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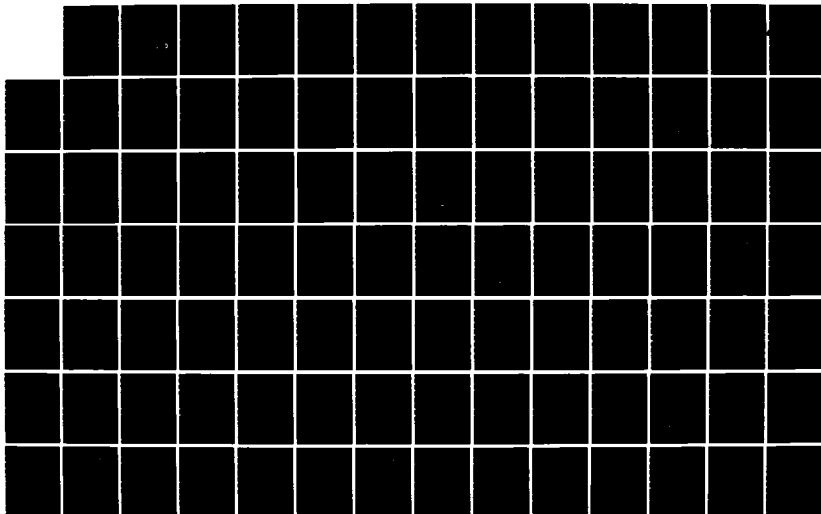
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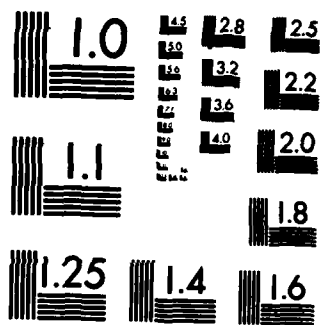
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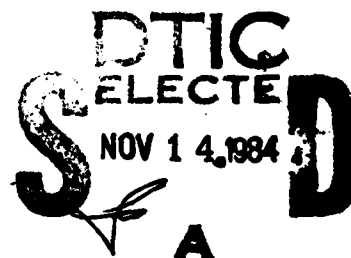
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ELECTRON DISTRIBUTION FUNCTION AND MOBILITY
IN III - V COMPOUNDS FOR HIGH AND LOW FIELDS

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September, 1984



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Electron distributions are calculated for low and high electric fields in III-V semiconductors with polar optical scattering. Approximate analytic expressions are derived by two analytic methods and compared to results found by a perturbation method and iteration method for low electric fields. Electron mobility is calculated. For high electric fields an expansion inversely in the electric field gives an approximate electron distribution function valid at low energies. A non-parabolic energy band and p-wave scattering (they are interdependent) do not affect the asymmetrical part of the distribution function within the central valley; hence, they do not cause the negative differential resistance.				
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LIST OF VARIABLES

a, a'	 coefficients in Overlap Integral	(pg.6)
b	 $\frac{4E^2(e^{\frac{\sigma}{\sigma_0}}-1)}{3E_0^2(e^{\frac{\sigma}{\sigma_0}}+1)}$	(pg.54)
c, c'	 coefficients for p-wave	(pg.6)
d	 $\frac{1(e^{\frac{\sigma}{\sigma_0}}+1)}{2(e^{\frac{\sigma}{\sigma_0}}-1)}$	(pg.54)
e	 charge of electron	(pg.7)
f	 distribution function	(pg.5)
f_0	 symmetrical part of distribution function	(pg.7)
f_1	 asymmetrical part of the distribution function	(pg.56)
f_0^j	 terms in f_0 expansion	(pg.56)
f_1^j	 terms in f_1 expansion	(pg.56)
ϵ	 f_1/k	(pg.7)
h	 Planck's constant	(pg.5)
j	 index in series expansions	(pg.18)
k, k'	 wave vector	(pg.5)
m	 superscript indicating order of term	(pg.20)
m^*	 effective mass of electron	(pg.6)
m_0	 mass of electron in free space	(pg.6)
n	 subscript indicating order of term	(pg.12)
n_0	 density of electron in conduction band	(pg.10)
x	 normalised electron energy ξ/σ	
y	 $\sqrt{x}$	(pg.35)
A'	 equals $aa' + \{cc'[k'^2 + k^2]/2kk'\}$	(pg.8)
A_n, B_n, C_n, D_n	 coefficients which are functions of ξ and σ	(pg.11)
$A_n^m, B_n^m, C_n^m, D_n^m$	 coefficients written in terms of δ	(pg.18)
E	 electric field	(pg.5)
E_0	 dielectric breakdown field	(pg.7)
$F(0)$	 integration constant	(pg.54)
F_0^2	 constants of integration	(pg.105)
$G(\sigma)$	 mobility function	(pg.43)
H'	 perturbing potential	(pg.5)
$H(x-x')$	 Step-function	(pg.11)
K	 $2E/(3E_0)$	(pg.55)
$L(k, k')$	 overlap integral	(pg.6)
N	 number of phonons	(pg.7)
ΔN	 change in the number of phonons	(pg.7)

LIST OF VARIABLES
(continue)

N_c	 effective density of state	(pg.10)
α, β	 coefficients in energy function	(pg.8)
δ	 $\exp(-\sigma)$	(pg.18)
ϵ	 smallness parameter	(pg.68)
ϵ_∞	 dielectric constant at infinite frequency	(pg.7)
ϵ_0	 dielectric constant at zero frequency	(pg.7)
γ	 function of asymmetrical part of distribution	(pg.11)
λ	 lower limit of integration	(pg.73)
μ	 drift mobility	(pg.43)
η	 energy function normalised to $\hbar\omega$	(pg.71)
σ	 optical phonon energy normalised by $k_B T$	(pg.8)
τ	 relaxation time for large x	(pg.53)
ω	 longitudinal optical phonon frequency	(pg.5)
ξ	 electron energy normalised to $k_B T$	(pg.11)
ξ	 new variable of ξ valid from 0 to σ	(pg.12)
Ω	 energy function	(pg.8)
ϵ	 electron energy	(pg.5)
θ	 angle between $\vec{k}$ and $\vec{k}'$	(pg.6)
ϕ	 angle between $\vec{k}$ and $\vec{E}$	(pg.87)
φ, φ'	 azimuthal angle of $\vec{E}$ and $\vec{k}'$ respectively	(pg.87)
ϕ	 represents $2\sinh^{-1}\sqrt{x}$	(pg.52)
ϵ_g	 effective mass band energy	(pg.6)
ϵ_p	 defined by $\epsilon_p = \epsilon_g [m_0/m^* - 1]$	(pg.6)
ϵ_0	 $\hbar^2 k^2 / (2m_0)$	(pg.6)

ELECTRON DISTRIBUTION FUNCTIONS AND MOBILITY IN III-V SEMICONDUCTOR COMPOUNDS FOR HIGH AND LOW ELECTRIC FIELDS

1. INTRODUCTION

Since the 1950s when the first paper was published about the semi-conducting properties of III-V compounds, this class of compounds has been used in various electronic devices such as infra-red detectors, semiconductor lasers, Gunn diodes, and light emitting diodes. The construction of these devices is based on various transport properties. One of the most important of these is the drift mobility, which will be discussed in this report.

Determination of the drift mobility of III-V semiconductors is useful in the development of high speed devices. In this report the electron drift mobility is found here as a moment integral over the momentum space of the electron distribution functions. The distribution function is calculated from the steady state Boltzmann equation which balances the frictional forces due to scattering of the carriers and the force on the carriers due to the applied electric field. The properties of the electron distribution functions are of interest itself as well as a means for calculating the drift mobility.

In the III-V semiconductor compounds polar-optical scattering is the dominant form of scattering mechanism over a considerable range of temperature[1-4]. Since polar scattering is not elastic, a relaxation time cannot be easily written. A relaxation time exists only when the carriers have energy which is much larger or much smaller than the polar optical phonon energy[5,6]. Therefore the relaxation time approach is not suitable for solving the Boltzmann equation in polar compounds. A few methods have been suggested. The earliest of these is the Variational Method [1,7,8]. Kohler[8] introduced this method in 1948; it was applied to polar mode scattering by Howarth and Sondheimer [7] for a spherical, parabolic energy band. Ehrenreich[3] later used the method for a non-parabolic Kane [9] band where he combined the polar scattering with acoustic and electron-hole scattering. In many cases the method was and still is mathematically complicated when results of sufficient accuracy are required. This is especially true when non-parabolicity of the bands has to be included. A more attractive method, the Drifted Maxwell-Boltzmann Distribution Method [10,11], was widely used. To some extent it is still used. This method assumes that the carrier distribution function can be approximated by a drifted Maxwellian function. However this, method is not applicable to some III-V semiconductor compounds at high electric fields [12].

At present two of the more popular methods are the Iteration Method [13-16] and the Monte Carlo Method [17-21]. In the Iteration Method the Boltzmann equation is reduced to an infinite set of algebraic equations which are truncated at some point and solved iteratively. This method has the advantage of self-correction of the error through iteration.

Very often the number of calculations required is smaller than in the Variational Method. Rode[15,16] made a series of calculations using this method and was able to show that its results agree very well with experimental data for a number of III-V compounds such as GaAs, GaSb, InAs and InSb [15].

With the availability of better and faster computers, the Monte Carlo Method [21] has become more popular. In this method the Boltzmann equation is not directly solved. The transport coefficients are evaluated by the simulation of electron trajectories in a computer. The solution to the transport equation can be approximated by estimating the characteristics of the carrier population as statistical averages over a large number of collisions. This method is especially useful in those cases where it is difficult to solve the transport equation by the other methods described. The main drawbacks of this method are the large amount of computer time required for the statistical averaging and the ad hoc assumption about the collision process.

In this report the calculations of the electron distribution function and drift mobility are made with some assumptions. These are that there is an applied DC electric field and that the carriers (electrons) are in a steady state. Under these conditions semi-classical statistics are often used. The Boltzmann equation can be used if the device is not too small and if the temperature is not too low. For very low temperature or for very small devices, quantum effects have to be considered. Calculations will be made for low electric field and high electric field cases. For the low electric field case two additional methods will be introduced. These two methods are extensions

of the Iteration Method used by Fortini, Diquet and Lugand [14]. The Iteration Method will be presented briefly to verify their results and to use them as a basis for comparison. It is part of the aim of this report to arrive at a method which is less tedious than the Iteration Method and the Monte Carlo Method. The Analytical Method is especially of interest here because it allows the expression of the asymmetric part of the electron distribution function in a simple and compact form. The analytic form gives a better picture of the different parameter dependences, such as temperature dependence, of the distribution function.

For high electric fields, a simple analytical expression for the electron distribution function has been derived by Stratton [5], and Conwell and Vassell [6]. However, one of the drawbacks of their analytical result is that it fails at low energy. In this report a Perturbation Method is suggested, which extends their result to include the low energy region. The effect of non-parabolicity of the central valley of the conduction band and p-wave admixture on the carrier distribution function at high electric fields is also examined. This is of interest because it has been claimed that non-parabolicity alone can cause negative differential mobility within the central valley for high electric fields [22].

1.1 BOLTZMANN EQUATION FOR ELECTRONS IN DC FIELDS.

In this section the steady-state Boltzmann equation for electrons in III-V semiconductor compounds with a non-parabolic energy band and p-wave scattering included is presented. For a steady state system the Boltzmann equation consists of two terms, a collision term and an

electric field term. The collision term represents the rate of change of the carrier distribution function due to the collisions and the electric field term is the rate of change of the carrier distribution function caused by the force on the carriers due to the applied electric field $\vec{E}$.

$$\left(\frac{df}{dt}\right)_E + \left(\frac{df}{dt}\right)_C = 0. \quad (1.1)$$

The electric field term and the collision term are indicated by the subscripts E and C respectively. In this report carriers are assumed to be electrons.

The collision integral consists of four terms that represent the scattering in momentum space of electrons in and out of the state with wave vector $\vec{k}$. The terms are the scattering in and out of state $\vec{k}$ with emission or absorption of a phonon.

$$\begin{aligned} \left[\frac{df}{dt}\right] = \frac{-2\pi}{\hbar} \sum_{\vec{k}'} & \left[| \langle N+1 | H' | N \rangle |^2 L(\vec{k}, \vec{k}') \delta(E_{\vec{k}} - E_{\vec{k}'} + \hbar\omega) f(\vec{k}') \right. \\ & + | \langle N-1 | H' | N \rangle |^2 L(\vec{k}, \vec{k}') \delta(E_{\vec{k}} - E_{\vec{k}'} - \hbar\omega) f(\vec{k}') \\ & - | \langle N+1 | H' | N \rangle |^2 L(\vec{k}, \vec{k}') \delta(E_{\vec{k}}, -E_{\vec{k}'} + \hbar\omega) f(\vec{k}) \\ & \left. - | \langle N-1 | H' | N \rangle |^2 L(\vec{k}, \vec{k}') \delta(E_{\vec{k}}, -E_{\vec{k}'} - \hbar\omega) f(\vec{k}) \right] \end{aligned} \quad (1.2)$$

$\hbar$ is the Planck's constant divided by 2π and ω is the frequency of the longitudinal optical phonon. $E_{\vec{k}}$ is the energy of the carrier associated with the state $\vec{k}$ and the difference in energy between $\vec{k}$ and $\vec{k}'$ is $\hbar\omega$. In (1.2) the terms represent scattering of carriers into $\vec{k}$ from $\vec{k}'$ by emission of $\hbar\omega$, into $\vec{k}$ from $\vec{k}'$ by absorption of $\hbar\omega$, out of $\vec{k}$ by emission of $\hbar\omega$ and out of $\vec{k}$ by absorption of $\hbar\omega$, respectively. $L(\vec{k}, \vec{k}')$

is the overlap integral which describes the degree of mixing of the s-wave and p-wave functions. For two electron states with momentum $\hbar\vec{k}$ and $\hbar\vec{k}'$, the overlap integral is given by [1,23]

$$L(\vec{k}, \vec{k}') = (aa' + cc' \cos \theta')^2 \quad (1.3)$$

where $a^2 + c^2 = 1$

and

$$a^2 = \frac{1}{2} \left[1 + \frac{\xi_g}{(\xi_g)^2 + 4\xi_0\xi_p} \right] \quad (1.4)$$

a and c are normalization coefficients for s-wave and p-wave functions respectively. They are discussed in references [1,23]. a' and c' are the coefficients evaluated at $\vec{k}'$. θ' is the angle between $\vec{k}$ and $\vec{k}'$. ξ_g is the effective mass band energy gap, and ξ_0 and ξ_p are defined as

$$\xi_0 = \frac{\hbar^2 k^2}{2m_0} \quad (1.5)$$

$$\xi_p = \xi_g \left[\frac{m_0}{m^*} - 1 \right] \quad (1.6)$$

where m_0 is the mass of electron in free space and m^* is the effective mass of the electron.

The term $\langle N+1 | H' | N \rangle$ in (1.2) is a matrix element describing the scattering events for the transition between the states $\vec{k}$ and $\vec{k}'$ brought about by a perturbing potential H' . For high mobility materials such as those considered here, the coupling between the electron and phonon is weak. Perturbation theory may be used to describe the interaction. The matrix element for this case has been derived by Frolich [24], Callen [25] and Ehrenreich [1] using perturbation theory. The matrix element for a wave function with s-symmetry can be written as [26]

$$|(N+1|H'|N)|^2 = \frac{2\pi\hbar^2 eE_0}{m^2 V |\vec{k}-\vec{k}'|^2} \left(N + \frac{1}{2} + \frac{\Delta N}{2} \right) \quad (1.7)$$

where N is the number of phonons with wave vector $\vec{k}'-\vec{k}$ and ΔN is the change in N in the process described by the matrix element. V is the volume of the crystal and eE_0 is given by [26]

$$eE_0 = \frac{m^* e^2 \hbar \omega}{\hbar^2} \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0} \right) \quad (1.8)$$

with e being the charge of an electron and ϵ_∞ and ϵ_0 are the dielectric constants in cgs units for infinite and zero frequencies respectively.

To reduce equation (1.2) to a more useful form, the summation over $\vec{k}'$ is transformed to an integration in spherical coordinates with $\vec{k}$ along the z -axis. For a weak electric field the electron distribution function is nearly spherically symmetric [27]. This electron distribution may be expanded into a series of Legendre polynomials that is made up of a dominant term and a series of small perturbed terms. It has been shown for the case considered here that retaining the dominant term and the first asymmetric term gives a good approximation to the electron distribution function [28]. The distribution function in the collision integral can therefore be approximated by the first two Legendre polynomials

$$f(\vec{k}) = f_0(\mathcal{E}) + k_E g(\mathcal{E}) \quad (1.9)$$

where $f_0(\mathcal{E})$ and $k_E g(\mathcal{E})$ are the symmetric and perturbed parts of the distribution function respectively. From now on the subscripts $\vec{k}$ and $\vec{k}'$ will be dropped. The notations $\mathcal{E}$ and $\mathcal{E}'$ will be used instead. k_E is

the component of $\vec{k}$ in the direction of the electric field $\vec{E}$. When equations (1.3)-(1.9) are substituted into the integral form of (1.2), it can be integrated as shown in Appendix 1. The result of the integration can be separated into a symmetrical part and an asymmetrical part. The two parts of the rate of change of the electron distribution due to collisions are shown below:

$$\left(\frac{df}{dt}\right)_{\text{sym}} = \frac{eE_0 N}{\hbar k} \left\{ (f_0(\xi') e^{\sigma} - f_0(\xi)) \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A' \cos' - a a' \cos' \right) \frac{d\Omega}{d\xi'} \right. \\ \left. + (f_0(\xi') - e^{\sigma} f_0(\xi)) \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A' \cos' - a a' \cos' \right) \frac{d\Omega}{d\xi'} \right\} \\ \xi' = \xi + \hbar \omega \\ \xi' = \xi - \hbar \omega \quad (1.10)$$

$$\left(\frac{df}{dt}\right)_{\text{asym}} = \frac{eE_0 N k_F}{\hbar k} \left\{ \left(\frac{e^{\sigma} g(\xi')}{k} \left(\frac{k'^2 + k^2}{k} \frac{A'}{2} \ln \left| \frac{k'+k}{k'-k} \right| - A' k' - \frac{(\cos')^2}{3} k' \right) \right. \right. \\ \left. - g(\xi) \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A' \cos' - a a' \cos' \right) \right) \frac{d\Omega}{d\xi'} \right. \\ \left. + \left(\frac{g(\xi')}{k} \left(\frac{k'^2 + k^2}{k} \frac{A'}{2} \ln \left| \frac{k'+k}{k'-k} \right| - A' k' - \frac{(\cos')^2}{3} k' \right) \right. \right. \\ \left. - e^{\sigma} g(\xi) \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A' \cos' - a a' \cos' \right) \right) \frac{d\Omega}{d\xi'} \right\} \\ \xi' = \xi + \hbar \omega \\ \xi' = \xi - \hbar \omega \quad (1.11)$$

where

$$A' = a a' + \frac{(k')^2 + k^2 \cos'}{2 k k'} \quad (1.12)$$

and

$$\sigma = \hbar \omega / k_B T; \quad (1.13)$$

k_B is the Boltzmann constant and T is the electron temperature. In (1.10) and (1.11) Ω is the energy function defined in terms of the effective mass.

$$\Omega = \hbar^2 k^2 / 2 m^* = (1 + \alpha \xi + \beta \xi^2 + \dots) \quad (1.14)$$

The coefficients $\alpha, \beta, \dots$ are constants that can be calculated for each

compound. The details are discussed in reference [6].

The Boltzmann collision terms (1.10) and (1.11) contain both a non-parabolic energy band for the central valley of the conduction band and s-wave and p-wave scattering. Many earlier treatments included non-parabolic energy bands with s-wave scattering. However, this approximation is inconsistent because a non-parabolic band exists only if p-wave and s-wave matrix elements are present; a parabolic energy band is consistent with only s-wave matrix elements.

The other term in (1.1), $(df/dt)_E$, which is the rate of change of the electron distribution function due to the electric field, has been derived by Conwell and Vassel[6] and is given by

$$\left(\frac{df}{dt}\right)_E = \frac{eE}{\hbar} \left[\frac{2}{3} \frac{\Omega}{\hbar\omega} \frac{d\mathcal{E}}{d\Omega} \frac{d\mathcal{E}}{d\mathcal{E}} + g + \frac{\hbar^2 k^2}{m^*} \frac{d\mathcal{E}}{d\Omega} \frac{df}{d\mathcal{E}} \right]. \quad (1.15)$$

2. ITERATION METHOD FOR LOW ELECTRIC FIELDS

In section 2, the electron distribution function is found from the steady-state Boltzmann equation for low electric fields by the Iteration Method.

For low electric fields, there is very little mixing of the s-wave with the p-wave. Therefore c and c' in (1.10) and (1.11) can be approximated by zeros. A' is reduced to unity in the same equations. It is assumed in this section that the energy band is spherical. Therefore Ω is equal to $\mathcal{E}$ and $d\Omega/d\mathcal{E}$ is equal to unity in (1.10)-(1.12). Under this condition the symmetrical part of the Legendre expansion of the electron distribution function, f_0 , can be approximated by a thermal equilibrium Maxwell-Boltzmann distribution.

$$f_0 = \frac{n_0}{N_c} e^{-(\mathcal{E}/k_B T)} \quad (2.1)$$

where N_c is the effective density of state and n_0 is the density of electrons in the conduction band valid for low electric fields. Classical statistics for the electrons are used here. Under certain conditions Fermi-Dirac statistics must be used.

With (2.1), (1.12), which is the asymmetrical part of the field equation, can be equated with the asymmetrical part of the collision term according to the steady-state Boltzmann equation to give

$$\begin{aligned} -n_0 \frac{E(e^\sigma - 1)}{E_0} \left[\frac{n^2}{2m^* \pi k_B T} \right]^{3/2} e^{-\xi} & \\ = 2 \left[\sinh^{-1}(\xi/\sigma)^{1/2} + e^\sigma H(\xi - \sigma) \sinh^{-1} \left(\frac{\xi - \sigma}{\sigma} \right)^{1/2} \right] kg(\xi) \end{aligned}$$

$$\begin{aligned}
 & +e^{\xi} \left[1 - \frac{(2\xi+\sigma)}{\sqrt{\xi(\xi+\sigma)}} \sinh^{-1}(\xi/\sigma) \right] kg(\xi+\sigma) \\
 & +H(\xi-\sigma) \left[1 - \frac{(2\xi-\sigma)}{\sqrt{\xi(\xi-\sigma)}} \sinh^{-1}\left(\frac{\xi-\sigma}{\sigma}\right)^{1/2} \right] kg(\xi-\sigma)
 \end{aligned}
 \tag{2.2}$$

where $H(\xi-\sigma)$ is the unit-step function, ξ is the normalised form of the energy, $\mathcal{E}/k_B T$. This equation is the same as equation (5) of the paper of Fortini et al [14] except there is a missprint in their paper. The square-root sign in the denominator is over $(\xi-\sigma)$ instead of the product of $\xi(\xi-\sigma)$. If a new function, $\gamma(\xi, \sigma)$, is defined as shown in equation (2.3), equation (2.2) can be rewritten in the form of equation (6) of the paper by Fortini et al [14]. This is shown in equation (2.4) below.

$$\gamma(\xi, \sigma) = \frac{kg(\xi)}{\frac{n_0 E_0 (e^{\xi} - 1)}{E_0} \left(\frac{n^2}{2m \pi k_B T} \right)^{3/2}}
 \tag{2.3}$$

$$A(\xi)\gamma(\xi-\sigma, \sigma) + B(\xi)\gamma(\xi, \sigma) + C(\xi)\gamma(\xi+\sigma, \sigma) = D(\xi)
 \tag{2.4}$$

where the coefficients $A(\xi)$, $B(\xi)$, $C(\xi)$, and $D(\xi)$ are defined as

$$A(\xi) = H(\xi-\sigma) \left[1 - \frac{(2\xi-\sigma)}{\sqrt{\xi(\xi-\sigma)}} \sinh^{-1}\left(\frac{\xi-\sigma}{\sigma}\right)^{1/2} \right]
 \tag{2.5}$$

$$B(\xi) = 2 \left[\sinh^{-1}(\xi/\sigma)^{1/2} + e^{\xi} H(\xi-\sigma) \sinh^{-1}\left(\frac{\xi-\sigma}{\sigma}\right)^{1/2} \right]
 \tag{2.6}$$

$$C(\xi) = e^{\xi} \left[1 - \frac{(2\xi+\sigma)}{\sqrt{\xi(\xi+\sigma)}} \sinh^{-1}(\xi/\sigma)^{1/2} \right]
 \tag{2.7}$$

and

$$D(\xi) = -\xi e^{-\xi}.
 \tag{2.8}$$

2.1 ITERATION EQUATIONS FOR THE ASYMMETRICAL DISTRIBUTION FUNCTION

One of the methods for solving the difference equation (2.4) involves iteration. This has been done by Fortini, Diquet and Lugand [14], and Grigorev, Dykman and Tomchuk [13]. In this section the method of Fortini et al [14] will be described briefly. Although the aim of this report is to find approximate analytical expressions for the electron distribution function, the Iteration Method is solved in this section in order to verify the results of Fortini et al. The results then can be used as a reference by which the results from the other methods may be compared.

A new variable $\tilde{\xi}$ is defined such that $\xi = \tilde{\xi} + n\sigma$ where the range of $\tilde{\xi}$ is from zero to σ . Therefore, for a given value of σ the distribution function consists of a set of functions $\gamma(\tilde{\xi}, \sigma)$, $\gamma(\tilde{\xi} + \sigma, \sigma)$, ..., $\gamma(\tilde{\xi} + n\sigma, \sigma)$. Each of these $\gamma(\tilde{\xi} + n\sigma, \sigma)$ is defined in the interval $n\sigma$ to $(n+1)\sigma$. For convenience, $\gamma(\tilde{\xi} + n\sigma, \sigma)$ will be denoted by γ_n . The function $\gamma(\xi, \sigma)$ may be found for the range of ξ needed by combining the set of γ_n for n equals to 0, 1, 2, ...etc. In terms of γ_n equation (2.4) can be written as an infinite set of coupled difference equations:

$$\begin{aligned} B_0 \gamma_0 + C_0 \gamma_1 &= D_0 \\ A_1 \gamma_0 + B_1 \gamma_1 + C_1 \gamma_2 &= D_1 \\ A_2 \gamma_1 + B_2 \gamma_2 + C_2 \gamma_3 &= D_2 \\ &\vdots \\ A_{n-1} \gamma_{n-2} + B_{n-1} \gamma_{n-1} + C_{n-1} \gamma_n &= D_{n-1} \\ A_n \gamma_{n-1} + B_n \gamma_n + C_n \gamma_{n+1} &= D_n \\ &\vdots \end{aligned} \tag{2.9}$$

where the notation for A, B, C and D is defined by

$$Q_n = Q(\xi + n\sigma). \quad (2.10)$$

A finite set of equations may be chosen from (2.9) to calculate $\gamma(\xi, \sigma)$. Since there is always one more unknown than algebraic equations, an approximation is needed. One of the most obvious approximations is the assumption that $C_n \gamma_{n+1}$ is small in the last equation of the chosen set and can be dropped. This assumption allows us to truncate the infinite set of coupled difference equations. The result is a set of $n+1$ equations with $n+1$ unknowns:

$$\begin{aligned} B_0 \gamma_0 + C_0 \gamma_1 &= D_0 \\ A_1 \gamma_0 + B_1 \gamma_1 + C_1 \gamma_2 &= D_1 \\ A_2 \gamma_1 + B_2 \gamma_2 + C_2 \gamma_3 &= D_2 \\ &\vdots \\ A_{n-1} \gamma_{n-2} + B_{n-1} \gamma_{n-1} + C_{n-1} \gamma_n &= D_{n-1} \\ A_n \gamma_{n-1} + B_n \gamma_n &= D_n. \end{aligned} \quad (2.11)$$

These equations can be solved simultaneously for the unknown $\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n$. For a given σ , $\gamma(\xi, \sigma)$ can be calculated once the set of γ_n has been found. This procedure is repeated with more equations (larger n) until the change in the γ_n of interest is small. This iteration procedure is expected to produce γ_n to any desired degree of accuracy. However, the calculation becomes very tedious as n increases.

2.2 RESULTS FOR $\gamma(\xi, \sigma)$ FROM THE ITERATION METHOD

With the term $C_n \gamma_{n+1}$ in the equation (2.8) dropped, the resulting set of $n+1$ equations with $n+1$ unknowns may be solved on a computer. Fortini et al [14] performed the iteration on an IBM computer up to an

order n where γ_{n+1} differs from γ_n by less than 5×10^{-4} . $\gamma(\xi, \sigma)$ was obtained in the range $0 < \xi < (n+1)\sigma$ from the set $(\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n)$. However no values for n were given to indicate the rate of convergence for the iteration process.

In this report the iteration is done on an Apple II computer and the results of $\gamma(\xi, \sigma)$ are plotted against ξ in figure 1. The plots are for values of σ equal 1, 2, 3, 4, and 5.45. These results are almost identical to those given by the Fortini et al [14] paper. In that paper only plots for σ equal 1, 2, 3 and 4 are given. The additional plot of $\sigma=5.45$ in this report corresponds to the temperature of liquid Nitrogen, 77°K . This value is chosen because of the interest in devices working at that temperature. The temperature equivalent to the optical phonon energy, $\hbar\omega$, is assumed to be 418°K as compared to 416°K used in the reference [14]. Room temperature will correspond to a σ of 1.393 and is expected to have a plot of $\gamma(\xi, \sigma)$ lying between $\sigma=1$ and $\sigma=2$. At values of ξ where σ is an integer, there are sharp breaks because of the emission of a longitudinal optical phonon. These sharp breaks are not discontinuities. They appear discontinuous because of the scale used in plotting $\gamma(\xi, \sigma)$. An expansion of the scale around these 'break points' will show that they are continuous.

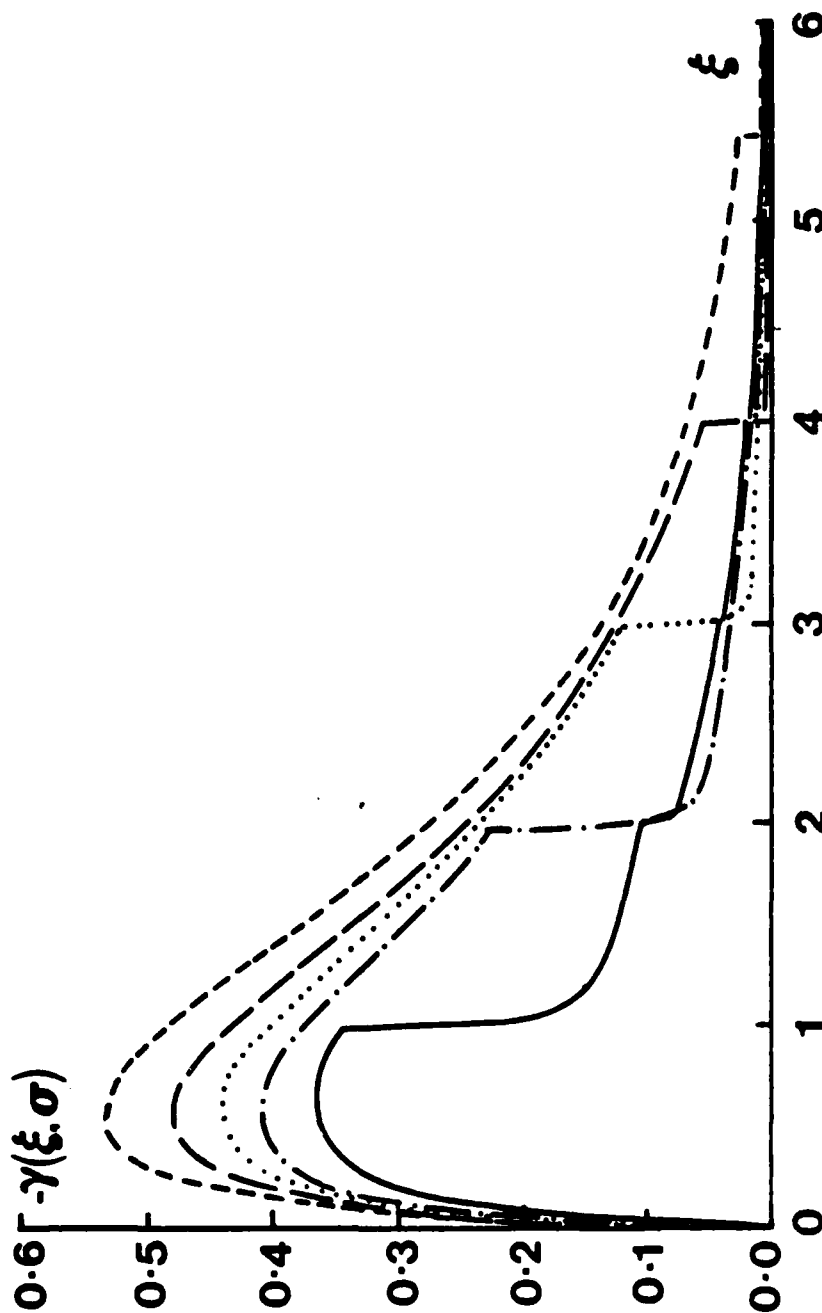


Figure 1. The perturbed electron distribution function $-\gamma(\xi, \sigma)$ versus ξ for the case when Iteration Method is used. The different type of curves represent different σ . The solid line is for $\sigma=1$, the dashed line is for $\sigma=2$, the dotted line is for $\sigma=3$, the large dashed line is for $\sigma=4$ and the small dashed line is for $\sigma=5.45$.

