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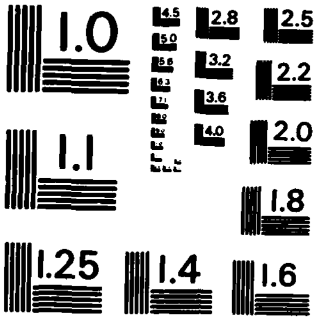

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INFINITE ORDER AUTOREGRESSIVE REPRESENTATIONS OF MULTIVARIATE STATIONARY STOCHASTIC PROCESSES

by

Mohsen Pourahmadi

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INFINITE ORDER AUTOREGRESSIVE REPRESENTATIONS OF MULTIVARIATE STATIONARY STOCHASTIC PROCESSES[1]

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Abstract. Consider a q-variate weakly stationary stochastic process $\{X_n\}$ with the spectral density W. The problem of autoregressive representation of $\{X_n\}$ or equivalently the autoregressive representation of the linear least squares predictor of X_n based on the infinite past is studied. It is shown that for every W in a large class of densities, the corresponding process has a mean convergent autoregressive representation. This class includes as special subclasses, the densities studied by Masani (1960) and the author (1984). As a consequence it is shown that the condition $W^{-1} \in L^1_{q\times q}$ or minimality of $\{X_n\}$ is dispensable for this problem. When W is not in this class or when W has zeros of order 2 or more, it is shown that in this case $\{X_n\}$ has a mean Abel summable or mean compounded Cesáro summable autoregressive representation.

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Keywords and Pharases: q-variate stationary processes, autoregressive and moving average representation, spectral density matrix, Abel and Cesáro summability.

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1. Introduction.

While it is well-known that every purely nondeterministic full rank q-variate weakly stationary stochastic process ~~(SSP)~~ $\{X_n\}$ with the spectral density W has an (infinite order) one-sided moving average representation, not every such process can have a mean convergent (infinite order) autoregressive representation (ARR) and the problem of ARR of such processes has not received the attention which it deserves. Due to the importance of ARR in prediction theory, and particularly in the statistical theory of multivariate time series, this paper is devoted to the problem of finding the weakest condition on W which guarantees the existence of an infinite order ARR for $\{X_n\}$.

To be more precise, we say that the SSP$\{X_n\}$ has a mean convergent (summable) ARR if there exists a sequence $\{A_k\}_{k=1}^{\infty}$ of constant $q \times q$ matrices such that, for each n,

$$X_n = \sum_{k=1}^{\infty} A_k X_{n-k} + \varepsilon_n, \tag{1.1}$$

where $\{\varepsilon_n\}$ is the innovation process of $\{X_n\}$ and the infinite series $\sum_{k=1}^{\infty} A_k X_{n-k}$ is to be convergent (summable) in the mean. This representation of the process $\{X_n\}$ as an infinite order stochastic difference equation can also be regarded as the inversion of the one-sided moving average respresentation of $\{X_n\}$ given in (2.6). Such inversion of the one-sided moving average representation of a q-variate SSP plays a vital role in the statistical estimation of the parameters of $\{X_n\}$. For the notation and definitions see Section 2.

It is obvious that the problem of ARR $\{X_n\}$, cf. (1.1), is equivalent to the problem of ARR of $\hat{X}_{n|n-1}$ (the linear least squares predictor of X_n based on $\{X_{n-k};\ k \geq 1\}$):

$$\hat{X}_{n|n-1} = \sum_{k=1}^{\infty} A_k X_{n-k} \, . \tag{1.2}$$

which has been studied by Wiener and Masani [16], and Masani [7].

It follows from the isomorphism between the time and spectral domains of $\{X_n\}$ that the infinite series (1.1) or (1.2) is mean convergent (summable), if and only if the isomorph of ε_n in $L^2(W)$ has a convergent (summable) Fourier series in the norm of $L^2(W)$. For a purely nondeterministic full rank SSP $\{X_n\}$ with the spectral density $W = \Phi\Phi^*$ and G the one-step ahead prediction error matrix, it is well-known [6, II, p.115] that the function $G^{\frac{1}{2}} \Phi^{-1} e^{in\theta}$ in $L^2(W)$ is the isomorph of ε_n in $M(X)$. Thus, the series in (1.1) is mean convergent (summable), if and only if the Fourier series of Φ^{-1} is convergent (summable) to Φ^{-1} in the norm of $L^2(W)$. Also, it can be shown that the $(\nu+1)$-step ahead $(\nu \geq 0)$ linear least squares predictor $\hat{X}_{n+\nu|n-1}$ based on $\{X_{n-k}; k \geq 1\}$ has a mean convergent (summable) ARR, if and only if the Fourier series of Φ^{-1} is convergent (summable) to Φ^{-1} in the norm of $L^2(W)$. (This latter assertion for the univariate processes is proved by Miamee and Salehi [11] in the spectral domain and by Bloomfield [3] in the time domain.)

