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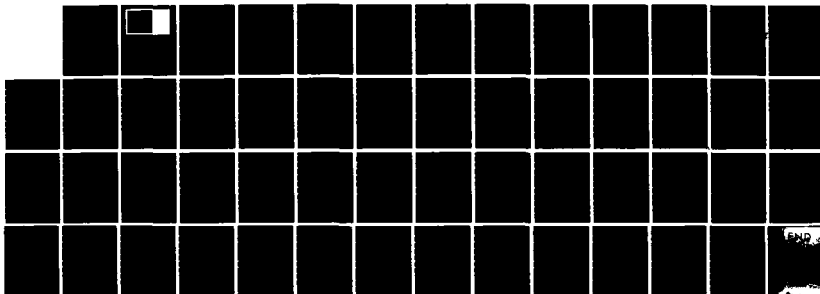
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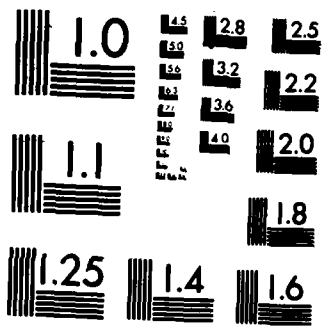
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Michael G. Crandall

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

This is a review paper which outlines the main points of the theory of nonlinear semigroups and evolution governed by accretive operators. The subject is now rather mature, so most of the principal ideas and results are not new. However, the presentation here is organized differently from that in other sources and does touch upon recent results. An attempt has been made to make this paper a pleasant route to a certain view of the subject. This manuscript represents the author's contribution to the proceedings of the Symposium on Nonlinear Functional Analysis and Applications held in Berkeley in July, 1983.

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NONLINEAR SEMIGROUPS AND EVOLUTION GOVERNED BY ACCRETIVE OPERATORS

Michael G. Crandall

In this review article we will outline some of the main points of the theory of nonlinear semigroups and evolution governed by accretive operators. As the subject has achieved a certain maturity, most of the principal ideas and results are not new. However, the current presentation is somewhat different from others in its style and choice of topics, and we have tried to make it pleasant reading for newcomers to the subject. It does touch upon some recent results, and we hope it will be of interest.

The material is organized into 8 sections. Section 1 contains preliminaries and introduces the subject via a "generation" theory point of view. Here one finds elementary notions about semigroups, their generators, strong solutions and accretive operators. Section 2 introduces the notion of a "mild solution" of an abstract initial-value problem, a notion which allows a certain unity and ease of expression in the following discussion which would otherwise be severely hampered by a lack of regular solutions. Mild solutions are, roughly, uniform limits of solutions of suitable difference approximations of the problem under consideration. Section 3 presents the basic convergence results which state that if suitable difference approximations can be solved, then their solutions will converge. These results lie at the heart of the theory and provide an ample supply of mild solutions.

Section 4 is off the usual track and presents something a bit more novel. Here, in a model case, a relationship between Kato's theory of quasilinear evolution and the results of Section 3 is exhibited.

Section 5 is also organized in an unusual way. Here we return to the generation question and explain some of the highlights as well as a couple of open problems. These considerations are used to introduce more subtle conditions under which it can be

proved that there are solvable difference approximations and recent remarks by Kobayashi on the question of necessary conditions. This section is not referred to in what follows it.

In Section 6, at last, we concede to the conventional and discuss the regularity of our mild solutions. Here the standard conditions guaranteeing the differentiability of mild solutions and the pointwise satisfaction of the equations are given. This is also a natural place to describe the inequalities which Benilan proved uniquely identify a mild solution when it exists.

Section 7 briefly describes the most useful auxiliary results of the theory. These results concern the continuous dependence of solutions on the equation, representation of solutions and compactness criteria of various sorts.

Few of the results stated here are proved, although some description of the line of argument is given from time to time. Similarly, we have omitted all references in the text proper, as comments as to who did what interrupt the flow and do not serve a browsing reader well. We partially correct this in Section 8 where further comments are made on the material of the previous sections and (incomplete) references are given. Here we also attempt to refer to some of the current activity in this area of which we are aware, but the field has become too vast to attempt any sort of completeness in describing either the old or the new in an article of moderate length. For example, we have not attempted to discuss questions of asymptotic behaviour, an area which has enjoyed a great deal of relatively recent activity, in the text (but we do give some references in Section 8). An even more profound omission concerns applications, which we at first thought to approach somehow. However, this idea was abandoned owing to our inability to come up with a satisfactory scheme that was not inferior to suggesting the reader refer to Crandall [26], Evans [43] and Barbu [3] (in that order). This will not yield an up-to-date view of the situation, but it will provide some simple examples and then an accurate impression of the nature and range of applications. More recent references are given in Section 8.

It should be mentioned that this review is affected by work the author has done with Benilan and Pazy on the small book [13], which is in a state of perpetual preparation. The author is also indebted to S. Oharu, A. Pazy and S. Reich for their comments on this paper.

Section 1. Preliminaries

The origins of this subject lie in questions posed by its pioneers about the "generation" of semigroups of transformations. We adopt this point of view as a pedagogical device, although a more "applied" attitude holds sway at the moment. Thus we begin by defining the class of semigroups under consideration and observing properties which their "generators" might be expected to have. This leads us naturally to the class of accretive operators.

Let X be a real Banach space with the norm $\|\cdot\|$. The norm of the dual space X^* of X will also be denoted by $\|\cdot\|$. If C is a subset of X , a semigroup on C will mean a collection $\{S(t): 0 < t\}$ of self-maps of C with the properties (i) below:

$$(i) \quad S(0)x = x \text{ and } S(t)S(\tau)x = S(t + \tau)x \text{ when } 0 < t, \tau \text{ and } x \in C.$$

Note that the value of $S(t)$ at $x \in C$ is written $S(t)x$ even if $S(t)$ is not a linear function. A semigroup S on C is a continuous semigroup if

$$(ii) \quad \text{The mapping } (0, \infty) \times C \ni (t, x) \rightarrow S(t)x \in C \text{ is continuous when } C \text{ carries the norm topology of } X.$$

We are mainly interested in the situation in which the continuity of $S(t)x$ in the "state variable" x is special. A continuous semigroup S on C which satisfies

$$(iii) \quad \|S(t)x - S(t)y\| < \|x - y\| \text{ for } 0 < t \text{ and } x, y \in C,$$

is said to be nonexpansive or a semigroup of contractions. More generally, if there is a number ω such that

$$(iv) \quad \|S(t)x - S(t)y\| < e^{\omega t} \|x - y\| \text{ for } 0 < t \text{ and } x, y \in C,$$

then we say that S is a quasicontractive (of type ω) semigroup on C . Of course, if either (iii) or (iv) hold, then S is continuous as soon as $t \rightarrow S(t)x$ is continuous for each $x \in C$.

The notion of a semigroup is an abstraction of the notion of a uniquely solvable initial-value problem of the form

$$\begin{aligned} \frac{du}{dt} + Au &= 0, \\ \text{(IVP)} \quad u(0) &= x, \end{aligned}$$

where A is a (nonlinear) operator $A: D(A) \subseteq X \rightarrow X$. If for each $x \in C$ (IVP) has a unique solution (in some sense!) $u(t)$ on $[0, \infty)$, then putting $S(t)x = u(t)$ should define a semigroup on C . Further properties of the semigroup should correspond to further properties of the operator A and the notion of solution involved, and ways of relating semigroups and initial-value problems (or related objects) we will refer to here as "generation theory".

The most obvious way to attempt to associate an initial-value problem (IVP) with a semigroup S on C is to compute the operator

$$A_S x = \lim_{t \rightarrow 0} \frac{x - S(t)x}{t}$$

whose domain $D(A_S)$ is the set of $x \in C$ such that the limit exists, and then hope "solving" (IVP) with $A = A_S$ will return S . The operator $-A_S$ is called the infinitesimal generator of S . Let us see how the quasicontractive property (iv) of a semigroup S would be reflected in its infinitesimal generator. If S satisfies (iv), then for $0 < \lambda, t$

$$\begin{aligned} \|x - \hat{x} + \lambda \left(\frac{x - S(t)x}{t} - \frac{\hat{x} - S(t)\hat{x}}{t} \right)\| &> (1 + \frac{\lambda}{t}) \|x - \hat{x}\| - \frac{\lambda}{t} \|S(t)x - S(t)\hat{x}\| \\ &> (1 + \lambda(1 - e^{-\omega t})/t) \|x - \hat{x}\| \end{aligned}$$

so if x, \hat{x} are in $D(A_S)$ we may pass to the limit to find that

$$\|x - \hat{x} + \lambda(A_S x - A_S \hat{x})\| > (1 - \lambda\omega) \|x - \hat{x}\| \text{ for } x, \hat{x} \in D(A_S).$$

We will refer to this property by saying that $A_S + \omega I$ is accretive. More precisely, if A is an operator then $A + \omega I$ is accretive if

$$(1.1) \quad \|x - \hat{x} + \lambda(Ax - A\hat{x})\| > (1 - \lambda\omega) \|x - \hat{x}\| \text{ for } x, \hat{x} \in D(A) \text{ and } \lambda > 0.$$

In the special case that $A + 0I$ is accretive we simply say that A is accretive. (There is a little subtlety here, and we leave it to the reader to check that $A + \omega I$ is

accretive if and only if $A + \omega I$ is accretive.) It follows from the above remarks that if S is a semigroup of contractions, then A_S is accretive.

If $z = x - \hat{x}$ and $w = Ax - A\hat{x}$ and $\omega = 0$, then (1.1) can be written as $0 < [z, w]_\lambda$ for $\lambda > 0$ where

$$(1.2) \quad [z, w]_\lambda = \frac{|z + \lambda w| - |z|}{\lambda}$$

defines $[z, w]_\lambda$ for $\lambda \neq 0$. Since $\lambda \rightarrow |z + \lambda w|$ is a convex function of λ we may define

$$(1.3) \quad [z, w] = \lim_{\lambda \rightarrow 0} [z, w]_\lambda = \inf_{\lambda > 0} [z, w]_\lambda$$

and then observe that an operator $A: D(A) \subseteq X \rightarrow X$ is accretive if and only if

$$(1.4) \quad 0 < [x - \hat{x}, Ax - A\hat{x}] \text{ for } x, \hat{x} \in D(A).$$

Let us list some properties of the bracket $[\cdot, \cdot]$ before continuing. One further concept, namely that of the "duality map" $J: X \rightarrow X^*$ is required. It is given by

$$(1.5) \quad J(x) = \{ x^* \in X^*: x^*(x) = |x| \text{ and } |x^*| \leq 1 \}$$

where $x^*(x)$ denotes the value of $x^* \in X^*$ at $x \in X$. For example, $J(0)$ is the closed unit ball in X^* .

Proposition 1. Let $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$. We have:

- (i) $[\cdot, \cdot]: X \times X \rightarrow \mathbb{R}$ is upper-semicontinuous.
- (ii) $[\alpha x, \beta y] = |\beta| [x, y]$ if $\alpha \beta > 0$.
- (iii) $[x, \alpha x + y] = \alpha |x| + [x, y]$.
- (iv) $|[x, y]| \leq |y|$ and $[0, y] = |y|$.
- (v) $-[x, -y] \leq [x, y]$.
- (vi) $[x, y + z] \leq [x, y] + [x, z]$.
- (vii) $|[x, y] - [x, z]| \leq |y - z|$.
- (ix) $[x, y] = \max\{x^*(y): x^* \in J(x)\}$.

Let us consider still another way to say that $A + \omega I$ is accretive. If we put $z = x + \lambda Ax$ and $\hat{z} = \hat{x} + \lambda A\hat{x}$ in (1.1). Then, formally, $x = (I + \lambda A)^{-1}z$ and $\hat{x} = (I + \lambda A)^{-1}\hat{z}$ and (1.1) may be reformulated as

$$(1.6) \quad |(I + \lambda A)^{-1}z - (I + \lambda A)^{-1}\hat{z}| \leq (1 - \lambda\omega)^{-1}|z - \hat{z}| \text{ for } z, \hat{z} \in R(I + \lambda A),$$

from which we see that $(I + \lambda A)^{-1}$ is indeed a function with $(1 - \lambda\omega)^{-1}$ as a Lipschitz constant if $\lambda\omega < 1$. Just as f^{-1} need not be a function if f is, A need not be a function in order that (1.6) hold, and we will continue the discussion in the multivalued generality that this suggests.