To get an idea of the rate of convergence of the iteration process, values of γ_0 at $\tilde{\xi}=0.7$ are calculated for increasing number of equations, n . The results are tabulated in table 1. The value $\tilde{\xi}=0.7$ is chosen because it is near the peak of the distribution function and the absolute difference is expected to be greatest there. From the table it is apparent that the convergence for the iteration is slow for $\sigma=1$, but as σ increases the rate of convergence increases sharply. For $\sigma=1$, five equations ($n=4$) are needed for the change of 0.3 percent in γ_n , but for $\sigma=4$ the same number of equations will give a difference of less than 5×10^{-5} percent in γ_0 .

In table 2, values of γ_n at $\tilde{\xi}=0.7$ for various σ are tabulated. This table gives a comparison of the magnitude for γ_n of various σ . This allows the estimation of the errors resulting from the truncation of the infinite set of equations into a set of n equations. From these two tables, it is apparent that for large values of σ , γ_0 and γ_1 alone will provide a good approximation for $\gamma(\tilde{\xi}, \sigma)$.

$\sigma \backslash n$	1	2	3	4	5.45
1	-	0.39263	0.43313	0.47068	0.52592
2	-	0.40291	0.43603	0.47142	0.52603
3	0.36417	0.40403	0.43613	0.47143	0.52603
4	0.36593	0.40415	0.43614	0.47143	0.52603
5	0.36638	0.40416	0.43614		
6	0.36649				
7	0.36652				
8	0.36653				

Table 1. Values of different γ_0 at $\xi=0.7$ when different number of equations are used. The number of equations used in each row is $n+1$.

$\sigma \backslash \gamma_n$	1	2	3	4	5.45
γ_0	0.3665	0.4041	0.4361	0.4714	0.5260
γ_1	0.1147	0.0370	0.0106	3.126×10^{-3}	5.880×10^{-4}
γ_2	0.0497	5.348×10^{-3}	4.822×10^{-4}	4.322×10^{-5}	1.531×10^{-6}
γ_3	0.0227	8.671×10^{-4}	2.638×10^{-5}	7.661×10^{-7}	5.114×10^{-9}
γ_4	0.0102	1.415×10^{-4}	1.533×10^{-6}	1.500×10^{-8}	1.998×10^{-11}

Table 2. Values of γ_n for different σ at $\xi=0.7$

3. PERTURBATION METHOD FOR THE ASYMMETRIC DISTRIBUTION FUNCTIONS

The Boltzmann equation for low electric fields is solved by the Perturbation Method in section 3. The Perturbation Method is introduced as a first step in finding an approximate analytical form for the electron distribution functions.

In order that a perturbation expansion can be done, a small expansion parameter is needed. $e^{-\sigma}$ can be used since it is smaller than unity for all positive σ . For most of the temperature range of interest, $e^{-\sigma}$ is small enough that rapid convergence can be expected. In this report $e^{-\sigma}$ will be represented by δ . It is assumed that each γ_n can be expanded into a power series in

$$\gamma_n = \sum_{j=0}^{\infty} \gamma_n^j \delta^j. \quad (3.1)$$

The superscript of γ_n^j indicates the order of the term in each expansion and not a power. The coefficients A_n , B_n , C_n and D_n are replaced by A_n^0 , $B_n^0 + B_n^{-1}/\delta$, C_n^{-1}/δ and $D_n^n \delta^n$ respectively. In this form the coefficients are shown below:

$$A_n^0 = H(\tilde{\xi} + (n-1)\sigma) \left[1 - \frac{[2\tilde{\xi} + (2n-1)\sigma]}{\sqrt{(\tilde{\xi} + n\sigma)(\tilde{\xi} + (n-1)\sigma)}} \sinh^{-1} \left(\frac{\tilde{\xi} + (n-1)\sigma}{\sigma} \right)^{1/2} \right] \quad (3.2)$$

$$B_n^0 = 2 \sinh^{-1} \left(\frac{\tilde{\xi} + n\sigma}{\sigma} \right)^{1/2} \quad (3.3)$$

$$B_n^{-1} = H(\tilde{\xi} + (n-1)\sigma) 2 \sinh^{-1} \left(\frac{\tilde{\xi} + (n-1)\sigma}{\sigma} \right)^{1/2} \quad (3.4)$$

$$C_n^{-1} = 1 - \frac{[2\tilde{\xi} + (2n+1)\sigma]}{\sqrt{(\tilde{\xi} + n\sigma)(\tilde{\xi} + (n+1)\sigma)}} \sinh^{-1} \left(\frac{\tilde{\xi} + n\sigma}{\sigma} \right)^{1/2} \quad (3.5)$$

$$D_n^n = -(\xi + n\sigma) e^{-\xi}. \quad (3.6)$$

When (3.1)-(3.6) are substituted into (2.11), the result is the set of equations shown in (3.7).

$$\begin{aligned} B_0^0 \sum_j \gamma_0^j \delta^j + C_0^{-1} \sum_j \gamma_1^j \delta^{j-1} &= D_0^0 \\ A_1^0 \sum_j \gamma_0^j \delta^j + B_1^0 \sum_j \gamma_1^j \delta^j + B_1^{-1} \sum_j \gamma_1^j \delta^{j-1} + C_1 \sum_j \gamma_2^j \delta^{j-1} &= D_1^1 \\ \vdots \\ A_n^0 \sum_j \gamma_{n-1}^j \delta^j + B_n^0 \sum_j \gamma_n^j \delta^j + B_n^{-1} \sum_j \gamma_n^j \delta^{j-1} &= D_n^n. \end{aligned} \quad (3.7)$$

Relations between the terms in the γ_n expansions may be obtained by equating the coefficients for various powers of δ . By equating the coefficients of δ^{-1} , a set of equations relating γ_n^0 for different n is obtained:

$$\begin{aligned} C_0^{-1} \gamma_1^0 &= 0 \\ B_1^{-1} \gamma_1^0 + C_1^{-1} \gamma_2^0 &= 0 \\ B_2^{-1} \gamma_2^0 + C_2^{-1} \gamma_3^0 &= 0 \\ \vdots \\ B_n^{-1} \gamma_n^0 &= 0. \end{aligned} \quad (3.8)$$

From the first equation in (3.8), γ_1^0 is zero. $\gamma_1^0 = 0$ implies all γ_n^0 with n greater than zero are zeros. This can be seen by substituting γ_1 into the second equation, finding $\gamma_2^0 = 0$ and then substituting γ_2^0 into the next equation, etc....

$$\gamma_n^0 = 0, \quad \text{for } n > 0. \quad (3.9)$$

A set of equations relating γ_n^0 and γ_n^1 can be obtained by equating the coefficients of δ^1 . This set of equation, which is shown in (3.10), can be reduced to the two equations in (3.11) by using the results for γ_n^0 obtained. The two equations in (3.11) can be solved simultaneously for γ_0^0 and γ_1^1 :

$$\begin{aligned} B_0^0 \gamma_0^0 + C_0^{-1} \gamma_1^1 &= D_0^0 \\ A_1^0 \gamma_0^0 + B_1^0 \gamma_1^0 + B_1^{-1} \gamma_1^1 + C_1^{-1} \gamma_2^1 &= 0 \\ &\vdots \\ A_n^0 \gamma_{n-1}^0 + B_n^0 \gamma_n^0 + B_n^{-1} \gamma_n^1 &= 0. \end{aligned} \quad (3.10)$$

$$\begin{aligned} B_0^0 \gamma_0^0 + C_0^{-1} \gamma_1^1 &= D_0^0 \\ A_1^0 \gamma_0^0 + B_1^{-1} \gamma_1^1 &= 0. \end{aligned} \quad (3.11)$$

The procedure for obtaining expressions for other γ_n^m is very similar to that use for γ_0^0 and γ_1^1 . In general, by equating the coefficients of δ^m , a set of equations relating γ_n^{m-1} and γ_n^m is obtained. This set of equations can be reduced to $m+2$ equations by using the results that have been found from solving the equations obtained by equating the coefficients of δ^{m-1} . The set of $m+2$ equation is shown in (3.12).

$$\begin{aligned} B_0^0 \gamma_0^m + C_0^{-1} \gamma_1^m &= 0 \\ A_1^0 \gamma_0^m + B_1^0 \gamma_1^m + B_1^{-1} \gamma_1^{m+1} + C_1^{-1} \gamma_2^{m+2} &= 0 \\ &\vdots \\ A_{m-1}^0 \gamma_{m-2}^{m-1} + B_{m-1}^0 \gamma_{m-1}^{m-1} + B_{m-1}^{-1} \gamma_{m-1}^m + C_{m-1}^{-1} \gamma_m^m &= 0 \\ A_m^0 \gamma_{m-1}^m + B_m^0 \gamma_m^m + B_m^{-1} \gamma_m^{m+1} + C_m^{-1} \gamma_{m+1}^{m+1} &= D_m^m \end{aligned} \quad (3.12)$$

$$A_{m+1}^0 \gamma_m^m + B_{m+1}^{-1} \gamma_{m+1}^{m+1} = 0.$$

These equations can be solved simultaneously to arrive at an expression for a given γ_n^m in terms of A_n^0 , B_n^0 , B_n^{-1} , C_n^{-1} , D_n^n and other lower order γ_n which have been found. The process becomes more complicated for higher orders of m and n . In Appendix 2, the calculations for the first few γ_n are carried through. Since the procedure for calculating any given γ_n^m is similar, general formula for a given γ_n^m in terms of other γ_n^m can be obtained quite easily. Once the first few γ_n^m are known, the general formulas can be used to calculate higher order terms in the expansions of various γ_n . The complete list of the formulas is shown below:

$$\gamma_0^0 = \frac{B_1^{-1} D_0^0}{B_1^{-1} B_0^0 - A_1^0 C_0^{-1}} \quad (3.13)$$

$$\gamma_n^m = 0, \quad n > m \quad (3.14)$$

$$\gamma_{n+1}^{n+1} = \frac{-A_{n+1}^0 \gamma_n^n}{B_{n+1}^{-1}} \quad (3.15)$$

$$\gamma_0^1 = \frac{D_1^1 C_0^{-1} - C_0^{-1} (B_1^0 \gamma_1^1 + C_1^{-1} \gamma_2^2)}{A_1^0 C_0^{-1} - B_0^0 B_1^{-1}} \quad (3.16)$$

$$\gamma_0^m = \frac{-C_0^{-1} (B_1^0 \gamma_1^m + C_1^{-1} \gamma_2^{m+1})}{A_1^0 C_0^{-1} - B_0^0 B_1^{-1}}, \quad m \geq 2 \quad (3.17)$$

$$\gamma_1^m = -B_0^0 \gamma_0^{m-1} / C_0^{-1}, \quad m \geq 2 \quad (3.18)$$

$$\gamma_n^{n+1} = \frac{D_n^n - (A_n^0 \gamma_{n-1}^n + B_n^0 \gamma_n^n + C_n^{-1} \gamma_{n+1}^{n+1})}{B_n^{-1}}, \quad n \geq 1 \quad (3.19)$$

$$\gamma_n^m = - \frac{(A_n \gamma_{n-1}^{m-1} + B_n^0 \gamma_n^{m-1} + C_n^{-1} \gamma_{n+1}^m)}{B_n^{-1}}, \quad n \geq 2, m \geq n+2. \quad (3.20)$$

Unlike the Iteration Method, the accuracy of any γ_n calculated by the Perturbation Method does not depend on the truncation process. The accuracy of the Perturbation Method depends on the number of terms summed for each γ_n . When equating the coefficients for a given δ^m in (3.7), so long as the number of equations n is picked such that $n \geq m+2$, the set of equations will reduce to just $m+2$ equations. With these $m+2$ equations, the $m+2$ unknowns in the γ_n expansions can be calculated exactly.

Theoretically the Perturbation Method allows us to obtain a given γ_n to any degree of accuracy by using (3.13)-(3.20). It is apparent that to calculate a certain term, a few terms of the preceding order have to be used. However each term needs at most three substitutions. Table 3 shows the terms collected after solving the set of equations obtained from equating each power of δ in (3.7). The terms in each column are solved simultaneously by using the terms in the column to the left.

$\gamma_n \backslash \delta^m$	δ^0	δ^1	δ^2	δ^3	δ^4	δ^5		δ^m
γ_0	γ_0^0	γ_0^1	γ_0^2	γ_0^3	γ_0^4	γ_0^5		γ_0^m
γ_1	γ_1^1	γ_1^2	γ_1^3	γ_1^4	γ_1^5	γ_1^6		γ_1^{m+1}
γ_2	-	γ_2^2	γ_2^3	γ_2^4	γ_2^5	γ_2^6		γ_2^{m+1}
γ_3	-	-	γ_3^3	γ_3^4	γ_3^5	γ_3^6		γ_3^{m+1}
γ_4	-	-	-	γ_4^4	γ_4^5	γ_4^6		γ_4^{m+1}
γ_5	-	-	-	-	γ_5^5	γ_5^6		γ_5^{m+1}
.								.
.							...	.
.							..	.
γ_n								γ_n^{n+1}

Table 3. The different γ_n in each column are the terms that can be calculated from the set of equations obtained by equating the coefficients for the δ shown above each column.

3.1 $\gamma(\xi, \sigma)$ FROM THE PERTURBATION METHOD.

Plots of $\gamma(\xi, \sigma)$ versus ξ obtained by the Perturbation Method are shown in figure 2. The plots are for $\sigma = 1, 2, 3, 4$ and 5.45 . These plots are almost identical to those obtained by using the Iteration Method except for ξ near zero. Different number of terms are summed for each γ_n . For $\sigma = 1$, the number of terms summed for γ_0 is nine, and for other σ , the number of terms summed is $10-n$. For $\sigma = 4$, three and two terms are summed for γ_0 and γ_1 respectively. When σ is large, the convergence is very rapid and only a few terms in each γ_n expansion are required to provide a good approximation.

Comparisons between the sizes of some of the terms are shown in table 4 and 5. The comparisons are done at $\tilde{\xi} = 0.7$. In table 4 one of the observable trends is that for $n > 0$, the dominant term is the second term in each expansion. Even though this is a little unusual in a convergent series, it is not unheard of. This type of behaviour can be found in some expansions in Bessel functions and some solutions to the Schrodinger equation. It is apparent that the first three terms in a given γ_n expansion will give a rather good approximation for γ_n .

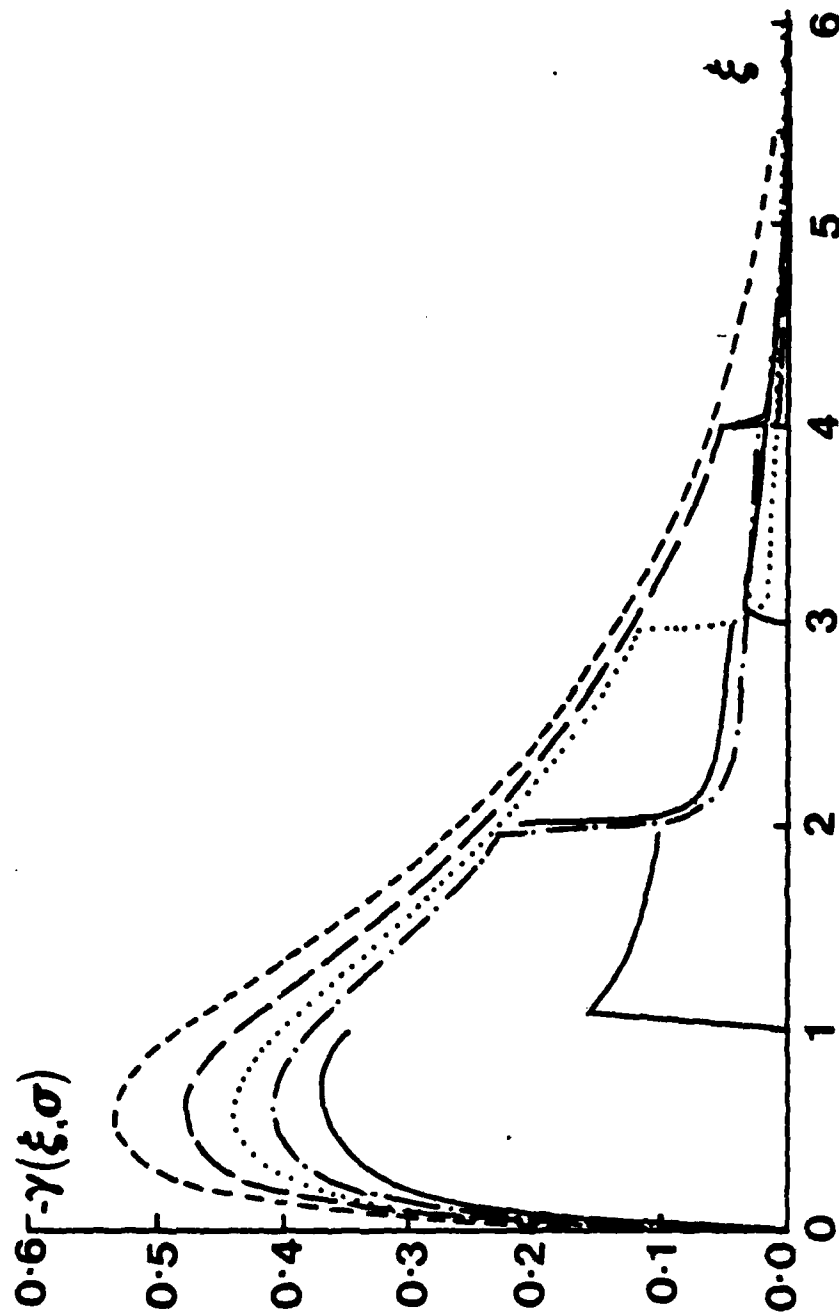


Figure 2. The perturbed electron distribution function $-\gamma(\xi, \sigma)$ for the case when Perturbation Method is used. The different types of curves represent different σ as described in Figure 1.

		$\sigma=1, \quad \xi=0.7$				
$\gamma_m \backslash n$	0	1	2	3	4	5
γ_0	-0.2841					
γ_1	-0.0877	-0.04631				
γ_2	-4.885×10^{-3}	-0.07287	-9.613×10^{-3}			
γ_3	6.285×10^{-4}	4.057×10^{-3}	-0.04344	-2.190×10^{-3}		
γ_4	1.174×10^{-4}	5.219×10^{-4}	3.168×10^{-3}	-0.0226	-5.255×10^{-4}	
γ_5	-1.440×10^{-5}	-9.750×10^{-5}	2.424×10^{-4}	2.066×10^{-3}	-0.0108	-1.304×10^{-4}
γ_6	5.266×10^{-6}	-1.195×10^{-5}	-7.556×10^{-5}	6.812×10^{-5}	1.133×10^{-3}	-4.992×10^{-3}
γ_7	2.551×10^{-7}	4.373×10^{-6}	-3.410×10^{-6}	-4.616×10^{-5}	-1.642×10^{-6}	6.201×10^{-4}
γ_8	-1.567×10^{-7}	2.118×10^{-7}	3.063×10^{-6}	9.328×10^{-7}	-2.367×10^{-5}	-1.823×10^{-5}
γ_9		-1.302×10^{-7}	7.599×10^{-9}	1.770×10^{-6}	2.009×10^{-6}	-1.044×10^{-5}
Ite'n γ_n	0.3665	0.1147	0.0497	0.2271	0.1025	

Table 4. Values of the different terms in the γ_n expansions. The values of γ_n from the Iteration Method have been included in the last row for comparison.

$\sigma=4$, $\tilde{\epsilon}=0.7$

γ_n	0	1	2	3	4	5
γ_0	-0.4580					
γ_1	-0.01361	-2.181×10^{-3}				
γ_2	2.028×10^{-4}	-9.589×10^{-4}	-2.060×10^{-5}			
γ_3	-4.613×10^{-6}	1.428×10^{-5}	-2.287×10^{-5}	2.243×10^{-7}		
γ_4	1.148×10^{-7}	-3.249×10^{-7}	2.529×10^{-7}	-5.468×10^{-7}	-2.165×10^{-9}	
γ_5	-2.794×10^{-9}	8.091×10^{-9}	-5.204×10^{-9}	4.933×10^{-9}	-1.271×10^{-8}	-3.18×10^{-11}
γ_6	6.70×10^{-11}	-1.97×10^{-10}	-1.33×10^{-10}	-8.43×10^{-11}	1.04×10^{-10}	-3.19×10^{-11}
γ_7	-1.61×10^{-11}	4.72×10^{-12}	-3.29×10^{-12}	2.18×10^{-12}	-1.41×10^{-12}	2.27×10^{-12}
It'n γ_n	.4714	3.126×10^{-3}	4.322×10^{-5}	7.661×10^{-7}	1.500×10^{-8}	

Table 5. Values of the different terms in the γ_n expansions calculated. The results for the γ_n from the Iteration Method are shown in the last row for comparison.

Table 6 shows the size of the errors when the first two terms and the first three terms are used to approximate γ_n . The results from the Iteration Method of section 2 have been used as standards. The comparisons are made for $\sigma = 1$ and $\sigma = 4$. For each γ_n , the results from summing the first two and three terms are divided by the results of the Iteration Method. For $\sigma = 1$ the error from using three terms is always less than one percent. The error from using two terms is greater. However in most cases the first two terms alone are sufficient to represent γ_n . Even though the error is about ten percent for γ_4 for the case when $\sigma = 1$, this will not really affect the mobility calculations very much. This is because the contribution to the mobility by each γ_n becomes smaller as σ gets larger. The main contributions to the drift mobility come from γ_0 and γ_1 .

Since the leading two terms are enough to give a good approximation for a given γ_n , the Perturbation Method is simpler than the Iteration Method.

Ratio \ γ_n	γ_0	γ_1	γ_2	γ_3	γ_4
$\frac{\text{Perturb'n } \gamma_n \text{ (2 terms)}}{\text{Iteration } \gamma_n}$ $\sigma = 1$	1.0146	1.039	1.067	1.092	1.113
$\frac{\text{Perturb'n } \gamma_n \text{ (3 terms)}}{\text{Iteration } \gamma_n}$ $\sigma = 1$	1.0013	1.0036	1.0033	1.0011	0.9981
$\frac{\text{Perturb'n } \gamma_n \text{ (2 terms)}}{\text{Iteration } \gamma_n}$ $\sigma = 4$	1.0004	1.0044			
$\frac{\text{Perturb'n } \gamma_n \text{ (2 terms)}}{\text{Iteration } \gamma_n}$ $\sigma = 4$	.99999	.99987			

Table 6. Ratio of the γ_n from the Perturbation Method to the γ_n calculated by Iteration Method. The ratios tabulated are for $\sigma = 1$ and 4, at $\tilde{\xi} = 0.7$. For each σ the results from using two and three terms in the expansions are compared.

3.2 FAILURE OF THE PERTURBATION METHOD

For $\tilde{\xi}$ near zero, the Perturbation Method fails to approximate γ_n . The magnitudes of the γ_n are very large when $\tilde{\xi}$ approaches zero. The failure is more apparent for $\sigma = -1$ but is also true for the other σ . The sign of the γ_n can either be positive or negative, and can be different when a different number of terms is summed. This suggests that the expansions for γ_n are divergent at $\tilde{\xi}$ near zero.

For $\tilde{\xi}$ small, δ is no longer small compared to $\tilde{\xi}$. The Perturbation Method fails when this happens because terms kept for the lowest order in γ_n are smaller than terms dropped. This result can be most easily seen by truncating equation (2.10) to two equations. The results for γ_0 and γ_1 from solving the two equations are

$$\gamma_0 = \frac{B_1 D_0 - C_0 D_1}{B_0 B_1 - C_0 A_1} \quad (3.21)$$

$$\gamma_1 = \frac{B_0 D_1 - A_1 D_0}{B_0 B_1 - A_1 C_0} \quad (3.22)$$

When both δ and $\tilde{\xi}$ are small, two expansions can be made of γ_0 and γ_1 : an expansion in δ and an expansion in $\tilde{\xi}$. If δ and $\tilde{\xi}$ are very small, only the leading term of these quantities in each expansion has to be kept. Since two expansions in a small parameter are made, the order in which the expansions are performed can make a difference. To check the difference an expansion first in δ and then one in $\tilde{\xi}$ is performed, and next the order in which the expansions are done is reversed. The check is done on γ_0 and γ_1 . To obtain the leading terms in δ , the limit as δ

approaches zero is taken for γ_0 and γ_1/δ . For the leading terms in ξ , the functions that approximate γ_0 and γ_1 are expanded in ξ and the leading terms are extracted. The results for the two different orders in which the leading terms are obtained are given in table 7.

	γ_0	γ_1
$\lim_{\xi \rightarrow 0} [\lim_{\delta \rightarrow 0} (\gamma_n)]$	$-\frac{\sqrt{\xi}\sigma}{2} \left[1 + \frac{1}{3\sinh^{-1}(1)} \right]$	$\frac{\xi}{3}$
$\lim_{\delta \rightarrow 0} [\lim_{\xi \rightarrow 0} (\gamma_n)]$	$\frac{-\sqrt{\xi}\sigma}{2}$	$\frac{-\sigma}{2\sinh^{-1}(1)}$

Table 7. The limiting forms of γ_0 and γ_1 when different order in limiting procedures are used.

Both γ_0 and γ_1 approach different limiting forms for the two limiting procedures. However for γ_0 the error is not so significant since in both limiting procedures γ_0 approaches zero. The difference in γ_1 is more significant since the first limiting procedure leads to zero instead of a constant. The Perturbation Method gives a result which corresponds to the limiting procedure in row one of table 7. For ξ near zero the limiting procedure in the second row is correct. Therefore the Perturbation Method fails near ξ equal zero. The behaviour of other γ_n can be examined in the same way. One would find that the Perturbation Method fails near ξ equal zero as for γ_0 and γ_1 .

The Perturbation Method fails when the leading terms of the two limiting procedures are of the same order. In the case of γ_1 , this occurs when

$$\tilde{\xi} = \left[\frac{3\delta}{2} \right]^{3/2} \sigma. \quad (3.23)$$

Since δ is a decreasing exponential of σ , as σ increases, the right hand side of (3.23) decreases rapidly. Therefore as σ increases, the point where the perturbation fails moves closer to $\tilde{\xi}=0$. This explains why the failure does not show up in the γ_n plots for σ greater than one. If the horizontal scale, ξ , is expanded near $\tilde{\xi}=0$, the failure is expected to appear.

Since the value of the mobility for the carriers is proportional to the area under the γ -curve, the Perturbation Method still can give a good approximation. This result follows because the region where the perturbation fails badly represents only a small fraction of $\tilde{\xi}$ between zero and σ . The small section can be omitted in the integration of γ_n without seriously affecting the values of the mobility calculated. As long as the integration does not include the point $\tilde{\xi}=0$, the results of the mobility should be good, especially when σ is large.

4 INTRODUCTION TO ANALYTICAL METHOD

Analytical expressions for $\gamma(\xi, \sigma)$ can be derived via two different approaches. These two approaches will be discussed in this section.

The first approach is to derive the different γ_n directly from the truncated set of equations in (2.10). For a given n , this set of equations can be solved simultaneously for each γ_n in terms of A_n , B_n , C_n , and D_n . Efforts can be made to simplify the result of each γ_n to a simple analytical form. One of the problems of this approach is the complexity involved when more equations in (2.10) are used. However, for accurate results a large number of equations have to be used. This will lead to an expression for γ_n which is very complicated and cannot be simplified easily.

The second approach is to derive the analytical formula from the result of the perturbation method in section 3. In the perturbation method, the formula for any γ_n^m in a given γ_n expansion has been derived and is given by one of the formulas in (3.13)-(3.20). Each of these γ_n^m is simplified separately and they are combined together for each γ_n . This method is expected to be more efficient because in many cases, only the first few γ_n^m are necessary for sufficient accuracy. Moreover, it is easier to simplify each γ_n^m than the γ_n in the first method.

4.1 ANALYTICAL FORMULA FROM THE ITERATION METHOD.

In this section the method of deriving analytical formulas for γ_n by solving the truncated set of equations, (2.10), is discussed. The initial step is to decide the number of equations to be used. In this

method the accuracy of the result for the electron distribution function depends on the number of γ_n used and the accuracy of each γ_n . Therefore, as many equations as possible should be included in solving for the different γ_n . However, the number of equations used is limited because a point is reached when the results for these γ_n found by solving the set of equations will be too complicated to be solved easily. In this report only the first four equations will be used.