From the previous discussion the convergence (summability) of the Fourier series of Φ^{-1} to Φ^{-1} in the norm of $L^2(W)$ emerges as the only spectral necessary and sufficient condition for the existence of a mean convergent (summable) ARR of $\{X_n\}$. Although this condition is not concrete in terms of W, it is extremely useful in obtaining some useful concrete sufficient conditions in terms of W for the ARR of $\{X_n\}$. These conditions are stated and proved in Section 3 by using some techniques from harmonic analysis. In the following we review and discuss the implication of these conditions for the problem of ARR of $\{X_n\}$.

In 1958, it was shown by Wiener and Masani [16] that the boundedness condition $cI \le W \le dI$, where $0 < c \le d < \infty$, is sufficient for the existence of a mean convergent ARR of $\{X_n\}$. Later, Masani [7, Theorem 5.2] weakened this severe boundedness condition considerably and replaced it by

$$(1.3) \qquad W \in L^{\infty}_{q\times q}, \quad W^{-1} \in L^{1}_{q\times q}.$$

It was pointed out by Masani [7, p. 143] that the condition $W \in L^{\infty}_{q\times q}$ in (1.3) is unduly strong and it would be worth while to relax it. (It was also conjectured by Feldman [4] that the condition $W \in L^{\infty}_{q\times q}$ is dispensable.)

In [13] the author has shown that, indeed, the restriction $W \in L^{\infty}_{q\times q}$ is dispensable. This is done by employing the equivalence between the convergence of Fourier series of all functions in $L^2(W)$ and the positivity of the angle Θ between the "past and present" subspace and the "future" subspace of $\{X_n\}$ cf. Theorem 3.1. Thus, we have from Theorem 3.1 that $\{X_n\}$ has a mean convergent ARR if

$$(1.4) \qquad \Theta > 0 \ (\text{ or } \rho(W) < 1).$$

For $q = 1$, i.e. a scalar density W, it is well-known [6] that $\rho(W) < 1$, if and only if $W = \exp(u + \tilde{v})$, where u and v are bounded real-valued functions with $\|v\|_{\infty} < \pi/2$. $\tilde{V}$ denotes the harmonic conjugate of V. From this characterization of scalar densities satisfying (1.4) it follows that a matrical density satisfying (1.4) is not necessarily bounded. For some partial characterizations of matrix-valued densities satisfying (1.4) see [2, 13].

It is shown in [13] that (1.3) and (1.4) offer two independent sufficient conditions for the existence of mean convergent ARR of $\{X_n\}$. Furthermore, conditions (1.3) and (1.4) both entail that $W^{-1} \in L^{1}_{q\times q}$, i.e. they require that the process $\{X_n\}$ to be minimal, cf. [7], which we feel is a strong restriction for the ARR of $\{X_n\}$ and it is desirable to relax it.

Our first new result in this paper shows that the condition $W^{-1} \varepsilon L^1_{q\times q}$ or minimality of $\{X_n\}$ in (1.3) is dispensable for the existence of a mean convergent ARR of $\{X_n\}$. Also this result gives a more general sufficient condition for the mean convergent ARR of $\{X_n\}$ which includes both (1.3) and (1.4) as special cases.

To state this result and for later use we denote the class of densities satisfying (1.3) by M, those satisfying (1.4) by A and define a new class $A \otimes M$ by $A \otimes M = \{W; W = W_1^{\frac{1}{2}} W_2 W_1^{\frac{1}{2}}, W_1 \varepsilon A$ and $W_2 \varepsilon M\}$, where $W_1^{\frac{1}{2}}$ denotes the positive square root of W_1. Note that by choosing $W_1 = I$ $(W_2 = I)$ we see that this new class has $M(A)$ as its proper subset. Now, we can state our first theorem which is an immediate consequence of Theorem 3.2. We note that this theorem is a multivariate extension of a similar (univariate) result of Bloomfield [3] and that the factorization $W = W_1 W_2$ used by Bloomfield is not suitable in the multivariate setting, since the product of two positive definite matrices is not necessarily a positive definite matrix.

<u>Theorem 1.1</u>. Let $\{X_n\}$ be a purely nondeterministic full rank SSP with the spectral density matrix W. If $W \varepsilon A \otimes M$, then $\{X_n\}$ has a mean convergent ARR.

It is easy to check that a W in $A \otimes M$ does not necessarily have the property $W^{-1} \varepsilon L^1_{q\times q}$. (As an example when $q = 1$, one can take $W = |1 - e^{i\theta}|^{\lambda}$, $1 < \lambda < 2$.) But, for $W \varepsilon A \otimes M$, W^{-1} is necessarily in $L^{\frac{1}{2}}_{q\times q}$. Note that the scalar density $W = |1 - e^{i\theta}|^2$ which corresponds to the univariate SSP $X_n = \varepsilon_n - \varepsilon_{n-1}$ does not belong to the univariate version of $A \otimes M$. Thus, Theorem 1.1 does not provide any information concerning the existence of a mean convergent ARR for this process. However, this process $\{X_n\}$ can not have a mean convergent ARR, since in this case the infinite series in (1.1), i.e.

