

The geometry of a symmetric dumbbell is completely determined by the center-to-center separation, R and sphere radii, a. Furthermore, if all distances in the problem are scaled with "a", then the possible dumbbell shapes are spanned by varying R/a over [2,∞). If the dumbbell rotates about its center of mass as $\underline{\omega}\times\underline{x}$, then the spheres move as:

$$\underline{U}_\alpha = \underline{U} + \underline{\omega}\times\underline{x}_\alpha, \quad \alpha=1,2. \tag{5.6}$$

We can now write the forces and torques on each sphere in terms of the two-sphere resistance functions. The torque on the dumbbell is the sum of the torques on each sphere plus the couple from the forces on the spheres. After some algebra, we arrive at:

$$T_i = C_{ij}(\omega_j-\Omega_j) - \tilde{H}_{ijk}E_{jk}, \tag{5.7}$$

where the dumbbell resistance functions can be written in terms of the two-sphere resistance functions as:

$$C_{ij} = X^C d_i d_j + Y^C(\delta_{ij} - d_i d_j). \tag{5.8}$$

with $\quad X^C = 2(X^C_{11}+X^C_{12})$

and $\quad Y^C = \frac{1}{2}R^2(Y^A_{11}-Y^A_{12}) + 2R(Y^B_{11}-Y^B_{12}) + 2(Y^C_{11}+Y^C_{12}).$

$$H_{ijk} = Y^H(d_j\epsilon_{kil}+d_k\epsilon_{jil})d_l. \tag{5.9}$$

with $\quad Y^H = \frac{1}{4}R^2(Y^A_{11}-Y^A_{12}) + \frac{1}{2}R(Y^B_{11}-Y^B_{12}) - R(Y^G_{11}-Y^G_{12}) + 2(Y^H_{11}+Y^H_{12}).$

If we set $\underset{\sim}{T} = \underset{\sim}{0}$ in (5.7), we can solve for the angular velocity as:

$$\underset{\sim}{\omega} = \underset{\sim}{\Omega} + \{2Y^H/Y^C\}\underset{\sim}{d}\times(\underset{\approx}{E}\cdot\underset{\sim}{d}). \tag{5.10}$$

A prolate spheroid with aspect ratio ρ rotates as given by (5.10), but with $2Y^H/Y^C$ replaced by $(\rho^2-1)/(\rho^2+1)$. Therefore, the dumbbell moves like a prolate spheroid with the aspect ratio,

$$\rho = [(1 + 2Y^H/Y^C)/(1 - 2Y^H/Y^C)]^{1/2}. \tag{5.11}$$

Plots of the "equivalent-spheroid" aspect ratio vs. the dumbbell shape parameter, R/a are shown in Figure 6.

Acknowledgements:

We thank Professor David Jeffrey (McGill) for making available early versions of his papers.