More precisely, we will call a mapping $A: X \rightarrow 2^X$ (the subsets of X) an "operator" in X . Functions with domain and range in X are identified with the corresponding single-valued operators, where a single-valued operator A is one whose values Ax are either singletons or the empty set. The effective domain $D(A)$ of an operator A is

$$D(A) = \{x \in X: Ax \text{ is not empty}\}.$$

If A is a single-valued operator (or the corresponding function) and $x \in D(A)$, we will use Ax , depending on the context, to denote either the singleton set or its corresponding element. If A and B are operators and $\lambda \in \mathbb{R}$ then we form new operators A^{-1} , λA , and $A + B$ in the expected ways. For example,

$$A^{-1}x = \{y \in X: y \in Ax\}.$$

We formulate the notion of accretiveness for operators. The equivalence of the four conditions given is clear from the above.

Definition 1. If A is an operator in X and $\omega \in \mathbb{R}$, then $A + \omega I$ is accretive (or, for short, $A \in \mathcal{A}(\omega)$) if the following equivalent conditions hold:

(i) $(1 - \lambda\omega)\|x - \hat{x}\| < \|x - \hat{x} + \lambda(y - \hat{y})\|$ for $y \in Ax$, $\hat{y} \in A\hat{x}$ and $\lambda > 0$.

(ii) $\|x - \hat{x}, y - \hat{y}\| > -\omega\|x - \hat{x}\|$ for $y \in Ax$ and $\hat{y} \in A\hat{x}$.

(iii) If $y \in Ax$ and $\hat{y} \in A\hat{x}$, then there is an $x^* \in J(x - \hat{x})$ such that

$$x^*(y - \hat{y}) > -\omega\|x - \hat{x}\|.$$

(iv) If $\lambda > 0$ and $\lambda\omega < 1$, then $(I + \lambda A)^{-1}$ is single-valued and has

$(1 - \lambda\omega)^{-1}$ as a Lipschitz constant.

In practice, it is usually (ii) which is used to verify accretiveness. We complete this section by recalling the notion of a strong solution of the inclusion

$$(DE)_f \quad u'(t) + Au(t) \ni f(t)$$

in which A is an operator in X and $f: [0, T] \rightarrow X$ is Bochner integrable with respect to

Lebesgue measure (that is $f \in L^1(0, T; X)$). The space $W^{1,1}(0, T; X)$ consists of those functions u which have the form

$$(1.7) \quad u(t) = u(0) + \int_0^t h(s) ds$$

for some $h \in L^1(0, T; X)$. It is well-known that $W^{1,1}(0, T; X)$ consists of exactly those absolutely continuous functions $u: [0, T] \rightarrow X$ which are differentiable a.e. on $[0, T]$ and that when (1.7) holds with $h \in L^1(0, T; X)$, then $u'(t) = h(t)$ a.e.. Moreover, if X is reflexive then every absolutely continuous $u: [0, T] \rightarrow X$ belongs to $W^{1,1}(0, T; X)$, while there are spaces X and absolutely continuous functions u which are nowhere differentiable.

Definition 2. A strong solution of $(DE)_\omega$ on $[0, T]$ is a $u \in W^{1,1}(0, T; X)$ such that $f(t) - u'(t) \in Au(t)$ almost everywhere on $[0, T]$.

As a sample exercise in the concepts we have introduced so far, let us prove that if $A + \omega I$ is accretive, then strong solutions of $(DE)_\omega$ are determined by their initial-values. More precisely:

Proposition 2. Let $f, \hat{f} \in L^1(0, T; X)$, $A \in \mathcal{A}(\omega)$ and u, \hat{u} be strong solutions of $u' + Au \ni f$, $\hat{u}' + A\hat{u} \ni \hat{f}$, respectively, on $[0, T]$. Then

$$(1.8) \quad \begin{aligned} \|u(t) - \hat{u}(t)\| &< e^{\omega t} \|u(0) - \hat{u}(0)\| + \int_0^t e^{\omega(t-s)} \|u(s) - \hat{u}(s), f(s) - \hat{f}(s)\| ds < \\ &< e^{\omega t} \|u(0) - \hat{u}(0)\| + \int_0^t e^{\omega(t-s)} \|f(s) - \hat{f}(s)\| ds. \end{aligned}$$

Proof. Let $f: [0, T] \rightarrow X$ be differentiable from the right at $s \in [0, T]$. Then for $h > 0$

$$\begin{aligned} \frac{\|f(s+h) - f(s)\|}{h} &= \frac{\|f(s) + hf'_R(s) - f(s)\|}{h} + o(1) \\ &= [f(s), f'_R(s)]_h + o(1) \end{aligned}$$

where $f'_R(s)$ denotes the right derivative of f at s . Upon letting $h \rightarrow 0$ we find that $\|f(t)\|$ has a right derivative at $t = s$ and

$$(1.9) \quad D_R \|f(t)\| \Big|_{t=s} = [f(s), f'_R(s)] ,$$

where D_R denotes the right derivative. Similarly,

$$(1.10) \quad D_L \|f(t)\| \Big|_{t=s} = - [f(s), -f'_L(s)] ,$$

and we conclude that if both f and $\|f(t)\|$ are differentiable at $t = s$, then

$$(1.11) \quad \frac{d}{dt} \|f(t)\| \Big|_{t=s} = [f(s), f'(s)] = -[f(s), -f'(s)].$$

If u and \hat{u} are absolutely continuous, then so is $g(t) = \|u(t) - \hat{u}(t)\|$, which is therefore differentiable a.e.. If u and \hat{u} are as in the proposition then u , \hat{u} and g are all differentiable at almost every t and, by the above, for such values of t we have

$$\begin{aligned} \frac{d}{dt} \|u(t) - \hat{u}(t)\| &= -[u(t) - \hat{u}(t), \hat{u}'(t) - u'(t)] = \\ &= -[u(t) - \hat{u}(t), (f(t) - u'(t)) - (\hat{f}(t) - \hat{u}'(t)) + (\hat{f}(t) - f(t))]. \end{aligned}$$

Since u and \hat{u} are strong solutions of their respective equations, we have $f(t) - u'(t) \in Au(t)$ and $\hat{f}(t) - \hat{u}'(t) \in A\hat{u}(t)$ a.e.. At such points t , by Proposition 1 (vi) and Definition 1(ii),

$$\begin{aligned} & [u(t) - \hat{u}(t), (f(t) - u'(t)) - (\hat{f}(t) - \hat{u}'(t)) + (\hat{f}(t) - f(t))] > \\ & > [u(t) - \hat{u}(t), (f(t) - u'(t)) - (\hat{f}(t) - \hat{u}'(t))] - [u(t) - \hat{u}(t), f(t) - \hat{f}(t)] \\ & > -\omega \|u(t) - \hat{u}(t)\| - [u(t) - \hat{u}(t), f(t) - \hat{f}(t)] \end{aligned}$$

We conclude that $g(t) = \|u(t) - \hat{u}(t)\|$ satisfies

$$g'(t) < \omega g(t) + [u(t) - \hat{u}(t), f(t) - \hat{f}(t)]$$

and the integration of this elementary inequality yields the first inequality of (1.8). The final inequality of (1.8) comes from Proposition 1(iv).

In particular, if the assumptions of Proposition 2 hold and $f = \hat{f}$, $u(0) = \hat{u}(0)$, then $u = \hat{u}$ and strong solutions of the initial-value problem are unique. Even more, they depend continuously on initial data and the forcing term f according to the estimate (1.8). If $f = \hat{f}$, the estimate of (1.8) amounts to the same quasicontractive estimate from which we motivated the condition " $A + \omega I$ " is accretive, and the circle is complete.

However, it is an unpleasant but very interesting fact that we cannot think only of strong solutions. Indeed, we will have to travel rather far from this notion to

accommodate the full range of phenomena in this subject. In particular, there is an example of a quasicontractive semigroup S on $X = C([0,1])$ such that $S(t)x$ is not differentiable for any $0 < t$ or $x \in X$ and there are natural accretive operators A arising in partial differential equations for which the problem $(DE)_f$ with $f = 0$ has no strong solutions on $[0, \infty)$, a phenomenon which corresponds to the development of "shocks" - that is, discontinuities develop in the solution or its derivatives in such a way as to render it subtle to well-posed the corresponding problem. We take up another, broader, notion of "solution" in the next section.

Section 2. Mild Solutions

Let us motivate the notion to be introduced the following way. The existence question for the classical Cauchy problem

$$(CP) \quad \begin{aligned} \frac{du}{dt}(t) + Au(t) &= f(t) \\ u(0) &= x, \end{aligned}$$

where A is a continuous function in some $\mathbb{R}^n = X$ is often approached via the method of "Euler lines". Now, this just amounts to solving (which is trivial matter when it is possible - see below) the explicit difference approximation

$$(2.1) \quad \begin{aligned} \frac{u_\lambda(i\lambda) - u_\lambda((i-1)\lambda)}{\lambda} + Au_\lambda((i-1)\lambda) &= f((i-1)\lambda), \\ u_\lambda(0) &= x, \end{aligned}$$

for the nodal values $u_\lambda(i\lambda)$ of the approximation u_λ , interpolating them linearly, and then studying the (subsequential) convergence of u_λ to a solution u . Of course, if linear interpolation produces approximate solutions u_λ which converge uniformly to a continuously differentiable solution, so will piecewise constant interpolation. Now the scheme (2.1) is often not a good one if A is a partial differential operator, since the formal solution of (2.1) is, in the case $f = 0$,

$$u_\lambda(i\lambda) = (I - \lambda A)u_\lambda((i-1)\lambda) = (I - \lambda A)^i u_\lambda(0) = (I - \lambda A)^i x$$

and one is applying high powers $(I - \lambda A)^i$ of a differential operator to a fixed x , and

this is a bad idea in general. On the other hand, if we replace (2.1) by

$$(2.2) \quad \frac{u_\lambda(i\lambda) - u_\lambda((i-1)\lambda)}{\lambda} + Au_\lambda(i\lambda) = f(i\lambda)$$

and $f = 0$ we formally have

$$u_\lambda(i\lambda) = (I + \lambda A)^{-1} u_\lambda((i-1)\lambda)$$

or

$$(2.3) \quad u_\lambda(i\lambda) = J_\lambda^i x$$

where

$$(2.4) \quad J_\lambda = (I + \lambda A)^{-1}.$$

Now, provided the inverses have an adequately large domain, this procedure is well defined and proceeds by applying powers of the inverses, which have a better chance of behaving well. The method (2.2) is called "implicit Euler". One is naturally led to consider the idea of accretive operators when approaching things from this point of view, since the method (2.3) has a better chance of success if the operators J_λ^i are equicontinuous for $i\lambda$ bounded. About the only way to guarantee something like this is to ask each factor J_λ to have a Lipschitz constant K_λ , and then J_λ^i has K_λ^i as a Lipschitz constant. If $K_\lambda = (1 - \lambda\omega)^{-1}$, then $K_\lambda^i \rightarrow e^{\omega t}$ as $i\lambda \rightarrow t$ and $i \rightarrow \infty$. Moreover, one is naturally led to place conditions on the domain of the resolvents J_λ . The best thing is to have them defined everywhere (at least for small λ).

Definition 3. Let $A \in \mathcal{A}(\omega)$. Then $A + \omega I$ is m-accretive if $R(I + \lambda A) = X$ for $\lambda > 0$ and $\lambda\omega < 1$.

The condition of m-accretivity is enjoyed by many important operators in applications. The simplest example of an A such that $A + \omega I$ is m-accretive is a Lipschitz continuous function A on all of X which has ω as a Lipschitz constant. Also every continuous function A on X which is accretive is m-accretive (although this is not so easy to prove). However, examples from differential equations will not be continuous and will typically have small domains.