$-\sum_{k=1}^{\infty} X_{n-k}$ does not converge in the mean. This example shows that processes $\{X_n\}$ for which W is not in $A \otimes M$ (or in other words if W has zeros of order 2 or more), can not have mean convergent ARR. In view of this it is natural to ask whether such processes can have an ARR with a weaker requirement of convergence, say summability, for the infinite series $\sum_{k=1}^{\infty} A_k X_{n-k}$ in (1.1). The next theorem shows that this is actually possible for a large class of processes when we replace the mean convergence of the series (1.1) by its mean Abel summability, and compounded Cesáro summability.

We say that $\{X_n\}$ has a mean Abel summable ARR if the infinite series in (1.1) is Abel summable in the mean, i.e. $\lim_{r \to 1^-} \sum_{k=1}^{\infty} r^k A_k X_{n-k}$ exists in the mean for each n. The next theorem is immediate from Theorem 3.4(c).

<u>Theorem 1.2</u>. Let $\{X_n\}$ be a purely nondeterministic full rank SSP with the spectral density W. Let P be a complex-valued trigonometric polynomial of some degree n and $W' \in A \otimes M$. If $W = |P|^2 W'$, then $\{X_n\}$ has a mean Abel summable ARR.

From practical point of view (specially for the purpose of prediction) the mean Abel summable ARR of $\{X_n\}$ is not particularly useful for the following reason: $\lim_{r \to 1} \sum_{k=1}^{\infty} r^k A_k X_{n-k}$ can be approximated by $\sum_{k=1}^{\infty} r_0^k A_k X_{n-k}$, where r_0 is near one. Then, since only a finite segment of the past is available, one has to further approximate the infinite sum $\sum_{k=1}^{\infty} r_0^k A_k X_{n-k}$ by $\sum_{k=1}^{N} r_0^k A_k X_{n-k}$, where N is the number of available observations from the past. This introduces two sources of error in computing, say, $\hat{X}_{n|n-1}$. In view of this it is desirable to replace the Abel summability of the series in (1.1) by another method of summability which reduces the two sources of error to just one. For this it seems that the Cesáro, or compounded Cesáro summability is more appropriate. In the following we provide a different reason (motivated by a computational problem in prediction theory) for the feasibility of the compounded Cesáro summability method in ARR of $\{X_n\}$.

Since $\hat{X}_{n|n-1} \in M_{-\infty}^{n-1}(x)$, it is well-known that

$$(1.5)\qquad \hat{X}_{n|n-1} = \lim_{N\to\infty} \sum_{k=1}^{N} A_{Nk} X_{n-k},$$

where the A_{Nk}'s are some $q \times q$ constant matrices to be determined. For a fixed N, the matrices A_{Nk}, $1 \le k \le N$, can be found by inverting an $(N+1)q\times(N+1)q$ matrix. However, as N increases this procedure requires inverting larger matrices and thus becomes less feasible computationally. This is caused mainly by the dependence of A_{Nk} on N. Thus to overcome this computational problem it is desirable to find a condition on W which guarantees that either the A_{Nk}'s in (1.5) can be replaced by A_k's or by $\sigma_{N,k}$ A_k's, where σ_{Nk} is a scalar depending only on k and N but not on W. Obviously, if the former case prevails then (1.5) reduces to (1.2) or (1.1), and if the latter case prevails we have

$$(1.6)\qquad X_n = \lim_{N\to\infty} \sum_{k=1}^{N} \sigma_{Nk} A_k X_{n-k} + \varepsilon_n .$$

In either case we have $A_k = -G^{\frac{1}{2}}D_k$, $k = 1,2,\ldots,$ where the D_k's are the Fourier (Taylor) coefficients of Φ^{-1}, cf. [9].

Let $\{\alpha_N\}_{N=1}^{\infty}$ be a fixed increasing sequence of positive numbers such that $\lim_{N\to\infty} \frac{\alpha_N}{N} = 0$. We say that $\{X_n\}$ has a mean compounded Cesáro summable ARR (corresponding to $\{\alpha_N\}$) if (1.6) holds with

$$\sigma_{Nk} = \binom{N+\alpha_N}{N}^{-1} \sum_{i=k}^{N} \binom{N-1+\alpha_N-1}{N-i}$$

$N = 1,2,\ldots,$ $k = 1,2,\ldots,N$. For more information on the compounded Cesaro summability and the σ_{Nk}'s see Section 3, Theorem 3.5 and the discussion preceeding it.

Now, the proof of the next theorem is immediate from Theorem 3.5(b).

Theorem 1.3. Let $\{X_n\}$ be a purely nondeterministic full rank SSP with the spectral density W. Let P be a complex-valued trigonometric polynomial of some degree n and $W' \in A \otimes M$. If $W = |P|^2 W'$, then the process $\{X_n\}$ has a mean compounded Cesáro summable ARR.

At the end, we would like to note that Theorems 3.4 and 3.5 show the existence of a mean Abel (compounded Cesaro) summable ARR of $\{X_n\}$ only when the spectral density matrix has a finite number of zeros (of any finite orders) on $[-\pi,\pi]$. As yet, we do not have any information on the ARR of a process whose spectral density has a zero of infinite order. For $q = 1$, $W(\theta) = \exp\{-|\theta|^{-\lambda}\}$, $0 < \lambda < 1$, provides a family of such densities.