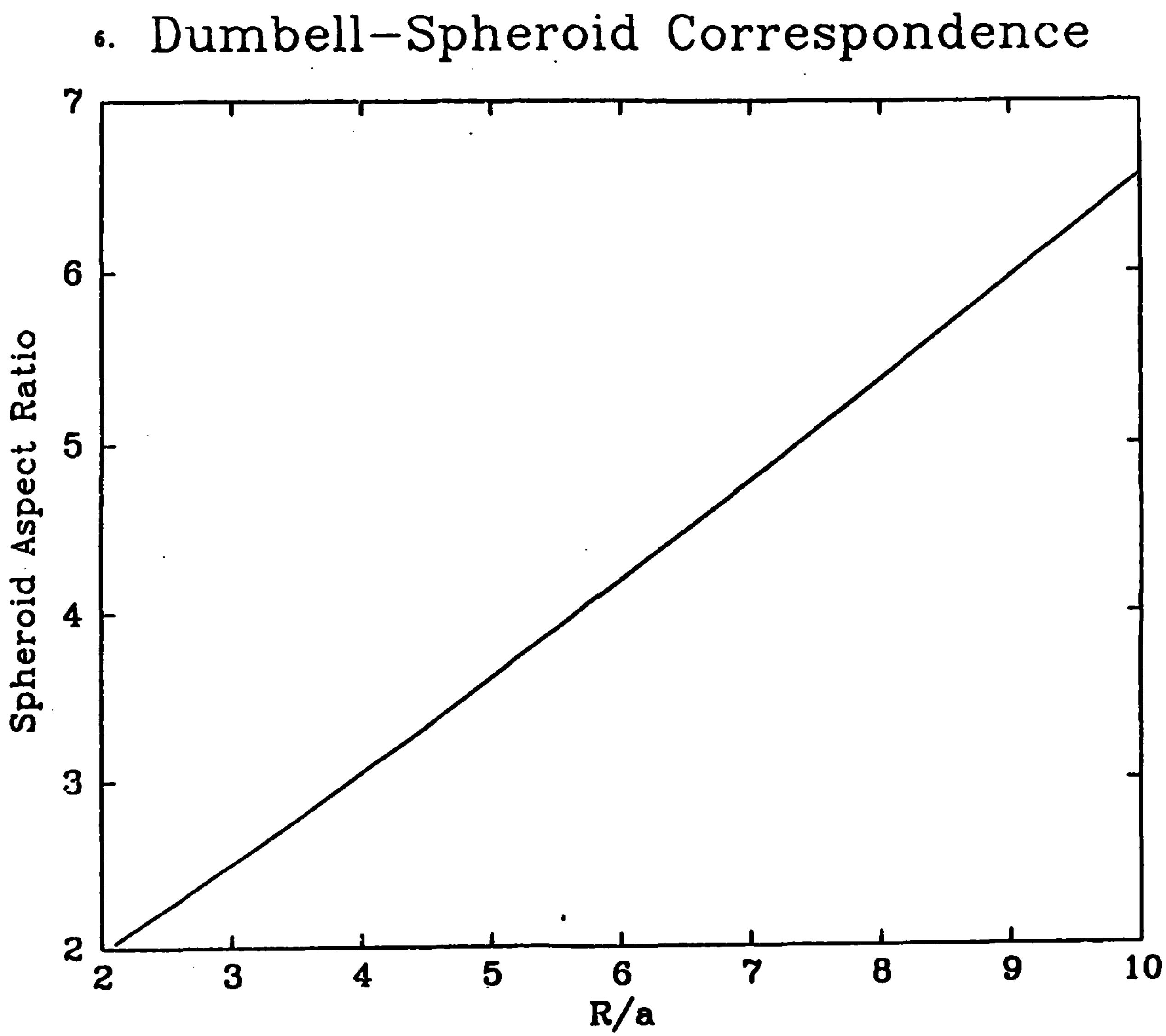


Figure 6. Rotation of a rigid dumbbell in s shear field.
Aspect ratio of the equivalent prolate spheroid.

Appendix

1. Notes on the Computer Programs

The following section contains listings of five programs. The "main" program, MANDR.FOR requires only two inputs: the number of collocation points and the scaled separation between the spheres, R/a. MANDR.FOR occupies five pages and this length may give the wrong impression that the algorithm is involved. However this program merely sets up the appropriate inputs for the entire collection of resistance functions and passes them to subroutine STOKES.FOR. The heart of the algorithm is contained in STOKES.FOR, which occupies only a single page. If one were interested in a particular resistance function, the correct set of input parameters for STOKES.FOR may be deduced by examining the appropriate module in the main program.

As mentioned in Section 2, axisymmetric problems reduce to smaller systems. In axisymmetric translational and straining problems, the resulting 2N×2N system consists of the upper-left blocks of the general system. Consequently, in STOKES.FOR we need only insert the line:

```
IF (M .EQ. 0) IDIM = 2*NPT
```

before calling the inversion (LINPAK) subroutines.

In axisymmetric swirl problems, the N×N system is not located in the upper-left portion of the general system. Consequently, the appropriate block cannot be obtained as easily from the general case. We have taken the simple remedy of taking the 2-3 block out of STOKES.FOR and creating the Degenerate subroutine, STOKESD.FOR.

The following parameters are passed in and out of STOKES.FOR:

R: Sphere-sphere separation (scaled by sphere radius).

NSTAR: Order of the lowest non-zero multipole. (e.g. if the hydrodynamic force on the sphere is nonzero, NSTAR=1. If the force is zero, then NSTAR is 2 in our application because of the dipole moment.

M: Parameter m from the resistance problem.

SGN: Symmetry parameter S.

NPT: Number of collocation points (N in the text).

RHS: One dimensional array which contains the RHS of the system of equations. It originates from the surface velocity evaluated at the N collocation points.

AMN: One dimensional output array of the a_{mn} (with n = NSTAR, ...).

BMN: One dimensional output array of the b_{mn} (with n = NSTAR, ...).

CMN: One dimensional output array of the c_{mn} (with n = NSTAR, ...).

Function subroutine PNS(N,M,X) calculates $P_n^m(X)[\sin\theta]^{-m}$. (PNS stands for P-No-Sine). PNSP(N,M,X) does the corresponding steps for $P_n^{m\prime}(X)$. Subroutine DMATIN.FOR is used when the mobility functions are calculated from the inverse of the resistance matrix.