If (2.10) is truncated at $n=3$, the four equations are shown below:

$$\begin{aligned} B_0\gamma_0 + C_0\gamma_1 &= D_0 \\ A_1\gamma_0 + B_1\gamma_1 + C_1\gamma_2 &= D_1 \\ A_2\gamma_1 + B_2\gamma_2 + C_2\gamma_3 &= D_2 \\ A_3\gamma_2 + B_3\gamma_3 &= D_3. \end{aligned} \tag{4.1}$$

These equations are solved simultaneously and the expressions for γ_0 , γ_1 , γ_2 , and γ_3 are shown in (4.2)-(4.5):

$$\gamma_0 = \frac{D_0[(B_2B_3 - C_2A_3)B_1 - B_3A_2] - C_0D_1(B_2B_3 - C_2A_3) + C_0C_1(B_3D_2 - C_2D_3)}{B_1B_0(B_2B_3 - C_2A_3) - B_0B_3A_2 - C_0A_1(B_2B_3 - C_2A_3)} \tag{4.2}$$

$$\gamma_1 = \frac{(B_2B_3 - C_2A_3)(D_1 - A_1\gamma_0) - C(B_3D_2 - C_2D_3)}{B_1(B_2B_3 - C_2A_3) - B_3A_2} \tag{4.3}$$

$$\gamma_2 = \frac{B_3D_2 - C_2D_3 - B_3A_2\gamma_1}{B_2B_3 - C_2A_3} \tag{4.4}$$

$$\gamma_3 = (D_3 - A_3\gamma_2)/B_3. \tag{4.5}$$

As an initial step in simplifying the γ_n , the terms in each expression

can be grouped in terms of δ . Consider the expression for γ_0 in (4.2). The A_n , B_n , C_n and D_n are replaced by A_n^0 , $B_n^0 + B_n^{-1}/\delta$, and C_n^{-1}/δ and D_n^n respectively. These factors have been defined in (3.2)-(3.6). Since δ is small for most of the range of σ , it is assumed that only those terms with δ^{-2} and δ^{-3} dominate. Other terms are assumed to be relatively small and dropped. This approximate form of γ_0 with terms grouped in power of δ is shown in (4.6).

$$\gamma_0 = \left\{ D_0^0 B_1^{-1} B_2^{-1} B_3^{-1} + \delta \left[D_0^0 (B_1^0 B_2^{-1} B_3^{-1} + B_1^{-1} B_2^{-1} B_3^0 - A_3^0 C_2^{-1} B_1^{-1}) - D_1^1 C_0^{-1} B_2^{-1} B_3^{-1} \right] \right. \\ \left. B_0^0 B_1^{-1} B_2^{-1} B_3^{-1} - A_1^0 B_2^{-1} B_3^{-1} C_0^{-1} + \delta \left[B_0^0 (B_1^0 B_2^{-1} B_3^{-1} + B_1^{-1} B_2^0 B_3^{-1} + B_1^{-1} B_2^{-1} B_3^0) \right. \right. \\ \left. \left. - A_3^0 C_2^{-1} (B_0^0 B_1^{-1} - C_2^{-1} A_3^0) - A_1^0 C_0^{-1} (B_2^0 B_3^{-1} + B_2^{-1} B_3^0) \right] \right\}^{-1} \quad (4.6)$$

Since the objective here is to arrive at a simple analytical expression for γ_0 , (4.6) has to be simplified further. A new variable $y = \sqrt{(\xi/\sigma)}$, is introduced for the convenience of the simplification process. The inverse hyperbolic sine functions in A_n^0 , B_n^0 , B_n^{-1} , and C_n^{-1} are expanded into Taylor series about $y=0$. Further simplifications are made so that they can be written in polynomials in powers of y . These polynomials are substituted into (4.6) and the numerator and the denominator are simplified separately. A relatively simple expression for γ_0 can be obtained by retaining the few terms with low order in y and δ . The final form of γ_0 is shown in (4.7). The details of the simplification process are shown in the Appendix 3.

$$\gamma_0 = \frac{-(\xi\sigma)^{1/2} e^{-\xi}}{2} \left[\frac{y(1.01 + 0.4912y^2) + 1.5637\delta}{y(1.01 - 0.1260y^2) + 0.8904\delta} \right] \quad (4.7)$$

The same procedure is used to obtain analytical expressions for γ_1 and

γ_2 .

$$\gamma_1 = \frac{-\sigma e^{-\tilde{\xi}-\sigma}}{4} \left[\frac{0.67336 + 0.4488y^3}{0.3366y + 0.29688} \right] \quad (4.8)$$

$$\gamma_2 = \frac{-\sigma e^{-\tilde{\xi}-3\sigma}}{3.57488 + 4.0542y} \quad (4.9)$$

Only the first three γ_n will be used in this report.

4.2 ANALYTICAL FORMULA FROM THE PERTURBATION METHOD.

In this section the analytical formula for γ_n will be derived from the results of the Perturbation Method. From section 3, it is apparent that for the range of σ of interest here, the first two terms or three terms in each γ_n expansion will provide a very good approximation for each γ_n . Therefore only the analytical expressions for the first few γ_n need to be found.

The first analytical expression to be solved is the leading term of the γ_0 expansion. The formula for γ_0^0 is given in (3.13). After the substitution of A, B, C, and D from (2.5)-(2.8), the γ_0^0 written in terms of y is shown below

$$\gamma_0^0 = \frac{2\sinh^{-1}(y)\tilde{\xi}e^{-\tilde{\xi}}}{[2\sinh^{-1}(y)]^2 - \left[1 - \frac{(2y^2+1)\sinh^{-1}(y)}{\sqrt{y^2(y^2+1)}} \right]^2} \quad (4.10)$$

The inverse hyperbolic sine is expanded into Taylor series about $y=0$, and the result can further be simplified as shown in the Appendix 3. The final form of γ_0^0 written as a polynomial in $\tilde{\xi}/\sigma$ is shown in (4.11)

$$\gamma_0^0 = -\frac{(\tilde{\xi}\sigma)^{1/2}}{2} e^{-\tilde{\xi}} \left(1 + \frac{11\tilde{\xi}}{18\sigma} \dots \right) \quad (4.11)$$

In this case only the first two non-zero terms have been shown. With γ_0 found, it can be used to find γ_1^1 by using equation (3.15). The A_1^0 and B_1^{-1} in that equation are expanded into Taylor series about $x=0$ as before. A polynomial similar to (4.11) is obtained for γ_1^1 by simplifying the resulting expression. This process can be repeated for $\gamma_2^2, \gamma_3^3, \dots, \gamma_n^n$ through equations (3.16)-(3.20). The results for γ_1^1, γ_2^2 and γ_3^3 are shown below:

$$\gamma_1^1 = -\tilde{\xi} e^{-\tilde{\xi}} (1/3 - 0.05925\tilde{\xi}/\sigma + \dots) \quad (4.12)$$

$$\gamma_2^2 = -\tilde{\xi} e^{-\tilde{\xi}} (0.1644 - 0.01716\tilde{\xi}/\sigma + \dots) \quad (4.13)$$

$$\gamma_3^3 = -\tilde{\xi} e^{-\tilde{\xi}} (0.09605 - 0.00524\tilde{\xi}/\sigma + \dots). \quad (4.14)$$

In order to obtain γ_0^1 from equation (3.16), the results for γ_1^1 and γ_2^2 may be used. As in the previous cases, the coefficients A_1^1, C_0^{-1}, B_0^0 and B_1^{-1} are expanded into Taylor series. When these series along with (4.12) and (4.13) are substituted into (3.16), the resulting expression can be simplified to the form shown in (4.15). The same procedure is repeated for γ_1^2 and γ_2^3 by using equations (3.18) and (3.19) respectively

$$\gamma_0^1 = -e^{-\tilde{\xi}} (\sigma/3 + 0.1967\tilde{\xi} + \dots) \quad (4.15)$$

$$\gamma_1^2 = -\sigma e^{-\tilde{\xi}} (1.346 + 0.2242\tilde{\xi}/\sigma + \dots) \quad (4.16)$$

$$\gamma_2^3 = -\sigma e^{-\tilde{\xi}} (0.8724 + 1.3052\tilde{\xi}/\sigma + \dots). \quad (4.17)$$

Before higher order γ_n^1 are calculated, it is useful to see how each γ_n^1 will contribute to the drift mobility calculation. From figure 1, it is apparent that γ_0 and γ_1 will contribute most significantly to the drift mobility. As n gets larger, the contribution from a given γ_n becomes smaller. When σ is greater than two, the contribution is mainly from

γ_0 alone. Therefore it is not necessary to calculate γ_n to more than the first few n . For each γ_n , it can be seen from Table 6 that if only the leading two terms in each expansion are used, the approximations are very good. In order to get simple and compact analytical formulas, only the two leading terms are used for each expansion of γ_0 , γ_1 , γ_2 and γ_3 . These analytical formula are shown below:

$$\gamma_0 = -\left(\frac{\tilde{\xi}}{\sigma}\right)^{1/2} e^{-\tilde{\xi}} \left(1 + \frac{11\tilde{\xi}}{18\sigma}\right) - e^{-\tilde{\xi}-\sigma} (\sigma/3 + 0.1967\tilde{\xi}) \quad (4.18)$$

$$\gamma_1 = -\tilde{\xi} e^{-\tilde{\xi}-\sigma} (.3333 - 0.0592\tilde{\xi}/\sigma) - \sigma e^{-\tilde{\xi}-2\sigma} (1.1346 + 0.2242\tilde{\xi}/\sigma) \quad (4.19)$$

$$\gamma_2 = -\tilde{\xi} e^{-\tilde{\xi}-2\sigma} (0.1644 - 0.01716\tilde{\xi}/\sigma) - \sigma e^{-\tilde{\xi}-3\sigma} (0.8724 + 1.305\tilde{\xi}/\sigma) \quad (4.20)$$

$$\gamma_3 = -\tilde{\xi} e^{-\tilde{\xi}-3\sigma} (0.09605 - 0.00524\tilde{\xi}/\sigma). \quad (4.21)$$

4.3 $\gamma(\xi, \sigma)$ FROM THE ANALYTICAL METHODS.

The resulting γ_n from the analytical formulas derived from the Iteration Method and the Perturbation Method are plotted in figures 3 and 4.

In figure 3, the γ_n are calculated from equation (4.7)-(4.9), and the results plotted against ξ for σ equal to 1, 2, 3, 4 and 5.45. There are some noticeable differences between the γ_n in figure 3 and those in figures 1 and 2. This is especially true for smaller σ . The analytical formulas that are derived from (2.10) can be good approximations when σ is large. However when it is small, the approximation is not as good.

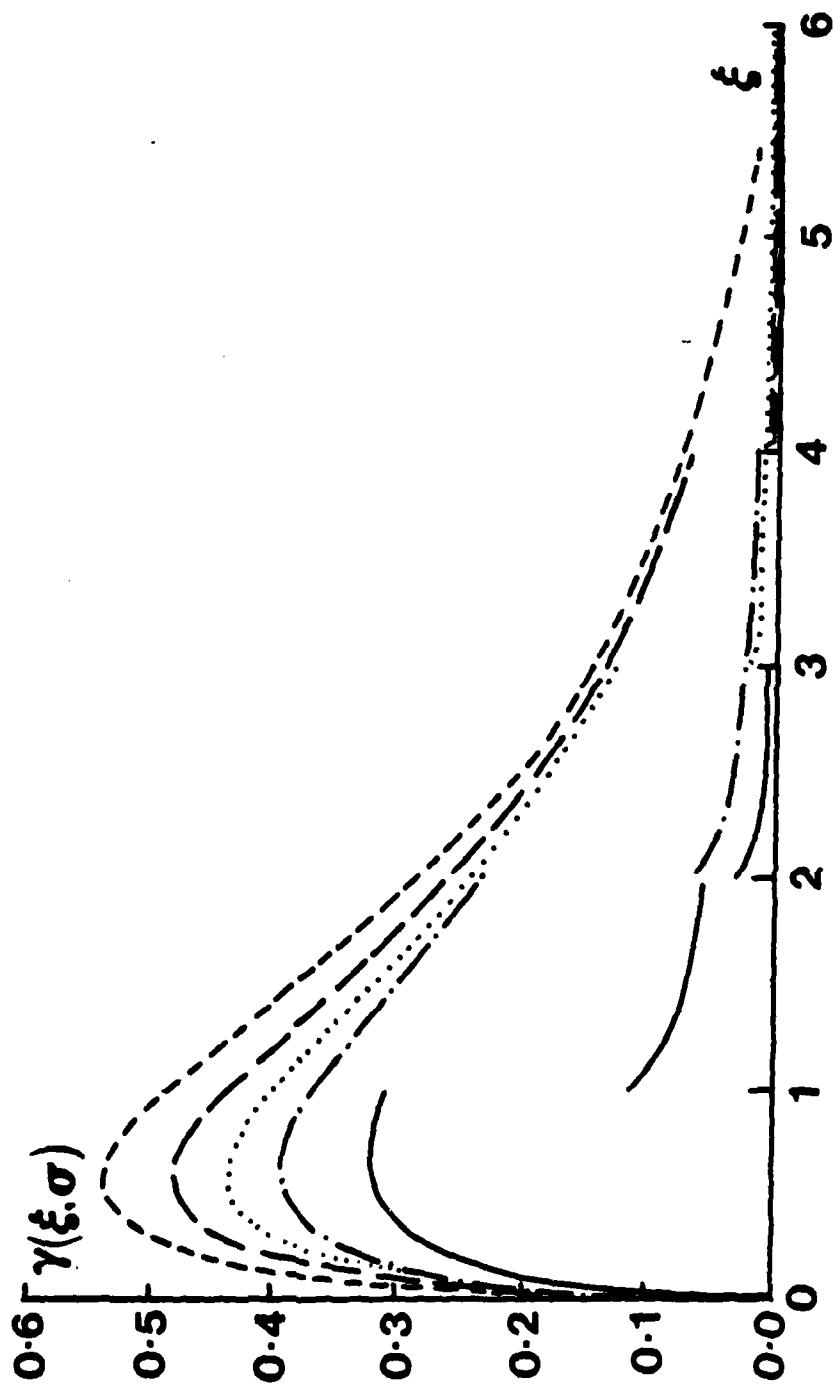


Figure 3. The perturbed distribution function $-\gamma(\xi, \sigma)$ for the case when the analytical formula derived from the Iteration Method is used. The representations of the different curves is the same as that in figure 1.

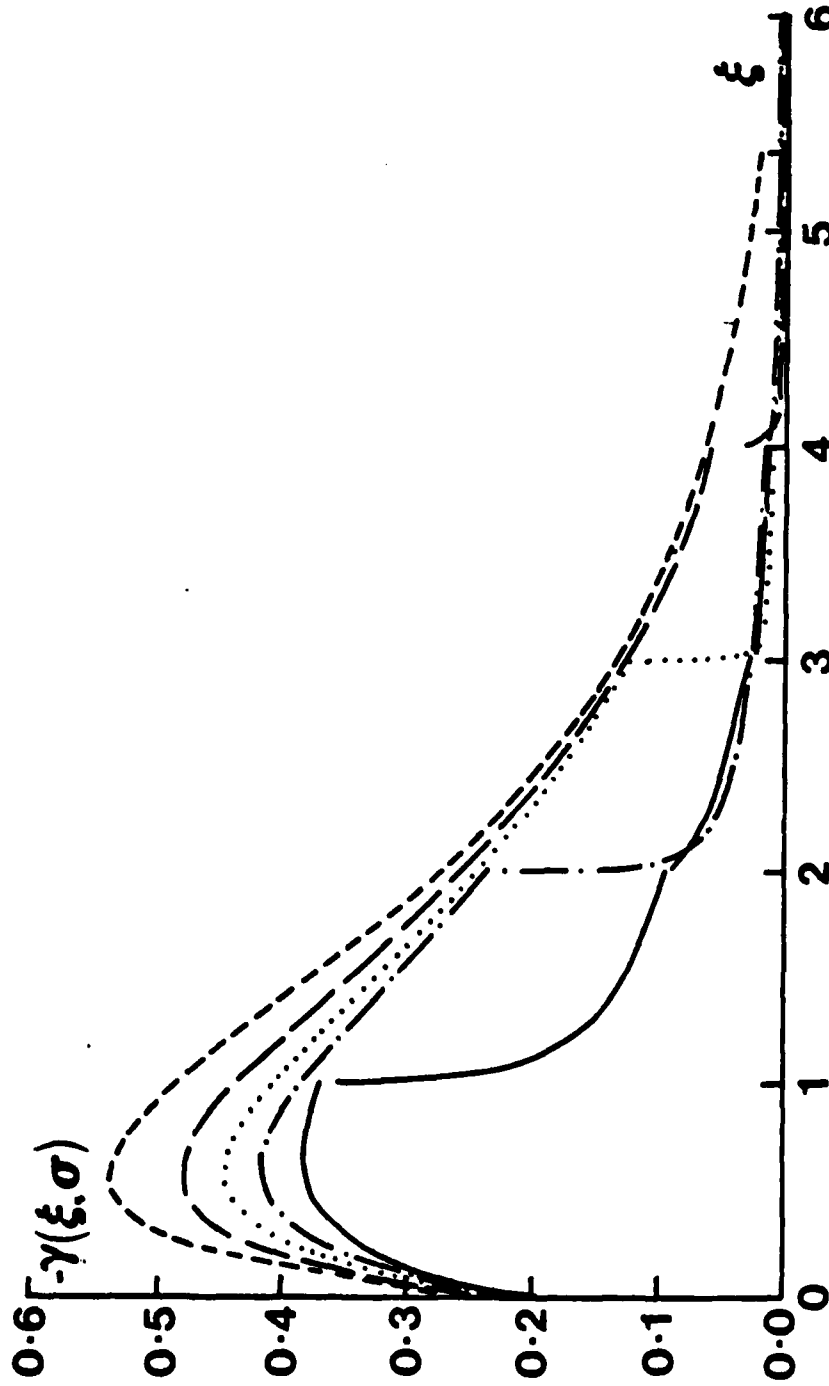


Figure 4. The perturbed distribution function $-\gamma(\xi, \sigma)$ for the case when the analytical formula derived from the Perturbation Method is used. The representations of the different curves is the same as that in figure 1.

The problem in trying to derive analytical formulas directly from the set of equations in (2.10) is the complexity involved in simplifying equations (4.2)-(4.5). As discussed earlier a larger number of equations in (2.10) are needed for more accurate γ_n , but it will lead to some very complicated expressions that are hard to simplify. In this case only four equations have been used and γ_0 is very complicated as given in (4.2). In reducing (4.2) to any simple analytical form, some of the terms dropped turn out to be significant. There appears to be no alternative because there is no single expansion parameter that will enable (4.2) to be simplified easily. Therefore simplicity is achieved at the expense of complexity.

In figure 4 the result of the γ_n calculated from the analytical formulas derived from the Perturbation Method are plotted against ξ for σ equals 1, 2, 3, 4 and 5.45. The results obtained from these analytical formulas are very close to those obtained in figures 1 and 2. The only noticeable difference is at $\xi=0$ where the γ_0 calculated from the analytical formula in this case does not go to zero. The failure of the Perturbation Method as discussed in section 3 is the root of this problem. Since the drift mobility is proportional to the area under the γ_n curve, the failure is not going to affect the drift mobility calculation significantly. The region of failure is too small to have any significant effect.

The analytical formulas which are derived from the Perturbation Method are simpler because each γ_n^1 can be calculated separately. The accuracy of each γ_n calculated is independent of the number of equations

in (2.10). The formulas for the different γ_n^1 given in (3.13)-(3.20) are relatively simple and easy to simplified. Another advantage of this method is that only the first two terms in each expansion are needed to provide a good approximation for each γ_n . The resulting analytical formulas are also more compact and simple.

5. FACTOR G AND MOBILITY.

Once $\gamma(\xi, \sigma)$ has been found, the electron mobility can be found by the relation [14]

$$\mu = \frac{-4\hbar(2\pi k_B T)^{1/2}(e^\sigma - 1)}{3\pi e m^*{}^{3/2}(1/\epsilon_\infty - 1/\epsilon_0)} \int_0^\infty \gamma(\xi) d\xi \quad (5.1)$$

For comparisons between the various method, a function, $G(\sigma)$, which is defined in the form shown in (5.2) will be used.

$$G(\sigma) = \int_0^\infty \gamma(\xi, \sigma) \xi d\xi. \quad (5.2)$$

This function has been used by Fortini et al. in their paper. A comparison between the factor $G(\sigma)$ obtained by the different methods is shown in figure 5. The values of some of the $G(\sigma)$ are tabulated in table 8. Some of the values obtained by Fortini et al. has been included for comparison. The number of equations used in the Iteration Method for obtaining the factor $G(\sigma)$ is twelve. In the Perturbation Method, the number of γ_n^m summed is different for different γ_n . For $\sigma=1$, eight (i.e. $n=0$ to 7) terms are used and the number is decreased as σ increases, such that for a given σ the γ_n used are $\gamma_0, \gamma_1, \gamma_2, \dots$ where $n=7-\sigma$ for each integer σ . The number of terms summed for each γ_n also vary with n as described in section 4.

From table 8, it can be seen that the results from the Iteration Method used in this report are smaller than that obtained by Fortini et al [14]. The only exception is when σ is equal to one. The results obtained in this report are smaller because fewer equations are used

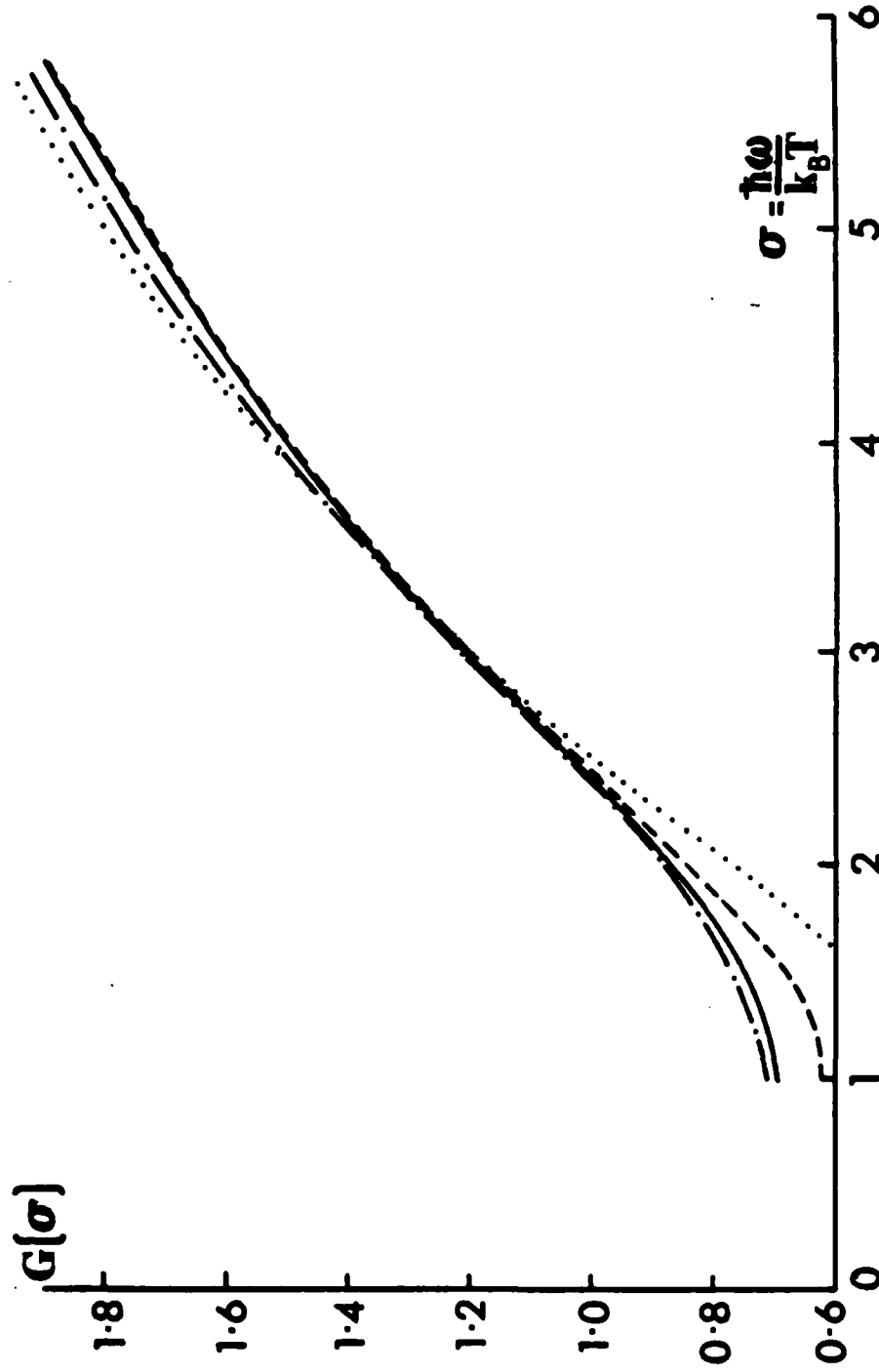


Figure 5. The mobility function $-G(\sigma)$ calculated from the four different methods plotted against σ . The result of the Iteration Method is represented by the solid line, the result of the Perturbation Method is represented by the dashed line, the analytical formula derived from the Iteration Method is shown by the dotted line and the analytical formula derived from the Perturbation Method is represented by the dash dotted line.

σ	Fortini's $G(\sigma)$	Iteration $G(\sigma)$	Pertur'n $G(\sigma)$	Analytic 1 $G(\sigma)$	Analytic 2 $G(\sigma)$
1	0.6770	0.7005	0.62668	0.27689	0.7006
1.5	0.8752	0.7571	0.6880	0.5226	0.7738
2	0.8752	0.8787	0.8411	0.7648	0.8793
2.5	1.0321	1.0304	1.0135	0.9917	1.0332
3.0	1.1908	1.1922	1.1845	1.1981	1.2050
3.5	1.3484	1.3469	1.3441	1.3816	1.3724
4.0	1.4926	1.4932	1.4824	1.5413	1.5246
4.5	1.6173	1.6190	1.6186	1.6784	1.6584
5.0	-	1.7322	1.7301	1.7953	1.7745

Table 8. The values of the function $G(\sigma)$ from the different methods for $\sigma = 1$ to 5 are shown. The results of Fortini et al. are included for comparison.

here for the calculations of $G(\sigma)$. The error that results from using fewer equations is more serious for smaller σ . The results improve as σ increases and the error is less than one percent beyond $\sigma = 2.5$. This is also an indication of the fewer equations needed for the Iteration Method when σ is large. The reason why the result obtained by the Iteration Method here is larger than the result of Fortini et al when $\sigma=1$, can also be explained in the same manner. If one looks at the plot of $G(\sigma)$ in the paper of Fortini et al which extended the $G(\sigma)$ to σ smaller than one, one notices that at $\sigma=1$, $G(\sigma)$ is near a minimum. Therefore it is conceivable that when not enough equations are used in calculating $G(\sigma)$, the minimum will be higher. In this report σ is not extended to values smaller than one because the mathematical approximation is poor for $\sigma < 1$.

The Perturbation Method has results that match closely those from the Iteration Method for the whole range of σ except for the small region near $\sigma = 1$. This is not surprising since it has already been shown that the Perturbation Method works best for larger σ . For σ greater than two the Perturbation Method gives results that differ from the Iteration Method by less than five percent. The differences drop to less than 1 percent beyond $\sigma = 3$.

The first Analytical Method in (Section 4.1) is worst in approximating the factor $G(\sigma)$. At $\sigma = 1$ the value of $G(\sigma)$ obtained by this method is about half of the expected value. However, the approximation becomes better as σ increases, and the best approximation is near $\sigma = 3$. For higher σ , this Analytical Method overestimates the function $G(\sigma)$ by about five percent. The values of $G(\sigma)$ have the largest deviation

from the expected result when σ is small because in that region, the analytical expressions as given in (4.20), (4.22) and (4.23), carry too few terms to make a good approximation. Moreover for smaller σ , the convergence of the γ_n expansions is slower and more than just the first three terms are needed in each expansion for a better approximation.

The analytical formula that is derived from the Perturbation Method (Section 4.2) gives a good approximation for the whole region of σ . The best results are between $\sigma = 2$ and 3.5. For most of the range the analytical formulas over estimate the factor $G(\sigma)$. However on the whole the result matches that of the Iteration Method's results rather well. The error is less than 5 percent for the whole range shown. The accuracy that is found for the region where σ is small is unexpected. Since the analytical formula is derived from the perturbation formula with some approximations, it is not expected to give results that are more accurate than the Perturbation Method. In this case it appears that the errors from the Perturbation Method and the errors from the approximations made when deriving the analytical formula compensate each other in the region where σ is small.

After the value of $G(\sigma)$ for a given σ has been calculated, the drift mobility, μ , at that temperature can be calculated by substituting into (5.1). The drift mobility in (5.1) can be written as

$$\mu = -C_m T^{1/2} (e^{\sigma} - 1) G(\sigma). \quad (5.3)$$

The factor C_m is a constant which depends on the measured parameter m^* , ϵ_m and ϵ_s . For GaAs Fortini et al use a value of 233 for C_m but it is found that with the values of the parameters given by them ($m^* = 0.073 m_0$,

$\epsilon_1=11.10$ and $\epsilon_2=13.13$), the value for C_m is 227. In figure 5, the drift mobility for GaAs is plotted against the temperature with $C_m=227$. As in the case of the plot of factor $G(\sigma)$ versus σ , the analytical formula derived from the Iteration Method shows the largest deviation from the Iteration Method.

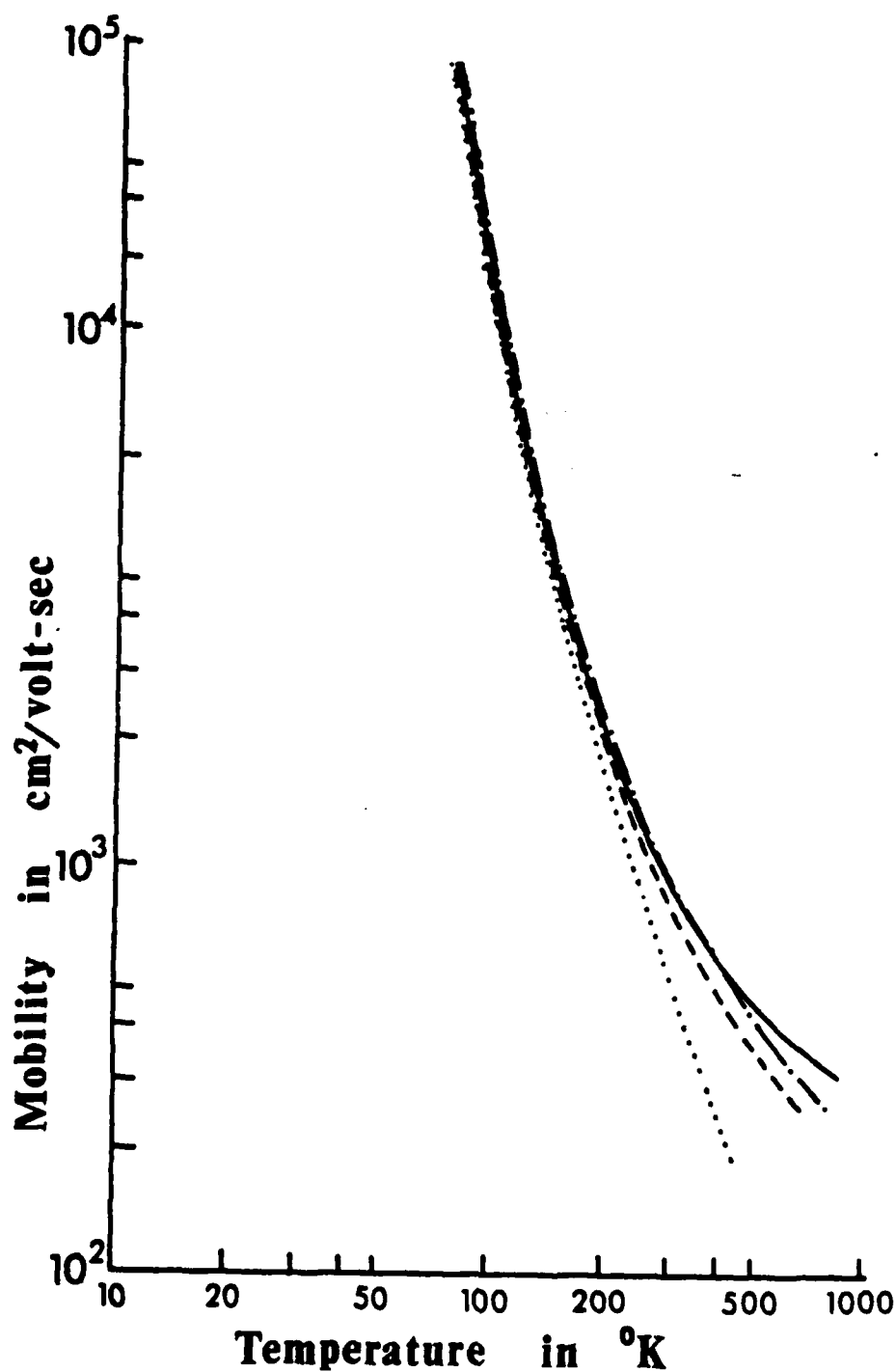


Figure 6. The drift mobility for GaAs calculated from the four methods are plotted against the temperature. The result of the Iteration Method is denoted by the solid line, the result of the Perturbation Method is denoted by the dashed line, the result of the analytical formulas derived from the Iteration Method and Perturbation Method are shown by the dotted line and dash-dotted line respectively.