2. Notation and Preliminaries.

Throughout this note for a $q \times q$ matrix $A = (a_{ij})$, $\mathrm{tr}A = \sum_{i=1}^{q} a_{ii}$, $A^* = (\bar{a}_{ji})$, det A stands for the determinant of A and A^{-1} for the inverse of A whenever it exists. For two $q \times q$ matrices A and B, $A \geq B$ means that $A - B \geq 0$, i.e. $A - B$ is a nonnegative definite matrix. Functions will be defined on $[-\pi,\pi]$ and we identify this interval with the unit circle in the complex plane in the natural way. Typical value of a function defined on $[-\pi,\pi]$ or on the unit circle will be denoted by $f(\theta)$ or $f(e^{i\theta})$. $\int$ stands for $\int_{-\pi}^{\pi}$ and dm for the normalized Lebesgue measure on $[-\pi,\pi]$, i.e. $dm(\theta) = \frac{d\theta}{2\pi}$. For $0 < P \leq \infty$, $L^P(H^P)$ denotes the usual Lebesgue (Hardy) space of functions on the unit circle, $L^P_{q\times q}(H^P_{q\times q})$ denotes the space of all $q \times q$ matrix-valued functions whose entries are in $L^P(H^P)$.

Let $(\Omega,B,\underline{P})$ be a probability space and $M = L^2_0(\Omega,B,\underline{P})$ the Hilbert space of all complex-valued random variables on Ω with zero expectation and finite variance. The inner product in M is given by $(x,y) = Ex\bar{y}$, x, $y \in M$. In the following, we introduce a few concepts which are needed in this study, for more discussion with proofs see [9, 16].

For an integer q, $1 \le q < \infty$, M^q denotes the cartesian product of M with itself q times, i.e. the set of all column vectors $X = (x_1, x_2, \ldots, x_q)$ with $x_i \in M$, $i = 1,2,\ldots,q$. M^q is endowed with a Gramian structure. For X and Y in M^q their Gramian is defined to be the $q \times q$ matrix $(X,Y) = [(x_i, y_j)]$. It is known that M^q is a Hilbert space under the inner product $((X,Y)) = tr(X,Y)$ and norm $\|X\| = \sqrt{((X,X))}$ provided the linear combinations are formed with constant $q \times q$ matrices as coefficients.

Let $\{X_n; n \in \mathbb{Z}\} \subset M^q$. $\{X_n; n \in \mathbb{Z}\}$ is said to be a q-variate stationary stochastic process (SSP) if the Gram matrix (X_m, X_n) depends only on $m-n$, for all integers m and n. Such a process has a spectral representation, i.e. for all integers n

$$(2.1) \qquad X_n = \int e^{-in\theta} dZ(\theta),$$

where $Z(.)$ is a countably additive orthogonally scattered M^q-valued measure [9].

The $q \times q$ nonnegative hermitian matrix-valued measure $F(.) = (Z(.), Z(.))$ is called the spectral measure of $\{X_n\}$. In case $F << m$, we say that $\{X_n\}$ has the spectral density $W = dF/dm$ on $[-\pi,\pi]$. It can be shown that $W(.)$ is a nonnegative hermitian matrix-valued function and for all integers m,n,

$$(2.2) \qquad (X_m, X_n) = \int e^{-i(m-n)\theta} W(\theta) dm(\theta).$$

For each SSP $\{X_n\}$, let $M_n^m(X) = \overline{sp}\{X_k; n \le k \le m\}$, $-\infty \le n \le m \le \infty$ where $\overline{sp}\{\ \}$ stands for the closed linear span of elements of $\{\ \}$ in the norm of M^q. The time domain of $\{X_n\}$ denoted by $M(X)$ is defined by $M(X) = M_{-\infty}^{\infty}(X)$. The spectral domain corresponding to the spectral density matrix W

is denoted by $L^2(W)$ and is defined by

$$L^2(W) = \{\Psi;\ \Psi \text{ a } q \times q \text{ matrix valued function with } \|\Psi\|_W^2 = \int \operatorname{tr}\Psi^*(\theta)W(\theta)\Psi(\theta)dm(\theta) < \infty\}.$$

It is known that $L^2(W)$ with inner product given by

$$((\Phi,\Psi))_W = \int \operatorname{tr}\, \Phi^*(\theta)W(\theta)\Psi(\theta)dm(\theta)$$

is a Hilbert space and, [9, Theorem 7.3], that the correspondence

$$T:\Psi \to \int \Psi(\theta)dZ(\theta) \tag{2.3}$$

is an isometric isomorphism on $L^2(W)$ onto $M(X)$. T is called the Kolmogorov isomorphism between the spectral and time domains. It follows from (2.2) and (2.3) that

$$\left(\sum_{j\in J} A_j X_j,\ \sum_{k\in K} B_k X_k \right) = \left(\sum_{j\in J} A_j e^{-ij\theta},\ \sum_{k\in K} B_k e^{-ik\theta} \right)_W , \tag{2.4}$$

where J and K are finite sets of integers and A_j, B_j are $q \times q$ constant matrices. An important problem in the prediction theory of an SSP $\{X_n\}$ is to find a (meaningful) series representation in the time domain which corresponds to a given Ψ in the spectral domain $L^2(W)$. The identity (2.4) shows that this can be done very easily when Ψ is a polynomial of finite degree. However, this problem is very complicated when Ψ is not of this form.