If these programs are installed on another computer, the sample run provided after the program listings may be used to test the installation.

```
C MANDR.FOR
C*************************************************************
C MOBILITY AND RESISTANCE FUNCTIONS FOR 2 IDENTICAL SPHERES    *
C     BY BOUNDARY COLLOCATION OF LAMB'S GENERAL SOLUTION       *
C                                                              *
C                         SANGTAE KIM                          *
C               DEPARTMENT OF CHEMICAL ENGINEERING             *
C          AND    MATHEMATICS RESEARCH CENTER                  *
C                  UNIVERSITY OF WISCONSIN                     *
C                                                              *
C                       RICHARD T. MIFFLIN                     *
C               DEPARTMENT OF CHEMICAL ENGINEERING             *
C                     PRINCETON UNIVERSITY                     *
C                                                              *
C VERSION 2: APRIL 24, 1984                                    *
C*************************************************************
C  PROGRAM WAS DEVELOPED AND TESTED ON A VAX 11/780, 1983-1984
C
C                            MAIN PROGRAM: MANDR
C            SUBROUTINES CALLED FROM MAIN: STOKES, STOKESD, DMATIN
C SUBROUTINES CALLED FROM STOKES, STOKESD: LINPAK ROUTINES DGECO,DGESL.
C                                          PNS,PNSP
C -----------------------------------------------------------------
C
C PROGRAM DESCRIPTION
C ------- -----------
C
C MAIN PROGRAM SETS PARAMETERS CORRESPONDING TO AMBIENT VELOCITY FIELD
C FOR EACH RESISTANCE FUNCTION. AFTER CALCULATION OF THE MULTIPOLES,
C MAIN ALSO SCALES THE RESULT ACCORDING TO THE NON-DIMENSIONALIZATION
C FOR EACH RESISTANCE FUNCTION. MAIN CALCULATES THE MOBILITY FUNCTION
C BY INVERTING THE GRAND RESISTANCE MATRIX.
C
C GIVEN THE PARAMETERS FOR THE AMBIENT VELOCITY, SUBROUTINE STOKES
C RETURNS MULTIPOLES COEFFICIENTS. THE SYSTEM OF EQUATIONS IN THIS
C SUBROUTINE ARE OBTAINED BY APPLYING THE BOUNDARY CONDITIONS AT EACH
C COLLOCATION POINT.
C
C SUBROUTINE STOKESD IS A SUBSET OF STOKES AND IS USED FOR THE
C DEGENERATE CASES INVOLVING THE RESISTANCE FUNCTIONS X11C AND X12C.
C
C PNS AND PNSP ARE DERIVED FROM THE ASSOCIATED LEGENDRE FUNCTIONS
C
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION RHS(300),AMN(100),BMN(100),CMN(100),
     *          COS1(100),AY(4,4),DUMMY(4,1)
C
C READ IN:
C  1) NUMBER OF COLLOCATION POINTS (EVEN INTEGER): IPTS
C  2) CENTER TO CENTER SEPARATION BETWEEN SPHERES: R
C
      READ (40,5) IPTS,R
    5 FORMAT (I10,F10.0)
C
C SET COLLOCATION POINTS:
C
      PI = 3.14159265358979D0
      DIPTS = DFLOAT(IPTS-1)
      DO 10 I=1,IPTS
      THETA = DFLOAT(I-1)/DIPTS*PI
   10 COS1(I) = DCOS(THETA)
C
C
```

```
C
C ********** CALCULATION OF X11A, X12A, X11G, X12G **********
C
      DO 20 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 1.D0
      RHS(I2) = 0.D0
   20 RHS(I3) = 0.D0
      CALL STOKES(R,1,0,-1.D0,IPTS,RHS,AMN,BMN,CMN)
      T1 = 2.D0/3.D0*AMN(1)
      T3 = -.5D0*AMN(2)