6. ELECTRON DISTRIBUTION FUNCTIONS FOR HIGH ELECTRIC FIELDS

For high electric fields, the carrier distribution function is found by solving the Boltzmann equation. The validity of using the Boltzmann equation for high electric fields has not been fully investigated. However, it seems for the range of the fields that is discussed in this report, the Boltzmann equation can be used.[26]

In III-V compounds, at high electric fields near room temperature, the scattering is still predominantly polar optical in nature as in the low field case. Other forms of scattering such as impurity scattering, space charge scattering and acoustic scattering are unimportant and will be neglected. For electron energies that are greater than the valley separation between the central and next higher valley in the conduction band, intervalley scattering has to be included. In this report intervalley scattering is assumed to be unimportant. Conwell [26], Conwell and Vassell [6] have discussed the various approaches that can be taken when intervalley and other forms of scatterings are to be included.

For a given electric field $\vec{E}$, the solution of the Boltzmann equation may be expanded in a series of Legendre polynomials, $P_n(\cos\theta)$, where θ is the angle between $\vec{E}$ and the wave vector $\vec{k}$. The distribution function is nearly spherically symmetric if the fraction of the carrier energy lost in a collision is small and if the scattering is not predominantly in a given direction [27]. It has been shown that under these conditions the first term in the series of Legendre polynomials is dominant, while the third and higher order terms in the series are very

small [28]. Therefore for a simple model, the distribution function can be approximated quite accurately by

$$f(\vec{k}) = f_0(\mathcal{E}) + f_1 \cos \theta = f_0(\mathcal{E}) + k_E g(\mathcal{E}) \quad (6.1)$$

where k_E is the component of $\vec{k}$ in the direction of $\vec{E}$.

In general polar optical scattering is non-elastic. For III-V compounds such as GaAs, at high electric fields, the carrier energy is much larger than the optical phonon energy and the scattering is reasonably elastic and isotropic. In most cases other forms of scattering such as impurity, acoustic and space charge scattering are also present. Some of these scattering processes are elastic and randomize the electrons with lower energies so that (6.1) still holds. In this report, the electron distribution function is assumed to be of the form shown in (6.1). Unlike the low field case, the spherically symmetric part of (6.1) is not the equilibrium Maxwell-Boltzmann distribution function but is a function in which the heating of the electrons by the strong electric field produces an elevated effective electron temperature.

6.1 SOLUTION FOR HIGH ELECTRIC FIELDS OF THE ELECTRON DISTRIBUTION FUNCTIONS

In this section the electron distribution function in high electric fields for polar optical scattering is calculated from the Boltzmann equation, (1.1). The method used by Conwell and Vassell [6] and Stratton [5] is discussed briefly. In this method the difference-differential equations are reduced to differential equations by Taylor

expansions of the distribution functions. A new method is presented in which the coupled difference-differential equations for the symmetric and asymmetric parts of the distribution function are derived by expansions in $1/E$. The solution from both methods is compared.

The collision term and electric field term of the Boltzmann equation for an arbitrary energy band have been derived by Conwell and Vassell [6]. The energy band is assumed to be parabolic in this section. The energy of the carrier electrons is assumed to be much larger than the longitudinal optical phonon energy. Since (1.11), (1.12) and (1.16) are for an arbitrary energy band, they can be simplified and substituted into the Boltzmann equation for the case of a parabolic energy band. For parabolic energy bands and with p-wave scattering neglected, Ω is equal ξ , and c and c' are zero. Therefore (1.10), (1.11) and (1.15) reduce to (6.2), (6.3) and (6.4) respectively:

$$\left(\frac{df_0}{dt}\right)_{\text{sym}} = \frac{eE_0 N}{\sqrt{(2m^* \hbar \omega x)}} \left\{ \left[e^{\sigma} f_0(x+1) - f_0(x) \right] \phi(x) + H(x-1) \left[f_0(x-1) - e^{\sigma} f_0(x) \right] \phi(x-1) \right\} \quad (6.2)$$

$$\left(\frac{dk_E g}{dt}\right)_{\text{asy}} = \frac{eE_0 \cos N}{\hbar} \left(e^{\sigma} g(x+1) \left[\frac{(2x+1)}{2\sqrt{x(x+1)}} - 1 \right] - g(x) \phi(x) + H(x-1) \left\{ g(x-1) \left[\frac{(2x-1)}{2\sqrt{x(x-1)}} - 1 \right] - g(x) e^{\sigma} \phi(x-1) \right\} \right) \quad (6.3)$$

$$\left(\frac{df}{dt}\right)_E = -eE \left[\frac{2}{3\hbar\sqrt{x}} \frac{d(x^{3/2}g)}{dx} + \frac{\hbar k \cos \theta}{m^*} \frac{df}{dx} \right] \quad (6.4)$$

where $H(x-1)$ is the unit-step function, and $\phi(x)$ and x are defined as

$$\phi(x) = 2 \sinh^{-1}(x)^{1/2} \quad (6.5)$$

$$x = \xi/\sigma \quad (6.6)$$

and $f_0(x)$ does not have the same argument as $f_0(\xi)$ but is used for simplicity. Equations (6.2)-(6.4) have been derived previously [6].

An analytical solution for the electron distribution function from equations (6.2)-(6.4) has been derived by Stratton [5] and Conwell and Vassell [6] for $x \gg 1$. They solved the equations by expanding $f_0(x+1)$ and $g(x+1)$ in Taylor series. This transforms the coupled difference-differential equations into differential equations. The inverse hyperbolic function, $\phi(x)$, is expanded in terms of $\ln(4x)$ and a series in $1/x$. By retaining terms up to second order for the Taylor expansion and by keeping terms up to $1/x^2$ in the expansion for $\phi(x)$, the symmetrical part and the asymmetrical part of the collision term are reduced to [5,6]

$$\left(\frac{df_0}{dt}\right)_{\text{sym}} = \frac{-eE_0}{\sqrt{2m^*\hbar\omega x}} \left\{ \frac{d}{dx} \left[\frac{1(e^\sigma + 1)\ln(4x)}{2(e^\sigma - 1)} \frac{df_0}{dx} \right] + f_0 \ln(4x) \right\} \quad (6.7)$$

$$\left(\frac{dk_{Eg}}{dt}\right)_{\text{asy}} = - \frac{eE_0}{\sqrt{2m^*\hbar\omega x}} \left(\frac{e^\sigma + 1}{e^\sigma - 1} \right) k_{Eg} \quad (6.8)$$

These equations are valid only for very large x . From (6.8) a relaxation time for the case when $x \gg 1$ can be written and is given by [5,6]

$$\tau = \frac{-k_{Eg}}{\left(\frac{dk_{Eg}}{dt}\right)_{\text{asy}}} = \frac{\sqrt{2m^*\hbar\omega x}}{eE_0} \left(\frac{e^\sigma - 1}{e^\sigma + 1} \right) \quad (6.9)$$

Equations (6.4), (6.7) and (6.8) can be substituted into the steady state Boltzmann equation to solve for $f_0(x)$ and $g(x)$. Two coupled

differential equations in $f_0(x)$ and $g(x)$ are obtained by equating the coefficients of the first two Legendre polynomials to zero separately.

$$\frac{-2eE}{3h\sqrt{x}} \frac{d(x^{3/2}g)}{dx} + \frac{eE_0}{\sqrt{2m^*h\omega x}} \left\{ \frac{d}{dx} \left[\frac{1(e^\sigma + 1)\ln(4x)}{2(e^\sigma - 1)} \frac{df_0}{dx} \right] + f_0 \ln(4x) \right\} = 0 \quad (6.10)$$

$$\frac{eEh\kappa}{m^*} \frac{df_0}{dx} + \frac{kg}{\tau} = 0 \quad (6.11)$$

From (6.11) $g(x)$ can be expressed in terms of df_0/dx and substituted into (6.10). The resulting exact second order differential equation can be integrated once to give a first order differential equation in $f_0(x)$. This first order differential equation is found to be separable and can be integrated to give [5,6]

$$f_0 = F(0) \exp \left\{ - \int \frac{\ln(4x)}{bx^2 + d\ln(4x)} dx \right\} \quad (6.12)$$

where $F(0)$ is a constant that results from the integration. The constants b and d are defined as

$$b = \frac{4E^2(e^\sigma - 1)}{3E_0^2(e^\sigma + 1)} \quad (6.13)$$

$$d = \frac{1}{2} \frac{(e^\sigma + 1)}{(e^\sigma - 1)} \quad (6.14)$$

The asymmetrical part $g(x)$ can be found by differentiating $f_0(x)$ and substituting the result into (6.11).

There are two main draw backs with the result obtained. The most

important is that $f_0(x)$ given in (6.12) is not normalizable. This problem is the result of neglecting intervalley scattering. Another drawback of (6.12) is that it cannot be extended to low electron energies. The result is expected to be valid at high energies because the Taylor expansions that are made for $f_0(x)$, $g(x)$, and $\phi(x)$ are for high electron energies. Both $f_0(x)$ and $g(x)$ fail around $x=0.25$ when $\ln(4x)$ is zero. For $x < 0.25$, $\ln(4x)$ becomes negative. In fact even when x is slightly larger than 0.25, a point is reached when $\ln(4x)$ is smaller than other terms in the $\phi(x)$ expansions which are dropped. Therefore calculated values of $f_0(x)$ and $g(x)$ have failed even at those points.

In this section an attempt is made to extend the result of Stratton [5], and Conwell and Vassell[6] to low energies. From (6.12) one notices there are two parameters, δ and E_0/E , which can be used as expansion parameters. For high electron energies, expansions of b and d in δ give leading terms which are independent of δ . Therefore, the expansion in δ is not expected to work well since it does not simplify (6.12) much. The other parameter, E_0/E , can be used as an expansion parameter since it is smaller than unity for high fields. In (6.11) for very large E the term with coefficient b will dominate. The coefficient b can be brought out of the integral and the exponent can be expanded in terms of $1/b$. This expansion which is equivalent to an expansion in $(E_0/E)^2$ appears to work even at lower energies. Therefore $2E/3E_0$ is chosen as an expansion parameter and is represented by K .

It is assumed that the electron distribution functions $f_0(x)$ and $kg(x)$ can be expanded in power series in $1/K$:

$$f_0(x) = \sum_j (N/K)^j f_0^j(x) \quad (6.15)$$

$$f_1(x) = \sum_j (N/K)^j f_1^j(x) \quad (6.16)$$

where $f_1(x)$ is used to represent $kg(x)$. The superscripts on f_0^j and f_1^j are not powers but are there to indicate the order of the terms in the expansions. These series forms of $f_0(x)$ and $f_1(x)$ are substituted into (6.2)-(6.4). The resulting equations when substituted into the steady state Boltzmann equation will allow us to find two coupled equations in $f_0(x)$ and $f_1(x)$. By equating the coefficients of the first and the second Legendre polynomials to zero we find the two equations are equation (A4.8) and (A4.9) in Appendix 4. From them equations relating the different terms in the expansions of $f_0(x)$ and $f_1(x)$ can be found by equating the coefficients of different powers of $1/K$. The first few of these equations are shown below:

$$x \frac{d[f_0^0(x)]}{dx} = 0 \quad (6.17)$$

$$\frac{d[xf_1^0(x)]}{dx} = 0 \quad (6.18)$$

$$\frac{d[xf_1^1(x)]}{dx} = N \left\{ \left[f_0^0(x) - e^\sigma f_0^0(x+1) \right] \phi(x) - H(x-1) \phi(x-1) \left[f_0^0(x-1) - e^\sigma f_0^0(x) \right] \right\} \quad (6.19)$$

$$x \frac{d[f_0^1(x)]}{dx} = \frac{N}{3} \left\{ \phi(x) f_1^0(x) - e^\sigma \left[\frac{(2x+1)\phi(x)}{\sqrt{x(x+1)}} - 1 \right] f_1^0(x+1) + H(x-1) \left[e^\sigma \phi(x-1) f_1^0(x) - \left[\frac{(2x-1)\phi(x-1)}{\sqrt{x(x-1)}} - 1 \right] f_1^0(x-1) \right] \right\} \quad (6.20)$$

where (6.17) and (6.18) are found from equating the coefficients of

$(1/K)^0$, and (6.19) and (6.20) are found from equating the coefficients of $(1/K)^1$. The equations found from equating higher powers of $1/K$ are given in Appendix 4. The various terms in the $f_0(x)$ and $f_1(x)$ expansions can be found by solving these equations. The functions $f_0(x+1)$, $f_1(x+1)$ and $\phi(x)$ are not expanded into Taylor series for solving the problem because the main interest here is to look at lower electron energies region (i.e. small x).

The first equation to be solved is (6.17). It is integrated and the result is $f_0^0(x)$ equal to a constant. This result is not surprising since it can be deduced from (6.12). For large x and large E the term bx in (6.12) will dominate the denominator of the exponent. In the limit as x and E become very large, the exponential is reduced to one, and $f_0(x)$ approaches $F(0)$. It is therefore possible to conclude that the leading term of the $f_0(x)$ expansion is $F(0)$,

$$f_0^0(x) = F(0). \quad (6.21)$$

The next equation to be solved is (6.18). It is integrated and by taking the limit as x approaches zero, it can be seen that the integration constant has to be zero or else $f_1^0(x)$ is infinite at $x=0$,

$$f_1^0(x) = 0. \quad (6.22)$$

This result and the result for $f_0^0(x)$ are substituted into (6.19) and (6.20) to solve for $f_1^1(x)$ and $f_0^1(x)$ respectively. To integrate (6.19), all the $f_0^0(x)$ are replaced by the constant $F(0)$. This first order differential equation can be reduced to an equation with integrals of inverse hyperbolic sine functions. The integrations are carried out in

Appendix 4 and the result is shown below,

$$f_1^1(x) = \frac{F(0)(1-e^\sigma)}{x} \left[(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)} - H(x-1) \left[(2x-1) \sinh^{-1} \sqrt{x-1} - \sqrt{x(x-1)} \right] \right]. \quad (6.23)$$

Since $f_1^0(x)$ is zero, integration of (6.20) results in $f_0^1(x)$ being a constant. The constant can be determined by matching the expansion of $f_0(x)$ in $1/K$ with (6.12). It has been shown that for large E and large x the leading term in (6.12) for an expansion in $1/b$ is $F(0)$. The next order term is of order $(E_0/E)^2$ or $1/K^2$. Since there is no term of order E_0/E , the $1/K$ term in (6.16) is expected to vanish, which implies that the constant $f_0^1(x)$ is zero.

Next the coupled equations found from equating coefficients of $(1/K)^2$ are integrated (see (A4.14) and (A4.15) in the Appendix 4). By using the result just found (i.e. $f_0^1=0$) the right side of (A4.14) vanishes and the resulting differential equation can be integrated. By taking the limit as x approaches zero it is found , as in the case of $f_1^1(x)$, that $f_1^2(x)$ is zero or else the distribution function is infinite at $x=0$. To solve for $f_0^2(x)$ in (A4.15), the result for $f_1^1(x)$ obtained in (6.23) is needed. The resulting equation obtained from substituting (6.23) into (A4.15) cannot be integrated easily. Therefore $f_0^2(x)$ is written in terms of an integral of $f_1^1(x)$ as shown in (A4.23). The constant of integration , $F_0^2(0)$, is included so that the final expansion of $f_0(x)$ will match the result of (6.12) for large x . This integration constant can be determined by looking at the condition on $f_0(x)$ as x approaches ∞ . For (6.15) to match (6.12) as $x \rightarrow \infty$, $f_0^2(x)$ needs to vanish. Therefore the constant of integration , $F_0^2(0)$, is an integral

from zero to infinity of some functions of $f_1^1(x)$, $\sinh^{-1}(x)$ and x , as shown in (A2.24). The final form of $f_0^2(x)$ is shown below,

$$f_0^2(x) = \frac{1}{3} \int_0^x \frac{dx}{x} \left\{ 2f_1^1(x) \sinh^{-1} \sqrt{x} - e^\sigma f_1^1(x+1) \left[\frac{(2x+1) \sinh^{-1} \sqrt{x}}{\sqrt{x(x+1)}} - 1 \right] \right. \\ \left. - H(x-1) \left\{ 2e^\sigma f_1^1(x) \sinh^{-1} \sqrt{x-1} - f_1^1(x-1) \left[\frac{(2x-1) \sinh^{-1} \sqrt{x-1}}{\sqrt{x(x-1)}} - 1 \right] \right\} \right\} \\ - \frac{1}{3} \int_0^x \frac{dx}{x} \left\{ 2f_1^1(x) \sinh^{-1} \sqrt{x} - e^\sigma f_1^1(x+1) \left[\frac{(2x+1) \sinh^{-1} \sqrt{x}}{\sqrt{x(x+1)}} - 1 \right] \right. \\ \left. + H(x-1) \left\{ 2e^\sigma f_1^1(x) \sinh^{-1} \sqrt{x-1} - f_1^1(x-1) \left[\frac{(2x-1) \sinh^{-1} \sqrt{x-1}}{\sqrt{x(x-1)}} - 1 \right] \right\} \right\} . \quad (6.24)$$

So far two non-zero terms in the $f_0(x)$ expansion have been obtained by integrating (A4.16). The result of the integration is an expression for $f_1^3(x)$ which is the product of $1/x$ and an integral with $f_0^2(x)$ as shown in (A4.26). The integral can be simplified from a multiple integrals to a single integral by using integration by parts as shown in Appendix 4. The resulting expression for $f_1^3(x)$ is shown in (6.25),

$$f_1^3(x) = 1/x \left\{ [f_0^2(x) - e^\sigma f_0^2(x+1)] [(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)}] \right. \\ \left. - H(x-1) [f_0^2(x-1) - e^\sigma f_0^2(x)] [(2x+1) \sinh^{-1} \sqrt{x-1} - \sqrt{x(x-1)}] \right. \\ \left. - \int_0^x dx [(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)}] \left[\frac{df_0^2(x)}{dx} - e^\sigma \frac{df_0^2(x+1)}{dx} \right] \right. \\ \left. + \int_1^x dx [(2x-1) \sinh^{-1} \sqrt{x-1} - \sqrt{x(x-1)}] \left[\frac{df_0^2(x-1)}{dx} - e^\sigma \frac{df_0^2(x)}{dx} \right] \right\} . \quad (6.25)$$

Even though $f_1^3(x)$ written in this form looks more complicated than in the form it had before the integration by parts was done (as shown in (A4.26)), it is actually in a better form for numerical integration. Written in this form, the first two terms can be calculated by straight substitutions once $f_0^2(x)$ has been calculated from (6.24). The factors

df_0^2/dx with the arguments, $x, x+1$ and $x-1$, in the last two terms are functions of $f_1^1(x)$ as shown in (A4.15). Therefore the process of calculating $f_1^3(x)$ and $f_0^2(x)$ involves a single integration of a function with $f_1^1(x)$. In this report it is assumed that the first two terms in the expansion (6.15) and (6.16) are good enough approximations for $f_0(x)$ and $f_1(x)$.

6.2 ASYMMETRIC PART OF THE DISTRIBUTION FUNCTIONS FOR HIGH ELECTRIC FIELDS

In figures 7, 8 and 9, the asymmetrical part of the electron distribution function, $f_1(x)/F(0)$ is plotted against x for both the methods discussed in section 6.2. For the method used by Stratton[5] and Conwell and Vassell[6], f_1 is obtained by differentiating (6.12) and substituting it into (6.11). This form of $f_1(x)$ obtained from (6.11) is shown in (6.26).

$$f_1(x) = \frac{-3K}{(2N+1)} \times \frac{df_0(x)}{dx} \quad (6.26)$$

$$f_1(x) = f_1^1(N/K) + f_1^3(N/K)^3. \quad (6.27)$$

Equation (6.27) shows the formula for calculating $f_1(x)$ that is derived from the new expansion method suggested. In this case it is assumed that the first two non-zero terms in the $1/K$ expansion of $f_1(x)$ are good enough to approximate it.

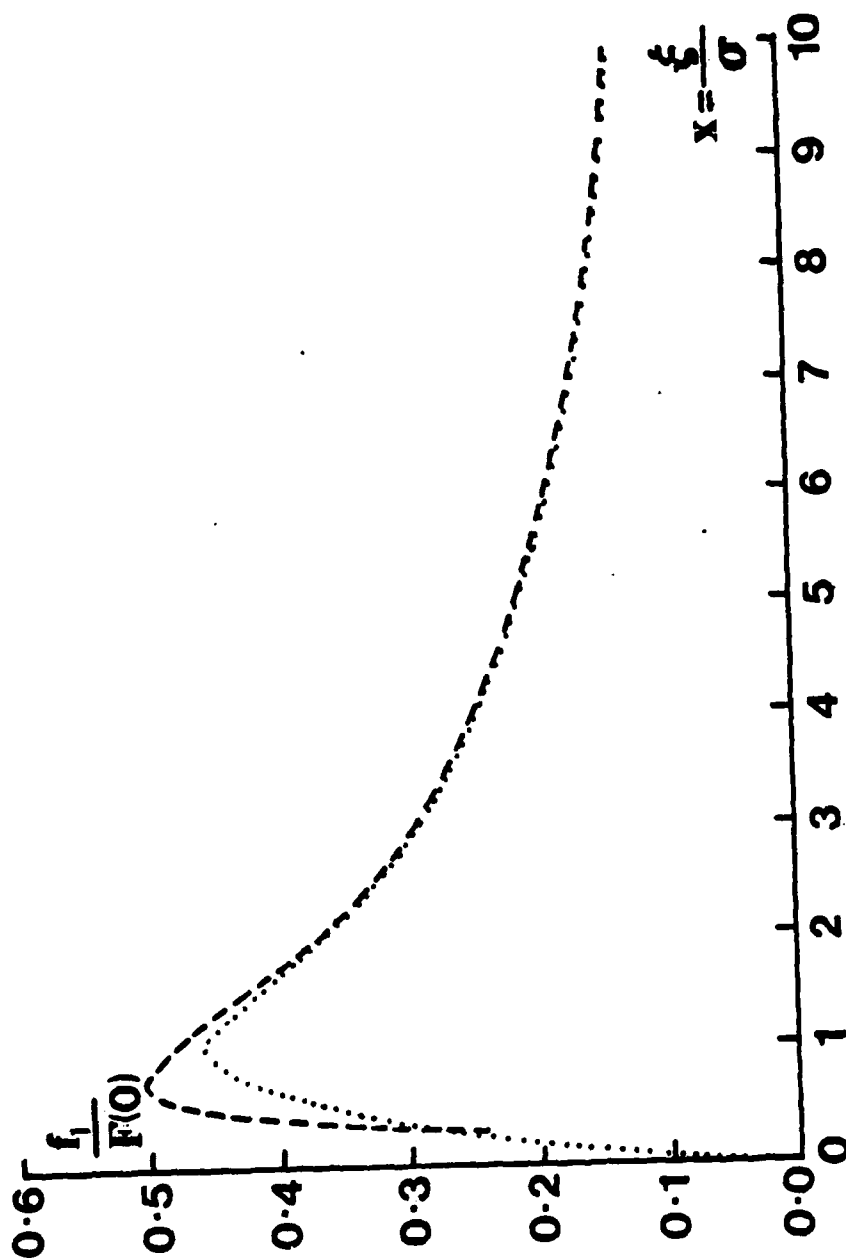


Figure 7. Graphs of $f_1(x)/F(0)$ from the two methods plotted against when $K=3$ and $\sigma=1$. The dashed line is from Stratton/Conwell's method. The dotted line is the result from the $1/K$ power expansion method.

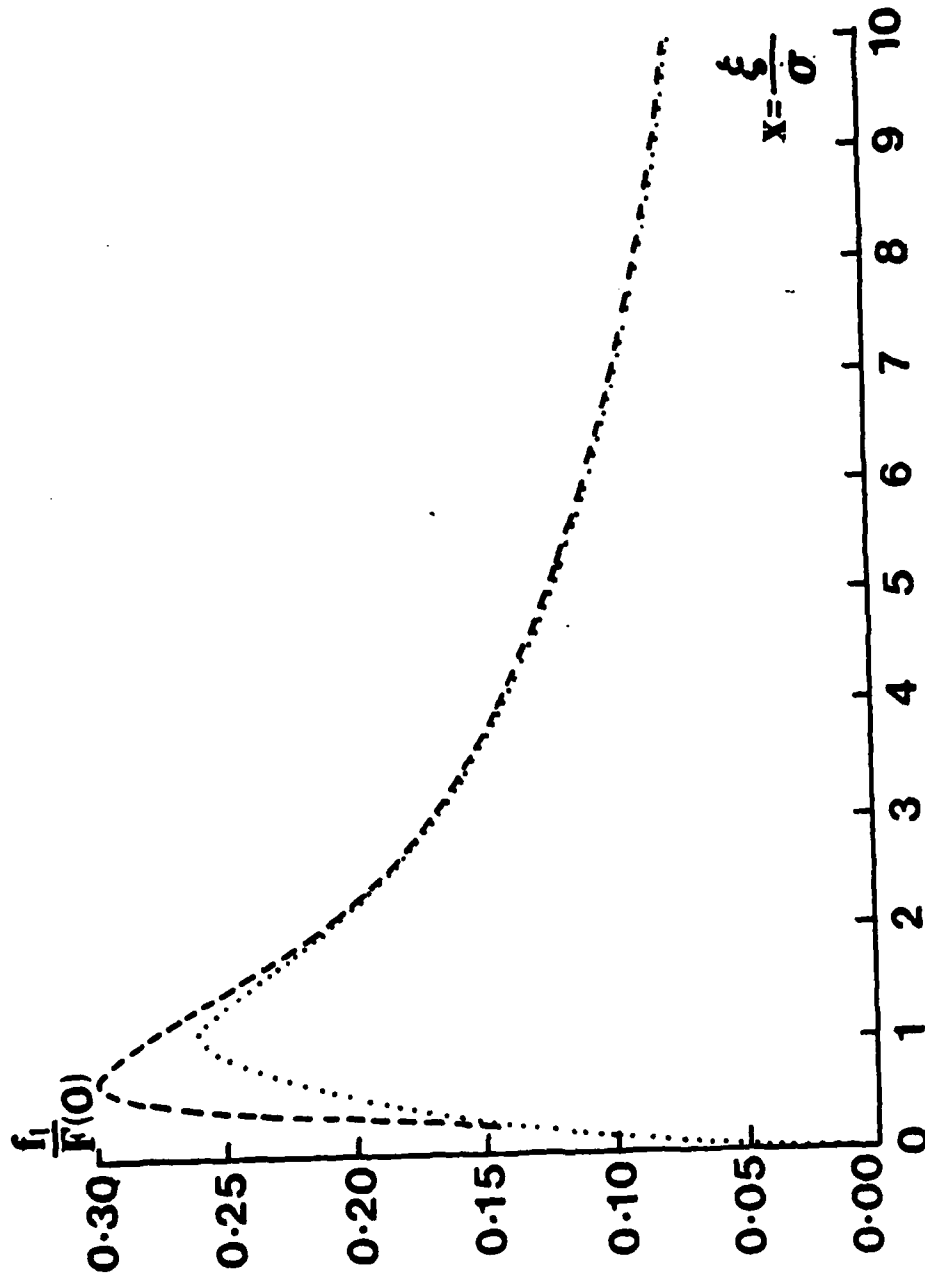


Figure 8. Graphs of $f_1(x)/F(0)$ from the two different methods are plotted against x for the case when $\sigma = 1$ and $K = 5$. The dashed line is for the method of Stratton/Conwell, and the dotted line is for the $1/K$ expansion method.

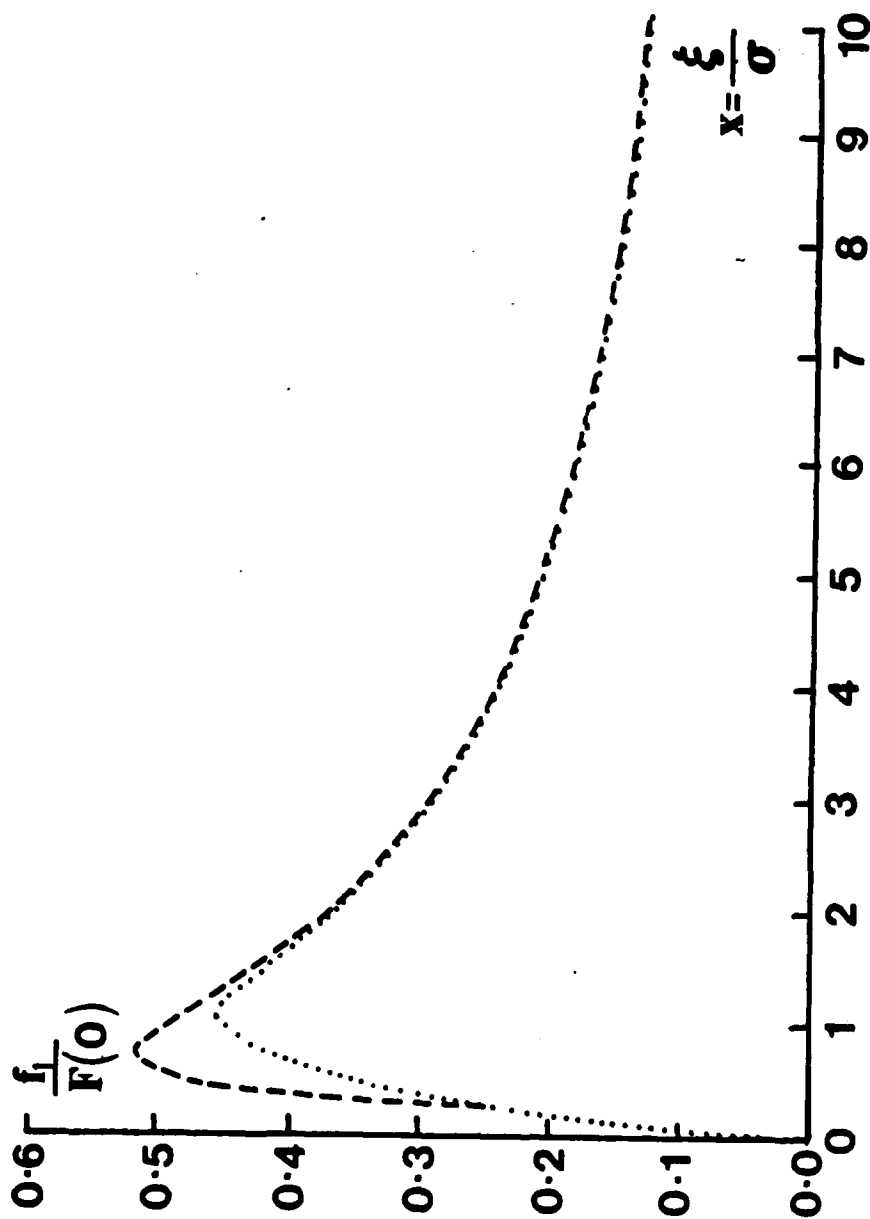


Figure 9. Graphs of the $f_1(x)/F(0)$ from the two different methods plotted against x for $K=3$ and $\sigma = 1.4266$ which corresponds to the σ for GaAs. The representations of the different lines are the same as in figure 7 and 8.

For figures 7 and 8, the value of σ used is 1 for both while K is equal to 3 in figure 7 and $K=5$ in figure 8. For figure 9 the value of σ and K are 1.462 and 3 respectively. The value of 1.462 is used for σ in this case because it corresponds to the case of GaAs at room temperature.

The $f_1(x)$ obtained by the $1/K$ expansion method is valid to $x=0$. The $f_1(x)$ obtained from differentiating (6.14) is not valid around $x=.25$. There is a very noticeable difference between the result obtained by the two methods for x less than 2. The difference becomes less as x increases. Beyond $x=9.5$, the difference is less than 5 percent. The largest difference occur around the maxima. In the case of $\sigma=1$ and $K=3$ the $f_1(x)$ obtained from the method of Stratton [5] and Conwell and Vassell[6] has a maximum around $x=.75$ and has a $f_1(x)/F(0)$ value of 0.5607. The $f_1(x)/F(0)$ obtained by the perturbation expansion in $1/K$ has its maximum around $x=1$ and has a value of 0.4396. Since the $f_0(x)$ in equation (6.13) is found by assuming that x is very much larger than the optical phonon energy, it is not expected to work well for small x . The $f_1(x)$ that is derived from it is therefore not expected to be very accurate for small x either. Therefore the curve of $f_1(x)/F(0)$ which is calculated from the $1/K$ expansion method is expected to give a better description of $f_1(x)$. For large K the expansion method is expected to give a better description of both the symmetrical and asymmetrical part of the distribution function for all x . This is true because the series expansion in $1/K$ depends on K and not on x .