The prediction problem of an SSP $\{X_n\}$ with the spectral density W can be stated as determining the matrices A_k (in terms of W), $0 < k < \infty$ such that for a fixed $\nu \geq 1$, the linear least squares predictor of $X_{n+\nu}$, i.e.

$\hat{X}_{n+\nu} = (X_{n+\nu} | M^n_{-\infty}(X))$ can be written as

(2.5) $$\hat{X}_{n+\nu} = \lim_{N\to\infty} \sum_{k=0}^{N} A_k X_{n-k} = \sum_{k=0}^{\infty} A_k X_{n-k},$$

where for a vector $X \in M^q$ and a subspace S of M^q, $(X \mid S)$ denotes the orthogonal projection of X on the subspace S of M^q.

Let $\varepsilon_n = X_n - (X_n | M^{n-1}_{-\infty}(X))$, $n \in \mathbb{Z}$. The stochastic process $\{\varepsilon_n; n \in \mathbb{Z}\}$ is called the <u>innovation</u> process of $\{X_n\}$. It is known that $(\varepsilon_m, \varepsilon_n) = \delta_{mn} G$, where $G = (\varepsilon_0, \varepsilon_0)$ is the prediction error matrix for lag 1. The SSP $\{X_n\}$ is said to be of <u>full rank</u> if the matrix G is nonsingular. Throughout this note we assume that $\{X_n\}$ is a purely nondeterministic full rank process. This is equivalent to assuming that $\{X_n\}$ has a spectral density W such that log log det $W \in L^1$ and $W = \Phi\Phi^*$, where $\Phi \in H^2_{q\times q}$ is an outer function with $\Phi(0) = G^{\frac{1}{2}}$. We refer to Φ as the <u>generating function</u> of the process $\{X_n\}$. Since $\{X_n\}$ has full rank, we can define a process $\{Y_n\}$, called the normalized innovation process of $\{X_n\}$, by $Y_n = G^{-\frac{1}{2}}\varepsilon_n$. Then $(Y_m, Y_n) = \delta_{mn} I$. By using the Wold's decomposition [9] we get an infinite order one-sided moving average representation for $\{X_n\}$:

(2.6) $$X_n = \sum_{k=0}^{\infty} C_k Y_{n-k},$$

$$\Phi(\theta) = \sum_{k=0}^{\infty} C_k e^{ik\theta}, \quad (C_0 = G^{\frac{1}{2}}).$$

As a measure of the angle between the past-present and future subspaces of the process $\{X_n\}$ we take its "cosine" defined by

$$\rho(W) = \sup\{|((Y,Z))|: Y \in M^0_{-\infty}(X),\ Z \in M_1^{\infty}(X)$$

$$\text{and } \|Y\| \le 1,\ \|Z\| \le 1\}.$$

It is clear that $0 \leq \rho(W) \leq 1$. The past-present and future are said to be at <u>positive angle</u> if $\rho(W) < 1$.

3. Mean Convergence of the Fourier Series of Φ^{-1} in $L^2(W)$.

As noted in Section 1, mean convergence of the Fourier series of Φ^{-1} in $L^2(W)$ emerges as the only necessary and sufficient condition for the autoregressive representation of a purely nondeterministic full rank process $\{X_n\}$ with the spectral density W. This section is devoted to finding useful sufficient conditions on W which guarantee the mean convergence of the Fourier series of Φ^{-1} and many more functions in $L^2(W)$.

We would like to note that for a general density W, Φ^{-1} is not necessarily in $L^1_{q\times q}$ and therefore the Fourier coefficients of Φ^{-1} are not well-defined. For the next theorem we need the natural assumption on W, that is W is such that $L^2(W) \subset L^1_{q\times q}$. Under this assumption the Fourier coefficients of every function in $L^2(W)$ is well-defined. It can be shown [10,13,14] that $L^2(W) \subset L^1_{q\times q}$ if $W^{-1} \in L^1_{q\times q}$, and only if $(\det W)^{-1/2q} \in L^1$. Thus this assumption is weaker than $\rho(W) < 1$.

The next theorem which is a multivariate extension of a deep theorem of Helson and Szegö [6, p. 131] provides a necessary and sufficient condition for the mean convergence of the Fourier series of every function in $L^2(W)$, c.f. [10, 13, 14].

<u>Theorem 3.1</u>. Let W be a $q \times q$ matrix-valued density function. Then $\rho(W) < 1$, if and only if $L^2(W) \subset L^1_{q\times q}$ and the Fourier series of any Ψ in $L^2(W)$ converges to Ψ in the norm of $L^2(W)$.

Next we find a weaker sufficient condition on W which guarantees the convergence of the Fourier series of every function in a small subclass S of $L^2(W)$. For a density W, we define $S = \{\Psi \in L^2(W);\ \Psi W \Psi^* \in L^\infty_{q\times q}\}$. Note that this class has Φ^{-1} as one of its elements. In the following theorem we show that if $W \in A \otimes M$, then the Fourier series of every Ψ in S converges to Ψ in the norm of $L^2(W)$. This theorem is a matricial extension of a similar univariate result due to Bloomfield [3].