C
      DO 30 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 1.D0
      RHS(I2) = 0.D0
   30 RHS(I3) = 0.D0
      CALL STOKES(R,1,0,1.D0,IPTS,RHS,AMN,BMN,CMN)
      T2 = 2.D0/3.D0*AMN(1)
      T4 = -.5D0*AMN(2)
C
      X11A = .5D0*(T1 + T2)
      X12A = .5D0*(T1 - T2)
      X11G = .5D0*(T3 + T4)
      X12G = .5D0*(T3 - T4)
C
C
C ****** CALCULATION OF Y11A, Y12A, Y11B, Y12B, Y11G, Y12G ******
C
      DO 40 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 0.D0
      RHS(I2) = 0.D0
   40 RHS(I3) = 1.D0
      CALL STOKES(R,1,1,-1.D0,IPTS,RHS,AMN,BMN,CMN)
      T1 = 2.D0/3.D0*AMN(1)
      T3 = 2.D0*CMN(1)
      T5 = -.5D0*AMN(2)
C
      DO 50 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 0.D0
      RHS(I2) = 0.D0
   50 RHS(I3) = 1.D0
      CALL STOKES(R,1,1,1.D0,IPTS,RHS,AMN,BMN,CMN)
      T2 = 2.D0/3.D0*AMN(1)
      T4 = 2.D0*CMN(1)
      T6 = -.5D0*AMN(2)
C
      Y11A = .5D0*(T1 + T2)
      Y12A = .5D0*(T2 - T1)
      Y11B = -.5D0*(T3 + T4)
      Y12B = .5D0*(T3 - T4)
      Y11G = .5D0*(T5 + T6)
      Y12G = .5D0*(T6 - T5)
C
C
C
C
```

```
C
C ************** CALCULATION OF X11C, X12C **************
C
      DO 60 I=1,IPTS
   60 RHS(I) = -1.D0
      CALL STOKESD(R,-1.D0,IPTS,RHS,CMN)
      T1 = CMN(1)
C
      DO 70 I=1,IPTS
   70 RHS(I) = -1.D0
      CALL STOKESD(R,1.D0,IPTS,RHS,CMN)
      T2 = CMN(1)
C
      X11C = .5D0*(T1 + T2)
      X12C = -.5D0*(T1 - T2)
C
C ********** CALCULATION OF Y11C, Y12C, Y11H, Y12H **********
C
      DO 80 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 1.D0
      RHS(I2) = 0.D0
   80 RHS(I3) = -COS1(I)
      CALL STOKES(R,1,1,-1.D0,IPTS,RHS,AMN,BMN,CMN)
      T1 = 2.D0/3.D0*AMN(1)
      T3 = CMN(1)
      T5 = -.25D0*AMN(2)
C
      DO 90 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 1.D0
      RHS(I2) = 0.D0
   90 RHS(I3) = -COS1(I)
      CALL STOKES(R,1,1,1.D0,IPTS,RHS,AMN,BMN,CMN)
      T2 = 2.D0/3.D0*AMN(1)
      T4 = CMN(1)
      T6 = -.25D0*AMN(2)
      Y11C = .5D0*(T3 + T4)
      Y12C = .5D0*(T3 - T4)
      Y11H = .5D0*(T5 + T6)
      Y12H = .5D0*(T5 - T6)
C
C ********** CALCULATION OF X11M+X12M, Y11+Y12M, Z11+Z12M **********
C
      DO 100 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 2.D0*COS1(I)
      RHS(I2) = -1.D0
  100 RHS(I3) = 0.D0
      CALL STOKES(R,1,0,1.D0,IPTS,RHS,AMN,BMN,CMN)
      X1112M = .1D0*AMN(2)
C
      DO 110 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 3.D0
      RHS(I2) = 0.D0
  110 RHS(I3) = 3.D0*COS1(I)
      CALL STOKES(R,1,1,-1.D0,IPTS,RHS,AMN,BMN,CMN)
      Y1112M = .1D0*AMN(2)
```

```
C
      DO 120 I=1,IPTS
      I2 = I+IPTS
      I3 = I2+IPTS
      RHS(I) = 0.D0
      RHS(I2) = 0.D0
  120 RHS(I3) = 6.D0
      CALL STOKES(R,2,2,1.D0,IPTS,RHS,AMN,BMN,CMN)
      Z1112M = .1D0*AMN(1)
C
C ———— END OF CALCULATIONS FOR THE RESISTANCE FUNCTIONS ————
C
      WRITE (41,500) IPTS,R
      WRITE (41,510)
      WRITE (41,520) X11A,Y11A,X12A,Y12A
      WRITE (41,530) Y11B,Y12B
      WRITE (41,540) X11C,Y11C,X12C,Y12C
      WRITE (41,550) X11G,Y11G,X12G,Y12G
      WRITE (41,560) Y11H,Y12H
      WRITE (41,570) X1112M,Y1112M,Z1112M