The spherically symmetric parts that are found from both methods are not normalizable. As x approaches infinity the values of f_0 from both the methods approach the constant $F(0)$ and do not go to zero. The

reason for this is that for large energy (large x) intervalley scattering has to be included. In order to account for intervalley scattering, an additional term has to be added to the collision term. With intervalley scattering included in the calculations, a normalizable electron distribution function is expected.

A better description of the distribution function is also found if a non-parabolic energy band is included. Conwell and Vassell[6] have shown that the effect from non-parabolicity of the energy band is not important for very strong electric fields. However, the inclusion of non-parabolicity in the energy band is of interest because it has been claimed that in some compounds with large separation between the central valley and the next satellite valley of the energy band, negative differential resistance can be caused by polar optical scattering in the central valley alone [22].

7. EFFECTS OF ENERGY BAND NON-PARABOLICITY AND P-WAVE SCATTERING ON THE ELECTRON DISTRIBUTION FUNCTION.

In this section the effects of non-parabolicity in the central valley and p-wave scattering on the electron distribution function and the drift mobility will be investigated. Both the effects have been investigated by Hauser, Glisson and Littlejohn[22]. The aim of this section is to check their results. Two of their findings are: (i) in certain ternary and quaternary III-V semiconductor compounds the negative differential resistance can be caused by the non-parabolicity in the central valley alone, (ii) the effects of p-wave scattering on the drift mobility for the III-V compounds investigated is negligible.

When looking at the effects of the non-parabolicity of the central valley, Hauser et al[22] assumed an ϵ -k relation of the form shown in the equation below.

$$(\hbar^2 k^2)/(2m^*) = \epsilon(1 + \alpha \epsilon) \quad (7.1)$$

where α is the first order non-parabolicity factor quoted from Conwell and Vassell[6]. This constant is different for different compounds and is related to the effective-mass energy gap and $\vec{k} \cdot \vec{p}$ interaction matrix P [9]. For more details on α one is referred to reference [6]. By using the energy and momentum equations, Hauser et al [22] were able to show that negative resistance can occur for electron scattering in the central valley alone. The drift mobility of $\text{Al}_{.25}\text{In}_{.75}\text{As}$ was calculated by using this simple model. They showed that the results from their analytical model agree very well with another set of results that they obtained from Monte Carlo calculations. In the Monte Carlo calcu-

lations the authors made two sets of calculations: one that includes p-wave scattering and another without the p-wave scattering. They were able to show that the effect of p-wave scattering is negligible.

There appears to be a contradiction between the results of their simple analytical model and the Monte Carlo calculations in relation to the effect of p-wave scattering. It is true that if the energy band is nearly parabolic then p-wave scattering is not important. However for a non-parabolic energy band, the p-wave mixing cannot be neglected because non-parabolic effect cannot be large if p-wave mixing is small [23]. Therefore it is inconsistent to assume a non-parabolic energy band, and no p-wave and s-wave mixing.

It is the aim of this section of the thesis to investigate how significant the p-wave mixing is in a non-parabolic energy band of the type given in (7.1). Calculations are made to see the magnitude of the p-wave contribution to the asymmetrical part of the electron distribution functions. As in the case of parabolic energy bands, the steady state Boltzmann equation will be used. The collision term in (1.1) has been derived in Appendix 1, and is given by (A1.10) and (A1.11). It is assumed that the scattering due to the s-type wave state is dominant. Therefore the magnitude of c and c' are small compared to a and a' in those equations. It is possible to drop $(cc')^2$ in A'^2 without much error. A' has been defined by equation (1.12) in this thesis. Since $a^2 = 1 - c^2$, all the terms with aa' can be replaced by $(1 - c^2)(1 - c'^2)$. This results in some terms with $c^2 c'^2$ which can be dropped. The symmetrical part of the distribution function after terms with $c^2 c'^2$ have been dropped is shown in equation (7.2),

$$\left(\frac{df}{dt}\right)_{\text{sym}} = \frac{eE_0 N}{\hbar k_0} \left\{ \left[f_0(\xi') e^{\sigma} - f_0(\xi) \right] \left[\ln \left| \frac{k'+k}{k'-k} \right| + c^2 \left(\frac{k'^2 + k^2}{kk'} - 2 \right) \ln \left| \frac{k'+k}{k'-k} \right| - 2 \right] \frac{d\Omega}{d\xi'} \right. \\ \left. + \left[f_0(\xi') - e^{\sigma} f_0(\xi) \right] \left[\ln \left| \frac{k'+k}{k'-k} \right| + c^2 \left(\frac{k'^2 + k^2}{kk'} - 2 \right) \ln \left| \frac{k'+k}{k'-k} \right| - 2 \right] \frac{d\Omega}{d\xi'} \right\} \quad (7.2)$$

$\xi' = \xi + \hbar\omega$
 $\xi' = \xi - \hbar\omega$

where the prime in ξ' and k' implies that the factors involved are to be evaluated at $\xi + \hbar\omega$ as indicated. The asymmetrical part of the distribution function is given by equation (A5.3) in Appendix 5. In the case for which there is no p-wave scattering, $c=0$ and the equation reduces to (A4.1) for a non-parabolic band. It is assumed that the symmetrical part of the collision term can be written as the sum of a dominant part associated with s-wave scattering and a perturbed part which is due to p-wave scattering

$$\left(\frac{df_0}{dt}\right)_{\text{sym}} = \left(\frac{df_0}{dt}\right)_{\text{s-w}} + \epsilon \left(\frac{df_0}{dt}\right)_{\text{p-w}} \quad (7.3)$$

where ϵ is a smallness parameter which will be set to unity later. In this form, the s-wave associated dominant part represents those terms without c^2 , while $(df_0/dt)_{\text{p-w}}$ consists of those terms with c^2 .

When using the collision term in the Boltzmann equation, it is convenient to express the k and k' in terms of the electron energy. For the form of the non-parabolic energy band in (7.1), approximations are made in order to arrive at a term which is considered simple enough to be used later on. The assumption used in making the approximations is $\xi \gg \hbar\omega$. This assumption is appropriate because a high electric

field, $\mathcal{E}$, leads to a high average electron energy. Moreover it appears that p-wave scattering is more pronounced at higher electron energies. The three approximations that are made involve the simplification of the term $(k'^2 + k^2)/kk'$, the expansion of $f_0(\mathcal{E})$ into Taylor series and the dropping the $\ln[(k'+k)/(k'-k)]$ terms. The details of the approximations are in Appendix 5. The simplified form of the symmetrical part of the collision term due to p-wave scattering is shown in (7.4).

$$\left(\frac{df_0}{dt}\right)_{\text{c sym}} = \frac{-2eE_0\omega^2}{\hbar k} \left\{ 4\alpha\hbar\omega f_0(\mathcal{E}) + \hbar\omega \frac{df_0(\mathcal{E})}{d\mathcal{E}} \left[1+2\alpha + (2N+1)2\alpha\hbar\omega \right] \right. \\ \left. + \frac{(\hbar\omega)^2}{2} \frac{d^2 f_0(\mathcal{E})}{d\mathcal{E}^2} \left[(1+2\alpha\mathcal{E})(2N+1) + 2\alpha\hbar\omega \right] \right\} \quad (7.4)$$

p-wave

From the steady state Boltzmann equation, (1.1), the rate of change of the symmetrical part of the electron distribution function in the lower valley can be written in the form shown in (7.5) by assuming no intervalley scattering

$$\frac{2e^2 E^2}{3m^*(1+2\alpha\mathcal{E})} \left[\frac{\hbar\omega}{\mathcal{E} + \alpha\mathcal{E}^2} \right]^{1/2} \frac{d}{d\mathcal{E}} \left[\left[\frac{\mathcal{E} + \alpha\mathcal{E}^2}{\hbar\omega} \right]^{3/2} \frac{\tau f_0}{1+2\alpha\mathcal{E}} \right] + \left(\frac{df_0}{dt}\right)_{\text{c-sw}} + \epsilon \left(\frac{df_0}{dt}\right)_{\text{c-pw}} = 0 \quad (7.5)$$

where the non-parabolicity in the energy band of the form in (7.1) has been used for the field term. The collision term has also been written as the sum of a s-wave scattering and a p-wave scattering terms as given in (7.3). The only difference between (7.5) and the equation in reference [6] is the extra term that has been added to account for the p-wave scattering. τ is the polar optical relaxation time for $\mathcal{E} \gg \hbar\omega$ as

discussed in section 6. For the non-parabolicity considered in this section, it is given by [6]

$$\tau = \left[\frac{2m^* \hbar \omega}{e E_0} \right]^{1/2} \frac{e^\sigma - 1}{e^\sigma + 1} \frac{(\xi + \alpha \xi^2)^{1/2}}{\sqrt{\hbar \omega} (1 + 2\alpha \xi)} \quad (7.6)$$

The collision term where no p-wave scattering is assumed has been derived for a non-parabolic band by Stenflo [29] and Conwell et al [6]. For convenience the form derived by Stenflo [29] will be used here. With the expression from (7.4) and (7.6), (7.5) can be written in the form shown below

$$\begin{aligned} & \frac{2\alpha^2 E^2}{3e E_0 m^* (1 + 2\alpha \xi)} \left[\frac{2m^*}{\xi + \alpha \xi^2} \right]^{1/2} \frac{e^\sigma - 1}{e^\sigma + 1} \frac{d}{d\xi} \left[\frac{(\xi + \alpha \xi^2) df_0}{(1 + 2\alpha \xi) d\xi} \right] \\ & + \frac{e E_0 (\hbar \omega)^2}{\hbar k (1 + 2\alpha \xi)} \frac{d}{d\xi} \left\{ \left[\frac{f_0}{\hbar \omega} + (N + 1/2) \frac{df_0}{d\xi} \right] (1 + 2\alpha \xi)^2 \ln \left[\frac{4 (\xi + \alpha \xi^2)}{\hbar \omega (1 + 2\alpha \xi)} \right] \right\} \\ & - \frac{\epsilon 2e E_0 c^2}{\hbar k} \left\{ 4\alpha \hbar \omega f_0 + \hbar \omega \frac{df_0}{d\xi} (1 + 2\alpha \xi + (2N + 1) 2\alpha \hbar \omega) \right. \\ & \left. + \frac{(\hbar \omega)^2}{2} \frac{d^2 f}{d\xi^2} \left((1 + 2\alpha \xi) (2N + 1) + 2\alpha \hbar \omega \right) \right\} = 0. \end{aligned} \quad (7.7)$$

It is assumed that the solution of (7.7) can be written as the sum of a dominant term $f_0^0(x)$ due to s-wave scattering and a small perturbed term $\epsilon f_0^1(x)$ caused by p-wave scattering.

$$f_0 = f_0^0 + \epsilon f_0^1 \quad (7.8)$$

where ϵ is a smallness parameter to be set to one later. When this sum is substituted into (7.7), the terms in the equation can be reordered in

terms of ϵ^0 and ϵ^1 . The coefficients of each power of ϵ can be set to zero and two equations are obtained:

$$\frac{2e^2 E^2}{3eE_0 m^* (1+2\alpha\epsilon)} \left[\frac{2m^*}{\epsilon + \alpha\epsilon^2} \right]^{1/2} \frac{(e^\sigma - 1)}{(e^\sigma + 1)} \frac{d}{d\epsilon} \left[\frac{(\epsilon + \alpha\epsilon^2)^2}{(1+2\alpha\epsilon)} \frac{df_0^0}{d\epsilon} \right]$$

$$\frac{eE_0 (\hbar\omega)^2}{\hbar k (1+2\alpha\epsilon)} \frac{d}{d\epsilon} \left[\frac{f_0^0}{\hbar\omega} + (N+1/2) \frac{df_0^0}{d\epsilon} \right] (1+2\alpha\epsilon)^2 \ln \left[\frac{4(\epsilon + \alpha\epsilon^2)}{\hbar\omega(1+2\alpha\epsilon)} \right] = 0$$

(7.9)

$$\frac{2e^2 E^2}{3eE_0 m^* (1+2\alpha\epsilon)} \left[\frac{2m^*}{\epsilon + \alpha\epsilon^2} \right]^{1/2} \frac{(e^\sigma - 1)}{(e^\sigma + 1)} \frac{d}{d\epsilon} \left[\frac{(\epsilon + \alpha\epsilon^2)^2}{(1+2\alpha\epsilon)} \frac{df_0^1}{d\epsilon} \right]$$

$$+ \frac{eE_0 (\hbar\omega)^2}{\hbar k (1+2\alpha\epsilon)} \frac{d}{d\epsilon} \left[\frac{f_0^1}{\hbar\omega} + (N+1/2) \frac{df_0^1}{d\epsilon} \right] (1+2\alpha\epsilon)^2 \ln \left[\frac{4(\epsilon + \alpha\epsilon^2)}{\hbar\omega(1+2\alpha\epsilon)} \right]$$

$$- \frac{2eE_0 \omega^2}{\hbar k} \left[4\alpha\hbar\omega f_0^0 + \hbar\omega \frac{df_0^0}{d\epsilon} (1+2\alpha\epsilon) + \frac{(\hbar\omega)^2}{2} \frac{d^2 f_0^0}{d\epsilon^2} (1+2\alpha\epsilon) (2N+1) \right] = 0.$$

(7.10)

The first of these equations is the one that is obtained by equating the coefficients of the first Legendre polynomial P_0 to zero in the Boltzmann equation with p-wave scattering neglected. This equation has been solved by Conwell et al [6] for an arbitrary energy band. The solution that they derived is shown below.

$$f_0^0 = F(0) \exp \left[- \int \frac{(\eta')^4 \ln(4\eta/\eta') dx}{b^2 \eta + d(\eta')^4 \ln(4\eta/\eta')} \right]$$

(7.11)

where $x = \epsilon/\hbar\omega$ is defined in section 6, b and d are defined by equations (6.14) and (6.15) in section 6 and η is the energy function Ω normalised by the $\hbar\omega$. For the type of non-parabolicity considered here,

$\eta = (\mathcal{E} + \alpha \mathcal{E}^2) / (\hbar m)$ and the prime in η indicates differentiation with respect to x . $F(0)$ is a constant of integration. In Conwell and Vassell's [6] paper, $F(0)$ has been omitted. For a non-parabolic energy band (i.e. $\alpha \neq 0$), (7.11) would reduce to (6.13) which is the symmetrical part of the electron distribution function when there is no mixing of p- and s-wave functions.

To see how much p-wave scattering affects the electron distribution function, equation (7.10) has to be solved for $f_0^1(x)$. There are three terms within the bracket that are multiplied by the factor c^2 . The last of these three terms is a product of the second derivative of $f_0^0(x)$ and $(\hbar m)^2$, which is small compared to the other two terms and is dropped. It has also been found to be convenient to work with the dimensionless variables x and η which are introduced in (7.11). Equation (7.10) can be integrated once to give a first order differential equation which still contains an integral term as shown below

$$\frac{df_0^1}{dx} + \frac{f_0^1 \eta'^4 \ln(4\eta/\eta')}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} = \frac{\eta'^2}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \int_0^x 2c^2 \frac{d[\eta'^2 f_0^0]}{dx} dx. \quad (7.12)$$

In this equation, c^2 is also a function of $\mathcal{E}$, and is written in terms of x so that the right hand side can be integrated. From (1.4) and (1.5), it is possible to obtain an expression for c^2 in terms of $\mathcal{E}_p$, $\mathcal{E}_g$ and $\mathcal{E}_0$. Only $\mathcal{E}_0$ is a function of k and is given by $\mathcal{E}_0 = \hbar^2 k^2 / (2m_0)$, where m_0 is the mass of electron in vacuum. Since m_0 and the effective mass, m^* are related through $m_0 = m^* (1 + \mathcal{E}_p / \mathcal{E}_g)$, c^2 can be expressed in terms of $\hbar^2 k^2 / (2m^*)$, $\mathcal{E}_p$, and $\mathcal{E}_g$. This expression can then be changed to into one that is written in terms of η' through the relations $\hbar^2 k^2 / (2m^*) = \mathcal{E} + \alpha \mathcal{E}^2$

and $\alpha = \xi_p^2 / [\xi_g(\xi_g + \xi_p)]$. The various substitutions mentioned above have been carried out in Appendix 5, and the final result for c^2 is a simple function of η' .

$$c^2 = \frac{1}{2}(1 - 1/\eta') \quad (7.13)$$

This expression is substituted into the right hand side of (7.12) and integration by parts is used. The right hand side of (7.12) can also be simplified a little by expressing the coefficients of f_0^1 in terms of the derivative of $\log [F(0)/f_0^0]$ from the result obtained earlier for f_0^0 in (7.11). Therefore (7.12) can be rewritten as

$$\frac{df_0^1}{dx} + f_0^1 \frac{d[\ln(F(0)/f_0^0)]}{dx} = \frac{\eta'^2}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} [\eta'^2 f_0^0 - f_0^0 \eta' - 2\alpha \hbar \omega \int_0^x f_0^0 dx] \quad (7.14)$$

It is possible to transform the right hand side of the above equation into the form of an exact integral by the integrating factor, $F(0)/f_0^0$. With the integrating factor, (7.14) can be integrated from a lower limit, λ , to x . This lower limit is chosen such that f_0^1/f_0^0 vanishes at $x=\lambda$. The lower limit is found to be one. After the integration, $F(0)$ cancels out from both side of the equation, and the result of f_0^1 expressed as a ratio of f_0^1 is

$$\frac{f_0^1}{f_0^0} = \int_1^x \left\{ \frac{\eta'^2}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \left[\eta'^2 - \eta' - \frac{2\alpha \hbar \omega}{f_0^0} \left(F(0)(1 - e^{1/d}) + \int_1^x f_0^0 dx \right) \right] \right\} dx \quad (7.15)$$

The factor $F(0)(1 - \exp(1/d))$ is an approximation for the integral from zero to 1 of f_0^0 . This has been used to overcome the singularity that f_0

has at $x=0.25$, as discussed in section 6.

The effect of the p-wave scattering on the mobility of electrons can be seen by looking at the asymmetrical part of the electron distribution function g , which is expressed in terms of f_0 from (6.12). However the difference here is that in this case f_0 consists of a term associated with s-wave scattering and an extra term associated with p-wave scattering.

$$g = \frac{-\hbar e E \tau}{m^* \hbar v \eta'} \left[\frac{df_0}{dx} + \epsilon \left(f_0^0 P + \frac{df_0^0}{dx} \int_1^x P dx \right) \right] \quad (7.16)$$

where the first term is due to s-wave scattering and the term with ϵ is due to p-wave scattering. P is the expression within the integral in (7.15)

$$P = \frac{\eta'}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \left[\eta'^2 - \eta' - 2\alpha \hbar v \left(F(0)(1 - e^{-1/d}) + \int_1^x f_0^0 dx \right) \right] \quad (7.17)$$

Since the mobility is proportional to the integral of the asymmetrical part of the electron distribution function, the effects of p-wave scattering on the mobility can be seen by looking at how much it affects the distribution function.

7.1 RESULTS WITH P-WAVE SCATTERING INCLUDED.

The ratio f_0^1/f_0^0 as given in (7.15) has been numerically integrated for GaAs and Al_{0.25}In_{0.75}As. The results are plotted in figures 10 and 11

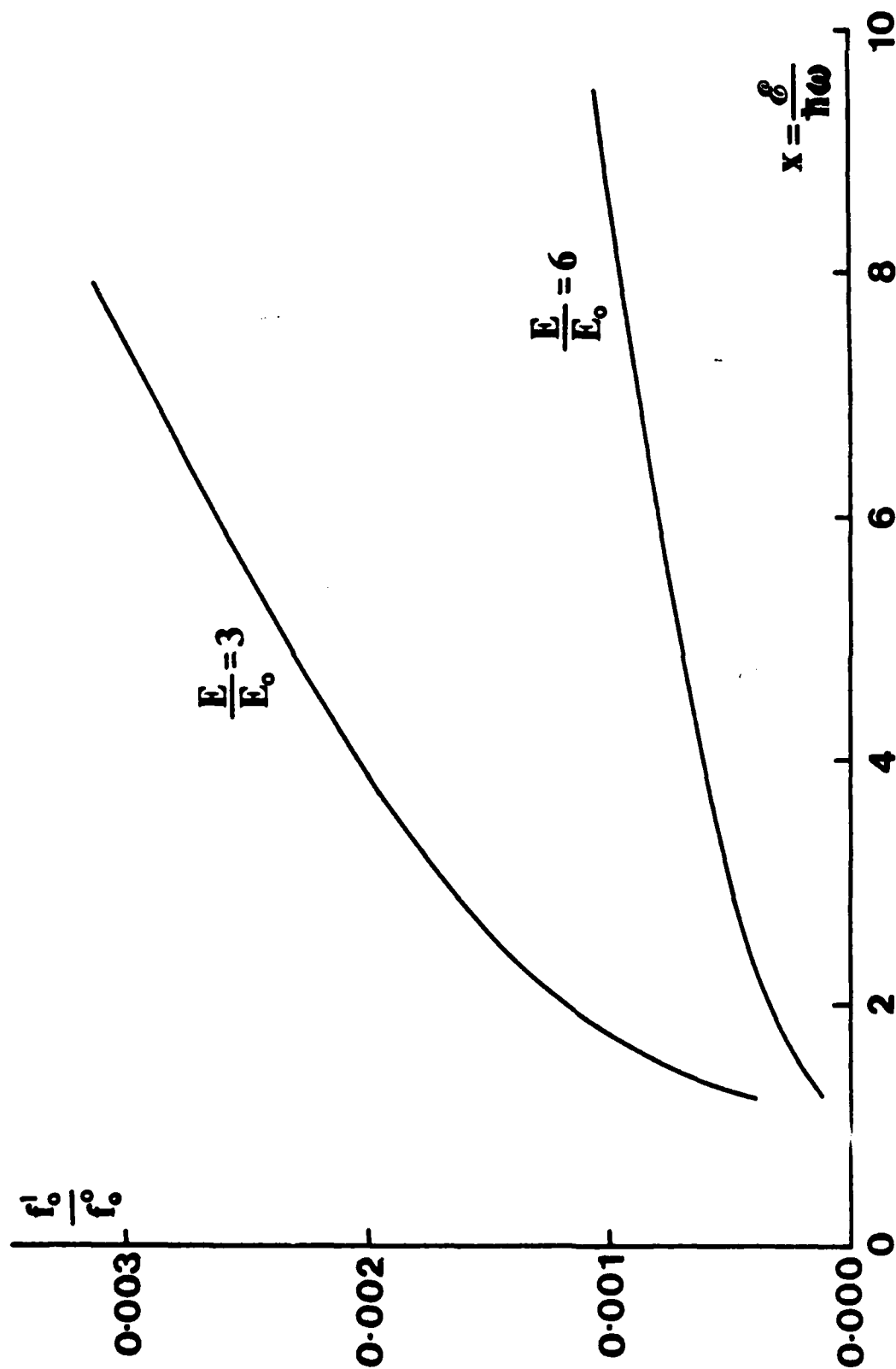


Figure 10. Plots of the normalised symmetrical part of the distribution function due to p-wave scattering, f_0^1/f_0 versus x for GaAs. Two different values for E/E_0 , 3 and 6, are used here as indicated. The value of α used here is 8.576 eV

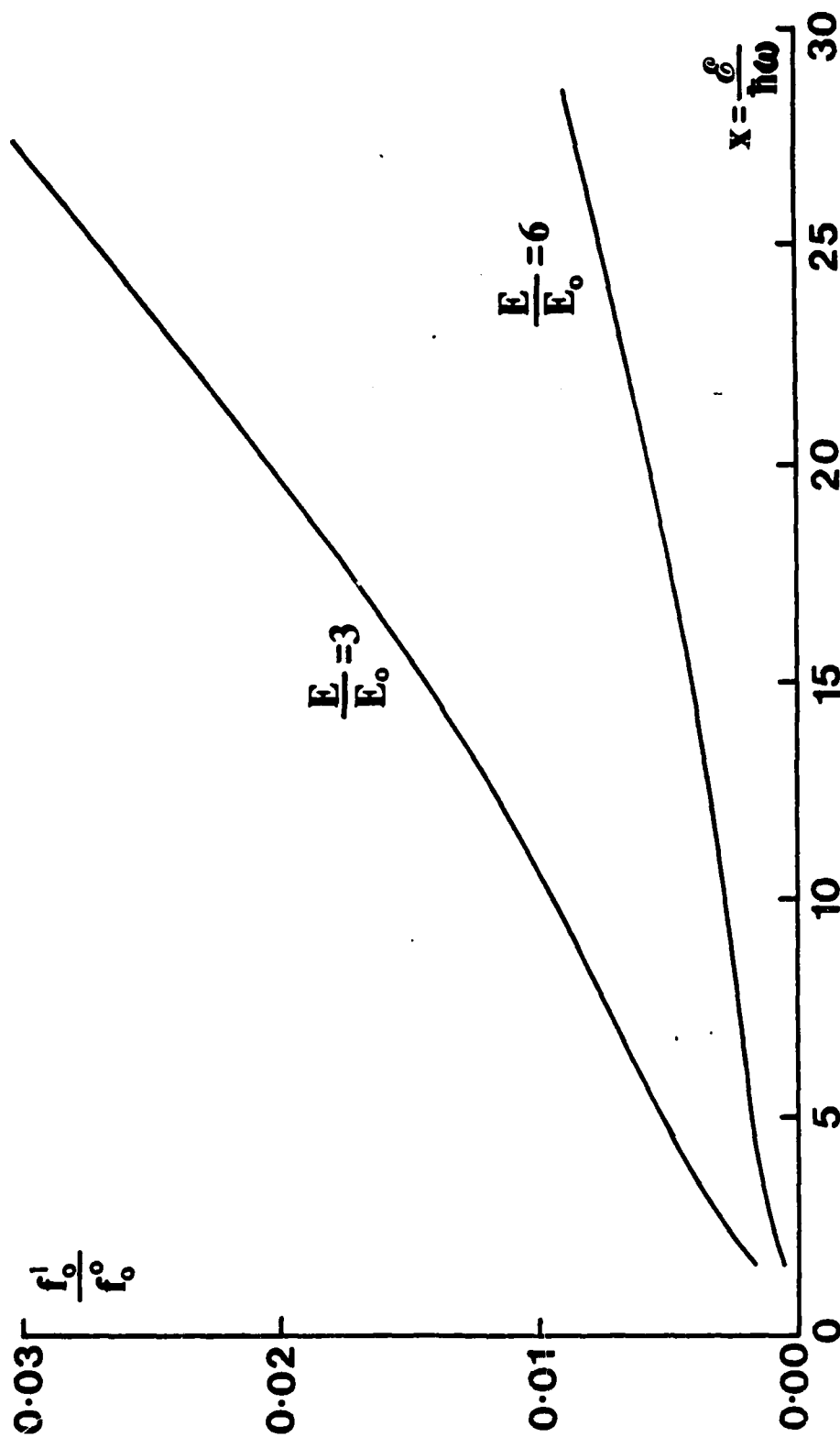


Figure 11. Plots of the p-wave associated symmetrical part of the distribution function, f_0^s against x for Al. $\frac{E}{E_0} = 3$ and 6. The value of τ used here is 1.04 eV^{-1} .

The integrations have been carried out on an Apple II computer with a simple numerical integration program. For each compound two values of E/E_0 are used in order to see the variation of f_0^1 with electric field. The values of α have been calculated from the known values of ξ_g and ξ_p for each compound. The range of the dimensionless energy, x , is chosen such that it is about equal to the difference in the energy of the central valley minimum and the next higher valley minimum. Therefore, the effect of p-wave scattering is studied for the energies below that for which the transferred-electron effect becomes important.

In Figure 10 is the result for GaAs. At room temperature $\hbar\omega$ is about 0.036eV and the valley separation between the minima of the central valley, (Γ) and that of the next higher valley(L) is about 0.31eV. Therefore for $x > 10$, intervalley scattering has to be included in the collision term. For a given electron energy (x), the effect of p-wave scattering decreases as the applied field, $\vec{E}$, is increased. For a given applied field the effects of the p-wave scattering increases as the energy of the electron increases. The increase is more rapid for lower electric fields. However the perturbed part is still very small for the whole range of x investigated since f_0^1 is always less than 0.3 percent of f_0^0 . The contribution of the p-wave scattering to the asymmetrical part of the distribution function is shown in figures 11 and 12. In these figures, the asymmetrical part f_1 and its s-wave component and p-wave component are plotted.

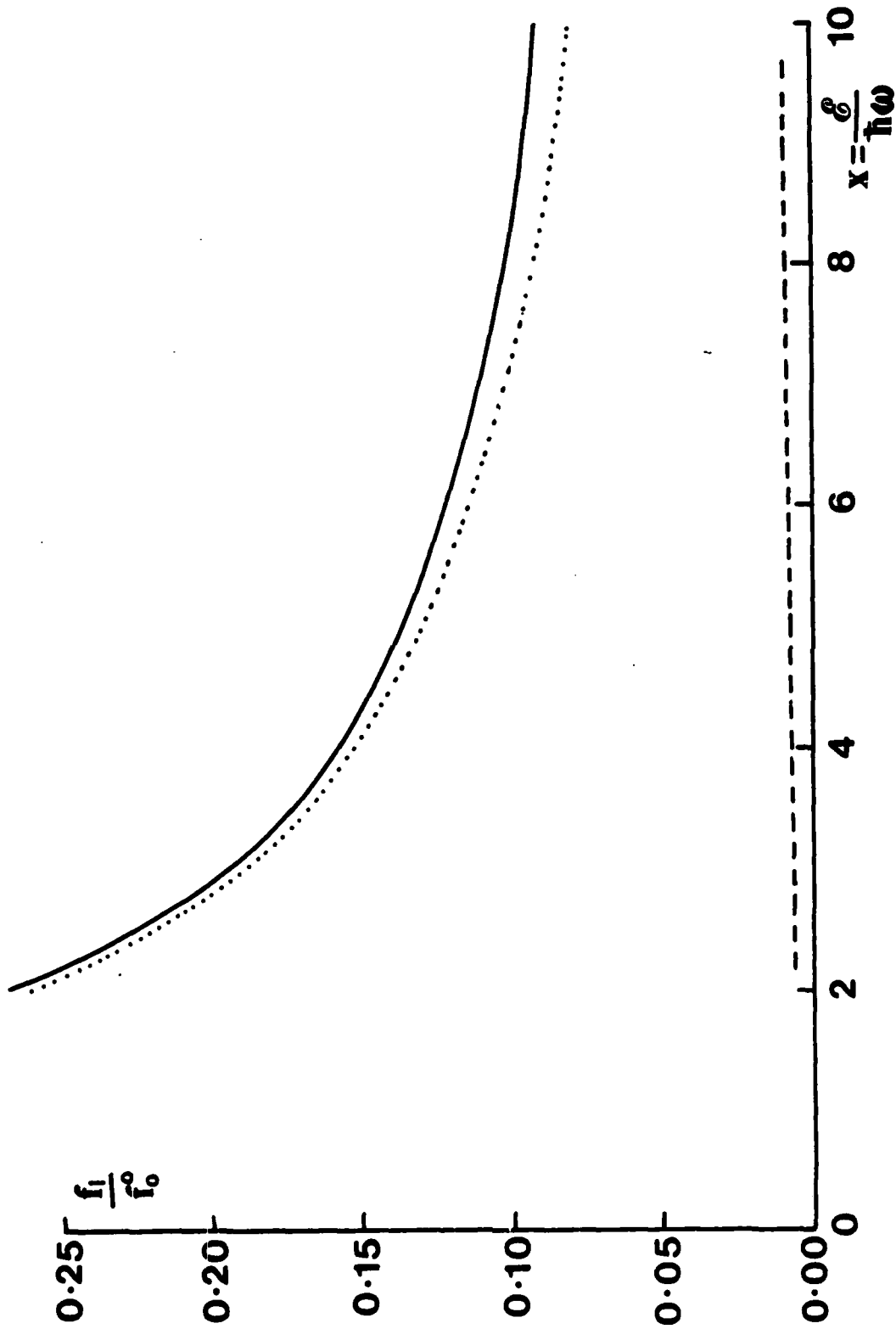


Figure 12. The asymmetrical part of the distribution function, f_1 , and its s-wave and p-wave components plotted against x_1 for GaAs for the case when $E/E_0=3$. The α used here is 0.576 eV^{-1} . The solid line is for f_1/f_0 , the dotted line is its s-wave component and the dashed line is its p-wave component.