With this prologue, we will adopt below a notion of solution which refers directly to the approximation method used to prove the existence of solutions, the implicit Euler method. There is ample evidence that this is a reasonable thing to do. It provides a great unity to the spectrum of examples developed over the last decade or so and we can speak of "mild solutions" of very different problems. On the abstract side, we find it a completely adequate notion for discussing $(DE)_f$ and the basic existence theory and generation theory involving accretive operators which has been developed. It has shortcomings when one attempts to deal with generalizations of $(DE)_f$, which we call a "quasi-autonomous equation", to more general time dependencies, e.g. $u' + A(t)u = 0$, as will be mentioned later.

We now define the notion. For a while, there will be no restrictions of accretiveness imposed and the discussion is completely general in this respect. In order to accommodate the natural generality $f \in L^1(0, T; X)$ and other results discussed later, we will need more general approximations than (2.3) - in particular, we will want to refer to variable meshes rather than the constant step size λ above.

Let $f \in L^1(0, T; X)$ and $\epsilon > 0$. An ϵ -discretization on $[0, T]$ of $u' + Au \ni f$ on $[0, T]$ consists of a partition $\{0 = t_0 < t_1 < t_2 < \dots < t_N\}$ of the interval $[0, t_N]$ and a finite sequence $\{f_1, f_2, \dots, f_N\} \subseteq X$ such that

$$(2.5) \quad \begin{aligned} & \text{(a) } t_i - t_{i-1} < \epsilon \text{ for } i = 1, \dots, N \text{ and } T - \epsilon < t_N < T, \\ & \text{and} \\ & \text{(b) } \sum_{i=1}^N \int_{t_{i-1}}^{t_i} |f(s) - f_i| ds < \epsilon. \end{aligned}$$

The inequalities (2.5) (a) require that the step sizes of the partition not exceed ϵ and the end point $t_N < T$ not miss T by more than ϵ . The inequality (b) requires that the forcing term f be approximated within ϵ in $L^1(0, t_N; X)$ by the piecewise constant function whose value on $(t_{i-1}, t_i]$ is f_i . These data do not refer to the operator A occurring in the equation. We will indicate it by writing $D_A(0=t_0, t_1, \dots, t_N; f_1, \dots, f_N)$ for the discretization.

A solution of a discretization $D_A(0=t_0, t_1, \dots, t_N; f_1, \dots, f_N)$ is a piecewise constant function $v: [0, t_N] \rightarrow X$ whose values v_i on $(t_{i-1}, t_i]$ satisfy

$$(2.6) \quad \frac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i \ni f_i \text{ for } i = 1, \dots, N.$$

The value $v_0 = v(0)$ is not otherwise restricted. An ϵ -approximate solution of an inclusion $u' + Au \ni f$ is a solution v of an ϵ -discretization $D_A(0=t_0, \dots, t_N; f_1, \dots, f_N)$.

Definition 4. Let A be an operator in X , $T > 0$, and $f \in L^1(0, T; X)$. Then a mild solution of $u' + Au \ni f$ on $[0, T]$ is a $u \in C[0, T; X]$ (the continuous functions from $[0, T]$ into X) with the property that for each $\epsilon > 0$ there is an ϵ -approximate solution v of $u' + Au \ni f$ on $[0, T]$ such that $|v(t) - u(t)| < \epsilon$ for t in the domain of v .

Less formally, mild solutions are the continuous uniform limits of approximate solutions. We reiterate that while this notion is just broad enough to provide an adequate extension of the class of strong solutions for our purposes, mild solutions represent a very considerable generalization the strong notion.

Let us mention that while we have treated the endpoints 0 and T of the interval $[0, T]$ asymmetrically in our definition of ϵ -discretizations and ϵ -approximate solutions, this difference disappears at the level of mild solutions (although this requires an argument). That is, the class of mild solutions is unchanged if one requires only $0 < t_0 < \epsilon$ in the definition of an ϵ -discretization and makes the obvious subsequent modifications. We do not know if the class of mild solutions is left unchanged in general if we require both $t_0 = 0$ and $t_N = T$. Without further ado, if $a < b$ and $f \in L^1(a, b; X)$ then one defines strong solutions on $[a, b]$, ϵ -discretizations of $u' + Au \ni f$ on $[a, b]$ and mild solutions on $[a, b]$ in the obvious way. If Q is an arbitrary interval and $f \in L^1_{loc}(Q; X)$ a mild (strong) solution of $u' + Au \ni f$ on Q is a $u \in C(Q; X)$ which is a mild (respectively, strong) solution on every compact subinterval of Q (and this is consistent with the original definitions, as must be checked).

The next result collects a variety of properties of mild solutions of differential equations.

Proposition 3. (Properties of mild solutions.) Let A be an operator in X ,

Q an interval in \mathbb{R} , and $f \in L^1_{loc}(Q; X)$.

(i) If u is a strong solution of $u' + Au \ni f$ on Q , then u is a mild solution of $u' + Au \ni f$ on Q .

(ii) If u is a mild solution of $u' + Au \ni f$ on Q , then $u(Q) \subseteq \overline{D(A)}$ (the closure of $D(A)$).

(iii) Let $Q = [0, T]$ and u be a mild solution of $u' + Au \ni f$ on $[0, T]$. If $D(A)$ is closed and A is single-valued and continuous, then

$$u(t) = u(0) - \int_0^t Au(s)ds + \int_0^t f(s)ds \text{ for } 0 < t < T.$$

In particular, u is a strong solution and it is a classical solution if f is continuous.

(iv) (Continuation property). Let $Q = Q_1 \cup Q_2$ where Q_1 is a subinterval of Q . If $u \in C(Q; X)$ is a mild solution of $u' + Au \ni f$ on each Q_i , then it is a mild solution on Q .

(v) (Closure property). Let $f_n \in L^1_{loc}(Q; X)$, $u_n \in C(Q; X)$, and u_n be a mild solution of $u'_n + Au_n \ni f_n$ for $n = 1, 2, \dots$. If f_n converges to f in L^1 and u_n converges to u uniformly on compact subsets of Q as $n \rightarrow \infty$, then u is a mild solution of $u' + Au \ni f$ on Q .

(vi) (Translation property). Let u be a mild solution of $u' + Au \ni f$ on Q and $h \in \mathbb{R}$. Then $v(t) = u(t + h)$ is a mild solution of $v' + Av \ni g$ on $Q - h$ where $g(t) = f(t + h)$.

(vii) (Perturbation property). Let p be a continuous mapping of $\overline{D(A)}$ into X . Then u is a mild solution of $u' + Au + p(u) \ni f$ on Q if and only if $u \in C(Q; X)$, u has its values in $\overline{D(A)}$, and u is a mild solution of $u' + Au \ni g$ on Q , where $g(t) = f(t) - p(u(t))$.

As the reader can see, many of the usual properties we associate with the solutions of ordinary differential equations remain true in the context of mild solutions, a comforting fact. One of the things we can do with the notion of a mild solution is define a semigroup from an arbitrary operator A . Indeed, given an operator A in X , put

$$D_A = \{x \in X: u' + Au \ni 0, u(0) = x, \text{ has a unique mild solution on } [0, \infty)\}$$

and define $S_A(t): D_A \rightarrow X$ by

$$(2.7) \quad S_A(t)x = u(t) \text{ where } u \text{ is the mild solution of } u' + Au \ni 0, u(0) = x.$$

Proposition 4. Let A be an operator in X . Then S_A is a semigroup on D_A .

We will call S_A the "semigroup generated by $-A$ ".

Section 3. Convergence of Approximate Solutions

In this section we outline the main facts concerning the existence of mild solutions of the initial-value problem

$$(IVP)_{x,f} \quad \begin{aligned} \frac{du}{dt} + Au &\ni f(t), \\ u(0) &= x, \end{aligned}$$

where $A + \omega I$ is accretive and $f \in L^1(0, T; X)$. Consider a discretization

$D_A(0=t_0, t_1, \dots, t_N; f_1, \dots, f_N)$ of $u' + Au \ni f$. The nodal values $v_i = v(t_i)$ of a solution v of this discretization satisfy

$$(3.1) \quad \frac{v_i - v_{i-1}}{t_i - t_{i-1}} + \lambda v_i \ni f_i \text{ for } i = 1, \dots, N$$

or, equivalently,

$$(3.2) \quad v_i = J_{\delta_i} (v_{i-1} + \delta_i f_i) \text{ for } i=1, \dots, N, \text{ where } \delta_i = t_i - t_{i-1} \text{ and } J_{\lambda} = (I + \lambda A)^{-1},$$

and we are assuming that J_{λ} is a function in the range of λ where we use it. If $A + \omega I$ is accretive, then this is the case in (3.2) provided that $\delta_i \omega < 1$ for $i = 1, \dots, N$. If

$\lambda + \omega I$ is also m -accretive, then the domain of J_λ is all of X for $\lambda \omega < 1$, and (3.2) uniquely determines the v_i for any choice of the discretization $D_\lambda(0=t_0, \dots, t_N; f_1, \dots, f_N)$ of small mesh and initial-value v_0 of v . Thus for a large and interesting class of operators λ , every discretization of small mesh of $u' + Au \ni f$ is uniquely solvable once the initial-value of the solution is specified. This adds interest to the next theorem. In the theorem, an ϵ -approximate solution of $(IVP)_{x,f}$ on $[0, T]$ means an ϵ -approximate solution of $u' + Au \ni f$ on $[0, T]$ which further satisfies

(3.3) $\|v(0) - x\| < \epsilon$. Theorem 1. (Convergence) Let $\lambda + \omega I$ be accretive, $x \in \overline{\text{cl}} D(\lambda)$ (the closure of $D(\lambda)$), and $f \in L^1(0, T; X)$. For each $\epsilon > 0$ let $(IVP)_{x,f}$ have an ϵ -approximate solution on $[0, T]$. Then the problem $(IVP)_{x,f}$ has a unique mild solution u . Moreover, there is a function $\kappa(\epsilon)$ such that $\kappa(0+) = 0$ and if v is an ϵ -approximate solution of $(IVP)_{x,f}$ then

$$\|u(t) - v(t)\| < \kappa(\epsilon) \text{ for } 0 < t < T - \epsilon.$$

The convergence theorem asserts that if ϵ -approximate solutions exist for each ϵ , then they converge as $\epsilon \rightarrow 0$ to a mild solution u and the difference between u and any ϵ -approximate solution is estimable in terms of ϵ . The estimate κ will depend on T , ω , the behaviour of λ near x (or simply $\|\lambda\|$ for $y \in \lambda x$ if $x \in D(\lambda)$) and the modulus of continuity of f in L^1 with respect to translations. In particular, if $\lambda + \omega I$ is m -accretive, and $f \in L^1(0, T; X)$, then $(IVP)_{x,f}$ has a unique mild solution for every $x \in \overline{\text{cl}} D(\lambda)$.

We will not give the details of the proof of the Convergence Theorem, but we will outline a simple and interesting attack which gives much more information than claimed so far.