Theorem 3.2. Let W be a $q \times q$ matrix-valued density function and $S = \{\Psi \in L^2(W); \Psi W \Psi^* \in L^\infty_{q \times q}\}$. If $W \in A \otimes M$, then the Fourier series of every function $\Psi \in S$ converges to Ψ in the norm of $L^2(W)$.

Proof. Let ω_1 and ω_q denote the smallest and largest eigenvalue of W_2. Then for $W \in A \otimes M$ we have

(3.1) $$\omega_1 W_1 \leq W \leq \omega_q W_1 \qquad \text{a.e.(Leb.)},$$

and for $\Psi \in S$ we have

$$\int \mathrm{tr}\Psi\, W_1 \Psi^* dm \leq \int \omega_1^{-1}\, \mathrm{tr}\Psi\, W\Psi^* dm \leq \|\mathrm{tr}\Psi\, W\Psi^*\|_\infty \int \omega_1^{-1}\, dm < \infty,$$

which proves that $\Psi \in S$ implies $\Psi \in L^2(W_1)$.

Since $\Psi \in L^2(W_1)$ and $\rho(W_1) < 1$, it follows from Theorem 3.1 (and the argument preceeding it) that the Fourier series of every $\Psi \in S$ converges to Ψ in the norm of $L^2(W_1)$, i.e. with S_n^Ψ denoting the symmetric n-th partial sum of the Fourier series of Ψ we have

(3.2) $$\|\Psi - S_n^\Psi\|_{W_1} \to 0 \quad \text{as} \quad n \to \infty.$$

To finish the proof we need to show that $\|\Psi - S_n^\Psi\|_W \to 0$ as $n \to \infty$. But this the consequence of (3.2) and the inequality

$$\|\Psi - S_n^\Psi\|_W^2 \leq \|\omega_q\|_\infty \|\Psi - S_n^\Psi\|_{W_1}^2$$

which follows from (3.1). Q.E.D.

As mentioned in Section 1, a density in $A \otimes M$ can not have a zero of order 2 or more. Thus Theorem 3.2 does not provide any information in regard to autoregressive representation of such processes. Also it was noted that such processes can not have mean convergent autoregressive representation. Therefore it is natural to replace the requirement of mean convergence of the series for the autoregressive representation by a weaker mode of convergence for the series, say, summability. In the following, first we study the Abel summability of the series involved. For this we need some notation.

For a density W we define $H^2(W) = \overline{sp}\{e^{in\theta}I;\ n \geq 0\}$ in $L^2(W)$. Note that $\Phi^{-1} \varepsilon H^2(W)$. Also we would like to note that for a general W, Φ^{-1} and other elements of $H^2(W)$ are note necessarily in $L^1_{q\times q}$ and therefore their Fourier coefficients are not well-defined. By an argument similar to that of Rosenblum [15, p.41] one can identify the elements of $H^2(W)$ with analytic matrix-valued functions, and thus here by the Fourier coefficients of elements of $H^2(W)$ we actually mean the Taylor coefficients of these analytic functions.

Let $P_r(.)$ denote the Poisson kernel, i.e. $P_r(\theta) = \frac{1 - r^2}{1-2r\cos\theta+r^2}$, $0 < r < 1$ and $-\pi < \theta \leq \pi$. For a function $\Psi \varepsilon H^2(W)$ with Fourier (Taylor) coefficients $\{\Psi_k\}_{k=0}^{\infty}$ the convolution of P_r and Ψ is defined (and denoted) by

$$(3.3) \qquad \Psi_r(e^{i\theta}) = \Psi(re^{i\theta}) = (P_r * \Psi)(e^{i\theta}) = \sum_{k=0}^{\infty} \Psi_k r^k e^{ik\theta}.$$

We say that the Fourier (Taylor) series of a function $\Psi \varepsilon H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$, if and only if

$$(3.4) \qquad \lim_{r\to 1^-} \|\Psi_r - \Psi\|_W = 0.$$

It follows from the isomorphism between the time and spectral domains that the autoregressive representation of $\{X_n\}$ is mean Abel summable, if and only if the Fourier (Taylor) series of Φ^{-1} is Abel summable to Φ^{-1} in the norm of $L^2(W)$.

Next, following the pattern of Theorems 3.1 and 3.2, we find a necessary and sufficient condition on W for the Abel summability of the Fourier (Taylor) series of every function in $H^2(W)$. For this the matrix-valued function Q defined (in terms of Φ) by

$$Q(\theta) = Q(r, \theta, \Phi) = \Phi^{-1}(re^{i\theta})\Phi(e^{i\theta}), \quad -\pi < \theta \le \pi , \tag{3.5}$$

plays an important role. The following theorem is actually a matricial extension of some of the (univariate) results due to Rosenblum [15, Theorem 1(ii)].

Theorem 3.3. Let $W = \Phi\Phi^*$ be a $q \times q$ matrix-valued density function with $\log \det W \in L^1$. Then the following statements are equivalent.

(a) The Fourier (Taylor) series of every function Ψ in $H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$.