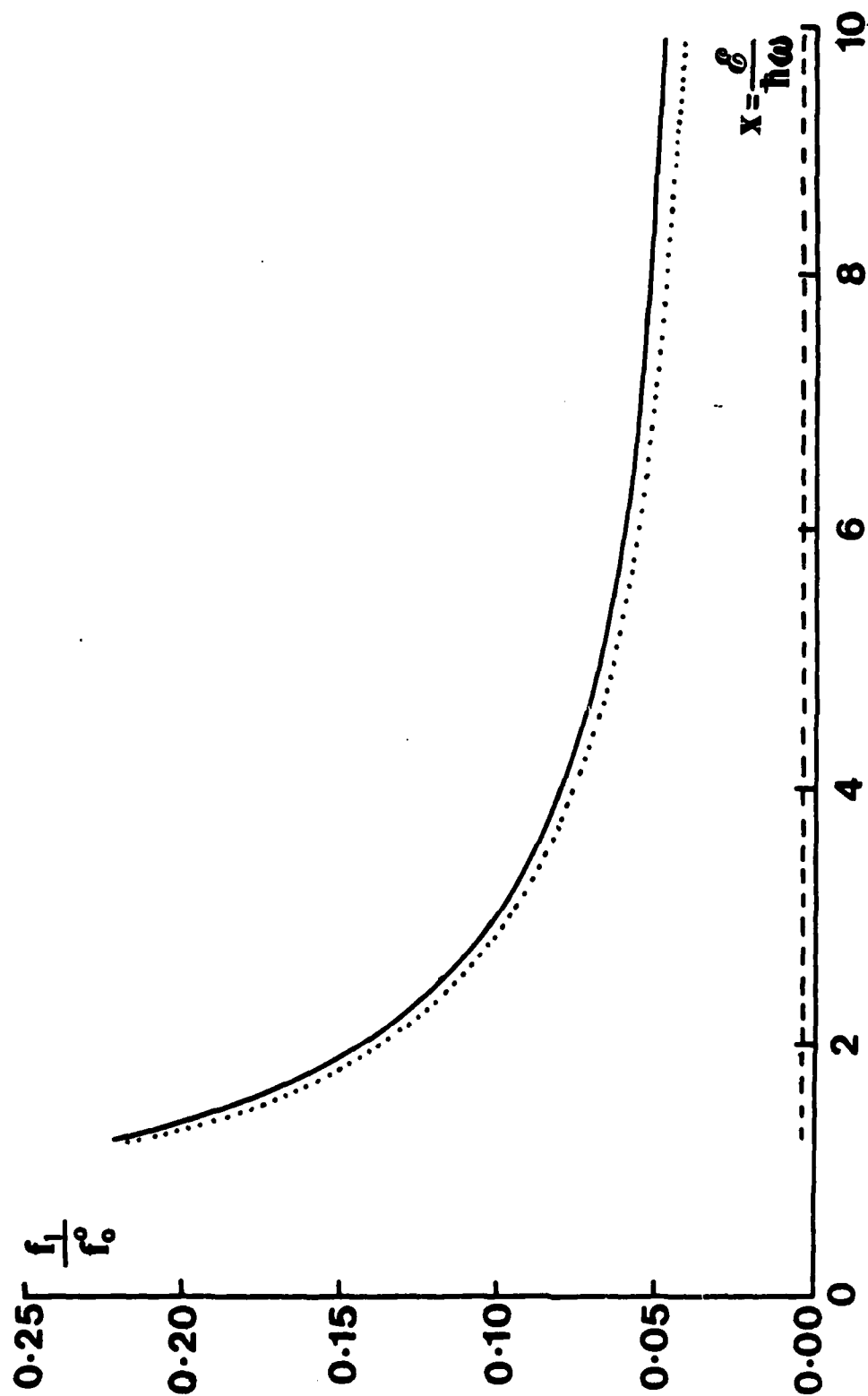


Figure 13. The asymmetrical part of the distribution function f_1/f_0 and its s-wave and p-wave components plotted against x for GaAs where $E/E_0=6$ and $\tau=0.576$ eV. The representation of the lines are the same as that in figure 12.

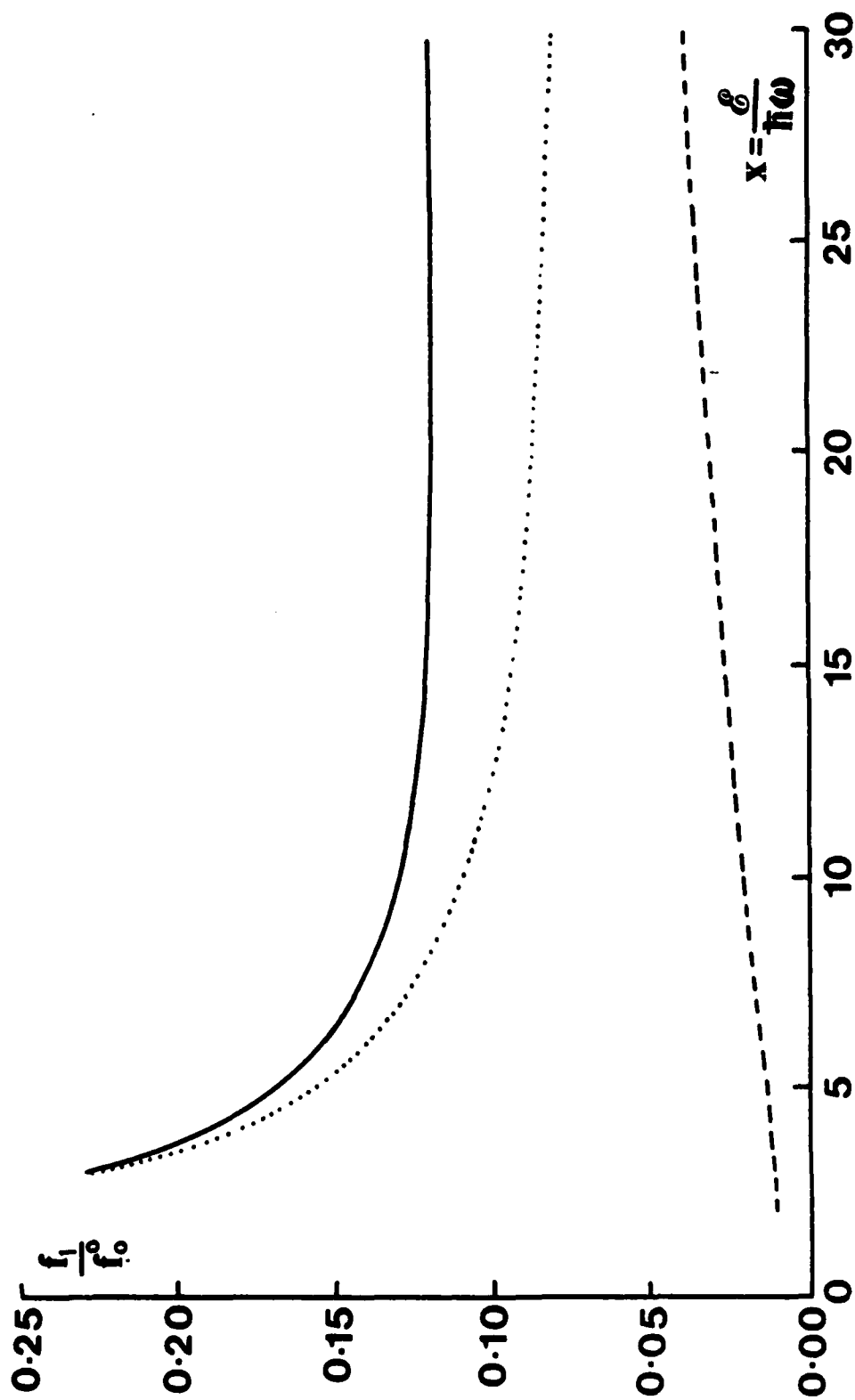


Figure 14. The asymmetrical part of the distribution function, f_1/f_0 along with its s-wave and p-wave components plotted against x for Al 25 In 75 As. This is the case for $E/E_0=3$ and $\alpha=1.04 \text{ eV}^{-1}$. The solid line represents the total asymmetrical part and the dotted line and the dashed line are the s-wave and p-wave components respectively.

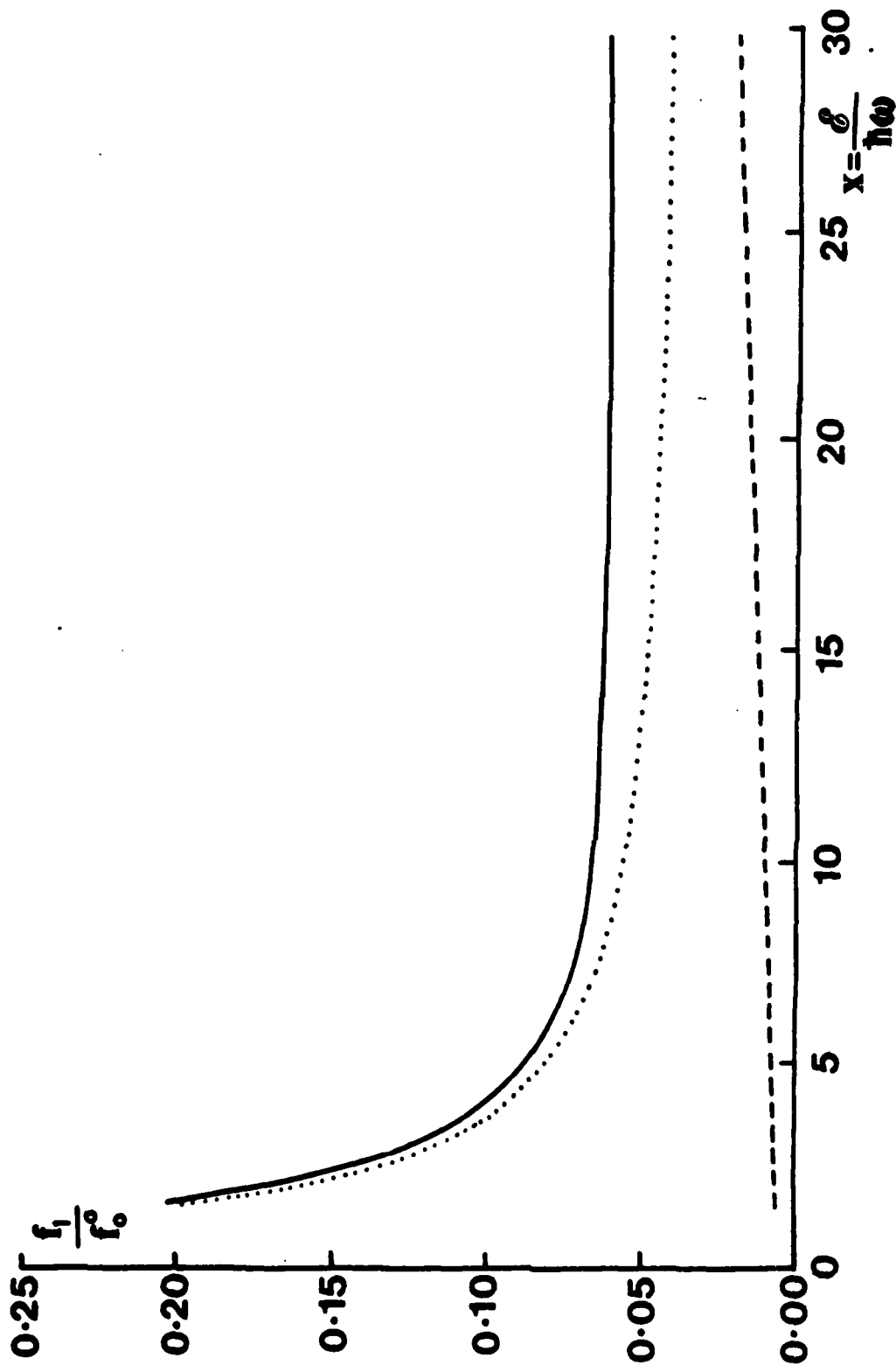


Figure 15. Plots of the asymmetric part, f_1/f_0^0 , and its s-wave and p-wave components versus x for Al $2p$ in As. This is for the case when $E/E_0 = 6$ and $\alpha = 1.04 \text{ eV}^{-1}$. The different line representations are as described in figure 14.

As can be seen, the p-wave contribution to f_1 , hence the mobility, is small for GaAs. Even at the high energy end where $x=10$, the p-wave accounts for only 10 percent of the scattered electrons at that energy. Therefore the contribution of p-wave scattering to the mobility is expected to be very much less than 10 percent since there are more electrons at the lower energies portion of the distribution function which is less affected by the p-wave scattering.

In $\text{Al}_{.25}\text{In}_{.75}\text{As}$, the effect of p-wave is more important as can be seen in figure 10. The size of f_0^1 varies from 0 to 3 percent of f_0^0 , which is larger than the case for GaAs. However the effect is still small. As in the case of GaAs, f_1 increases with increasing x and decreases with increasing E . The fraction of the asymmetrical part of the distribution function due to p-wave scattering is shown in figure 13 and 14 for $E/E_0=3$ and 6 respectively. The effect of the p-wave scattering is more important here than in the case of GaAs. At $x=30$, about a third of the electrons at that energy are scattered by p-wave. However, as discussed earlier, the effects on the mobility is small since there are more electrons in the lower energy portion of the distribution function.

7.2 DISCUSSION OF THE RESULTS WITH P-WAVE SCATTERING INCLUDED

Various approximations have been made in deriving (7.15) and (7.16). These give an estimate of the effect on the electron distribution function due to p-wave scattering. The conclusion is that p-wave mixing does not have a significant effect on the electron distribution function and hence the drift mobility. It can be seen from figures 11-14 that the contribution to the asymmetrical part of the

electron distribution function is less than 5 percent. Since the p-wave scattering does not affect the drift mobility significantly, the effect of the non-parabolicity in the band structure can not be as important as has been claimed by Hauser et al [22]. Their suggestion that the negative differential mobility in certain ternary and quaternary III-V compounds occurs for scattering within the central valley only if the energy band is non-parabolic band may be wrong.

Rather, the calculations for $\text{Al}_{.25}\text{In}_{.75}\text{As}$ seem to indicate that the negative differential resistance for electric fields near the peak for the velocity-electric field curve arises from electron scattering in the central valley of a parabolic energy band. This conclusion is reached not only based on the findings here that the effect of p-wave scattering on the electron distribution is negligible but from their Monte Carlo calculations that p-wave scattering, hardly alters the velocity-electric field curve.

To understand the possibility of negative differential resistance in III-V semiconductor compounds due to scattering in the central valley alone for a parabolic energy band one can compare the results for GaAs and $\text{Al}_{.25}\text{In}_{.75}\text{As}$. For GaAs the peak of the velocity-electric field curve occurs at a value much smaller than $E=3E_0/2=9.3$ kV/cm whereas for $\text{Al}_{.25}\text{In}_{.75}\text{As}$ the peak of the velocity-electric field curve occurs at roughly $E=3E_0/2=5.2$ kV/cm. The value of $E=3E_0/2$ is significant since from the equation for polar optical scattering we have found for $K \ll 1$ an expansion in K is suitable and for $K \gg 1$ an expansion in $1/K$ is suitable for the perturbed distribution function that determines the drift mobility. From the form of the two series one can reasonably expect a

maximum in the drift mobility for $K=1$. For GaAs the energy gap between the central valley and the closest lying upper valley of the conduction band is relatively small. Therefore the maximum is not reached at $E=9.3$ kV/cm because a significant number of electrons are transferred to the upper valley at much lower electric fields. On the other hand, for $\text{Al}_{.25}\text{In}_{.75}\text{As}$, the energy separation between the central valley and the closest upper valley is considerably larger than it is in GaAs and additionally the physical parameters of $\text{Al}_{.25}\text{In}_{.75}\text{As}$ give a smaller value of $3E_0/2$ (5.2 kV/cm). Hence, for $\text{Al}_{.25}\text{In}_{.75}\text{As}$ the peak in the velocity-electric field curve occurs for an electric field such that few electrons are transferred to the upper valley.

For the physical mechanism that causes the negative differential resistance, consider the hypothesis that the electrons are heated by the electric field to the extent that the increase in the drift mobility with increasing electric field is offset by the decrease in the drift mobility with increasing effective electron temperature. From (7.15) and (7.16) one can verify that it seems plausible the denominator in the integral is effectively a temperature that increases as the electric field increases.

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9. APPENDICES

9.1 APPENDIX 1:- SYMMETRICAL AND ASYMMETRICAL PARTS OF THE ELECTRON DISTRIBUTION FUNCTION.

The collision operator for a non-parabolic band is given by

$$\begin{aligned} \frac{df}{dt} = \frac{2\pi}{h} \sum_{\vec{k}} & \left[|(N+1)H'|N| |^2 L(\vec{k}, \vec{k}') \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) f(\vec{k}') \right. \\ & + |(N-1)H'|N| |^2 L(\vec{k}, \vec{k}') \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) f(\vec{k}') \\ & - |(N+1)H'|N| |^2 L(\vec{k}, \vec{k}') \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) f(\vec{k}) \\ & \left. - |(N-1)H'|N| |^2 L(\vec{k}, \vec{k}') \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) f(\vec{k}) \right] \end{aligned} \quad (A1.1)$$

The summation can be changed to an integral over k-space in spherical coordinates. The scattering matrix is given in section 1.

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{2\pi m^*} \int_0^\pi \int_0^{2\pi} \int_0^\pi & \left\{ \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) [f(\vec{k}') e^\sigma - f(\vec{k})] \right. \\ & \left. + \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) [f(\vec{k}') - e^\sigma f(\vec{k})] \right\} \frac{L(\vec{k}, \vec{k}') k'^2 \sin \theta' d\theta' d\varphi' dk'}{|\vec{k} - \vec{k}'|^2} \end{aligned} \quad (A1.2)$$

Since the overlap integral is given by $L(\vec{k}, \vec{k}') = (aa' + cc' \cos \theta')^2$ where θ' is the angle between $\vec{k}$ and $\vec{k}'$, the rate of change of the distribution function can be written as

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{2\pi m^*} \int_0^\pi \int_0^{2\pi} \int_0^\pi & \left\{ \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) [f(\vec{k}') e^\sigma - f(\vec{k})] \right. \\ & \left. + \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) [f(\vec{k}') - e^\sigma f(\vec{k})] \right\} \frac{(aa' - cc' \cos \theta')^2 k'^2 \sin \theta' d\theta' d\varphi' dk'}{k'^2 + k^2 - 2kk' \cos \theta'} \end{aligned} \quad (A1.3)$$

It is assumed that the distribution function $f(\vec{k})$ can be written in the form $f(\vec{k}) = f_0(\epsilon) + k_E g(\epsilon)$ where k_E is the component of $\vec{k}$ in the direction of $\vec{E}$. If θ is the angle between $\vec{k}$ and $\vec{E}$, $f(\vec{k}')$ can then be written as

$$f(\vec{k}') = f_0(\epsilon') + k' g(\epsilon') [\sin \theta' \sin \theta \cos(\varphi - \varphi') + \cos \theta \cos \theta']. \quad (A1.4)$$

φ and φ' are the azimuthal angles of the electric field and $\vec{k}'$ vector respectively. The integration of φ' over 2π results in the term with

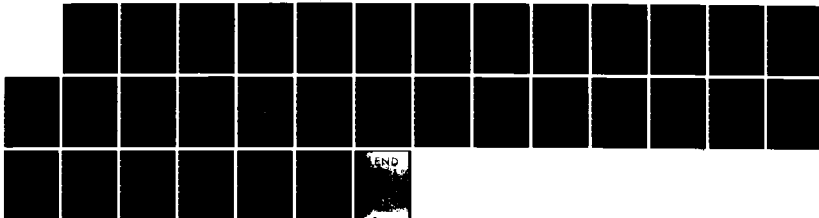
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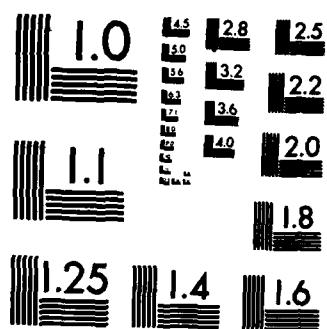
ELECTRON DISTRIBUTION FUNCTION AND MOBILITY IN III-V
COMPOUNDS FOR HIGH A. (U) WASHINGTON UNIV ST LOUIS MO
DEPT OF ELECTRICAL ENGINEERING S H SOH ET AL SEP 84
WU/EE-64463-1 N00014-82-K-0692 F/G 20/12

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$\cos(\varphi - \varphi')$ equals to zero.

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{m^2} \int_0^{\pi} \left\{ \delta(\varepsilon_k - \varepsilon_{k'}) + \hbar\omega \left[f(\varepsilon') e^{\sigma} + e^{\sigma} k' k_E \cos g(\varepsilon') - f_0(\varepsilon) - k_E g(\varepsilon) \right] \right. \\ \left. + \delta(\varepsilon_k - \varepsilon_{k'}) - \hbar\omega \left[f(\varepsilon') + k' k_E \cos g(\varepsilon') - e^{\sigma} f_0(\varepsilon) - e^{\sigma} k_E g(\varepsilon) \right] \right\} \\ \frac{(aa' + cc' \cos \theta)^2 k'^2}{k'^2 + k^2 - 2kk' \cos \theta} d(-\cos \theta) dk' \end{aligned} \quad (A1.5)$$

Integration of θ' from 0 to π gives

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{m^2} \int_0^{\pi} \left\{ \delta(\varepsilon_k - \varepsilon_{k'}) + \hbar\omega \left[f_0(\varepsilon') e^{\sigma} - f_0(\varepsilon) - k_E g(\varepsilon) \right] \right. \\ \left. + \delta(\varepsilon_k - \varepsilon_{k'}) - \hbar\omega \left[f_0(\varepsilon') - e^{\sigma} f_0(\varepsilon) - e^{\sigma} k_E g(\varepsilon) \right] \right\} \\ \left(\left\{ \frac{(aa')^2}{kk'} \ln \left| \frac{k'+k}{k'-k} \right| - 2aa'cc' \left[\frac{1}{kk'} - \frac{k'^2 + k^2}{(2kk')^2} \ln \left| \frac{k'+k}{k'-k} \right| \right] \right\} \right. \\ \left. + (cc')^2 \left\{ \frac{(k'+k)^4 - (k-k')^4}{2(2kk')^3} - 2 \frac{(k'^2 + k^2)}{(2kk')^3} \left[(k'+k)^2 - (k-k')^2 - \ln \left| \frac{k'+k}{k'-k} \right| \right] \right\} \right) \\ + \left[\delta(\varepsilon_k - \varepsilon_{k'}) + \hbar\omega e^{\sigma} k' k_E g(\varepsilon') / k + d(\varepsilon_k - \varepsilon_{k'}) k' k_E g(\varepsilon') / k \right] \\ \left(\left\{ -(aa')^2 \left[\frac{1}{kk'} - \frac{2(k'^2 + k^2)}{(2kk')^2} \ln \left| \frac{k'+k}{k'-k} \right| \right] - \frac{(cc')^2}{3kk'} \right. \right. \\ \left. \left. + \frac{(cc')^2 (k'^2 + k^2)}{2kk'} \left[\frac{-8k'k^3 - 8kk'^3}{2(2kk')^3} + 2(k'^2 + k^2) \ln \left| \frac{k'+k}{k'-k} \right| \right] \right\} \right. \\ \left. + 2aa'cc' \left[\frac{-8k'k^3 - 8k^3 k'}{2(2kk')^3} + 2(k'^2 + k^2) \ln \left| \frac{k'+k}{k'-k} \right| \right] \right) k'^2 dk' \end{aligned} \quad (A1.6)$$

A k' in the denominator can be factored and cancels part of k'^2 in the numerator. Then

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{m^2} \int_0^{\pi} d(k'^2) \left(\left\{ \delta(\varepsilon_k - \varepsilon_{k'}) + \hbar\omega \left[f_0(\varepsilon') e^{\sigma} - f_0(\varepsilon) - k_E g(\varepsilon) \right] \right. \right. \\ \left. \left. + \delta(\varepsilon_k - \varepsilon_{k'}) - \hbar\omega \left[f_0(\varepsilon') - e^{\sigma} f_0(\varepsilon) - e^{\sigma} k_E g(\varepsilon) \right] \right\} \right. \\ \left. \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A'cc' - aa'cc' \right) \right. \\ \left. + k_E / k \left[\delta(\varepsilon_k - \varepsilon_{k'}) + \hbar\omega e^{\sigma} g(\varepsilon') + \delta(\varepsilon_k - \varepsilon_{k'}) - \hbar\omega g(\varepsilon) \right] \right. \\ \left. \left[\frac{k'^2 + k^2}{2k} \ln \left| \frac{k'+k}{k'-k} \right| A' - A'^2 k' - \frac{(cc')^2 k'}{3} \right] \right) \end{aligned} \quad (A1.7)$$

9.1 APPENDIX 1:- SYMMETRICAL AND ASYMMETRICAL PARTS OF THE ELECTRON DISTRIBUTION FUNCTION.

The collision operator for a non-parabolic band is given by

$$\begin{aligned} \frac{df}{dt} = \frac{2\pi}{h} \sum_{\vec{k}} & \left[|(N+1)\langle H' | N \rangle|^2 L(\vec{k}, \vec{k}') \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) f(\vec{k}') \right. \\ & + |(N-1)\langle H' | N \rangle|^2 L(\vec{k}, \vec{k}') \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) f(\vec{k}') \\ & - |(N+1)\langle H' | N \rangle|^2 L(\vec{k}, \vec{k}) \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}} + \hbar\omega) f(\vec{k}) \\ & \left. - |(N-1)\langle H' | N \rangle|^2 L(\vec{k}, \vec{k}) \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}} - \hbar\omega) f(\vec{k}) \right] \end{aligned} \quad (A1.1)$$

The summation can be changed to an integral over k-space in spherical coordinates. The scattering matrix is given in section 1.

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{2\pi m^*} \int_0^\pi \int_0^\pi \int_0^\pi & \left\{ \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) [f(\vec{k}') e^\sigma - f(\vec{k})] \right. \\ & \left. + \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) [f(\vec{k}') - e^\sigma f(\vec{k})] \right\} \frac{L(\vec{k}, \vec{k}') \epsilon_{\vec{k}}^2}{|\vec{k} - \vec{k}'|^2} \sin \theta d\theta d\varphi d\vec{k}' \end{aligned} \quad (A1.2)$$

Since the overlap integral is given by $L(\vec{k}, \vec{k}') = (aa' + cc' \cos \theta)^2$ where θ is the angle between $\vec{k}$ and $\vec{k}'$, the rate of change of the distribution function can be written as

$$\begin{aligned} \frac{df}{dt} = \frac{\hbar e E_0 N}{2\pi m^*} \int_0^\pi \int_0^\pi \int_0^\pi & \left\{ \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} + \hbar\omega) [f(\vec{k}') e^\sigma - f(\vec{k})] \right. \\ & \left. + \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega) [f(\vec{k}') - e^\sigma f(\vec{k})] \right\} \frac{(aa' - cc' \cos \theta')^2 \epsilon_{\vec{k}}^2}{k'^2 + k^2 - 2kk' \cos \theta'} \sin \theta d\theta d\varphi d\vec{k}' \end{aligned} \quad (A1.3)$$

It is assumed that the distribution function $f(\vec{k})$ can be written in the form $f(\vec{k}) = f_0(\epsilon) + k_E g(\epsilon)$ where k_E is the component of $\vec{k}$ in the direction of $\vec{E}$. If θ is the angle between $\vec{k}$ and $\vec{E}$, $f(\vec{k}')$ can then be written as

$$f(\vec{k}') = f_0(\epsilon') + k' g(\epsilon') [\sin \theta' \sin \theta \cos(\varphi - \varphi') + \cos \theta' \cos \theta]. \quad (A1.4)$$

φ and φ' are the azimuthal angles of the electric field and $\vec{k}'$ vector respectively. The integration of φ' over 2π results in the term with

where $A' = aa' + \frac{k'^2 + k^2}{2kk'}$ cc'. It is assumed that the ϵ - k relation is given by

$$\frac{\hbar^2 k^2}{2m^*} = \epsilon$$

with $d(k'^2) = \frac{2m^*}{\hbar^2} \frac{d\epsilon}{d\epsilon'} d\epsilon'$, we find

$$\begin{aligned} \frac{df}{dt} = \frac{eE_0 N}{\hbar k} \left\{ \int d\epsilon' \frac{d\epsilon}{d\epsilon'} \left\{ \delta(\epsilon_k - \epsilon_{k'}, +\hbar\omega) [f_0(\epsilon') e^\sigma - f_0(\epsilon) - k_E g(\epsilon)] \right. \right. \\ \left. \left. + \delta(\epsilon_k - \epsilon_{k'}, -\hbar\omega) [f_0(\epsilon') - e^\sigma f_0(\epsilon) - e^\sigma k_E g(\epsilon)] \right\} \right. \\ \left. \left\{ A'^2 \ln |(k'+k)/(k'-k)| - 2A'cc' - aa'cc' \right\} \right. \\ \left. + \frac{k_E}{k} \left[\delta(\epsilon_k - \epsilon_{k'}, +\hbar\omega) e^\sigma g(\epsilon) + \delta(\epsilon_k - \epsilon_{k'}, -\hbar\omega) g(\epsilon) \right] \right. \\ \left. \left\{ \frac{k'^2 + k^2}{k} \frac{A' \ln |(k'+k)/(k'-k)| - A'^2 k' - (cc')^2 k'}{3} \right\} \right\} \quad (A1.8) \end{aligned}$$

This when integrated gives

$$\begin{aligned} \frac{df}{dt} = \frac{eE_0 N}{\hbar k} \left\{ \left[f_0(\epsilon') e^\sigma - f_0(\epsilon) - k_E g(\epsilon) \right] \left[A'^2 \ln |(k'+k)/(k'-k)| - A'cc' - aa'cc' \right] \right. \\ \left. + \frac{k_E}{k} e^\sigma g(\epsilon) \left(\frac{k'^2 + k^2}{k} \frac{A' \ln |(k'+k)/(k'-k)| - A'^2 k' - (cc')^2 k'}{3} \right) \right\} \frac{d\epsilon}{d\epsilon'} \\ \left. + \left[f_0(\epsilon') - e^\sigma f_0(\epsilon) - e^\sigma k_E g(\epsilon) \right] \left[A'^2 \ln |(k'+k)/(k'-k)| - A'cc' - aa'cc' \right] \right. \\ \left. + \frac{k_E}{k} g(\epsilon) \left(\frac{k'^2 + k^2}{k} \frac{A' \ln |(k'+k)/(k'-k)| - A'^2 k' - (cc')^2 k'}{3} \right) \right\} \frac{d\epsilon}{d\epsilon'} \quad (A1.9) \end{aligned}$$