First Step. Let v be a solution of a discretization $D_\lambda(0=s_0, s_1, \dots, s_M; f_1, \dots, f_M)$ and w be a solution of a discretization $D_\lambda(0=t_0, t_1, \dots, t_N; g_1, \dots, g_N)$ with nodal values v_i and w_j respectively. Put

$$a_{i,j} = \|v_i - w_j\|, \quad \gamma_i = s_i - s_{i-1}, \quad \delta_j = t_j - t_{j-1}.$$

Then

$$(3.4) \quad \left(1 - \frac{\gamma_i \delta_j}{\gamma_i + \delta_j}\right) a_{i,j} < \frac{\delta_j}{\gamma_i + \delta_j} a_{i-1,j} + \frac{\gamma_i}{\gamma_i + \delta_j} a_{i,j-1} + \frac{\gamma_i \delta_j}{\gamma_i + \delta_j} |f_{i,j}|$$

for $1 < i < M$, $1 < j < N$. Moreover, for any $x \in D(A)$ and $y \in Ax$

$$(3.5) \quad \begin{aligned} a_{i,0} &< \alpha_{i,1} |v_0 - x| + |w_0 - x| + \sum_{k=1}^i \alpha_{i,k} \gamma_k (|f_k| + |y|) \text{ for } 0 < i < M, \\ \text{and} \\ a_{0,j} &< \beta_{j,1} |w_0 - x| + |v_0 - x| + \sum_{k=1}^j \beta_{j,k} \delta_k (|g_k| + |y|) \text{ for } 0 < j < N, \end{aligned}$$

where

$$(3.6) \quad \alpha_{i,k} = \prod_{m=k}^i (1 - \omega \gamma_m)^{-1} \quad \text{and} \quad \beta_{j,k} = \prod_{m=k}^j (1 - \omega \delta_m)^{-1}.$$

The inequalities (3.4) and (3.5) are elementary consequences of the assumptions. The proof of the Convergence Theorem then reduces to estimating solutions of (3.4) and (3.5). One way to recognize the behaviour of solutions of these inequalities is outlined next.

Second Step. Consider real-valued functions $\psi(s, \tau)$, $\phi(s, \tau)$ of two real variables s, τ which are related by the differential equation

$$(3.7) \quad \psi_s(s, \tau) + \psi_\tau(s, \tau) - \omega \psi(s, \tau) = \phi(s, \tau) \text{ for } 0 \leq s \leq T \text{ and } 0 \leq \tau \leq T.$$

Let us introduce a grid

$$(3.8) \quad \Delta = \{(s_i, t_j) : 0 = s_0 < \dots < s_M \leq T; 0 = t_0 < \dots < t_N \leq T\}$$

and approximate (3.7) on this grid by the difference relations

$$(3.9) \quad \frac{v_{i,j} - v_{i-1,j}}{\gamma_i} + \frac{v_{i,j} - v_{i,j-1}}{\delta_j} - \omega v_{i,j} = \phi_{i,j} \text{ for } i=1, \dots, M, j=1, \dots, N,$$

where $\gamma_i = s_i - s_{i-1}$ and $\delta_j = t_j - t_{j-1}$. When (3.9) is solved for $v_{i,j}$ (given the values $v_{i,j}$ for $i=0$ or $j=0$) we might hope $|v_{i,j} - \psi(s_i, \tau_j)|$ is small when it is small on the "boundary" $i=0$ or $j=0$, the mesh $\max\{\gamma_i, \delta_j\}$ is small, and the values $\phi_{i,j}$ approximate ϕ on $(s_{i-1}, s_i) \times (\tau_{j-1}, \tau_j]$ in some good way. Now (3.9) becomes, upon rearranging,

$$(3.10) \quad \left(1 - \frac{\gamma_i \delta_j}{\gamma_i + \delta_j}\right) v_{i,j} = \frac{\delta_j}{\gamma_i + \delta_j} v_{i-1,j} + \frac{\gamma_i}{\gamma_i + \delta_j} v_{i,j-1} + \frac{\gamma_i \delta_j}{\gamma_i + \delta_j} \phi_{i,j}$$

and the relationship with (3.5) becomes apparent. The terms corresponding to $\phi_{i,j}$ in (3.4) are $|f_i - g_j|$ where the f_i and g_j are nodal values of piecewise constant approximations of functions $f, g \in L^1(0, T; X)$. It is not exactly clear how such $\phi_{i,j}$ approximate this ϕ , but they do so in the sense required to apply the convergence result described next.

Let $b \in C([-T, S])$. The integration of (3.6) subject to the boundary condition $\psi(s, \tau) = b(s - \tau)$ if $s = 0$ or $\tau = 0$ leads to the following formulae for $\psi = G(b, \phi)$:

$$(3.11) \quad G(b, \phi) = \begin{cases} e^{\omega \tau} b(s - \tau) + \int_0^{\tau} e^{\omega(\tau - \alpha)} \phi(s - \tau + \alpha, \alpha) d\alpha & \text{for } 0 < \tau < s < T, \\ e^{\omega s} b(s - \tau) + \int_0^s e^{\omega(s - \alpha)} \phi(\alpha, \tau - s + \alpha) d\alpha & \text{for } 0 < s < \tau < T. \end{cases}$$

In what follows ϕ is such that $G(b, \phi)$ is well defined and continuous.

We want to discuss a solution operator for the corresponding discrete problem and begin by introducing the norm in which approximation of ϕ will be required. To this end, if $\Omega = [0, S] \times [0, T]$ and $\phi: [0, S] \times [0, T] \rightarrow \mathbb{R}$ put

$$(3.12) \quad \|\phi\|_{\Omega} = \inf \left\{ \|f\|_{L^1(0, S)} + \|g\|_{L^1(0, T)} : |\phi(s, \tau)| < |f(s)| + |g(\tau)| \text{ a.e. on } \Omega \right\},$$

where it is understood that the inf of the empty set is $-\infty$. Next, if Δ is the grid (3.8), put

$$\Omega(\Delta) = [0, s_M] \times [0, t_N].$$

Let

$$B: [-t_N, s_M] \rightarrow \mathbb{R} \text{ and } \phi: \Omega(\Delta) \rightarrow \mathbb{R}$$

be piecewise constant on Δ , i.e. there are constants $B_{i,j}, \phi_{i,j}$ such that $B(0) = B_{0,0}$ and

$$B(x) = B_{i,j} \text{ for } i = 0 \text{ and } -t_j < x < -t_{j-1} \text{ or } j = 0 \text{ and } s_{i-1} < x < s_i,$$

and

$$\phi(s, \tau) = \phi_{i,j} \text{ for } (s, \tau) \in (s_{i-1}, s_i] \times (\tau_{j-1}, \tau_j].$$

If the mesh $m(\Delta) = \max\{\gamma_i, \delta_j\}$ of Δ satisfies $m(\Delta)\omega < 1$, the equations (3.9) are obviously uniquely solvable for $V_{i,j}$ given that

$$(3.13) \quad V_{i,j} = B_{i,j} \text{ for } i = 0 \text{ or } j = 0.$$

Let $V = H_\Delta(B, \phi)$ denote the piecewise constant function on Δ obtained by solving (3.9) subject to (3.13).

Theorem 2. Let $b \in C([-T, S])$ and $\phi: [0, T] \times [0, T] \rightarrow \mathbb{R}$. Then

$$\|G(b, \phi) - H_\Delta(B, \phi)\|_{L^1(\Omega(\Delta))} \rightarrow 0 \text{ as } m(\Delta) \rightarrow 0, \quad \|b - B\|_{L^\infty(-T, S)} + \|\phi - \phi\|_{\Omega(\Delta)} \rightarrow 0.$$

There is a lot packed into this result, and we have formulated it in a sort of sneaky way. In particular, no assumptions were made on ϕ in the theorem. This was possible, because the result asserts nothing unless we can approximate ϕ with piecewise constant functions in the norms (3.12), and not every function can be so approximated. In particular, not every bounded and measurable function on $[0, S] \times [0, T]$ can be so approximated. However, functions of the form $|f(s) - g(\tau)|$ with integrable f and g can.

Let us sketch the application of Theorem 2 to Theorem 1. Let $f, g \in L^1(0, T; X)$, $x_0, \hat{x}_0 \in \overline{D(A)}$, v be an ϵ -approximate solution of

$$(3.14) \quad u' + Au \ni f, \quad u(0) = x_0,$$

and w be an ϵ -approximate solution of

$$u' + Au \ni g, \quad u(0) = \hat{x}_0,$$

and the discretizations solved by v and w be the ones in the first step. The piecewise constant function B on $[-t_N, s_M]$ whose values $B_{i,j}$ for $i = 0$ or $j = 0$ are the right hand sides of (3.5) tends, as $\epsilon \rightarrow 0$, uniformly to the function

$$(3.15) \quad b(s) = e^{\omega s} \|x_0 - x\| + \int_0^s e^{\omega(s-a)} (|f(a)| + |y|) da + \|x - \hat{x}_0\| \text{ for } 0 \leq s \leq T,$$

$$b(-\tau) = e^{\omega \tau} \|\hat{x}_0 - x\| + \int_0^\tau e^{\omega(\tau-a)} (|g(a)| + |y|) da + \|x - x_0\| \text{ for } 0 \leq \tau \leq T.$$

To prove this one uses the fact that the functions whose nodal values are the f_i and g_j

differ from f and g in L^1 by at most ϵ and elementary estimates. Moreover, if

$\phi(s, \tau) = |f(s) - g(\tau)|$ and θ is the piecewise constant function on Δ given by

$\theta_{i,j} = |f_i - g_j|$, then

$$(3.16) \quad \|\phi - \theta\|_{\Omega(\Delta)} < 2\epsilon.$$

Finally, since the $a_{i,j}$ satisfy the inequalities (3.5) and (3.6) they may be estimated above by the solution $v_{i,j} = N(B, \theta)_{i,j}$ of the corresponding equalities.

Recalling the meaning of the $a_{i,j}$ and Theorem 2 we conclude that for any $\eta > 0$

$$(3.17) \quad |v(s) - w(\tau)| < G(b, \phi)(s, \tau) + \eta$$

as soon as ϵ is small enough.

We use (3.17) in three situations. If $f = g$, $x_0 = \hat{x}_0$, and $s = \tau = t$, we compute $G(b, \phi)(t, t) = 2|x_0 - x|$ and conclude that

$$|v(t) - w(t)| < 2|x_0 - x| + \eta,$$

as soon as ϵ is small. Since $x_0 \in \overline{D(A)}$ and $x \in D(A)$ is arbitrary, this verifies the Cauchy criterion for the net of ϵ -approximate solutions of (3.14). Let u be the limit

of the ϵ -approximate solutions of (3.14) as $\epsilon \rightarrow 0$. Now we take the limit in (3.17)

with $s = t + h$, $f = g$ and $x_0 = \hat{x}_0$ to conclude that

$$(3.18) \quad \begin{aligned} |u(t+h) - u(t)| &< G(b, \phi)(t+h, t) = e^{\omega t} (e^{\omega h} + 1) |x_0 - x| + \\ &+ \int_0^h e^{\omega(h-a)} (|f(a)| + |y|) da + \int_0^t e^{\omega(t-a)} |f(a+h) - f(a)| da, \end{aligned}$$

for every $y \in Ax$ and $x \in D(A)$. It follows easily that u is continuous. In a similar way, (3.15) in the general case implies that if u and \hat{u} are mild solutions of problems $u' + Au \ni f$ and $\hat{u}' + A\hat{u} \ni \hat{f}$, then

$$(3.19) \quad |u(t) - \hat{u}(t)| < e^{\omega t} |u(0) - \hat{u}(0)| + \int_0^t e^{\omega(t-a)} |f(a) - \hat{f}(a)| da.$$

(The change in notation was made because we ran out of suitable letters.) The

inequality (3.19) reproduces the extreme inequalities of (1.8) for mild solutions.

Given the convergence, Theorem 2 also quickly implies the validity of the next

proposition.

Proposition 5. Let $A + \omega I$ be accretive, $f, \hat{f} \in L^1(0, T; X)$ and u, \hat{u} be mild solutions of $u' + Au \ni f$ and $\hat{u}' + A\hat{u} \ni \hat{f}$ respectively. Then

$$(3.20) \quad \|u(t) - \hat{u}(t)\| \leq e^{\omega(t-s)} \|u(s) - \hat{u}(s)\| + \int_s^t e^{\omega(t-\tau)} \| [u(\tau) - \hat{u}(\tau), f(\tau) - \hat{f}(\tau)] \| d\tau$$

for $0 \leq s < t < T$.

The convergence theorem does not address the question of when approximate solutions exist. Let us point out a couple of simple situations when this is not a problem. Recall that if $A + \omega I$ is m -accretive, then every discretization $D_\lambda(0=t_0, t_1, \dots, t_N; f_1, \dots, f_N)$ of small mesh is uniquely solvable when an initial value is specified. We summarize the situation as regards the case in which $A + \omega I$ is m -accretive.