(b) There exists a constant K_1, $0 < K_1 < \infty$, such that for all functions $\Psi \in H^2(W)$ we have

$$\| P_r * \Psi \|_W \le K_1 \| \Psi \|_W, \quad 0 < r < 1. \tag{3.6}$$

(c) There exists a constant K_2, $0 < K_2 < \infty$, such that

$$(P_r * \operatorname{tr} QQ^*)(\theta) \le K_2, \quad 0 < r < 1 \text{ and } -\pi < \theta \le \pi. \tag{3.7}$$

Proof. (a) $\Longrightarrow$ (b) follows from the uniform boundedness principle.

(b) $\Longrightarrow$ (a) follows from an argument similar to that given in Rosenblum [15, pp. 32-33].

To prove that (b) implies (c), we note that for each $0 < r < 1$ and $-\pi < x \leq \pi$ the function $\Psi(\theta) = \Psi(\theta, r, x) = (1 - re^{i(\theta-x)})^{-1} \Phi^{-1}(\theta)$ is in $H^2(W)$. This is a consequence of the closure theorem for $H^2_{q\times q}$, cf. [8, p. 288]. By using the simple inequality $4(1-r^2)|1-r^2e^{i\theta}|^{-2} \geq P_r(\theta)$, and applying (3.6) to this function Ψ we get (3.7):

$$\int_{-\pi}^{\pi} P_r(\theta-x)\ \mathrm{tr}\Phi^{-1}(re^{i\theta})\ \Phi(\theta)\ \Phi^*(\theta)\ \Phi^*(re^{i\theta})dm(\theta)$$

$$\leq 4(1-r^2) \int_{-\pi}^{\pi} |1-r^2 e^{i(\theta-x)}|^{-2}\ \mathrm{tr}\ \Phi^{-1}(re^{i\theta})W(\theta)\ \Phi^*(re^{i\theta})dm(\theta)$$

$$\leq 4(1-r^2)\ K_1 \int_{-\pi}^{\pi} |1-re^{i(\theta-x)}|^{-2}\ \mathrm{tr}\ \Phi^{-1}(\theta)W(\theta)\ \Phi^{*-1}(re^{i\theta})dm(\theta)$$

$$= 4q\ K_1 \int_{-\pi}^{\pi} P_r(\theta-x)\ dm(\theta) = 4q\ K_1.$$

To prove (c) implies (b) we note that for any $\Psi \in H^2(W)$ [using the Cauchy-Schwartz inequality, Fubini's theorem and (3.7)] that

$$||P_r * \Psi||_W = \int_{-\pi}^{\pi} \mathrm{tr}\ \Psi(re^{i\theta})\Phi(\theta)\ \Phi^*(\theta)\ \Psi^*(re^{i\theta})dm(\theta)$$

$$= \int_{-\pi}^{\pi} \mathrm{tr}\ \Psi(re^{i\theta})\Phi(re^{i\theta})Q(\theta)Q^*(\theta)\ [\Psi(re^{i\theta})\Phi(re^{i\theta})]^* dm(\theta)$$

$$\leq \int_{-\pi}^{\pi} \mathrm{tr}\ QQ^* \cdot\ \mathrm{tr}\ \Psi(re^{i\theta})\Phi(re^{i\theta})\ [\Psi(re^{i\theta})\Phi(re^{i\theta})]^* dm(\theta)$$

$$= \sum_{k,\ell=1}^{q} \int_{-\pi}^{\pi} \mathrm{tr}\ QQ^*\ |\sum_{j=1}^{q} \Psi_{kj}(re^{i\theta})\Phi_{jk}(re^{i\theta})|^2 dm(\theta)$$

$$= \sum_{k,\ell=1}^{q} \int_{-\pi}^{\pi} \mathrm{tr}\ QQ^*\ |\int_{-\pi}^{\pi} P_r(\theta-x)(\sum_{j=1}^{q} \Psi_{kj}\ \Phi_{j\ell})(x)dm(x)|^2 dm(\theta)$$

$$\leq \sum_{k,\ell=1}^{q} \int_{-\pi}^{\pi} \mathrm{tr}\ QQ^*\ (\int_{-\pi}^{\pi} P_r(\theta-x)\ (\sum_{j=1}^{q} \Psi_{kj}\ \Phi_{j\ell})(x)|^2 dm(x))dm(\theta)$$

$$= \int_{-\pi}^{\pi} \mathrm{tr}\ \Psi\ W\ \Psi^*\ (\int_{-\pi}^{\pi} P_r(x-\theta)\mathrm{tr}\ QQ^*(\theta)dm(\theta))dm(x)$$

$$\leq K_2\ ||\Psi||_W\ .$$

Q.E.D.

Although Theorem 3.3 provides two equivalent necessary and sufficient conditions for the Abel summability of the Fourier (Taylor) series of every function in $H^2(W)$, these conditions are hard to apply and are not explicit in terms of the components of W. In the next theorem by using Theorems 3.1, 3.2 and 3.3 we provide sufficient conditions for the Abel summability of the Fourier (Taylor) series of all functions in $H^2(W)$ and in particular Φ^{-1}, which are easy to apply and more explicit in terms of the components of W.