C
C        PART II OF THE PROGRAM CALCULATES MOBILITY FUNCTIONS
C        ____________________________________________________
C
      Y11A = 6.D0*Y11A
      Y12A = 6.D0*Y12A
      Y11B = 4.D0*Y11B
      Y12B = 4.D0*Y12B
      Y11C = 8.D0*Y11C
      Y12C = 8.D0*Y12C
C
      AY(1,1) = Y11A
      AY(1,2) = Y12A
      AY(2,1) = Y12A
      AY(2,2) = Y11A
C
      AY(3,1) = Y11B
      AY(3,2) = Y12B
      AY(4,1) = -Y12B
      AY(4,2) = -Y11B
C
      AY(1,3) = Y11B
      AY(1,4) = -Y12B
      AY(2,3) = Y12B
      AY(2,4) = -Y11B
C
      AY(3,3) = Y11C
      AY(3,4) = Y12C
      AY(4,3) = Y12C
      AY(4,4) = Y11C
C
      DO 700 K=1,4
  700 DUMMY(K,1) = 0.D0
      CALL DMATIN(AY,4,DUMMY,1,DETERM,4)
C
      Y11AM = AY(1,1)*6.D0
      Y12AM = AY(1,2)*6.D0
      Y11BM = AY(3,1)*4.D0
      Y12BM = AY(3,2)*4.D0
      Y11CM = AY(3,3)*8.D0
      Y12CM = AY(3,4)*8.D0
C
C
```

```
C
      X11AM = X11A/(X11A*X11A - X12A*X12A)
      X12AM = -X12A/(X11A*X11A - X12A*X12A)
      X11CM = X11C/(X11C*X11C - X12C*X12C)
      X12CM = -X12C/(X11C*X11C - X12C*X12C)
C
      Y11HM = -Y11G*Y11BM - Y12G*Y12BM + Y11H*Y11CM + Y12H*Y12CM
      Y12HM = Y11G*Y12BM + Y12G*Y11BM + Y11H*Y12CM + Y12H*Y11CM
C
      X11GM = (X11G*X11AM + X12G*X12AM)/3.D0
      X12GM = (X11G*X12AM + X12G*X11AM)/3.D0
      Y11GM = (Y11G*Y11AM + Y12G*Y12AM)/3.D0 -Y11H*Y11BM+Y12H*Y12BM
      Y12GM = (Y11G*Y12AM + Y12G*Y11AM)/3.D0 -Y11H*Y12BM+Y12H*Y11BM
C
      X1MM = -X1112M - .8D0*(X11G-X12G)*(X12GM-X11GM)
      Y1MM = -Y1112M - 2.4D0*(Y11G-Y12G)*(Y12GM-Y11GM)
     *               + 2.4D0*(Y11H+Y12H)*(Y11HM+Y12HM)
      Z1MM = -Z1112M
C
C ---------- END OF CALCULATIONS FOR THE MOBILITY FUNCTIONS ----------
C
C       COMPUTE BATCHELOR AND GREEN'S K, L, M AND J FUNCTIONS
C       -----------------------------------------------------
C
      BGK = -Z1MM - 1.D0
      BGL = -Y1MM - 1.D0 - BGK
      BGM = -1.5D0*( X1MM + 1.D0 + BGK + 4.D0*BGL/3.D0 )
      BGJ = -1.0 - 0.2*X1MM - 0.4*(Y1MM+Z1MM)
C
C
      WRITE (41,510)
      WRITE (41,620) X11AM,Y11AM,X12AM,Y12AM
      WRITE (41,630) Y11BM,Y12BM
      WRITE (41,640) X11CM,Y11CM,X12CM,Y12CM
      WRITE (41,650) X11GM,Y11GM,X12GM,Y12GM
      WRITE (41,660) Y11HM,Y12HM
      WRITE (41,670) X1MM,Y1MM,Z1MM
      WRITE (41,680) BGK,BGL,BGM,BGJ
C
  500 FORMAT (15X,'NUMBER OF POINTS: ',I4,10X,'R = ',F8.3,//)
  510 FORMAT (23X,'X11',11X,'Y11',11X,'X12',11X,'Y12',/)
  520 FORMAT (5X,'RES. FNCS. A',F11.6,3F14.6,/)
  530 FORMAT (5X,'RES. FNCS. B',F25.6,F28.6,/)
  540 FORMAT (5X,'RES. FNCS. C',F11.6,3F14.6,/)
  550 FORMAT (5X,'RES. FNCS. G',F11.6,3F14.6,/)
  560 FORMAT (5X,'RES. FNCS. H',F25.6,F28.6,/)
  570 FORMAT (5X,'RES. FCNS. X11M+X12M,Y11M+Y12M,Z11M+Z12M:',
     *        F11.6,2F10.6,//)
  620 FORMAT (5X,'MOB. FNCS. A',F11.6,3F14.6,/)
  630 FORMAT (5X,'MOB. FNCS. B',F25.6,F28.6,/)
  640 FORMAT (5X,'MOB. FNCS. C',F11.6,3F14.6,/)
  650 FORMAT (5X,'MOB. FNCS. G',F11.6,3F14.6,/)
  660 FORMAT (5X,'MOB. FNCS. H',F25.6,F28.6,/)
  670 FORMAT (5X,'MOB. FNCS. X1M, Y1M AND Z1M:',F13.6,2F12.6,//)
  680 FORMAT (5X,'MOB. FCNS. K,L,M AND J:',4F12.5)
      STOP