Theorem 3. Let $A + \omega I$ be m -accretive, $x \in \overline{D(A)}$ and $f \in L^1(0, T; X)$. Then $u' + Au \ni f, u(0) = x$ has a unique mild solution on $[0, T]$. Moreover, if u and \hat{u} are, respectively, mild solutions of $u' + Au \ni f$ and $\hat{u}' + A\hat{u} \ni \hat{f}$ on $[0, T]$, then (3.20) holds.

Another simple situation arises when considering the problem

$$(3.21) \quad u' + Au \ni 0, u(0) = x.$$

We know that the solution u_λ of the discretization $D_\lambda(0, \lambda, 2\lambda, \dots, N\lambda; 0, \dots, 0)$ which satisfies $u_\lambda(0) = x$ (if it exists) is given by $u_\lambda(1\lambda) = J_\lambda^{-1}x$ (see (2.3) and (2.4)). Thus if $\lambda \in A(\omega)$ and

$$(3.22) \quad R(I + \lambda A) \supset \overline{D(A)} \text{ for small } \lambda > 0,$$

then (3.21) has a mild solution u for every $x \in \overline{D(A)}$. Moreover, u is given by

$$(3.23) \quad J_\lambda^{-1}x + u(t) \text{ as } \lambda \rightarrow 0 \text{ and } 1\lambda \rightarrow t.$$

When $\lambda \in A(\omega)$ satisfies (3.22), then its closure satisfies the stronger condition

$$(3.24) \quad R(I + \lambda A) \supset \overline{D(A)} \text{ for small } \lambda > 0,$$

and (3.21) is solvable for $x \in \overline{D(A)}$ and (3.23) still holds. In particular, if the range condition (3.24) holds, we have the exponential formula

$$(3.25) \quad S_\lambda(t)x = \lim_{n \rightarrow \infty} (I + (t/n)A)^{-n}x$$

for the semigroup generated by $-A$ on $\overline{D(A)}$.

Section 4. A Quasilinear Equation

In this section we sketch an application of the results of Section 3 to a (over simplified) quasilinear problem related to those previously considered by Kato. This section, while self-contained, is primarily intended for readers with some knowledge of Kato's theory. It demonstrates a strong relationship between his existence results and the results we have sketched so far.

Here X and Y denote reflexive Banach spaces with Y densely and continuously imbedded in X . The norms of X and Y are denoted by

$\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively. The problem of interest has the form

$$(4.1) \quad u' + B(u)u = 0, \quad u(0) = \phi,$$

in which $B(u)$ is a linear operator in X for each suitable u . We detail properties of B shortly, but first we must introduce a little notation.

In what follows linear operators are single-valued. If $C: D(C) \subset X \rightarrow X$ is a linear operator, C_Y , the part of C in Y , is the restriction of C to $\{y \in D(C) \cap Y: Cy \in Y\}$. If Z is a Banach space and C is a densely defined linear operator in Z such that $C + \omega I$ is m -accretive we write $C \in N(\omega, Z)$. The Hille-Yosida theorem (which we will not use) implies that $C \in N(\omega, Z)$ exactly when $-C$ is the infinitesimal generator of a continuous semigroup e^{-Ct} of linear operators which satisfies $\|e^{-Ct}\|_Z \leq e^{-\omega t}$.

In the assumptions below $r > 0$, $w_0 \in Y$ and

$$W = \{y \in Y: \|y - w_0\|_Y < r\}$$

is the closed r -ball centered at w_0 in Y . We assume that:

(B1) There is a $\theta \in \mathbb{R}$ and for each $w \in W$ an operator $B(w) \in N(\theta, X)$ such that $D(B(w)) \supset Y$ and $B(w)_Y \in N(\theta, Y)$.

(B2) There are $L, \gamma > 0$ such that for $w, \hat{w} \in W$ and $y \in Y$

$$\|B(w) - B(\hat{w})\|_Y \|y\|_X \leq L \|w - \hat{w}\|_X \|y\|_Y \quad \text{and} \quad \|B(w)_Y\|_X \leq \gamma \|y\|_Y.$$

(B3) There is a $\mu > 0$ such that if $w \in W$ then $B(w)w_0 \in Y$ and

$$\|B(w)w_0\|_Y < \mu.$$

In Kato's theory, (B1) is deducible from more subtle assumptions involving a linear isomorphism $S: Y \rightarrow X$ and conditions on $B(w)$ and the commutators $SB(w) - B(w)S$. Let us try to solve (4.1) via the discretisation

$$(4.2) \quad \frac{x_i - x_{i-1}}{\lambda} + B(x_{i-1})x_i = 0, \quad i = 1, \dots, N,$$

$$x_0 = \varphi.$$

Assume that $T > 0$ and (4.2) is solvable for each small $\lambda > 0$ for $x_i \in W$ for $i = 0, 1, \dots, N$ where $T < N\lambda$. Then put

$$(4.3) \quad \begin{aligned} u_\lambda(t) &= x_i \text{ for } (i-1)\lambda < t < i\lambda, \quad i = 1, \dots, N \\ \text{and } u_\lambda(0) &= \varphi. \end{aligned}$$

We claim that then u_λ converges strongly in X and weakly in Y uniformly on $[0, T]$ to function $u: [0, T] \rightarrow W$ which is Lipschitz continuous into X and weakly continuous into Y . Moreover, u is weakly continuously differentiable into X and satisfies $u'(t) + B(u(t))u(t) = 0$ for $0 < t < T$. In particular, it is a strong solution of $u' + \lambda(u) = 0$ where $\lambda(u) = B(u)u$. We sketch the proof of these claims and then the proof that (4.3) is solvable.

For each small $\lambda > 0$, let (4.2) be satisfied, $T < N\lambda$, $x_i \in W$, and u_λ be given by (4.3). Then, by (B2) and $x_i \in W$,

$$(4.4) \quad \|x_i - x_{i-1}\|_X = \lambda \|B(x_{i-1})x_i\|_X < \lambda \gamma \|x_i\|_Y < \lambda \gamma (x + \|w_0\|_Y).$$

Now put

$$(4.5) \quad A(x) = B(x)x \text{ for } x \in D(A) = W.$$

Clearly u_λ is a solution of the discretisation $D_\lambda(0, \lambda, \dots, N\lambda; \varepsilon_1, \dots, \varepsilon_N)$ of $u' + Au \ni 0$ where

$$(4.6) \quad \varepsilon_i = (B(x_i) - (B(x_{i-1})))x_i.$$

Using (4.4) and (B2) again, we see that the "errors" ε_i satisfy

$$\|\varepsilon_i\|_X < L \|x_i - x_{i-1}\|_X \|x_i\|_Y < \lambda \gamma (x + \|w_0\|_Y)^2,$$

and thus tend to zero in L^∞ and a fortiori in L^1 . Finally we check that

$A + \omega I$ is accretive in X where $\omega = \theta + L(r + \|w_0\|_Y)$. Indeed, by (B1) and (B2), if x and $\hat{x} \in D(A) = W$

$$\begin{aligned}
 \|x - \hat{x} + \lambda(A(x) - A(\hat{x}))\|_X &= \|x - \hat{x} + \lambda B(x)(x - \hat{x}) + \lambda(B(x) - B(\hat{x}))\hat{x}\|_X > \\
 (4.7) \quad &> \|x - \hat{x} + \lambda B(x)(x - \hat{x})\|_X - \lambda \|B(x) - B(\hat{x})\|_X \|\hat{x}\|_X > (1 - \lambda\theta) \|x - \hat{x}\|_X - \lambda L \|x - \hat{x}\|_X \|x\|_Y > \\
 &> (1 - \lambda(\theta + L(r + \|w_0\|_Y))) \|x - \hat{x}\|_X,
 \end{aligned}$$

and the claim is proved. The convergence of u_λ in X uniformly in t to a continuous limit u now follows from the results described in Section 3. Since each u_λ takes values in W , which is weakly closed in Y , and convergence in X boundedly in Y implies weak convergence in Y by the assumptions on X and Y , the convergence claims are established. Clearly $u(t)$ is weakly continuous into Y and $B(u(t))u(t)$ is weakly continuous into X . Moreover, it is easy to pass to the weak (in X) limit as $\lambda \rightarrow 0$ and $j\lambda \rightarrow t$ in the relation

$$u_\lambda(j\lambda) = \varphi + \int_\lambda^{j\lambda} B(u_\lambda(s - \lambda))u_\lambda(s)ds,$$

which follows from summing (4.2) from $i = 1$ to j , to find

$$u(t) = \varphi + \int_0^t B(u(s))u(s)ds,$$

which proves the claims about u being a weakly continuously differentiable strong solution of (4.1).

It remains to discuss the solvability of (4.2). By the assumption (B1), if $\lambda > 0$ and $\lambda\theta < 1$, then given $x_{i-1} \in W$ and any z in X we can uniquely solve

$$(4.8) \quad x + \lambda B(x_{i-1})x = z$$

for $x = (I + \lambda B(x_{i-1}))^{-1}z$ and $x \in Y$ if $z \in Y$ and

$$(4.9) \quad \|(I + \lambda B(x_{i-1}))^{-1}\|_Z < (1 - \lambda\theta)^{-1} \text{ for } Z = X \text{ or } Z = Y.$$

Hence, so long as we keep x_{i-1} in W we can compute x_i in Y . We estimate the range of i for which this is possible. Without loss of generality assume $0 < \theta$. We will keep

$\lambda\theta < 1/2$ so that

$$(1 - \lambda\theta)^{-1} < e^{2\lambda\theta}.$$

By (4.9) and (B3), so long as $x_{i-1} \in W$,

$$\begin{aligned} \|x_i - w_0\|_Y &< (1 - \lambda\theta)^{-1} \|x_i - w_0 + \lambda(B(x_{i-1})x_i - B(x_{i-1})w_0)\|_Y = \\ &= (1 - \lambda\theta)^{-1} \|x_{i-1} - (w_0 + \lambda B(x_{i-1})w_0)\|_Y < \\ &< (1 - \lambda\theta)^{-1} (\|x_{i-1} - w_0\|_Y + \lambda\|B(x_{i-1})w_0\|_Y) < \\ &< (1 - \lambda\theta)^{-1} (\|x_{i-1} - w_0\|_Y + \lambda\mu). \end{aligned}$$

Using the above inequalities and $x_0 = \varphi$, one finds easily that

$$\|x_i - w_0\|_Y < e^{2i\lambda} (\|\varphi - w_0\|_Y + i\lambda\mu),$$

so we conclude that if

$$e^{2(T+\lambda)\theta} (\|\varphi - w_0\|_Y + (T + \lambda)\mu) < r,$$

which will hold for λ and T small enough provided that φ lies in the interior of W , then (4.2) is solvable for $x_i \in W$ when $T < N\lambda < T + \lambda$, completing the discussion.

Section 5. Generation Theorems and Kobayashi's Existence Criterion.

In this section we introduce results of two kinds. On the one hand, if we are given a mapping from data (x, f) with the properties expected of the solution operator of the problem

$$(IVP)_{x,f} \quad u' + Au \ni f, \quad u(0) = x,$$

when $A + \omega I$ is accretive, we ask if that mapping indeed arises from an A in this way - this is generation theory in the spirit of the first section. Secondly, we will discuss more refined questions concerning the solvability of $(IVP)_{x,0}$ than have been posed so far.

To begin, let us recall that if $A + \omega I$ were m -accretive, $f \in L^1_{loc}([0, \infty): X)$ and $x \in D(A)$, then $(IVP)_{x,f}$ would have a unique mild solution $u \in C([0, \infty): X)$ which we will denote by $u = E_A(x, f)$. Moreover, E_A would enjoy certain properties which we now

abstract. Let K be a closed and nonempty subset of X , L be a translation invariant subspace of $L^1_{loc}([0, \infty); X)$ and consider the following properties of a mapping $E: K \times L \rightarrow C([0, \infty); X)$:

(E1) For $x \in K$ and $f \in L$, $E(x, f)(0) = x$ and $E(x, f)(t) \in K$ for $0 < t$.