Theorem 3.4. Let W be a $q \times q$ matrix-valued density function with $\log \det W \in L^1$ and P be a complex-valued trigonometric polynomial of some degree n.

(a) If $W \in A$, then the Fourier (Taylor) series of every function $\Psi \in H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$.

(b) Let $W' \in A$. If $W = |P|^2 W'$, then the Fourier (Taylor) series of every function $\Psi \in H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$.

(c) Let $S = \{\Psi \in H^2(W);\ \Psi W \Psi^* \in L^\infty_{q\times q}\}$ and $W' \in A \otimes M$. If $W = |P|^2 W'$, then the Fourier (Taylor) series of every function $\Psi \in S$ is Abel summable to Ψ in the norm of $L^2(W)$.

Proof. In view of Theorems 3.1 and 3.3, proofs of (a) and (b) are the same as the proof of Lemma 6 in [12]. (c) follows from (b) by using the method of proof of Theorem 3.2 and replacing S_n^Ψ by Ψ_r. Q.E.D.

Now, we turn to the problem of Cesaro summability of the Fourier (Taylor) series of functions in $H^2(W)$. Let $\Psi \in H^2(W)$ with Fourier (Taylor) coefficients $\{\Psi_k\}_0^\infty$ and partial sums $S_n(\theta) = \sum_{k=0}^{n} \Psi_k e^{ik\theta}$. For $\alpha > 0$, we say that the Fourier (Taylor) series of Ψ is $(C, \alpha)_-$ summable to Ψ in the norm of $L^2(W)$ if

$$(3.8)\qquad \lim_{n\to\infty} \left\| \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} \binom{n+\alpha}{n}^{-1} S_k - \Psi \right\|_W = 0.$$

For scalar sequences it is well-known that the strength of $(C-\alpha)$ methods increases with α. However, there are series which are Abel summable but not (C,α)-summable for any $\alpha > 0$, cf. [5, p. 108]. Because of this and in view of the importance of relations like (3.8) in prediction of $\{X_n\}$, cf. Section 1, we consider the stronger method of compounded Cesáro summability method: Let $\{\alpha_n\}_0^\infty$ be a (fixed) monotone increasing sequence of positive numbers. We say that the Fourier (Taylor) series of $\Psi \in H^2(W)$ is compounded Cesáro summable to Ψ if

$$\lim_{n\to\infty} \left\| \sum_{k=0}^{n} \binom{n-k+\alpha_n-1}{n-k}\binom{n+\alpha_n}{n}^{-1} S_k - \Psi \right\|_W = 0.$$

It is known [1] that the compounded Cesáro summability method is regular, if and only if $\lim\limits_{n\to\infty} \frac{\alpha_n}{n} = 0$. For more information on the subject of summability and the definition of undefined terms the reader may refer to [5].

The next theorem establishes analogue of Theorem 3.4 for the compounded Cesáro summability.

Theorem 3.5. Let W be a $q \times q$ matrix-valued density function with $\log \det W \in L^1$, P be a complex-valued trigonometric polynomial of some degree n and $\{\alpha_n\}$ a (fixed) monotone increasing sequence of positive numbers with $\lim\limits_{n\to\infty} \frac{\alpha_n}{n} = 0$.

(a) Let $W' \in A$. If $W = |P|^2 W'$, then the Fourier (Taylor) series of every function $\Psi \in H^2(W)$ is compounded Cesáro summable to Ψ in the norm of $L^2(W)$.

(b) Let $W' \in A \otimes M$ and S be as in part (c) of Theorem 3.4. If $W = |P|^2 W'$, then the Fourier (Taylor) series of every $\Psi \in S$ is compounded Cesáro summable to Ψ in the norm of $L^2(W)$.

Proof. We prove only part (a) since the proof of part (b) is the same as the proof of Theorem 3.4(c).

To prove (a) first we show that every $\Psi \in H^2(W)$ has radial limits a.e. (Leb.) Let ϕ and Φ_1 be the optimal factors of $|P|^2$ and W', respectively. Then Φ the optimal factor of $W = |P|^2 W'$ is given by $\phi\Phi_1$. Since $W' \in A$ it follows that $\Phi_1^{-1} \in H^{\frac{1}{2}}_{q\times q}$, and since ϕ is an analytic polynomial of finite degree with no zeros in the open unit disc we have that $\phi^{-1} \in H^{\delta}$, $0 < \delta < \frac{1}{2}$. Thus $\Phi^{-1} \in H^{p_1}_{q\times q}$, $\frac{1}{p_1} = 2 + \frac{1}{\delta}$. Now, we note that every $\Psi \in H^2(W)$ has a representation of the form $\Psi = h\,\Phi^{-1}$, where $h \in H^2_{q\times q}$, and this entails that $\Psi \in H^p_{q\times q}$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{p_1}$. Therefore, every $\Psi \in H^2(W)$ belongs to $H^p_{q\times q}$, for some $p > 0$, which implies that Ψ has radial limits a.e. The rest of the proof follows from Theorem 3.4(b) and adopting the method of proof of Theorems 6.1 and 6.3 [1]. Q.E.D.

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