      END
```

```
      SUBROUTINE STOKES(R,NSTAR,M,SGN,NPT,RHS,AMN,BMN,CMN)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(300,300),RHS(1),IPVT(300),Z(300),
     *          AMN(1),BMN(1),CMN(1)
      DATA LDA/300/
C DIMENSION OF MATRIX A MUST EXCEED OR EQUAL 3*NPT.
      DM = DFLOAT(M)
      DO 10 K = 1,NPT
      K2 = K + NPT
      K3 = K2 + NPT
      N = K + NSTAR - 1
      FN1 = DFLOAT(N+1)/DFLOAT(4*N-2)
      FN2 = DFLOAT(N-2)/DFLOAT( N*(4*N-2) )
        DO 10 I = 1,NPT
      I2 = I + NPT
      I3 = I + 2*NPT
      THETA = DFLOAT(I-1)/DFLOAT(NPT-1)*3.14159265358979D0
      CO1 = DCOS(THETA)
      SI1 = DSQRT(1.D0-CO1*CO1)
      R2 = DSQRT(SI1*SI1 + (R+CO1)**2)
      CO2 = -(R+CO1)/R2
      PNS1 = PNS(N,M,CO1)
      PNS2 = PNS(N,M,CO2)
      PPNS1 = PNS(N,M+1,CO1)
      PPNS2 = PNS(N,M+1,CO2)
      PNSP1 = PNSP(N,M,CO1)
      PNSP2 = PNSP(N,M,CO2)
C RATIOS OF SIN1/SIN2 FROM FACTORING OF SINES IN EQUATIONS
      RZ = ( 1.D0/R2 )**M
      RR = RZ/R2
      RPHI = RZ*R2
      A(I,K) = FN1*( CO1*PNS1 - RZ*SGN*CO2*PNS2/R2**N )
     *       - FN2*( PNSP1 - RZ*SGN*PNSP2/R2**N )
      A(I,K2) = -DFLOAT(N-M+1)*( PNS(N+1,M,CO1)
     *                 - RZ*SGN*PNS(N+1,M,CO2)/R2**(N+2) )
      A(I,K3) = DM*( PNS1 - RZ*SGN*PNS2/R2**(N+1) )
C
      A(I2,K) = ( FN1 + FN2*DM )*( PNS1 + RR*SGN*PNS2/R2**N )
     *          + FN2*( CO1*PPNS1 + RR*SGN*CO2*PPNS2/R2**N )
      A(I2,K2) = -DFLOAT(N+M+1)*( PNS1 + RR*SGN*PNS2/R2**(N+2) )
     *           -CO1*PPNS1 - RR*SGN*CO2*PPNS2/R2**(N+2)
      A(I2,K3) = -PPNS1 - RR*SGN*PPNS2/R2**(N+1)
C
      A(I3,K) = -DM*FN2*( PNS1 + SGN*RPHI*PNS2/R2**N )
      A(I3,K2) = DM*( PNS1 + RPHI*SGN*PNS2/R2**(N+2) )
      A(I3,K3) = PNSP1 + RPHI*SGN*PNSP2/R2**(N+1)
   10 CONTINUE
C
      IDIM = 3*NPT
      IF ( M .EQ. 0 ) IDIM = 2*NPT
      CALL DGECO(A,LDA,IDIM,IPVT,RCON,Z)
      CALL DGESL(A,LDA,IDIM,IPVT,RHS,0)
C
      DO 340 J=1,NPT
      J1 = J + NPT
      J2 = J1 + NPT
      AMN(J) = RHS(J)
      BMN(J) = RHS(J1)
  340 CMN(J) = RHS(J2)
      RETURN
      END
```

```
      SUBROUTINE STOKESD(R,SGN,NPT,RHS,CMN)
      IMPLICIT DOUBLE PREC..ION (A-H,O-Z)
      DIMENSION A(100,100),RHS(1),CMN(1),IPVT(100)
      DATA LDA/100/
C DIMENSION OF MATRIX A MUST EXCEED OR EQUAL NPT.
      DO 10 K = 1,NPT
      N = K
      FN1 = DFLOAT(N+1)/DFLOAT(4*N-2)
      FN2 = DFLOAT(N-2)/DFLOAT( N*(4*N-2) )
        DO 10 I = 1,NPT
      THETA = DFLOAT(I-1)/DFLOAT(NPT-1)*3.14159265358979D0
      CO1 = DCOS(THETA)
      SI1 = DSQRT(1.D0-CO1*CO1)
      R2 = DSQRT(SI1*SI1 + (R+CO1)**2)
      CO2 = -(R+CO1)/R2
      PPNS1 = PNS(N,1,CO1)
      PPNS2 = PNS(N,1,CO2)
C RATIOS OF SIN1/SIN2 FROM FACTORING OF SINES IN EQUATIONS
      RR = 1.D0/R2
      A(I,K) = -PPNS1 - RR*SGN*PPNS2/R2**(N+1)
C
   10 CONTINUE
C
      IDIM = NPT
      CALL DGEFA(A,LDA,IDIM,IPVT,INFO)
      CALL DGESL(A,LDA,IDIM,IPVT,RHS,0)
      DO 340 J=1,NPT
  340 CMN(J) = RHS(J)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION PNS(N,M,X)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      IF (M .LE. N) GO TO 1
      PNS = 0.D0
      RETURN
    1 N1 = N-M