(E2) E is translation invariant in the sense that

$$E(x, f)(t+\tau) = E(E(x, f)(\tau), f(\cdot + \tau))(t) \text{ for } x \in K, f \in L \text{ and } 0 < t, \tau.$$

(E3) If $x, \hat{x} \in K$, $f, \hat{f} \in L$ and $u = E(x, f)$, $\hat{u} = E(\hat{x}, \hat{f})$, then

$$\|u(t) - \hat{u}(t)\| \leq e^{\omega t} \|u(0) - \hat{u}(0)\| + \int_0^t e^{\omega(t-\tau)} \|u(\tau) - \hat{u}(\tau), f(\tau) - \hat{f}(\tau)\| d\tau$$

for $0 < t$.

For example, if $\omega = 0$ and $L = \{0\}$, then (E1)-(E3) reduce to the requirement that $S(t)x = E(x, 0)(t)$ defines a nonexpansive semigroup on K . Next we list some results which, under various circumstances, represent operators E satisfying (E1)-(E3) as arising from solving an initial-value problem (IVP) $_{x, f}$ with $A + \omega I$ accretive. In the first result we see that if L is large enough, then the situation is rather nice.

(i) Let (E1)-(E3) hold and L contain all the constant functions. Then there is a unique A such that $A + \omega I$ is m -accretive and E is the restriction of E_A to $K \times L$. Moreover, $\overline{D(A)} = K$.

It is also easy to see that the mapping $A \rightarrow E_A$ is one-to-one on

$$\{A: A + \omega I \text{ is } m\text{-accretive}\}$$

(essentially because $y \in Ax$ is equivalent to the constant function x being a solution of $u' + Au \ni y$ when $A + \omega I$ is m -accretive). When $L = L^1_{loc}([0, \infty); X)$ this provides us with a biunique correspondence between mappings E with the properties (E1)-(E3) and operators A with $A + \omega I$ m -accretive; this is a perfect result. The situation in the semigroup case, that is $L = \{0\}$, is considerably more complicated and there remain interesting unsolved problems. We will restrict our attention to the case $\omega = 0$, but all the results below remain valid in the general. We begin with the compact case.

(ii) If S is a nonexpansive semigroup on a closed convex set K in X and K is locally compact, then there is an accretive operator A in X with $\overline{D(A)} = K$,

$$K = \bigcap \{R(I + \lambda A) : \lambda > 0\}$$

and $S = S_A$ (equivalently, S is obtained from A via the exponential formula). In particular, if $X = \mathbb{R}^n$ is finite dimensional, then every nonexpansive semigroup on a closed convex set arises in this way. However, even if $X = \mathbb{R}^2$ with the maximum norm, there are distinct m -accretive operators A and B with domains all of X for which $S_A = S_B$.

The next results do not require compactness but restrict the geometry of X instead. In the event that $X = H$ is a Hilbert space, the notion of an accretive operator coincides with another notion, that of a monotone operator. Moreover, it is known that an operator is m -accretive if and only if it is accretive and not a proper restriction of another accretive operator - i.e., if it is maximal accretive (equivalently, maximal monotone). This is the origin of the " m " in m -accretive.

(iii) If $X = H$ is a Hilbert space, K is a closed and convex subset of X and S is a nonexpansive semigroup on K , then there is a unique m -accretive operator A in X such that $S = S_A$ and $\overline{D(A)} = K$. Moreover, this correspondence is biunique, the infinitesimal generator of S_A is $-A^0$ where A^0 is the minimal section of A . That is, for $x \in D(A)$, Ax is a closed convex set and A^0x is the projection of 0 on this set (its element of least norm).

The results above provide a perfect generation theorem for nonexpansive semigroups in Hilbert spaces which is really quite rich in structure. Moreover, it is nontrivial even in the case $X = \mathbb{R}$! A generalization of (iii) holds which places less severe geometrical restrictions on X , but more on K .

(iv) If the norm of X is uniformly Gateaux differentiable and the norm of X^* is Frechet differentiable, then the relation $S = S_A$ establishes a biunique correspondence between nonexpansive semigroups on closed convex nonexpansive retracts of X and m -accretive operators.

We will defer further remarks in this vein to the comments section. For now we content ourselves with the final remark that it is still an unsolved problem to determine whether or not an arbitrary nonexpansive semigroup S on a convex K can be represented in the form $S = S_A$ for an accretive A . It seems likely that if this is to be proved not true, then this will be done by presenting a nonexpansive semigroup S on a closed convex set K such that $\|S(t)x - x\|/t \rightarrow \infty$ as $t \rightarrow 0+$ for every $x \in K$. If this is to be proved true, it will likely involve some totally new arguments - a statement which leads us to a few comments on the arguments used to prove (i) - (iv).

In order to prove (i), one proceeds according to the following idea: Assuming that E is indeed of the form E_A , we fix $z \in X$ and try to build the solutions of $u' + Au + (\omega + 1)u \ni z$, $u(0) = x$ from E . If $A + \omega I$ is accretive, the time t mapping $x \rightarrow u(t)$ so defined is a strict contraction and by a fixed point argument we conclude that the problem has a constant solution $u \equiv x$. Then $x \in D(A)$ and $z - (\omega + 1)x \in Ax$. This leads to the following construction of A . First extend E to $K \times P$ where P is the space of piecewise constant functions. This is easy owing to (E2). E.g., if $f = z$ on $0 < t < a$ and $f = w$ on $a < t$, put $E(x, f)(t) = E(x, z)(t)$ for $0 < t < a$ and $E(x, f)(t + a) = E(x, E(x, z)(a))(t)$ if $0 < t$. Next use (E3) and the density of P in $L^1_{loc}(0, \infty; X)$ to extend E to all of $K \times L^1_{loc}(0, \infty; X)$. Next fix $z \in X$ and $x \in K$ and solve $u = E(x, -(\omega + 1)u + z)$ by iterating: $u_0 \equiv x$, and $u_n = E(x, -(\omega + 1)u_{n-1})$. Observe that $x \rightarrow e^t u(t)$ is nonexpansive and so there is a unique element of K fixed by the map $x \rightarrow u(t)$ for all t . That is, $x = E(x, -(\omega + 1)x + z)$ has a solution. Defining A by $z - (\omega + 1)x \in Ax$ yields an operator A with the desired properties.

In order to prove the results (ii) - (iv) a quite different path is taken. One attempts to produce A by defining

$$(5.1) \quad (I + \lambda A)^{-1} = \lim_{t \rightarrow 0} \left(I + \lambda \frac{I - S(t)}{t} \right)^{-1},$$

and the main work is to show the existence of a suitable (perhaps subsequential) limit. See the comments section.

The result (i) is a strong indication that if $A + \omega I$ is accretive and the problem $(IVP)_{x,f}$ has a mild solution on $[0, \infty)$ for every $x \in \overline{D(A)}$ and constant function f , then A is probably m -accretive. However, it does not quite say this, and there is an apparently open problem here. A variant of the question involved here is the problem of trying to give sufficient conditions and necessary conditions for the solvability of $(IVP)_{x,0}$ for arbitrary $x \in \overline{D(A)}$.

For example, the following is an interesting sufficient condition: Let $A + \omega I$ be accretive and

$$(5.2) \quad \liminf_{\lambda \rightarrow 0} \frac{d(R(I + \lambda A), x)}{\lambda} = 0 \text{ for } x \in \overline{D(A)},$$

where $d(C, x)$ denotes the distance from the set C to x . Then the problem

$$(5.3) \quad u' + Au \ni 0, u(0) = x$$

has a mild solution on $[0, \infty)$ for every $x \in \overline{D(A)}$. We call the condition (5.2)

"Kobayashi's criterion". This is obviously a generalization of the range condition.

It is also a sort of tangency condition: In the event that A is a continuous function on $\overline{D(A)}$ it can be shown to be equivalent to the assumption that

$$(5.4) \quad \liminf_{\lambda \rightarrow 0} \frac{d(x - \lambda Ax, \overline{D(A)})}{\lambda} = 0 \text{ for } x \in \overline{D(A)}.$$

If the limit inferior is replaced by the limit above, the statement just says that departing from x in the direction of $-Ax$ will leave $\overline{D(A)}$ at zero velocity.

In fact, necessary and sufficient conditions are known for the solvability of (5.3). For example, the following are equivalent if $A + \omega I$ is accretive:

- (a) (5.3) has a mild solution on $[0, \infty)$ for every $x \in \overline{D(A)}$.
- (b) For every $\varepsilon > 0$ and $x_0 \in \overline{D(A)}$ there is a $\delta \in (0, \varepsilon)$, an integer N , and $y_i \in Ax_i$, $h_i > 0$ for $i = 1, \dots, N$ such that

$$\sum_{i=1}^N h_i = \delta, \quad \sum_{i=1}^N \|x_i - x_{i-1} + h_i y_i\| < \varepsilon \delta.$$

That (a) \rightarrow (b) is trivial. The other implication requires interesting arguments which we will not sketch here. Do notice that Kobayashi's criterion is just the case (b) in the particular situation $N = 1$.

Section 6. Regularity of Mild Solutions

In the generality we have been discussing, if $A \in \mathcal{A}(\omega)$ the only strong solutions of the problems $u' + Au \ni f$ which are known to exist are the trivial ones, the constants. That is $u = x$ and $f = y$ for all t where $y \in Ax$. However, mild solutions are a satisfactory extension of the notion of strong solutions, since mild solutions are unique and strong solutions are mild. We have not addressed the other part of this consistency question, namely if a mild solution turns out to be "smooth", is it a strong solution? Similarly, we have not given conditions under which mild solutions are smooth. This we will do now. We do emphasize, before this, that even in applications one does not want to be limited to strong solutions, since there are important partial differential equations which simply do not have strong solutions.

A basic fact is the following consistency between the property $A \in \mathcal{A}(\omega)$ and differentiability of mild solutions of $u' + Au \ni f$.

Theorem 4. Let $A \in \mathcal{A}(\omega)$, $f \in L^1(0, T; X)$ and u be a mild solution of $u' + Au \ni f$ on $(0, T)$. Let u have a right derivative $u_R^1(\tau)$ at $\tau \in (0, T)$ and

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\tau}^{\tau+h} |f(t) - f(\tau)| dt = 0,$$

that is, τ is a right Lebesgue point of f . Then the operator \hat{A} given by

$$\hat{A}x = Ax \text{ for } x \neq u(\tau)$$

(6.1)

and

$$\hat{A}u(\tau) = Au(\tau) \cup \{f(\tau) - u_R^1(\tau)\}$$

satisfies $\hat{A} \in \mathcal{A}(\omega)$.

If $A + \omega I$ is m -accretive and $\hat{A} \in \mathcal{A}(\omega)$ is an extension of A (as is the case for the \hat{A} given by (6.1)), then $\hat{A} = A$. This maximality property arises because if \hat{A} were strictly bigger than A , then for $\lambda > 0$ and $\lambda\omega < 1$, $(I + \lambda\hat{A})^{-1}$ would be a function strictly extending the everywhere defined $(I + \lambda A)^{-1}$, and this is impossible. It thus follows at once that if $A + \omega I$ is m -accretive and $u \in W^{1,1}(0,T;X)$ is a mild solution of $u' + Au \ni f$, then u is a strong solution. When is a mild solution in $W^{1,1}(0,T;X)$? The principal conditions guaranteeing this are given by:

Proposition 6. Let $A \in \mathcal{A}(\omega)$, $f: [0,T] \rightarrow X$ be of bounded variation and $x \in D(A)$. If u is a mild solution of $u' + Au \ni f$ on $[0,T]$, then u is Lipschitz continuous. Moreover,

$$e^{|\omega|T} \|f(0+) - y\| + V(f,T) + |\omega| \int_0^T e^{|\omega|(T-\tau)} V(f,\tau) d\tau,$$

where

$$V(f,t) = \limsup_{h \rightarrow 0} \int_0^{t-h} \frac{\|f(\tau+h) - f(\tau)\|}{h} d\tau$$

is the variation of f over $[0,t]$, is a Lipschitz constant for u . If X is also reflexive, then $u \in W^{1,1}(0,T;X)$.