This rate of change of the distribution function due to collisions can be separated into symmetrical and asymmetrical parts where the symmetrical part is given by

$$\left(\frac{df}{dt}\right)_{c-sym} = \frac{eE_0 N}{\hbar k} \left\{ \left(f_0(\xi') e^\sigma - f_0(\xi) \right) \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A'cc' - aa'cc' \right) \frac{d\Omega'}{d\xi'} \right. \\ \left. + \left(f_0(\xi') - e^\sigma f_0(\xi) \right) \left(A'^2 \ln \left| \frac{k'+k}{k'-k} \right| - A'cc' - aa'cc' \right) \frac{d\Omega'}{d\xi'} \right\} \\ \xi' = \xi + \hbar\omega \\ \xi' = \xi - \hbar\omega \quad (A1.10)$$

and the asymmetrical part is

$$\left(\frac{df}{dt}\right)_{c-asym} = \frac{eE_0 N k_E}{\hbar k} \left\{ \frac{e^\sigma g(\xi')}{k} \left(\frac{k'^2 + k^2}{k} \frac{A' \ln \left| \frac{k'+k}{k'-k} \right| - A'k' - (cc')^2 k'}{2} \right) \right. \\ \left. - g(\xi) \left(A'^2 \ln \left| (k'+k)/(k'-k) \right| - A'cc' - aa'cc' \right) \frac{d\Omega'}{d\xi'} \right\} \\ \xi' = \xi + \hbar\omega \\ + \frac{g(\xi')}{k} \left(\frac{k'^2 + k^2}{k} \frac{A' \ln \left| \frac{k'+k}{k'-k} \right| - A'k' - (cc')^2 k'}{2} \right) \\ - e^\sigma g(\xi) \left(A'^2 \ln \left| (k'+k)/(k'-k) \right| - A'cc' - aa'cc' \right) \frac{d\Omega'}{d\xi'} \left\{ \right. \\ \xi' = \xi - \hbar\omega \quad (A1.11)$$

9.2. APPENDIX 2:- PERTURBATION METHOD

In the Perturbation Method, $e^{-\sigma} = \delta$ is used as the expansion parameter. The γ_n in equation (2.10) is expanded in the form shown below.

$$\gamma_n = \sum_{j=0}^{\infty} \gamma_n^j \delta^j \quad (A2.1)$$

for $n=1,2,3,\dots$

The superscripts of γ_n^j indicates the order of the terms in each expansion. The coefficients in (2.10) are also written in terms of δ . Therefore A_n , B_n , C_n , and D_n become A_n^0 , $B_n^0 + B_n^{-1}/\delta$, C_n^{-1} and $D_n^n \delta^n$ respectively. Each superscript indicates the power of δ by which the given coefficient is multiplied. In terms of the new representations, (2.10) can be written as

$$\begin{aligned} B_0^0 \sum_j \gamma_0^j + C_0^{-1} \sum_j \gamma_1^j &= D_0^0 \\ A_1^0 \sum_j \gamma_0^j + B_1^0 \sum_j \gamma_1^j \delta^j + B_1^{-1}/\delta \sum_j \gamma_1^j \delta^j + C_1^{-1}/\delta \sum_j \gamma_2^j \delta^j &= D_1^1 \delta \\ &\vdots \\ A_n^0 \sum_j \gamma_{n-1}^j \delta^j + B_n^0 \sum_j \gamma_n^j \delta^j + B_n^{-1}/\delta \sum_j \gamma_n^j \delta^j &= D_n^n \delta^n \end{aligned} \quad (A2.2)$$

By equating the coefficients of the different powers of δ , the set of equations obtained from the coefficients of δ^{-1} is given in (A2.3).

$$\begin{aligned} C_0^{-1} \gamma_0^0 &= 0 \\ B_2^{-1} \gamma_1^0 + C_1^{-1} \gamma_2^0 &= 0 \end{aligned}$$

$$\begin{aligned} B_2^{-1} \gamma_2^0 + C_2^{-1} \gamma_2^{-1} &= 0 \\ B_n^{-1} \gamma_n^0 &= 0 \end{aligned} \quad (A2.3)$$

It has been assumed that the infinite set of equations in (A2.2) is truncated at the n^{th} equation. The result from solving (A2.3) is

$$\gamma_n^0 = 0 \text{ for } n > 1. \quad (A2.4)$$

The next set of equations can be found by equating the coefficients of δ^0 in (A2.2).

$$\begin{aligned} B_0^0 \gamma_0^0 + C_0^{-1} \gamma_1^1 &= D_0^0 \\ A_1^0 \gamma_0^0 + B_1^0 \gamma_1^0 + B_1^{-1} \gamma_1^1 + C_1^{-1} \gamma_2^1 &= 0 \\ A_2^0 \gamma_1^0 + B_2^0 \gamma_2^0 + B_2^{-1} \gamma_2^1 + C_2^{-1} \gamma_3^1 &= 0 \\ &\vdots \\ A_n^0 \gamma_{n-1}^0 + B_n^0 \gamma_n^0 + B_n^{-1} \gamma_n^1 &= 0 \end{aligned} \quad (A2.5)$$

When (A2.4) is substituted into (A2.5), the set of equations is reduced to just two equations, (A2.7a) and (A2.7b). The general result from solving (A2.5) is

$$\gamma_n^1 = 0 \text{ for } n > 2 \quad (A2.6)$$

$$B_0^0 \gamma_0^0 + C_0^{-1} \gamma_1^1 = D_0^0 \quad (A2.7a)$$

$$A_1^0 \gamma_0^0 + B_1^{-1} \gamma_1^1 = 0 \quad (A2.7b)$$

(A2.7) can be solved simultaneously and the results are,

$$\gamma_0^0 = \frac{B_1^{-1} D_0^0}{B_1^{-1} B_0^0 - A_1^0 C_0^{-1}} \quad (A2.8)$$

$$\gamma_1^1 = \frac{-A_1^0 \gamma_0^0}{B_1^{-1}} \quad (A2.9)$$

The set of equations obtained from equating the coefficients of δ in the truncated set, (A2.2) is

$$\begin{aligned} B_0^0 \gamma_0^1 + C_0^{-1} \gamma_1^2 &= 0 \\ A_1^0 \gamma_0^1 + B_1^0 \gamma_1^1 + B_1^{-1} \gamma_1^2 + C_1^{-1} \gamma_2^2 &= D_1^1 \\ A_2^0 \gamma_1^1 + B_2^0 \gamma_2^1 + B_2^{-1} \gamma_2^2 + C_2^{-1} \gamma_3^2 &= 0 \\ &\vdots \\ A_n^0 \gamma_{n-1}^1 + B_n^0 \gamma_n^1 + B_n^{-1} \gamma_n^2 &= 0 \end{aligned} \quad (A2.10)$$

When the result in (A2.6) is substituted into (A2.10) it is found that

$$\gamma_n^2 = 0 \text{ for } n > 3 \quad (A2.11)$$

and the equations are reduced to three algebraic equations:

$$\begin{aligned} B_0^0 \gamma_0^1 + C_0^{-1} \gamma_1^2 &= 0 \\ A_1^0 + B_1^0 \gamma_1^1 + B_1^{-1} \gamma_1^2 + C_1^{-1} \gamma_2^2 &= D_1^1 \\ A_2^0 \gamma_1^1 + B_2^{-1} \gamma_2^2 &= 0. \end{aligned} \quad (A2.12)$$

These equations can be solved simultaneously and the results are

$$\gamma_2^2 = - (A_2^0 \gamma_1^1) / B_2^{-1} \quad (A2.13)$$

$$\gamma_0^1 = \frac{D_1^1 C_0^{-1} - C_0^{-1} (B_1^0 \gamma_1^1 + C_1^{-1} \gamma_2^2)}{A_1^0 C_0^{-1} - B_0^0 B_1^{-1}} \quad (A2.14)$$

$$\gamma_1^2 = - (B_0^0 \gamma_0^1) / C_0^{-1}. \quad (A2.15)$$

Another set of equations can be obtained by equating the coefficients of δ^2 and the set is shown below.

$$\begin{aligned} B_0^0 \gamma_0^2 + C_0^{-1} \gamma_1^3 &= 0 \\ A_1^0 \gamma_0^2 + B_1^0 \gamma_1^2 + B_1^{-1} \gamma_1^3 + C_1^{-1} \gamma_2^3 &= 0 \\ A_2^0 \gamma_1^2 + B_2^0 \gamma_2^2 + B_2^{-1} \gamma_2^3 + C_2^{-1} \gamma_3^3 &= D_2^2 \\ &\vdots \\ A_n^0 \gamma_{n-1}^2 + B_n^0 \gamma_n^2 + B_n^{-1} \gamma_n^3 &= 0 \end{aligned} \quad (A2.16)$$

When the result of (A2.11) is substituted into (A4.16), a generalized result similar to (A2.11) is obtained for γ_n^m where $m=3$.

$$\gamma_n^3 = 0 \quad \text{for } n > 4 \quad (A2.17)$$

Then the set (A2.16) is reduced to just four equations as shown below.

$$\begin{aligned} B_0^0 \gamma_0^2 + C_0^{-1} \gamma_1^3 &= 0 \\ A_1^0 \gamma_0^2 + B_1^0 \gamma_1^2 + B_1^{-1} \gamma_1^3 + C_1^{-1} \gamma_2^3 &= 0 \end{aligned}$$

$$A_2^0 \gamma_1^2 + B_2^0 \gamma_2^2 + B_2^{-1} \gamma_2^3 + C_1^{-1} \gamma_2^3 = D_2^2$$

$$A_3^0 \gamma_2^2 + B_3^{-1} \gamma_3^3 = 0 \quad (A2.18)$$

When these equations are solved simultaneously, the results are;

$$\gamma_2^3 = -(A_1^0 \gamma_2^2) / B_3^{-1} \quad (A2.19)$$

$$\gamma_2^3 = \frac{D_2^2 - (A_2^0 \gamma_1^2 + B_2^0 \gamma_2^2 + C_2^{-1} \gamma_3^3)}{B_2^{-1}} \quad (A2.20)$$

$$\gamma_0^2 = \frac{-C_0^{-1} (B_1^0 \gamma_1^2 + C_1^{-1} \gamma_2^3)}{A_1^0 C_0^{-1} - B_0^0 B_1^{-1}} \quad (A2.21)$$

$$\gamma_1^3 = - (B_0^0 \gamma_0^2) / C_0^{-1} \quad (A2.22)$$

Similarly from equating the coefficients of δ and applying the result in (A2.17), a general result for γ_n^4 .

$$\gamma_n^4 = 0 \quad \text{for } n > 5 \quad (A2.23)$$

The set of simultaneous equations obtained is

$$B_0^0 \gamma_0^3 + C_0^{-1} \gamma_1^4 = 0$$

$$A_1^0 \gamma_0^3 + B_1^0 \gamma_1^3 + B_1^{-1} \gamma_1^4 + C_1^{-1} \gamma_2^4 = 0$$

$$A_2^0 \gamma_1^3 + B_2^0 \gamma_2^3 + B_2^{-1} \gamma_2^4 + C_2^{-1} \gamma_3^4 = 0$$

$$A_3^0 \gamma_2^3 + B_3^0 \gamma_3^3 + B_3^{-1} \gamma_3^4 + C_3^{-1} \gamma_4^4 = D_3^3$$

$$A_4^0 \gamma_3^3 + B_4^{-1} \gamma_4^4 = 0 \quad (A2.24)$$

These equations when solved simultaneously give

$$\gamma_4^4 = -(A_4^0 \gamma_3^3) / B_4^{-1} \quad (A2.25)$$

$$\gamma_3^4 = \frac{D_3^3 - (A_3^0 \gamma_2^3 + B_3^0 \gamma_3^3 + C_3^{-1} \gamma_4^4)}{B_3^{-1}} \quad (A2.26)$$

$$\gamma_2^4 = \frac{-(A_2^0 \gamma_1^3 + B_2^0 \gamma_2^3 + C_2^{-1} \gamma_3^4)}{B_n^{-1}} \quad (A2.27)$$

$$\gamma_0^3 = \frac{-C_0^{-1} (B_1^0 \gamma_1^3 + C_1^{-1} \gamma_2^4)}{A_1^0 C_0^{-1} - B_0^0 B_1^{-1}} \quad (A2.28)$$

$$\gamma_1^4 = -(B_0^0 \gamma_0^3) / C_0^{-1} \quad (A2.29)$$

The same procedure can be repeated for obtaining higher order γ_n^m .
In general, by equating the coefficients of δ , it will be found that

$$\gamma_n^m = 0 \quad \text{for } m > n \quad (A2.30)$$

and the set of equations that is left to be solved will always be found to contain $m+2$ equations with an equal number of unknowns as shown (A2.31).

$$\begin{aligned} B_0^0 \gamma_0^m + C_0^{-1} \gamma_1^{m+1} &= 0 \\ A_1^0 \gamma_0^m + B_1^0 \gamma_1^m + B_1^{-1} \gamma_1^{m+1} + C_1^{-1} \gamma_2^{m+1} &= 0 \\ A_2^0 \gamma_1^m + B_2^0 \gamma_2^m + B_2^{-1} \gamma_1^{m+1} + C_2^{-1} \gamma_3^{m+1} &= 0 \\ &\vdots \\ A_m^0 \gamma_{m-1}^m + B_m^0 \gamma_m^m + B_m^{-1} \gamma_m^{m+1} + C_m^{-1} \gamma_{m+1}^{m+1} &= D_m^m \end{aligned} \quad (A2.31)$$

$$A_{m+1}^0 \gamma_m^{m+1} + B_{m+1}^{-1} \gamma_{m+1}^{m+1} = 0$$

This set of equations when solved gives

$$\gamma_{m+1}^{m+1} = -(A_{m+1}^0 \gamma_m^m) / B_{m+1}^{-1} \quad (A2.32)$$

$$\gamma_m^{m+1} = \frac{D_m^m - (A_m^0 \gamma_{m-1}^{m+1} + B_m^0 \gamma_m^m + C_m^{-1} \gamma_{m+1}^{m+1})}{B_m^{-1}}, \quad m > 1 \quad (A2.33)$$

$$\gamma_n^m = \frac{-(A_n^0 \gamma_{n-1}^{m-1} + B_n^0 \gamma_n^{m-1} + C_n^{-1} \gamma_{n+1}^m)}{B_n^{-1}}, \quad m > 4, \quad n > m-2 \quad (A2.34)$$

$$\gamma_0^m = \frac{-C_0^{-1} (B_1^0 \gamma_1^m + C_1^{-1} \gamma_2^{m+1})}{A_1^0 C_0^{-1} - B_0^0 B_1^{-1}}, \quad m > 2 \quad (A2.35)$$

$$\gamma_1^{m+1} = -(B_0^0 \gamma_0^m) / C_0^{-1} \quad (A2.36)$$

9.3 APPENDIX 3:- ANALYTICAL METHOD

If the truncated set of equations in (2.10) is assumed to have four equations (i.e. $n=3$), they are

$$B_0\gamma_0 + C_0\gamma_1 = D_0 \quad (A3.1)$$

$$A_1\gamma_0 + B_1\gamma_1 + C_1\gamma_2 = D_1 \quad (A3.2)$$

$$A_2\gamma_1 + B_2\gamma_2 + C_2\gamma_3 = D_2 \quad (A3.3)$$

$$A_3\gamma_2 + B_3\gamma_3 = D_3 \quad (A3.4)$$

where the coefficients A, B, C and D are defined in section 2. These set of equations can be solved simultaneously and the results are;

$$\gamma_0 = \frac{D_0[(B_2B_3 - C_2A_3)B_1 - B_3A_2] - C_0D_1(B_2B_3 - C_2A_3) + C_0C_1(B_3D_2 - C_2D_3)}{B_1B_0(B_2B_3 - C_2A_3) - B_0B_3A_2 - C_0A_1(B_2B_3 - C_2A_3)} \quad (A3.5)$$

$$\gamma_1 = \frac{(B_2B_3 - C_2A_3)(D_1 - A_1\gamma_0) - C_1(B_3D_2 - C_2D_3)}{(B_2B_3 - C_2A_3)B_1 - B_3A_2} \quad (A3.6)$$

$$\gamma_2 = \frac{B_2D_3 - C_2D_3 - B_3A_2\gamma_1}{B_2B_3 - C_2A_3} \quad (A3.7)$$

$$\gamma_3 = \frac{D_3 - A_2\gamma_2}{B_3} \quad (A3.8)$$

The coefficients A_n , B_n , C_n and D_n by A_n^0 , $B_n^0 + B_n^{-1}/\delta$, C_n^{-1}/δ and $D_n^n\delta^n$ respectively. Those terms with power of δ greater than -2 are assumed small and dropped. The results are shown in (A3.9)

$$\gamma_0 = \left\{ \delta^{-3} (D_0^0 B_1^{-1} B_2^{-1} B_3^{-1}) + \delta^{-2} [D_0^0 (B_1^0 B_2^{-1} B_3^{-1} + B_1^{-1} B_2^{-1} B_3^0 - A_3^0 C_2^{-1} B_1^{-1}) - D_1^1 C_0^{-1} B_2^{-1} B_3^{-1}] \right. \\ \left. \delta^{-3} [B_0^0 B_1^{-1} B_2^{-1} B_3^{-1} - A_1^0 C_0^{-1} B_2^{-1} B_3^{-1}] + \delta^{-2} [B_0^0 (B_1^0 B_2^{-1} B_3^{-1} + B_1^{-1} B_2^0 B_3^{-1} + B_1^{-1} B_2^{-1} B_3^0) \right. \\ \left. - A_3^0 C_2^{-1} (B_0^0 B_1^{-1} - C_2^{-1} A_3^0) - A_1^0 C_0^{-1} (B_2^0 B_3^{-1} + B_2^{-1} B_3^0)] \right\}^{-1} \quad (A3.9)$$

$$\gamma_1 = \left\{ \delta^{-1} A_1^0 \gamma_0 [C_2^{-1} A_3^0 - B_2^0 B_3^{-1} - B_2^{-1} B_3^0 - B_2^{-1} B_3^{-1} \delta^{-1}] + B_2^{-1} B_3^{-1} D_1^1 \delta^{-1} \right\} \\ \left\{ \delta^{-3} (B_1^{-1} B_2^{-1} B_3^{-1}) + \delta^{-2} (B_1^0 B_2^{-1} B_3^{-1} + B_1^{-1} B_2^0 B_3^{-1} + B_1^{-1} B_2^{-1} B_3^0 - B_1^{-1} C_2^{-1} A_3^0) \right. \\ \left. + \delta^{-1} (B_1^0 B_2^{-1} B_3^0 + B_1^{-1} B_2^0 B_3^0 + B_1^0 B_2^0 B_3^{-1} - B_1^0 C_2^{-1} A_3^0) \right\}^{-1} \quad (A3.10)$$

$$\gamma_2 = \frac{B_3^{-1} D_2^2 \delta - \gamma_1 (B_3^{-1} A_3^0 + B_3^0 A_3^0)}{B_2^{-1} B_3^{-1} \delta^{-2} + (B_2^0 B_3^{-1} + B_3^0 B_2^{-1} - C_3^{-1} A_3^0) \delta^{-1}} \quad (A3.11)$$

$$\gamma_3 = (D_3^3 - A_3^0 \gamma_2) / (B_3^0 + B_3^{-1}) \quad (A3.12)$$

If a new variable, y , is defined such that

$$y = (\xi/\sigma)^{1/2} \quad (A3.13)$$

then the coefficients can be rewritten as

$$A_1^0 = C_0^{-1} = 1 - \frac{(2y^2+1) \sinh^{-1} y}{\sqrt{(y^2+2)(y^2+1)}} \quad (A3.14)$$

$$A_2^0 = C_1^{-1} = 1 - \frac{(2y^2+3) \sinh^{-1} (1+y^2)^{1/2}}{\sqrt{(y^2+1)(y^2+2)}} \quad (A3.15)$$

$$A_3^0 = C_2^{-1} = 1 - \frac{(2y^2+5) \sinh^{-1} (y^2+2)^{1/2}}{\sqrt{(y^2+2)(y^2+3)}} \quad (A3.16)$$

$$B_0^0 = B_1^{-1} = 2 \sinh^{-1} y \quad (A3.17)$$

$$B_1^0 = B_2^{-1} = 2 \sinh^{-1} (y^2+1)^{1/2} \quad (A3.18)$$

$$B_2^0 = B_3^{-1} = 2 \sinh^{-1}(y^2+2)^{1/2} \quad (A3.19)$$

$$B_3^0 = 2 \sinh^{-1}(y^2+3)^{1/2} \quad (A3.20)$$

$$D_n^n = -\sigma(y^2+n)e^{-\xi} \quad (A3.21)$$

Taylor expansions can be used to expand the inverse hyperbolic functions. In this case the expansions are made about $y=0$.

$$\sinh^{-1}(y^2+n)^{1/2} = \sinh^{-1}(n)^{1/2} + \frac{y^2}{(2/n^2+1)} - \frac{y^4 3(2n+1)}{(24(n^2+1)^{3/2})} + \dots \quad (A3.22)$$

By using the expansions, the A_n^0 , B_n^0 , B_n^{-1} and C_n^{-1} can be written as a series in power of y as shown.

$$A_1^0 = C_0^{-1} = 4y^2(-1/3 + y^2/5 + \dots) \quad (A3.23)$$

$$A_2^0 = C_1^{-1} = -0.8696 - 0.5942y^2 + \dots \quad (A3.24)$$

$$A_3^0 = C_2^{-1} = -1.3396 - 0.4174y^2 + \dots \quad (A3.25)$$

$$B_0^0 = B_1^{-1} = 2y[1 - y^2/6 + 3y^4/40 + \dots] \quad (A3.26)$$

$$B_1^0 = B_2^{-1} = 2[\sinh^{-1}(1) + y^2/(2\sqrt{2}) + \dots] \quad (A3.27)$$

$$B_2^0 = B_3^{-1} = 2[\sinh^{-1}\sqrt{2} + y^2/(2\sqrt{3}) + \dots] \quad (A3.28)$$

$$B_3^0 = 2[\sinh^{-1}\sqrt{3} + y^2/10 + \dots] \quad (A3.29)$$

These expansions can be substituted into the expressions for g_0 , g_1 and g_2 and the final approximations are shown in equations (4.7), (4.8) and (4.9) of section 4.

When deriving the analytical formula the formulas derived from

Perturbation Method, the coefficients are expanded in terms of y using equations (A3.22-A3.27). In this case the different terms in γ_n expansions used are given in equations (3.13-3.20).

9.4 APPENDIX 4:- EXPANSIONS FOR HIGH ELECTRIC FIELDS

For a parabolic band $\Omega = \mathcal{E}$ and if there is no p-wave scattering, $c=0$ and $a=1$ in equations (A1.10) and (A1.11). These equations then reduce to equation (5.3.7) and (5.3.8) in Conwell's book [26].

$$\left(\frac{df_0}{dt}\right)_{c \text{ sym}} = \frac{eE_0 N}{\sqrt{(2m^* \hbar \omega x)}} \left\{ \left[e^\sigma f_0(x+1) - f_0(x) \right] \phi(x) + H(x-1) \left[f_0(x-1) - e^\sigma f_0(x) \right] \phi(x-1) \right\} \quad (A4.1)$$

$$\left(\frac{dk_E}{dt}\right)_{c \text{ asy}} = \frac{eE_0 \cos \theta N}{\hbar} \left(e^\sigma g(x+1) \sqrt{\frac{x+1}{x}} \left[\frac{(2x+1)\phi(x)}{2\sqrt{x(x+1)}} - 1 \right] - g(x) \phi(x) + H(x-1) \left\{ g(x-1) \sqrt{\frac{x-1}{x}} \left[\frac{(2x-1)\phi(x-1)}{2\sqrt{x(x-1)}} - 1 \right] - g(x) e^\sigma \phi(x-1) \right\} \right) \quad (A4.2)$$

where the $\ln \left| \frac{k'+k}{k'-k} \right|$ terms have been replaced by $\phi(x)$. The rate of change of the distribution function due to the applied electric field, (A1.12), is given by

$$\left(\frac{df(x)}{dt}\right)_E = -eE \left[\frac{2}{3\hbar\sqrt{x}} \frac{d(x^{3/2}g)}{dx} + \frac{\hbar k \cos \theta}{m^*} \frac{df}{dx} \right] \quad (A4.3)$$

When (A4.1) is equated with the symmetric part of (A4.3), and (A4.2) equated with the asymmetric part of (A4.3), the resulting equations are shown below.

$$\frac{d[xf_1(x)]}{dx} = N/K \left\{ \left[f_0(x) - e^\sigma f_0(x+1) \right] \phi(x) - H(x-1) \left[f_0(x-1) - e^\sigma f_0(x) \right] \phi(x-1) \right\} \quad (A4.4)$$

$$x \frac{df_0(x)}{dx} = \frac{N}{3K} \left(f_1(x) \phi(x) - e^\sigma f_1(x+1) \left[\frac{(2x+1)\phi(x)}{2\sqrt{x(x+1)}} - 1 \right] + H(x-1) \left\{ f_1(x) e^\sigma \phi(x-1) - f_1(x-1) \left[\frac{(2x-1)\phi(x-1)}{2\sqrt{x(x-1)}} - 1 \right] \right\} \right) \quad (A4.5)$$

where $f_1(x) = kg$ and $K = 2E/(3E_0)$. It is assumed that the electron distribution functions, $f_0(x)$ and $f_1(x)$ can be expanded into power series in $1/K$ where

$$f_0(x) = \sum_j (N/K)^j f_0^j(x) \quad (A4.6)$$

$$f_1(x) = \sum_j (N/K)^j f_1^j(x) \quad (A4.7)$$

Therefore (A4.4) and (A4.5) become

$$\begin{aligned} \frac{d}{dx} \left[x \sum_j (1/K)^j f_1^j(x) \right] = & N/K \left\{ \left[\sum_j (1/K)^j f_0^j(x) - e^{\sigma} \sum_j (1/K)^j f_0^j(x+1) \right] \phi(x) \right. \\ & \left. - H(x-1) \left[\sum_j (1/K)^j f_0^j(x-1) - e^{\sigma} \sum_j (1/K)^j f_0^j(x) \right] \phi(x-1) \right\} \quad (A4.8) \\ x \frac{d}{dx} \left[\sum_j (1/K)^j f_0^j(x) \right] = & \frac{N}{3K} \left\{ \phi(x) \left[\sum_j (1/K)^j f_1^j(x) - e^{\sigma} \left\{ \frac{(2x+1)\phi(x)}{2\sqrt{x(x+1)}} - 1 \right\} \sum_j (1/K)^j f_1^j(x+1) \right] \right. \\ & \left. + H(x-1) \left[e^{\sigma} \phi(x-1) \sum_j (1/K)^j f_1^j(x) \right. \right. \\ & \left. \left. - \left\{ \frac{(2x-1)\phi(x-1)}{2\sqrt{x(x-1)}} - 1 \right\} \sum_j (1/K)^j f_1^j(x-1) \right] \right\} \quad (A4.9) \end{aligned}$$

where the superscripts of $(1/K)$ is its power while the superscripts on $f_0(x)$ and $f_1(x)$ are just there to indicate the order of each terms in the expansions. The coefficients of the different are equated and the equations that are found for the first few power of $(1/K)$ are shown below.

For $(1/K)^0$

$$x \frac{d[f_0^0(x)]}{dx} = 0 \quad (A4.10)$$

$$\frac{d[x f_1^0(x)]}{dx} = 0 \quad (A4.11)$$

For $(1/K)^1$

$$\begin{aligned} \frac{d[x f_1^1(x)]}{dx} = & N \left\{ \left[f_0^0(x) - e^{\sigma} f_0^0(x+1) \right] \phi(x) \right. \\ & \left. - H(x-1) \phi(x-1) \left[f_0^0(x-1) - e^{\sigma} f_0^0(x) \right] \right\} \quad (A4.12) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} [f_0^0(x)] = \frac{N}{3} & \left\{ \phi(x) f_1^0(x) - e^\sigma \left[\frac{(2x+1)\phi(x)}{\sqrt{x(x+1)}} - 1 \right] f_1^0(x+1) \right. \\ & \left. + H(x-1) \left\{ e^\sigma \phi(x-1) f_1^0(x) - \left[\frac{(2x-1)\phi(x-1)}{\sqrt{x(x-1)}} - 1 \right] f_1^0(x-1) \right\} \right\} \end{aligned} \quad (A4.13)$$

For $(1/K)^2$

$$\frac{d}{dx} [x f_1^2(x)] = N \left\{ \left(f_0^1(x) - e^\sigma f_0^1(x+1) \right) \phi(x) - H(x-1) \left[\phi(x-1) \left(f_0^1(x-1) - e^\sigma f_0^1(x) \right) \right] \right\} \quad (A4.14)$$

$$\begin{aligned} \frac{d}{dx} (f_0^2(x)) = \frac{N}{3} & \left\{ \phi(x) f_1^1(x) - e^\sigma \left[\frac{(2x+1)\phi(x)}{2\sqrt{x(x+1)}} - 1 \right] f_1^1(x+1) \right. \\ & \left. + H(x-1) \left\{ e^\sigma \phi(x-1) f_1^1(x) - \left[\frac{(2x-1)\phi(x-1)}{2\sqrt{x(x-1)}} - 1 \right] f_1^1(x-1) \right\} \right\} \end{aligned} \quad (A4.15)$$

For $(1/K)^3$

$$\frac{d}{dx} [x f_1^3(x)] = N \left\{ \left[f_0^2(x) - e^\sigma f_0^2(x+1) \right] \phi(x) - H(x-1) \phi(x-1) \left[f_0^2(x-1) - e^\sigma f_0^2(x) \right] \right\} \quad (A4.16)$$

$$\begin{aligned} \frac{d}{dx} (f_0^3(x)) = \frac{N}{3} & \left\{ \phi(x) f_1^2(x) - e^\sigma \left[\frac{(2x+1)\phi(x)}{2\sqrt{x(x+1)}} - 1 \right] f_1^2(x+1) \right. \\ & \left. + H(x-1) \left\{ e^\sigma \phi(x-1) f_1^2(x) - \left[\frac{(2x-1)\phi(x-1)}{2\sqrt{x(x-1)}} - 1 \right] f_1^2(x-1) \right\} \right\} \end{aligned} \quad (A4.17)$$

From (A4.10), for the equation to be true for all x , f_0^0 has to be a constant and is assumed to be

$$f_0^0(x) = F(0) \quad (A4.18)$$

From (A4.11), it is found after integration that $x f_1^0(x)$ is a constant. By taking the limit as x approaches zero, it can be seen that the integration constant has to be zero or else $f_1^0(x)$ is infinite at $x=0$.

$$f_1^0(x) = 0 \quad (A4.19)$$

This result is substituted into (A4.14) and it is reduced to

$$x \frac{d f_0^1(x)}{dx} = 0 \quad (A4.20)$$

For this equation to be valid for all x , f_0^1 has to be a constant. For $f_0(x)$ to match the result of Stratton [5], and Conwell and Vassell [6] at very large E and x , $f_0^0(\infty) = F(0)$. This last result implies that all $f_0^m(\infty) = 0$ for all $m > 0$. Therefore the constant $f_0^1(x)$ must be zero.

$$f_0^1(x) = 0 \quad (A4.21)$$

Equation (A4.18) can be substituted into (A4.12) and integrated for an expression of $f_1^1(x)$ in terms of $F(0)$ and x

$$f_1^1(x) = \frac{F(0)(1-e^\sigma)}{x} \left\{ 2 \int_0^x \sinh^{-1} \sqrt{x} \, dx - 2 \int_1^x \sinh^{-1} \sqrt{x-1} \, dx \right\} \quad (A4.22)$$

$f_1^1(x)$ is zero at the lower limit, $x=0$. In the second integral the lower limit is the result of the step function. The method of substitution is used to integrate the inverse hyperbolic sine function.