      IF (M .EQ. 0) GO TO 100
      DM = DFLOAT(M)
      COEFF = 1.D0
      DO 5 I=1,M
    5 COEFF = COEFF*DFLOAT(2*I-1)
      P0 = COEFF
      IF (N .GT. M) GO TO 7
      PNS = P0
      RETURN
    7 P1 = P0*X*DFLOAT(2*M+1)
      IF (N .GT. M+1) GO TO 9
      PNS = P1
      RETURN
    9 P = P1
      PMINUS = P0
        DO 10 I=2,N1
        PPLUS = 2.D0*X*P - PMINUS
     *       + DFLOAT(2*M-1)*( X*P-PMINUS )/DFLOAT(I)
        PMINUS = P
   10   P = PPLUS
      PNS = PPLUS
      RETURN
C
  100 P0 = 1.D0
      IF (N .GT. M) GO TO 107
      PNS = P0
      RETURN
  107 P1 = X
      IF (N .GT. M+1) GO TO 109
      PNS = P1
      RETURN
  109 PMINUS = P0
      P = P1
        DO 110 I=2,N1
        PPLUS = 2.D0*X*P - PMINUS - (X*P - PMINUS)/DFLOAT(I)
        PMINUS = P
  110   P = PPLUS
      PNS = PPLUS
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION PNSP(N,M,X)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      N1 = N-M
      IF (M .EQ. 0) GO TO 100
      DM = DFLOAT(M)
      COEFF = 1.D0
      DO 5 I=1,M
    5 COEFF = COEFF*DFLOAT(2*I-1)
      P0 = COEFF
      PP0 = -DM*X*P0
      IF (N .GT. M) GO TO 7
      PNSP = PP0
      RETURN
    7 P1 = P0*X*DFLOAT(2*M+1)
      PP1 = DFLOAT(2*M+1)*( P0*(1.D0-X**2) + X*PP0 )
      IF (N .GT. M+1) GO TO 9
      PNSP = PP1
      RETURN
    9 P = P1
      PP = PP1
      PMIN = P0
        DO 10 I=2,N1
        PPLUS = 2.D0*X*P - PMIN + DFLOAT(2*M-1)*(X*P-PMIN)/DFLOAT(I)
        PPPLS = ( DFLOAT(I+2*M)*(1.D0-X**2) + DM*X**2 )*P
     *          - DM*X*PPLUS + X*PP
        PMIN = P
        P = PPLUS
   10   PP = PPPLS
      PNSP = PPPLS
      RETURN
C
  100 P0 = 1.D0
      PP0 = 0.D0
      IF (N .GT. 0) GO TO 107
      PNSP = 0.D0
      RETURN
  107 P1 = X
      PP1 = 1.D0 - X*X
      IF (N .GT. 1) GO TO 109
      PNSP = PP1
      RETURN
  109 PMIN = P0
      P = P1
      PP = PP1
        DO 110 I=2,N1
        PPLUS = 2.D0*X*P - PMIN - ( X*P-PMIN )/DFLOAT(I)
        PPPLS = DFLOAT(I)*PP1*P + X*PP
        PMIN = P
        P = PPLUS
  110   PP = PPPLS
      PNSP = PPPLS
      RETURN
      END
```

```
      SUBROUTINE DMATIN(A,N,B,M,DETERM,NMAX)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION INDEX
      DIMENSION A(NMAX,N),B(NMAX,M),PIVOT(100),INDEX(100)
      DETERM=1.D0
      DO 20 I=1,N
      PIVOT(I)=0.D0
   20 INDEX(I)=0.D0
      DO 550 I=1,N
      AMAX=0.D0
      DO 105 J=1,N
      IF (PIVOT(J).NE.0.D0) GO TO 105
      DO 100 K=1,N
      IF (PIVOT(K).NE.0.D0) GO TO 100
      TEMP=DABS(A(J,K))
      IF (TEMP.LT.AMAX) GO TO 100
      IROW=J
      ICOLUM=K
      AMAX=TEMP
  100 CONTINUE
  105 CONTINUE
      INDEX(I)=4.096D3*DFLOAT(IROW)+DFLOAT(ICOLUM)
      J=IROW
      AMAX=A(J,ICOLUM)
C SUPPRESS CALCULATION OF DETERMINANT **********
C     DETERM=AMAX*DETERM
      IF (DETERM.EQ.0.D0) GO TO 600
      PIVOT(ICOLUM)=AMAX
      IF (IROW.EQ.ICOLUM) GO TO 260
      DETERM=-DETERM
      DO 200 K=1,N
      SWAP=A(J,K)
      A(J,K)=A(ICOLUM,K)
  200 A(ICOLUM,K)=SWAP
      IF (M.LE.0) GO TO 260
      DO 250 K=1,M
      SWAP=B(J,K)
      B(J,K)=B(ICOLUM,K)
  250 B(ICOLUM,K)=SWAP