That is, we have a regularity of u under the stated conditions which is independent of X , namely Lipschitz continuity. However, it is only under further conditions on X (e.g., reflexivity) that this guarantees differentiability and hence $u \in W^{1,1}(0,T;X)$. In particular we have:

Corollary 1. Let $A + \omega I$ be m -accretive, $f: [0,T] \rightarrow X$ be of bounded variation, $x \in D(A)$ and X be reflexive. Then the mild solution of $u' + Au \ni f$, $u(0) = x$ is a Lipschitz continuous strong solution.

Under further restrictions on X more refined statements about regularity can be made, but they do not offer essential improvements over the information above and we omit them here. Likewise, conditions like Kobayashi's criterion can be used to replace m -accretivity of $A + \omega I$ in the case $f = 0$ to deduce results like Corollary 1.

A question related to the regularity considerations above is the following: Since the only known strong solutions are the constants in general, are they enough to determine (somehow) the class of all mild solutions? Another motivation for this question is the observation that the definition of a mild solution is not very "checkable". That is, given a function u , how can we tell if it is a mild solution of $u' + Au \ni f$? In general, we cannot simply compute u' and see if the relation is satisfied. Since $y \in Ax$ implies $u = x$ solves $u' + Au \ni y$, we know by Theorem 3 that any mild solution of $u' + Au \ni f$ satisfies

$$(6.2) |u(t) - x| \leq \int_0^t e^{\omega(t-s)} |u(s) - x| ds + \int_0^t e^{\omega(t-\tau)} |u(\tau) - x, f(\tau) - y| d\tau \text{ for } 0 \leq t \leq T.$$

for $y \in Ax$. In fact, this family of inequalities can be taken to define a class of solutions called integral solutions. However, as opposed to mild solutions, the notion depends on the norm of X via the bracket and is appropriate only if $A \in \mathcal{A}(\omega)$.

Moreover, it is not a good notion in general in the sense that it is easier to be an integral solution for a restriction of A than for A itself. However, it is a uniqueness criterion provided mild solutions are known to exist (which guarantees that A is "big enough" for the notion to be satisfactory). More precisely:

Theorem 5. Let $A \in \mathcal{A}(\omega)$, $f \in L^1(0, T; X)$ and v be a mild solution of $v' + Av \ni f$ on $[0, T]$. If $u \in C[0, T; X]$ satisfies (6.2) for every $y \in Ax$ and $u(0) = v(0)$, then $v = u$.

Hence if the existence of a mild solution is known, then one can determine if a given function is this mild solution or not according as to whether or not the relations (6.2) hold.

Section 7. Auxiliary Results: Continuity, Trotter Products and Compactness

In this section we formulate a variety of auxiliary results in the subject which give additional useful information. First among these addresses the problem of the dependence of the solution of

$$(IVP)_{x,f} \quad u' + Au \ni f, \quad u(0) = x,$$

on A . In order to formulate the results in a multivalued generality we recall the

notion of the limit inferior of a sequence A_n of operators. The operator $\liminf A_n$ is defined by $y \in \liminf A_n x$ if and only if there is a sequence $y_n \in A_n x_n$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. We will meet the condition $A \subset \liminf A_n$ below, and it will make things a bit clearer if we recall the following equivalent condition in the m -accretive case.

Proposition 7. Let $A_n + \omega I$ be m -accretive for $n = 1, 2, \dots, \infty$ (with ∞ explicitly included). Then $A_\infty \subset \liminf A_n$ if and only if

$$(7.1) \quad \lim_{n \rightarrow \infty} (I + \lambda A_n)^{-1} x = (I + \lambda A_\infty)^{-1} x \text{ for } x \in X$$

for $\lambda > 0$ and $\lambda \omega < 1$. Moreover, (7.1) holds for all such λ if it holds for one such λ .

We call the condition (7.1) "resolvent convergence". Now let us formulate the continuous dependence theorem in some generality.

Theorem 6. Let $A_n \in \mathcal{A}(\omega)$, $f_n \in L^1(0, T; X)$ for $n = 1, 2, \dots, \infty$. Let u_n be a mild solution of $u_n' + A_n u_n \ni f_n$ on $[0, T]$ for $n = 1, 2, \dots, \infty$. Let $A_\infty \subset \liminf A_n$ and

$$\lim_{n \rightarrow \infty} \int_0^T \|f_n(t) - f_\infty(t)\| dt + \|u_n(0) - u_\infty(0)\| = 0.$$

Then $u_n \rightarrow u_\infty$ uniformly on $[0, T]$.

This result, in the m -accretive case, says that if initial data converge, the forcing terms converge in $L^1(0, T; X)$, and the resolvents of the A_n converge, then the solutions converge. More generally, it makes the same claim provided only that the solutions exist. The method of proof involves observing that, by definition, $A_\infty \subset \liminf A_n$ implies that given any neighborhood of an approximate solution of $u_\infty' + A_\infty u_\infty \ni f_\infty$ then for n large enough we can find an approximate solution of $u_n' + A_n u_n \ni f_n$ in this neighborhood and then using the estimates in the proof of the convergence theorem. The utility of such a result is clear. For example, one may use it to prove the approximation result described next. If $A + \lambda I$ is m -accretive we may define the Yosida approximation A_η of A for small $\eta > 0$ by $A_\eta = \eta^{-1}(I - (I + \eta A)^{-1})$. Clearly A_η

is Lipschitz continuous with $2\eta^{-1}$ as a Lipschitz constant and it is easy to see that $A_\eta + \omega/(1 - \eta\omega)I$ is accretive. Since, as the reader could check, $(I + \lambda A_\eta)^{-1} + (I + \lambda A)^{-1}$ as $\eta \rightarrow 0$, the continuous dependence theorem implies that the solution u_η of $u'_\eta + A_\eta u_\eta = f$, $u_\eta(0) = x$ converges uniformly to the solution u of the corresponding problem with A replacing A_η as $\eta \rightarrow 0$. Since A_η is Lipschitz continuous, this is a natural way to approximate u by regular functions in a fashion closely related to the original problem.

Another sort of result of wide applicability can be motivated as follows: Suppose we want to solve

$$(7.2) \quad u' + Au + Bu = 0, \quad u(0) = x,$$

and that we know the solutions of the Cauchy problems for the separate equations

$$(7.3) \quad u' + Au = 0, \quad v' + Bv = 0,$$

in the form of the semigroups S_A and S_B and that A and B are functions. Assuming a large (and totally unreasonable) amount of regularity one computes

$$\frac{d}{dt} S_A(t)S_B(t)x \Big|_{t=0} = Ax + Bx.$$

That is, infinitesimally $F(t) = S_A(t)S_B(t)$ looks like $S_{A+B}(t)$ should look. Moreover, $F(t)x$ is well behaved as a function of x . Can we not then represent S_{A+B} in terms of $F(t)$? One has the following theorem to this effect:

Theorem 7. Let $A \in \mathcal{A}(\omega)$ satisfy the range condition (3.24) and $C = \overline{D(A)}$ be convex.

For each $t > 0$ let $F(t): C \rightarrow C$ and F satisfy:

$$(i) \quad \|F(t)x - F(t)y\| \leq e^{Lt} \|x - y\| \text{ for } x, y \in C \text{ and } 0 < t < 1.$$

$$(ii) \quad \lim_{t \rightarrow 0} (I + \lambda F(t))^{-1} x = (I + \lambda A)^{-1} x \text{ for } x \in C \text{ and } \lambda > 0, \lambda \max(\omega, L) < 1.$$

Then for each $x \in C$, $S_{A+B}(t)x = \lim_{n \rightarrow \infty} F(t/n)^n x$ uniformly on compact t -sets.

It is part of the proof that the inverses used in (ii) exist. This result applies in the "A + B" case above provided that one can verify the resolvent condition (ii) given A and B . In this event, the conclusion is called a "Trotter product" formula. However, there are many other circumstances under which one can verify the hypotheses

of the theorem. The proof consists of using the estimates of the convergence theorem together with another interesting approximation result, which we state for interest's sake in a special case.

Proposition 8. Let $C \subset X$ be closed, $F: C \rightarrow C$ be nonexpansive and $h > 0$. If the initial-value problem

$$u' + h^{-1}(I - F)u = 0, u(0) = x \in C$$

has a mild solution on $[0, T]$, then

$$\|F^n x - u(t)\| \leq 2\|x - z\| + \left(\left(n - \frac{t}{h} \right)^2 + n \right)^{1/2} \|z - Fz\|$$

holds for every $z \in C$, $0 < n$ and $0 < t \leq T$. In particular, choosing $x = z$ and $t = nh$ we have

$$\|F^n x - u(nh)\| \leq \sqrt{n} \|x - Fx\|.$$

The last sort of auxiliary result we discuss here concerns compactness. We fix an operator A with $A + \alpha I$ m -accretive and consider the initial-value problem

$$(IVP)_{x,f} \quad u' + Au \ni f, u(0) = x,$$

whose mild solution u we denote by $E(x, f)$. If $f = 0$, then $u(t) = E(x, 0)(t) = S_A(t)x$ where S_A is the semigroup generated by $-A$, and we will use this notation below. We ask when the images of various sets under E are compact in various senses. The simplest question concerns the semigroup case. A function in X is called compact if it maps bounded subsets of its domain into precompact sets in X and a semigroup S is compact if each $S(t)$ is compact for $t > 0$.

Theorem 8. Let $A + \alpha I$ be m -accretive and S the semigroup on $\overline{D(A)}$ generated by $-A$. Then $S(t)$ is compact if and only if the following two conditions are satisfied:

- (i) For each small $\lambda > 0$ the operator J_λ is compact.
- (ii) For each bounded subset B of $\overline{D(A)}$ and $s > 0$

$$\lim_{t \rightarrow s} S(t)x = S(s)x \text{ holds uniformly for } x \in B.$$

In applications to partial differential equations, compactness of $S(t)$ tends to arise from regularizing properties, that is $S(t)x$ will lie in a more regular class of functions for $t > 0$ than at $t = 0$. Another sort of compactness one is interested in is

the compactness of the trajectory

$$\text{tr}(x) = \{E(x,f)(t): 0 \leq t\}$$

of the solution of (IVP)_{x,f}. Compactness of trajectories is useful in making dynamical systems type arguments concerning the asymptotic behaviour of u. Concerning this property one has:

Theorem 9. Let A be m-accretive, $0 \in R(A)$, $f \in L^1([0, \infty): X)$ and $x \in \overline{D(A)}$. In addition, let $(I + \lambda A)^{-1}$ be compact for some $\lambda > 0$. Then $\text{tr}(x)$ is precompact.

The first conditions in the Theorem guarantee that $\text{tr}(x)$ is bounded and the compactness comes from the assumption on J_λ .

Next we look at the solution operator for (IVP) and consider when it is compact as a function of f for fixed $x \in \overline{D(A)}$. There arises the question of what topologies to use in the domain and range space here, and the next result contains an answer.

Theorem 10. Let $A + \omega I$ be m-accretive and $S(t)$ be the semigroup generated by $-A$. Fix $x \in \overline{D(A)}$ and $p > 1$. Let $Q: L^p(0, T; X) \rightarrow C[0, T; X]$ be given by $Q(f) = E(x, f)$. If $S(t)$ is a compact semigroup, then Q is a compact operator.

This result is unsatisfactory in that it does not allow the natural generality of $f \in L^1(0, T; X)$. It is possible to treat this case if we are willing to weaken our requirements in the range space. Moreover, we can then vary x as well.

Theorem 11. Under the assumptions of Theorem 10, if $S(t)$ is a compact semigroup then the solution operator E is compact as a mapping

$$E: \overline{D(A)} \times L^1(0, T; X) \rightarrow L^p(0, T; X)$$

for $1 < p < \infty$.