Let $y = \sqrt{x}$, therefore $dy = \frac{1}{2} \frac{dx}{\sqrt{x}}$

$$\begin{aligned} \text{Then } \int_0^x \sinh^{-1} \sqrt{x} \, dx &= 2y \int_0^{\sqrt{x}} \sinh^{-1} y \, dy \\ &= (x+1/2) \sinh^{-1} \sqrt{x} - \frac{\sqrt{x(x+1)}}{2} \end{aligned} \quad (A4.23)$$

Equation (A4.22) after the integration is given by,

$$\begin{aligned} f_1^1(x) = \frac{F(0)(1-e^\sigma)}{x} \left\{ (2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)} \right. \\ \left. - H(x-1) \left[(2x-1) \sinh^{-1} \sqrt{x-1} - \sqrt{x(x-1)} \right] \right\} \end{aligned} \quad (A4.24)$$

This equation can be substituted into (A4.15) and the resulting equation

can be integrated.

$$r_0^2(x) = \frac{1}{3} \int_0^x \frac{dx}{x} \left\{ 2f_1^1(x) \sinh^{-1} \sqrt{x} - e^\sigma f_1^1(x+1) \left[\frac{(2x+1) \sinh^{-1} \sqrt{x}}{\sqrt{x(x+1)}} - 1 \right] \right. \\ \left. - H(x-1) \left\{ 2e^\sigma f_1^1(x) \sinh^{-1} \sqrt{x-1} - f_1^1(x-1) \left[\frac{(2x-1) \sinh^{-1} \sqrt{x-1}}{\sqrt{x(x-1)}} - 1 \right] \right\} \right\} \\ + F_0^2(0) \quad (A4.25)$$

where F_0^2 is a constant of integration that will reduce $r_0^2(x)$ to zero at $x=\infty$. If $r_0^2(\infty)=0$, then

$$F_0^2(x) = -\frac{1}{3} \int_0^\infty \frac{dx}{x} \left\{ 2f_1^1(x) \sinh^{-1} \sqrt{x} - e^\sigma f_1^1(x+1) \left[\frac{(2x+1) \sinh^{-1} \sqrt{x}}{\sqrt{x(x+1)}} - 1 \right] \right. \\ \left. - H(x-1) \left\{ 2e^\sigma f_1^1(x) \sinh^{-1} \sqrt{x-1} - f_1^1(x-1) \left[\frac{(2x-1) \sinh^{-1} \sqrt{x-1}}{\sqrt{x(x-1)}} - 1 \right] \right\} \right\} \quad (A4.26)$$

Therefore,

$$r_0^2(x) = \frac{1}{3} \int_0^x \frac{dx}{x} \left\{ 2f_1^1(x) \sinh^{-1} \sqrt{x} - e^\sigma f_1^1(x+1) \left[\frac{(2x+1) \sinh^{-1} \sqrt{x}}{\sqrt{x(x+1)}} - 1 \right] \right. \\ \left. - H(x-1) \left\{ 2e^\sigma f_1^1(x) \sinh^{-1} \sqrt{x-1} - f_1^1(x-1) \left[\frac{(2x-1) \sinh^{-1} \sqrt{x-1}}{\sqrt{x(x-1)}} - 1 \right] \right\} \right\} \\ - \frac{1}{3} \int_0^\infty \frac{dx}{x} \left\{ 2f_1^1(x) \sinh^{-1} \sqrt{x} - e^\sigma f_1^1(x+1) \left[\frac{(2x+1) \sinh^{-1} \sqrt{x}}{\sqrt{x(x+1)}} - 1 \right] \right. \\ \left. + H(x-1) \left\{ 2e^\sigma f_1^1(x) \sinh^{-1} \sqrt{x-1} - f_1^1(x-1) \left[\frac{(2x-1) \sinh^{-1} \sqrt{x-1}}{\sqrt{x(x-1)}} - 1 \right] \right\} \right\} \quad (A4.27)$$

From equation (A4.16), $r_1^3(x)$ can be written in the integral form

$$r_1^3(x) = \frac{1}{x} \int_0^x dx \left\{ [r_0^2(x) - e^\sigma r_0^2(x+1)] \phi(x) \right. \\ \left. - H(x-1) [r_0^2(x-1) - e^\sigma r_0^2(x)] \phi(x-1) \right\} \quad (A4.28)$$

This equation can be simplified by integration by parts. Consider the first part with $\phi(x)$ replaced by $2\sinh^{-1} \sqrt{x}$,

$$I = \int_0^x dx \left[f_0^2(x) - e^{\sigma} f_0^2(x+1) \right] 2 \sinh^{-1} \sqrt{x}$$

Let $dv = 2 \sinh^{-1} \sqrt{x} dx$

and $u = [f_0^2(x) - e^{\sigma} f_0^2(x+1)]$

therefore,

$$I = \left[f_0^2(x) - e^{\sigma} f_0^2(x+1) \right] \left[(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)} \right] \Big|_{x=0}^x - \int_0^x [(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)}] \left[\frac{df_0^2}{dx}(x) - e^{\sigma} \frac{df_0^2}{dx}(x+1) \right] dx \quad (A4.29)$$

where the result from (A4.23) has been used for the integration of the inverse hyperbolic sine functions. Therefore (A4.28) can be written as

$$f_1^3(x) = 1/x \left\{ \left[f_0^2(x) - e^{\sigma} f_0^2(x+1) \right] \left[(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)} \right] - H(x-1) \left[f_0^2(x-1) - e^{\sigma} f_0^2(x) \right] \left[(2x+1) \sinh^{-1} \sqrt{x-1} - \sqrt{x(x-1)} \right] - \int_0^x [(2x+1) \sinh^{-1} \sqrt{x} - \sqrt{x(x+1)}] \left[\frac{df_0^2}{dx}(x) - e^{\sigma} \frac{df_0^2}{dx}(x+1) \right] + \int_1^x [(2x-1) \sinh^{-1} \sqrt{x-1} - \sqrt{x(x-1)}] \left[\frac{df_0^2}{dx}(x-1) - e^{\sigma} \frac{df_0^2}{dx}(x) \right] \right\} \quad (A4.30)$$

9.5 APPENDIX 5:- p-WAVE SCATTERING.

In the case where p-wave scattering is not significant, c is much smaller than a . If A' in A1.10 is replaced by $[aa' + (k'^2 + k^2)cc' / (2kk')]$ and if a^2 is further replaced by $1 - c^2$, the resulting equation contains terms with $(cc')^2$ and c^4 . These terms are assume to be small and dropped. The equation after these terms are dropped is shown below.

$$\begin{aligned} \frac{df_0(\xi)}{dt} \Big|_{\text{sym}} &= \frac{eE_0 N}{\hbar k} \left\{ [f_0(\xi') e^\sigma - f_0(\xi)] \left[\ln \left| \frac{k'+k}{k'-k} \right| + c^2 \left(\frac{k'^2 + k^2}{kk'} - 2 \right) \ln \left| \frac{k'+k}{k'-k} \right| - 2 \right] \right\} \frac{d\Omega}{d\xi'} \\ &\quad + [f_0(\xi) - e^\sigma f_0(\xi')] \left\{ \ln \left| \frac{k'+k}{k'-k} \right| + c^2 \left[\left(\frac{k'^2 + k^2}{kk'} - 2 \right) \ln \left| \frac{k'+k}{k'-k} \right| - 2 \right] \right\} \frac{d\Omega}{d\xi'} \end{aligned}$$

$\xi' = \xi + \hbar\omega$
(A5.1)

The same procedure can be used to reduce the asymmetrical part of the distribution function, (A1.11), and the result is shown below.

$$\begin{aligned} \frac{df_1(\xi)}{dt} &= \frac{eE_0 \hbar k_E}{\hbar k} \left\{ \frac{e^\sigma g(\xi')}{k} \left[\frac{k'^2 + k^2}{2k} \ln \left| \frac{k'+k}{k'-k} \right| - k' \right] - g(\xi) \ln \left| \frac{k'+k}{k'-k} \right| \right. \\ &\quad + \frac{c^2 e^\sigma g(\xi')}{k} \left[\frac{k'^2 + k^2}{2k} \left(\frac{k'^2 + k^2}{2k'k} - 1 \right) \ln \left| \frac{k'+k}{k'-k} \right| - k' \left(\frac{k'^2 + k^2}{kk'} - 2 \right) \right] \\ &\quad \left. - c^2 g(\xi) \left[\left(\frac{k'^2 + k^2}{kk'} - 2 \right) \ln \left| \frac{k'+k}{k'-k} \right| - 2 \right] \right\} \frac{d\Omega}{d\xi} \\ &\quad + \frac{g(\xi')}{k} \left[\frac{k'^2 + k^2}{2k} \ln \left| \frac{k'+k}{k'-k} \right| - k' \right] - e^\sigma g(\xi) \ln \left| \frac{k'+k}{k'-k} \right| \\ &\quad + \frac{c^2 g(\xi')}{k} \left[\frac{k'^2 + k^2}{kk'} \left(\frac{k'^2 + k^2}{2kk'} - 1 \right) \ln \left| \frac{k'+k}{k'-k} \right| - k' \left(\frac{k'^2 + k^2}{kk'} - 2 \right) \right] \\ &\quad \left. - c^2 e^\sigma g(\xi) \left[\left(\frac{k'^2 + k^2}{kk'} - 2 \right) \ln \left| \frac{k'-k}{k'+k} \right| - 2 \right] \right\} \frac{d\Omega}{d\xi} \end{aligned}$$

$\xi' = \xi + \hbar\omega$
(A5.2)

In the two equations above terms that are associated with p-wave scattering are terms with α^2 . If there is no p-wave scattering, then $\alpha=0$, and the equations will reduce to (A4.1) and (A4.2).

It is assumed that the non-parabolicity of the band is of the form

$$\Omega = \xi + \alpha \xi^2 \quad (\text{A5.3})$$

therefore $\frac{d\Omega}{d\xi} = 1 + 2\alpha\xi$

The next step is to express the k and k' in terms of ξ and $\hbar\omega$ by using the relation

$$\frac{\hbar^2 k^2}{2m^*} = \xi + \alpha \xi^2 \quad (\text{A5.4})$$

$$\begin{aligned} \left. \frac{k'^2 + k^2}{k'k} \right|_{\xi' = \xi + \hbar\omega} &= \frac{\xi + \hbar\omega + \alpha(\xi + \hbar\omega)^2 + \xi + \alpha\xi^2}{\sqrt{(\xi + \alpha\xi^2)(\xi + \hbar\omega + \alpha(\xi + \hbar\omega)^2)}} \\ &= \frac{2\xi + 2\alpha\xi^2 + 2\alpha\hbar\omega\xi + \alpha(\hbar\omega)^2 + \hbar\omega}{(\xi + \alpha\xi^2) \sqrt{1 + 2\alpha\hbar\omega\xi + \alpha(\hbar\omega)^2 + \hbar\omega}} \\ &= 2 + \frac{1}{4} \left[\frac{2\alpha\hbar\omega\xi + \alpha(\hbar\omega)^2 + \hbar\omega}{\xi + \alpha\xi^2} \right]^2 + \dots \end{aligned}$$

For $\xi \gg \hbar\omega$ this can be approximated by

$$\left. \frac{k'^2 + k^2}{k'k} \right|_{\xi' = \xi + \hbar\omega} = 2 + \frac{1}{4} \left[\frac{2\alpha\hbar\omega\xi + \hbar\omega}{\xi + \alpha\xi^2} \right]^2 \quad (\text{A5.5})$$

Similarly for the case when $\xi' = \xi - \hbar\omega$, the same procedure gives

$$\left. \frac{k'^2 + k^2}{kk'} \right|_{\xi' = \xi - \hbar\omega} = 2 + \frac{1}{4} \left(\frac{-2\alpha\hbar\omega\xi - \hbar\omega}{\xi + \alpha\xi^2} \right)^2 \quad (\text{A5.6})$$

The term $\ln \left| \frac{k' + k}{k' - k} \right|$ can be written in the form of inverse hyperbolic sine functions and are given by

$$\ln \left| \frac{k'+k}{k'-k} \right| = 2 \sinh^{-1} \left(\frac{k}{\sqrt{k'^2 - k^2}} \right) \quad (A5.7)$$

$\xi' = \xi + \hbar\omega$

$$\text{and} \quad \ln \left| \frac{k'+k}{k'-k} \right| = 2 \sinh^{-1} \left(\frac{k'}{\sqrt{k^2 - k'^2}} \right) \quad (A5.8)$$

$\xi' = \xi - \hbar\omega$

for the case when $\xi' = \xi + \hbar\omega$

$$\frac{k}{\sqrt{k'^2 - k^2}} = \frac{\sqrt{\xi + a\xi^2}}{\sqrt{\xi + \hbar\omega + a(\xi + \hbar\omega)^2 - (\xi + a\xi^2)}}$$

$\xi' = \xi + \hbar\omega$

for $\xi \gg \hbar\omega$

$$= \sqrt{\frac{\xi}{\hbar\omega} \frac{1+a\xi}{1+2a\xi}} \quad (A5.9)$$

Similarly for the case when $\xi' = \xi - \hbar\omega$,

$$\frac{k'}{\sqrt{k^2 - k'^2}} = \left[\frac{\xi}{\hbar\omega} \frac{(1+a\xi)}{(1+2a\xi)} - 1 \right]^{1/2} \quad (A5.10)$$

$\xi' = \xi - \hbar\omega$

when equation (A5.4)-(A5.10) are substituted into (A5.1), and only the terms due to p-wave scattering are considered, the equation can be written as

$$\begin{aligned} \frac{df_0(\xi)}{dt} = & \frac{eNE_0 c^2}{\hbar k} \left\{ \left[(1+2a\xi+2a\hbar\omega) (f_0(\xi+\hbar\omega) e^\sigma - f_0(\xi)) \right] \right. \\ & \left. \left\{ \frac{1}{2} \left[\frac{2a\hbar\omega\xi+\hbar\omega}{\xi+a\xi^2} \right]^2 \sinh^{-1} \left[\frac{\xi}{\hbar\omega} \frac{(1+a\xi)}{(1+2a\xi)} \right]^{1/2} - 2 \right\} \right. \\ & \left. + \left[(1+2a\xi-2a\hbar\omega) (f_0(\xi-\hbar\omega) - e^\sigma f_0(\xi)) \right] \right. \\ & \left. \left\{ \frac{1}{2} \left[\frac{2a\hbar\omega+\hbar\omega}{\xi+a\xi^2} \right]^2 \sinh^{-1} \left[\frac{\xi}{\hbar\omega} \frac{(1+a\xi)}{(1+2a\xi)} - 1 \right]^{1/2} - 2 \right\} \right\} \quad (A5.11) \end{aligned}$$

Let $x = \frac{\xi}{\hbar\omega} \frac{(1+a\xi)}{(1+2a\xi)}$, then the two inverse hyperbolic sine terms can be written as $\frac{\sinh^{-1}\sqrt{x}}{2x^2}$ and $\frac{\sinh^{-1}\sqrt{x-1}}{2x^2}$. Since the case considered here is for $\xi \gg \hbar\omega$, x is quite large compared to unity. However for large x , $\frac{\sinh^{-1}\sqrt{x}}{2x^2}$ is rather small compared to 2.

$$\begin{aligned} \frac{df_0(\xi)}{dt} = & -\frac{2eE_0 N c^2}{\hbar k} \left\{ (1+2\alpha\xi+2\alpha\hbar\omega) (f_0(\xi+\hbar\omega)e^\sigma - f_0(\xi)) \right. \\ & \left. + (1+2\alpha\xi-2\alpha\hbar\omega) (f_0(\xi-\hbar\omega) - e^\sigma f_0(\xi)) \right\} \end{aligned} \quad (A5.12)$$

Taylor expansions are made about $f_0(\xi+\hbar\omega)$ and if only the first three terms are retained, the equation can be simplified to

$$\begin{aligned} \left(\frac{df_0(\xi)}{dt} \right)_{\text{c sym}} = & -\frac{2eE_0 c^2}{\hbar k} \left\{ 4\alpha\hbar\omega f_0(\xi) + \hbar\omega \frac{df_0(\xi)}{d\xi} [1+2\alpha\xi+(2N+1)2\alpha\hbar\omega] \right. \\ \text{p-wave} & \left. + (\hbar\omega)^2 \frac{d^2 f_0(\xi)}{d\xi^2} [(1+2\alpha\xi)(2N+1)+2\alpha\hbar\omega] \right\} \end{aligned} \quad (A5.13)$$

When p-wave scattering is included, the Boltzmann equation describing the symmetrical part of the distribution function in the lower valley, assuming no intervalley scattering can be written as [6]

$$\begin{aligned} \frac{2e^2 E^2}{3m^* (1+2\alpha\xi)} \left[\frac{\hbar\omega}{\xi+\alpha\xi^2} \right]^{1/2} \frac{d}{d\xi} \left[\left[\frac{\xi+\alpha\xi^2}{\hbar\omega} \right]^{3/2} \frac{\tau f_0}{1+2\alpha\xi} \right] + \left(\frac{df_0}{dt} \right)_{\text{c-sw}} + \epsilon \left(\frac{df_0}{dt} \right)_{\text{c-pw}} = 0 \end{aligned} \quad (A5.14)$$

where the non-parabolicity in the energy band of the form in (A5.3) has been used for the field term. The collision term has also been written as the sum of a s-wave scattering and a p-wave scattering terms as given in (7.3). τ is the polar optical relaxation time for $\xi \gg \hbar\omega$ as discussed in section 6. For the non-parabolicity considered in this section, it is given by [6]

$$\tau = \left[\frac{2m^* \hbar\omega}{eE_0} \right]^{1/2} \frac{e^\sigma - 1}{e^\sigma + 1} \frac{(\xi+\alpha\xi^2)^{1/2}}{\sqrt{\hbar\omega} (1+2\alpha\xi)} \quad (A5.15)$$

The collision term $\frac{df_0}{dt}(\xi)$ is identical to the result when no p-wave scattering is considered. For the type of non-parabolicity considered here it can be written as [29]

$$\frac{df_0}{dt} = \frac{eE(\hbar\omega)^2}{\hbar k(1+2\alpha\xi)} \frac{d}{d\xi} \left\{ \left[\frac{f_0}{\hbar\omega} + (N+1/2) \frac{df_0}{d\xi} \right] (1+2\alpha\xi) \ln \left[\frac{4}{\hbar\omega} \frac{\xi + \alpha\xi^2}{1+2\alpha\xi} \right] \right\} \quad (A5.16)$$

Equation (A5.14) can, therefore, be written as

$$\begin{aligned} & \frac{2\alpha^2 E^2}{3eE_0 m^* (1+2\alpha\xi)} \left[\frac{2m^*}{\xi + \alpha\xi^2} \right]^{1/2} \left(\frac{e^{\sigma} - 1}{e^{\sigma} + 1} \right) \frac{d}{d\xi} \left[\frac{(\xi + \alpha\xi^2) df_0}{(1+2\alpha\xi) d\xi} \right] \\ & + \frac{eE_0(\hbar\omega)^2}{\hbar k(1+2\alpha\xi)} \frac{d}{d\xi} \left\{ \left[\frac{f_0}{\hbar\omega} + (N+1/2) \frac{df_0}{d\xi} \right] (1+2\alpha\xi)^2 \ln \left[\frac{4}{\hbar\omega} \frac{(\xi + \alpha\xi^2)}{(1+2\alpha\xi)} \right] \right\} \\ & - \frac{e2eE_0\omega^2}{\hbar k} \left[4\alpha\hbar\omega f_0 + \hbar\omega \frac{df_0}{d\xi} (1+2\alpha\xi + (2N+1)^2 \alpha\hbar\omega) \right. \\ & \left. + \frac{(\hbar\omega)^2}{2} \frac{d^2 f_0}{d\xi^2} \left((1+2\alpha\xi)(2N+1) + 2\alpha\hbar\omega \right) \right]_{=0} \end{aligned} \quad (A5.17)$$

It is assumed that the solution of A5.17 can be written as the sum of a dominant term $f_0^0(x)$ due to s-wave scattering and a small perturbed term $\epsilon f_0^1(x)$ caused by p-wave scattering.

$$f_0 = f_0^0 + \epsilon f_0^1 \quad (A5.18)$$

where ϵ is a smallness parameter to be set to one later. When this sum is substituted into A5.17, the terms in the equation can be reordered in terms of ϵ^0 and ϵ^1 . The coefficients of each power of ϵ can be set to zero and two equations are obtained.

$$\frac{2e^2 E^2}{3eE_0 m^* (1+2\alpha\xi)} \left[\frac{2m^*}{\xi + \alpha\xi^2} \right]^{1/2} \left(\frac{e^\sigma - 1}{e^\sigma + 1} \right) \frac{d}{d\xi} \left[\frac{(\xi + \alpha\xi^2)^2}{(1+2\alpha\xi)^2} \frac{df_0^0}{d\xi} \right] \\ + \frac{eE_0 (\hbar\omega)^2}{\hbar k (1+2\alpha\xi) d\xi} \left\{ \left[\frac{f_0^0}{\hbar\omega} + (N+1/2) \frac{df_0^0}{d\xi} \right] (1+2\alpha\xi)^2 \ln \left[\frac{4 (\xi + \alpha\xi^2)}{\hbar\omega (1+2\alpha\xi)} \right] \right\} = 0 \quad (A5.19)$$

$$\frac{2e^2 E^2}{3eE_0 m^* (1+2\alpha\xi)} \left[\frac{2m^*}{\xi + \alpha\xi^2} \right]^{1/2} \frac{e^\sigma - 1}{e^\sigma + 1} \frac{d}{d\xi} \left[\frac{(\xi + \alpha\xi^2)^2}{(1+2\alpha\xi)^2} \frac{df_0^1}{d\xi} \right] \\ + \frac{eE_0 (\hbar\omega)^2}{\hbar k (1+2\alpha\xi) d\xi} \left\{ \left[\frac{f_0^1}{\hbar\omega} + (N+1/2) \frac{df_0^1}{d\xi} \right] (1+2\alpha\xi)^2 \ln \left[\frac{4 (\xi + \alpha\xi^2)}{\hbar\omega (1+2\alpha\xi)} \right] \right\} \\ - \frac{2eE_0 \omega^2}{\hbar k} \left[4\alpha\hbar\omega f_0^0 + \hbar\omega \frac{df_0^0}{d\xi} (1+2\alpha\xi) + \frac{(\hbar\omega)^2}{2} \frac{d^2 f_0^0}{d\xi^2} (1+2\alpha\xi) (2N+1) \right] = 0 \quad (A5.20)$$

The first of these equations is the one that is obtained by equating the coefficients of the first Legendre polynomial P_0 to zero in the Boltzmann equation with p-wave scattering neglected. This equation has been solved by Conwell et al [6] for an arbitrary energy band. The solution that they derived is shown below.

$$f_0^0 = F(0) \exp \left[- \int \frac{(\eta')^4 \ln(4\eta/\eta') dx}{b\eta^2 + d(\eta')^4 \ln(4\eta/\eta')} \right] \quad (A5.21)$$

where $x = \xi/\hbar\omega$ as defined in section 6, b and d are defined by equation (6.14) and (6.15) in section 6. η is the energy function Ω normalised by the $\hbar\omega$. For the type of non-parabolicity considered here, $\eta = (\xi + \alpha\xi^2)/(\hbar\omega)$ and the prime in η indicates differentiation with respect to x . $F(0)$ is a constant of integration.

To see how much p-wave scattering affects the electron distribution function, equation (A5.20) has to be solved for $f_0^1(x)$. There are three terms within the bracket that are multiplied by the factor c^2 . The last of these three terms is a product of the second derivative of $f_0^0(x)$ and $(\hbar\omega)^2$, which is small compared to the other two terms and is dropped.

$$\begin{aligned} & \frac{2e^2 E^2}{3eE_0 m^* (1+2\alpha\mathcal{E})} \left[\frac{2m^*}{\mathcal{E} + \alpha\mathcal{E}^2} \right]^{1/2} \frac{e^\sigma - 1}{e^\sigma + 1} \frac{d}{d\mathcal{E}} \left[\frac{(\mathcal{E} + \alpha\mathcal{E}^2)^2}{(1+2\alpha\mathcal{E})^2} \frac{df_0^1}{d\mathcal{E}} \right] \\ & + \frac{eE_0 (\hbar\omega)^2}{\hbar k (1+2\alpha\mathcal{E})} \frac{d}{d\mathcal{E}} \left\{ \left[\frac{f_0^1}{\hbar\omega} + (N+1/2) \frac{df_0^1}{d\mathcal{E}} \right] (1+2\alpha\mathcal{E})^2 \ln \left[\frac{4(\mathcal{E} + \alpha\mathcal{E}^2)}{\hbar\omega(1+2\alpha\mathcal{E})} \right] \right\} \\ & - \frac{2eE_0 c^2}{\hbar k} \left[4\alpha\hbar\omega f_0^0 + \hbar\omega \frac{df_0^0}{d\mathcal{E}} (1+2\alpha\mathcal{E}) \right] = 0. \end{aligned} \quad (A5.22)$$

This equation can be written in a more compact form.

$$\begin{aligned} & \frac{4E^2}{3E_0} \frac{e^\sigma - 1}{e^\sigma + 1} \frac{d}{d\mathcal{E}} \left[\frac{\mathcal{E} + \alpha\mathcal{E}^2}{(1+2\alpha\mathcal{E})^2} \frac{df_0^1}{d\mathcal{E}} \right] + (\hbar\omega)^2 \frac{d}{d\mathcal{E}} \left\{ \left[\frac{f_0^1}{\hbar\omega} + (N+1/2) \frac{df_0^1}{d\mathcal{E}} \right] (1+2\alpha\mathcal{E})^2 \ln \left[\frac{4(\mathcal{E} + \alpha\mathcal{E}^2)}{\hbar\omega(1+2\alpha\mathcal{E})} \right] \right\} \\ & - 2\hbar\omega c^2 \frac{d}{d\mathcal{E}} \left[(1+2\alpha\mathcal{E})^2 f_0^0 \right] = 0. \end{aligned} \quad (A5.23)$$

In terms of the variables x and η (A5.23) can be written as

$$\frac{d}{dx} \left[\frac{b\eta^2}{\eta'} \frac{df_0^1}{dx} \right] + \frac{d}{dx} \left[f_0^1 + \frac{d}{dx} \frac{df_0^1}{dx} \right] \eta'^2 \ln(4\eta/\eta') - 2c^2 \frac{d(\eta'^2 f_0^0)}{dx} = 0 \quad (A5.24)$$

This equation can be integrated to give

$$\frac{df_0^1}{dx} + \frac{f_0^1 \eta'^4 \ln(4\eta/\eta')}{b\eta^2 + d\eta'^4 \ln(4\eta/\eta')} = \frac{\eta'^2}{b\eta^2 + d\eta'^4 \ln(4\eta/\eta')} \int_0^x 2c^2 d[\eta'^2 f_0^0] dx \quad (A5.25)$$

In this equation, c^2 is also a function of $\mathcal{E}$, and has to be written in terms of x so that the right hand side can be integrated. c^2 can be

written as [23]

$$c^2 = \frac{1}{2} \left[1 - \frac{\xi_g}{[\xi_g^2 + 4\xi_0\xi_p]^{1/2}} \right] \quad (A5.26)$$

where ξ_0 is given by

$$\xi_0 = \frac{\hbar^2 k^2}{2m_0} \quad (A5.27)$$

where m_0 is the mass of electron in vacuum. Since m_0 is related to m^* by $m_0 = m^*(1 + (\xi_p/\xi_g))$, c^2 can be written as

$$c^2 = \frac{1}{2} \left[1 - \frac{1}{\sqrt{1 + \frac{\hbar^2 k^2}{2m^*} \cdot \frac{4\xi_p}{\xi_g(\xi_g + \xi_p)}}} \right] \quad (A5.29)$$

Since $\hbar^2 k^2 / (2m^*)$ is just $\varepsilon + n\varepsilon^2$ for the type of non-parabolicity considered here, c^2 can be rewritten in the form

$$c^2 = \frac{1}{2} \left\{ 1 - \frac{1}{\left[\frac{1 + (\varepsilon + n\varepsilon^2)4\xi_p^2(\xi_p + \xi_g)}{\xi_g(\xi_g + \xi_p)\xi_p} \right]^{1/2}} \right\} \quad (A5.29)$$

For most of the compounds considered here ξ_p is much larger than ξ_g .

Therefore the $(\xi_p + \xi_g)/\xi_p$ can be approximated by unity. The factor $\xi_p^2/[\xi_g(\xi_g + \xi_p)]$ is just a [6]. Therefore c^2 can be approximated by

$$c^2 = \frac{1}{2}(1 - 1/\eta') \quad (A5.30)$$

This expression is substituted into the right hand side of (A5.25) and integration by parts is used to integrate the left side.

$$\frac{d}{dx} \left(\frac{f_0^1}{f_0^0} \right) + \frac{f_0^1 \eta'^4 \ln(4\eta/\eta')}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} = \frac{\eta'^2}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \left[\eta'^2 f_0^0 - \eta' f_0^0 - 2a\eta' \int_0^x f_0^0 dx \right] \quad (A5.31)$$

From (A5.21) it is found that

$$\frac{d}{dx} \left\{ \ln(F(0)/f_0^0) \right\} = \left[\frac{\eta' \ln(4\eta/\eta')}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \right] \quad (A5.32)$$

Therefore,

$$\frac{d}{dx} \left(\frac{f_0^1}{f_0^0} \right) + \frac{f_0^1}{f_0^0} \frac{d}{dx} \left\{ \ln(F(0)/f_0^0) \right\} = \frac{\eta'^2}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \left(\eta'^2 f_0^0 - f_0^0 \eta' - 2a\eta' \int_0^x f_0^0 dx \right) \quad (A5.33)$$

It is possible to transform the right hand side of the above equation into the form of an exact integral by the integrating factor, $F(0)/f_0^0$. With the integrating factor, (A5.33) can be integrated from a lower limit, λ , to x . This lower limit is chosen such that f_0^1/f_0^0 vanishes at $x=\lambda$. The lower limit is found to be one. After the integration, $F(0)$ cancels out from both side of the equation. and the result of $f_0^1(x)$ expressed as a ratio of f_0^1 is

$$\frac{f_0^1}{f_0^0} = \int_1^x \left[\frac{\eta'^2}{b\eta'^2 + d\eta'^4 \ln(4\eta/\eta')} \left(\eta'^2 - \eta' - \frac{2a\eta'}{f_0^0} \int_0^x f_0^0 dx \right) \right] dx \quad (A5.34)$$

For x smaller than one, $b\eta'^2$ is much smaller than $d\eta'^4 \ln(4\eta/\eta')$.

Therefore for the region $0 < x < 1$, f_0^0 can be approximated by

$$\begin{aligned} f_0^0 &= F(0) \exp - \int_0^x 1/d \, dx \\ &= F(0) d(1 - \exp(-1/d)). \end{aligned}$$

This allows (A5.34) to be written as

$$\frac{f_0^1}{f_0^0} = \int_1^x \frac{\eta'^2}{(b\eta^2 + d\eta'^4) \ln(4\eta/\eta')} \left\{ \eta'^2 - \eta' - \frac{2a\hbar\omega}{f_0^0} \left(F(0) \left(1 - e^{-1/d} \right) + \int_1^x f_0^0 dx \right) \right\} dx \quad (A5.35)$$

For large electric field and x , the asymmetrical part of the electron distribution function in terms of f_0 is [3,6]

$$f_1 = \frac{-\hbar e E \tau}{m^* \hbar m \eta'} \frac{df_0}{dx} \quad (A5.36)$$

where τ is given in (A5.16). In terms of f_0^0 and f_0^1 , f_1 is given by

$$f_1 = \frac{-\hbar e E \tau}{m^* \hbar m \eta'} \left[\frac{df_0^0}{dx} + s \left(f_0^0 P + \frac{df_0^0}{dx} \int_1^x P dx \right) \right] \quad (A5.37)$$

where the first term is due to s-wave scattering and the term with s is due to p-wave scattering. P is given by

$$P = \frac{\eta'}{b\eta^2 + d\eta'^4 \ln(4\eta/\eta')} \left(\eta'^2 - \eta' - 2a\hbar\omega [F(0) \left(1 - e^{-1/d} \right) + \int_1^x f_0^0 dx] \right) \quad (A5.38)$$

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