  260 K=ICOLUM
      A(ICOLUM,K)=1.D0
      DO 350 K=1,N
  350 A(ICOLUM,K)=A(ICOLUM,K)/AMAX
      IF (M.LE.0) GO TO 380
      DO 370 K=1,M
  370 B(ICOLUM,K)=B(ICOLUM,K)/AMAX
  380 DO 550 J=1,N
      IF (J.EQ.ICOLUM) GO TO 550
      T=A(J,ICOLUM)
      A(J,ICOLUM)=0.D0
      DO 450 K=1,N
  450 A(J,K)=A(J,K)-A(ICOLUM,K)*T
      IF (M.LE.0) GO TO 550
      DO 500 K=1,M
  500 B(J,K)=B(J,K)-B(ICOLUM,K)*T
  550 CONTINUE
  600 DO 710 I=1,N
      I1=N+1-I
      K=IDINT(INDEX(I1)/4.096D3)
      ICOLUM=IDINT(INDEX(I1)-4.096D3*DFLOAT(K))
      IF (K.EQ.ICOLUM) GO TO 710
      DO 705 J=1,N
      SWAP=A(J,K)
      A(J,K)=A(J,ICOLUM)
  705 A(J,ICOLUM)=SWAP
  710 CONTINUE
      RETURN
      END
```

```
                NUMBER OF POINTS:   12          R =    4.000

                         X11          Y11          X12          Y12

RES. FNCS. A       1.169470     1.043303    -0.427212    -0.204477
RES. FNCS. B                   -0.020331                  0.098152
RES. FNCS. C       1.000296     1.004226    -0.015631     0.008458
RES. FNCS. G       0.110028     0.003564    -0.255392    -0.008486
RES. FNCS. H                   -0.000331                  0.019701
RES. FCNS. X11M+X12M,Y11M+Y12M,Z11M+Z12M:    1.094851  0.971180  0.998097

                         X11          Y11          X12          Y12

MOB. FNCS. A       0.986769     0.999716     0.360471     0.195314
MOB. FNCS. B                    0.000120                 -0.031251
MOB. FNCS. C       0.999948     0.998914     0.015625    -0.007769
MOB. FNCS. G       0.005504     0.000020    -0.070784    -0.002604
MOB. FNCS. H                   -0.000750                  0.019569
MOB. FNCS. X1M, Y1M AND Z1M:     -1.072550   -0.970229   -0.998097

MOB. FCNS. K,L,M AND J:    -0.00190    -0.02787     0.16741     0.00184
```

Sample output from MANDR.FOR

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The resistance and mobility functions which completely characterize the linear relations between the force, torque and stresslet and the translational and rotational velocities of two spheres in low-Reynolds-number flow have been calculated using a boundary collocation technique. The ambient velocity field is assumed to be a superposition of a uniform stream and a linear (vorticity and rate-of-strain) field. This is the first compilation of accurate expressions for the entire set of functions. Our calculations are in agreement with earlier results for all functions for which such results are

ABSTRACT (cont.)

available. Our technique is successful at all sphere-sphere separations except at the almost-touching (gaps of less than .005 diameter) configuration.

New results for the stresslet functions have been used to determine Batchelor and Green's (1972) order c^2 coefficient in the bulk-stress (7.1 instead of their 7.6). The two-sphere functions have also been used to determine the motion of a rigid dumbbell in a linear field. We also show that certain functions have extrema. The source (FORTRAN) code is furnished in the appendix.

END

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