Section 8. Comments and References

In this section we amplify on the main text a bit and provide some basic references for the interested reader. No attempt has been made to be complete, and nothing like completeness has been achieved. However, the references we do quote together with the references they contain should suffice to accurately represent the

situation. Let us begin by noting that there are several books in the general area. The theory in Hilbert spaces is well developed in Brezis [17]. The general case is treated in Da Prato [36], Barbu [5] and Pavel [72], [73]. The books of Martin [64] and Browder [24] also treat aspects of the theory. The references provided by these works are not subsumed here and the reader will find many applications to partial differential equations in Barbu's book.

On Section 1. Early attempts to represent nonexpansive semigroups were made by Neuberger [69], Oharu [70] and Komura [59], [60]. Komura's dramatic ideas were a main stimulus for the rapid development which followed (e.g., Kato [49], [50], Crandall and Pazy [34], Dorroh [41] and Browder [23]).

The bracket $[,]$ and the duality map J are well known in functional analysis. However, nomenclature and notation are inconsistent. For example, in Reich [87] of this volume, $J(x)$ denotes what would be written $|x|J(x)$ in our notation. Sato [88] provides specific computations of the duality map, but the reader can work out what J and $[,]$ are for the common spaces. Workers in this subject learned Proposition 1 (ix) in Kato [49]. If J is not single-valued, a stronger condition than Definition 1 (iii) arises when the conclusion is required to hold for all $x^* \in J(x - \hat{x})$. An operator with this property is sometimes called s -accretive (or "Browder accretive", since this notion was taken to define accretive in Browder [23]). If J is single-valued on $X \setminus \{0\}$ the notions coincide. Interest in s -accretivity arises from facts like $A + B$ is accretive whenever A and B are and at least one of them is s -accretive.

Komura [59] is sometimes cited in this subject for a proof of the fact that if X is reflexive and $f: [0, T] \rightarrow X$ is Lipschitz continuous, then $f \in W^{1,1}(0, T; X)$, but theorems of Radon - Nikodym type for reflexive spaces were already proved by Phillips [77] and Dunford and Pettis [42]. Reflexive spaces are but examples of spaces with the Radon - Nikodym property.

On Section 2.

The notion of a mild solution is already suggested in Crandall and Liggett [32], although it was too early at that time to institutionalize the idea. The term "mild"

appears in Browder [21] in another context, but in a way consistent with our usage. The notation and presentation here are taken from [13], but variants of this natural idea appeared in various places (under various names), e.g. Kenmochi and Charu [52], Kobayashi [53], and Pierre [78]. There are many questions one can ask about mild solutions which have not been seriously approached. For example, it is known that a mild solution in the current sense cannot necessarily be approximated by solutions of discretisations with uniform steps - but it is not known if this is true when, e.g., A is accretive. It is known that if $f = 0$ and $X = \mathbb{R}$ and A is accretive, then uniform steps are enough. (Unpublished results of Crandall and Pierre). In most applications these issues are not serious, as A is either m -accretive or satisfies a variant of the range condition (3.22). R. Martin [63] proved that continuous accretive operators are m -accretive.

It is also known that mild solutions defined, as we have done, in the implicit way (2.6) (i.e., A is evaluated at v_1) differ from those defined in the explicit way in which Av_1 is replaced by $Av_{1-\tau}$ in (2.6). Indeed, it is easy to see that $u(t)$ is the limit of solutions of explicit approximations of $u' + Au \ni 0$ iff $v(t) = u(-t)$ is a mild solution of $v' + Av \ni 0$. The case of a single conservation law, a partial differential equation whose relevant solutions are not reversible and which can be accommodated in the theory ([29]) provides a significant counterexample. Proposition 3 is selected from [13].

The notion of a "strong solution" is standard, but sometimes people prefer to weaken it to require the continuity of the solution on $[0, T]$ and what we have called a strong solution on $[c, T]$ for each $c > 0$. This accommodates more examples and still allows one to do "calculus" without undue worry about the validity of the manipulations.

Examples of badly behaved nonlinear semigroups occur in [32], Plant [82] and Webb [92, 93], but mild solutions which are not strong solutions are familiar even in the linear theory when initial data do not lie in $D(A)$ or f is not sufficiently regular (in which case the variation of parameters formula typically provides such solutions).

On Section 3.

The first proof that solutions of difference approximations converge (in a more restricted context, but with general X) was in [32]. Rasmussen [83] provided a useful proof. Takahashi [89] gave a more general convergence proof with variable steps. Benilan [7] proved the existence and uniqueness of mild solutions for $u' + Au \ni f$ when A is m -accretive. The existence also follows from results of Crandall and Pazy [35].

The full Theorem 1 was first proved in Crandall and Evans [31] by the fun method sketched here. One finds appropriate error estimates in [31] as well. The result in the case $f = 0$ was obtained by Y. Kobayashi [53] who formulated his results for quasi-accretive operators, a notion which generalizes the accretive case and which was introduced by Takahashi [89], but for which significant nonaccretive examples are lacking. Kobayashi's method was different (and simpler) in the case $f = 0$, but it becomes more complex in the general case. See also Takahashi [89], [90]. The reader may consult K. Kobayasi [56] and K. Kobayasi, Y. Kobayashi, and S. Oharu [57] for even more general results by this method.

Indeed, there are a variety of generalizations of the above to time dependent equations of the form $u' + A(t)u \ni f$, although it is not easy to be satisfied with any particular set of technicalities or definitions in this case (as is already true in the linear setting). We mention that Kato [50] and Crandall and Liggett [32] already allowed some time dependence, while Crandall and Pazy [35] is more general. The case of "integrable" time dependence was handled in Evans [43] using the results of [31] (in essence, Theorem 2), and there is also the elegant and different approach of Pierre [79]. The recent works [57], which was mentioned above, Iwayami, Oharu and Takahashi [47], and K. Kobayasi and S. Oharu [58] extend the theory in various significant ways. It would be interesting to know the precise relationship between the convergence assertions in these works and Theorem 2.

On Section 4.

The theory of Kato referred to is the simplest context described in his survey [51] in this volume, and we refer the reader to this paper for appropriate references. The relationship with the nonlinear theory sketched here is noted in Crandall and Souganidis [37], and can be greatly generalized. See also Hazan [46] in this regard. If the assumptions of this Section are strengthened by requiring the existence of an operator $S:Y \rightarrow X$ with the properties described in [51], then the conclusions of this Section can be strengthened to assert that the difference approximations converge in the strong topology of Y uniformly in t and the proof can be adapted to prove the continuity of the solution as a Y -valued function (and the assumption (B3) dropped). This is done in Crandall and Souganidis [38]. Another work, which is in a more preliminary stage, extends these results to the variable norm setting explained in Kato's article in this volume.

On Section 5.

The result (i) is due to Benilan [7]. The result (ii) follows from Crandall and Liggett [32]. The biunique correspondence of (iii) was proved by Crandall and Pazy [34]. The existence of an m -accretive A such that $S = S_A$ in Hilbert spaces in this generality involves Minty's theorem, which essentially states the equivalence of "maximal monotone" and " m -accretive" in Hilbert spaces, and this result fails in general. (See Crandall and Liggett [33] and Calvert [27].)

The idea to obtain A via (5.11) is Komura's ([60]), and so is the first proof of the existence of this limit in Hilbert spaces. This result was the hardest step in the proof of (iii). New ideas had to be introduced to extend this convergence result outside of Hilbert spaces, and this was done by Baillon [3]. Reich sharpened this line of the theory, and (iv) can be found in [84]. We refer to Reich [85,86] for further references and discussion. See also [87], Theorem 1.6. By the way, the results of

[33] to the effect that the limit (5.11) always exist if X is two dimensional but may fail to exist if X is three dimensional show that the success of this approach must involve geometrical considerations.

The sufficiency of Kobayashi's criterion (5.2) for the solvability of (5.3) was a fascinating result of [53]. This result allowed very slick proofs of m -accretivity if operators of the form $A + B$ where A is m -accretive and B is continuous and accretive, generalizing results of Martin [63], Webb [93], and Barbu [6]. The equivalence of (5.2) and (5.4) is remarked in [30]. Numerous people, including Kaplan and Yorke [48] and Takahashi [89], contributed to the development of this line of thought. The sufficiency of (b) is an unpublished result of Y. Kobayashi. He also shows (b) is equivalent to another condition related to the sufficient conditions of Pierre [80]. One also finds examples indicating the distinction between various conditions in [80].

Section 6.

The results on regularity of mild solutions we will regard as being "from the community", but let us mention that the main facts were not so obvious in the beginning. It was mentioned in Section 2 that a strong solution is a mild solution - this is not entirely obvious. Theorem 4 is the heart of results in the other direction - it implies that differentiable mild solutions satisfy the equation pointwise if A is "big enough" in the sense that the operator of (6.1) cannot properly extend A (and so mild solutions are strong if they are regular enough). Theorem 5 is a simple version of Benilan's uniqueness theorem ([7]).

Section 7.

Theorem 6, in this general formulation, may not appear in the literature. (We are using a formulation from [13]). See, however, Miyadera and Kobayashi [66], and results in this spirit in general Banach spaces go back to Benilan [7], Brezis and Pazy [20], Kurtz [62] and Goldstein [45]. For examples of substantial applications of this result in pde see, e.g., Benilan and Crandall [10] or Kenmochi and Oharu [52].

Theorem 7, the conclusion of which is called the nonlinear Chernoff formula, is due to Brezis and Pazy [20]. An earlier version and Proposition 8 are due to Miyadera and Oharu [68]. Theorem 7 has many applications in pde - see, e.g., Berger, Brezis and Rogers [16], Coron [28], Kenmochi and Oharu [52], and Oharu and Kobayasi [58]. There has been a fair amount of recent activity concerning results of this general type in special circumstances. See, e.g., Benilan and Ismail [15], Reich [85,86], Kobayashi [54], [55] and their references. M. Pierre [81] has recently obtained quite interesting results (both positive and negative) on the validity of more general formulae involving nonuniform steps.

One can ask to what extent the implications in Theorems 6 and 7 are reversible and be led thereby to the question of convergence versus resolvent consistency. Since the conclusion of Theorem 7 always holds if $F(t) = S(t)$, this links up with the problem of the existence of the limit (5.1). See Reich [86] for recent results and references.

The various compactness results are proved in Brezis [18], Dafermos and Slemrod [39] and Baras [4]. See Konishi [61] for an early result of this type and Brezis and Friedman [19] for an application in pde.

Asymptotic Behaviour.

We have omitted the topic of asymptotic behaviour. The works of Bruck [25] and Baillon [2] stimulated a large amount of subsequent work on these lines of research, and the area remains quite active. The survey article Bruck [26] is a recent source on this topic and we refer the reader to it. Other recent sources on aspects of this question include Pazy [74], [75], [76], Reich [87], Miyadera [65], and - in a somewhat different vein - Alikakos and Rostamian [1].

Regularizing Effects. A final topic we mention is that of regularizing effects. These concern questions of regularity - interpreted in various ways - of $S(t)x$ for $t > 0$ that x itself may not enjoy. There is no general theory available yet, but the phenomenon is widespread and of considerable interest when it is present. On the abstract side, the best known examples are the regularizing effects of linear analytic semigroups and semigroups generated by subdifferentials of convex functions in Hilbert spaces (see

[17]). In the general nonreflexive case we have some examples, e.g. those of Benilan [8,9], Veron [91], Benilan and Crandall [12], and Crandall and Pierre [36]. A new regularizing semigroup is also discussed in Reich [87]. One wonders if there is an informative unifying point of view which might relate these various examples.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This is a review paper which outlines the main points of the theory of non- linear semigroups and evolution governed by accretive operators. The subject is now rather mature, so most of the principal ideas and results are not new. How- ever, the presentation here is organized differently from that in other sources and does touch upon recent results. An attempt has been made to make this paper a pleasant route to a certain view of the subject. This manuscript represents the author's contribution to the proceedings of the Symposium on Nonlinear Functional Analysis and Applications held in Berkeley in July, 1983.		

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