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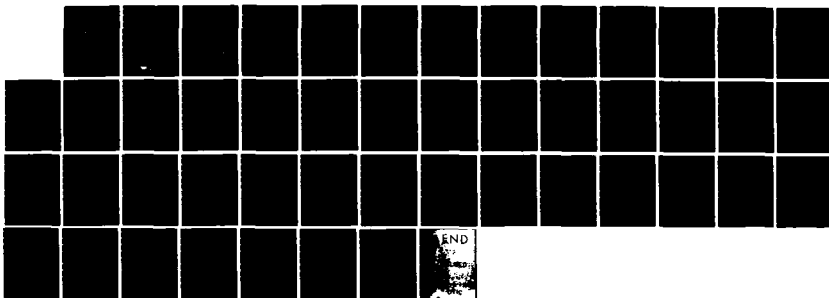
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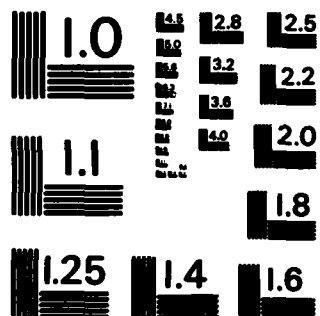
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Gérard Cornuéjols*

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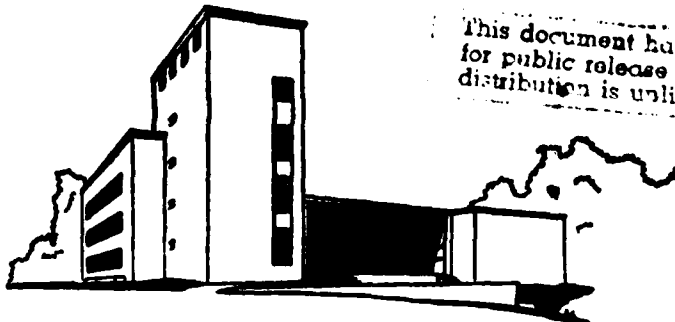
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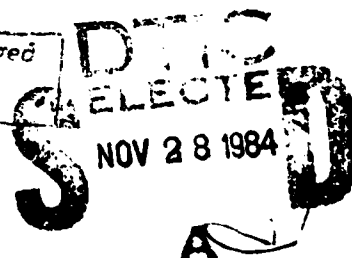
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Management Science Research Report No. MSRR 510

THE TRAVELING SALESMAN PROBLEM ON A GRAPH
AND SOME RELATED INTEGER POLYHEDRA

by

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Abstract: Given a graph $G = (N, E)$ and a ^{certain} length function $l : E \rightarrow \mathbb{R}$, the Graphical Traveling Salesman Problem is that of finding a minimum length cycle going at least once through each node of G . This formulation has advantages over the traditional formulation where each node must be visited exactly once. ^{The authors} ~~We~~ give some facet inducing inequalities of the convex hull of the solutions to that problem. Some related integer polyhedra are also investigated. Finally, an efficient algorithm is given when G is a series-parallel graph.



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1. Introduction

Consider a graph $G = (N, E)$ and a function $l: E \rightarrow \mathbb{R}$ which associates the length $l(e)$ to each edge $e \in E$. The classical Traveling Salesman Problem, denoted by TSP, is that of finding a Hamilton cycle (N, H) of G such that $l(H) \equiv \sum_{e \in H} l(e)$ is minimum. (A Hamilton cycle of G is a cycle going exactly once through each node of G .) The Traveling Salesman Problem derives its name from the following interpretation: the nodes of G represent cities that must be visited by a salesman and the edges represent roads or other transportation links connecting the cities. One of the cities is the traveling salesman's hometown from which he starts his tour and to which he must return.

Two difficulties arise in stating the TSP as above. First, the graph G may not be Hamiltonian (i.e., G may not have a Hamilton cycle.) Second, even when G is Hamiltonian, the shortest way to visit all the nodes of G may not be to follow a Hamilton cycle. Instead, it may be shorter to go through some nodes more than once and/or use some edges more than once.

The traditional way to overcome these difficulties is to transform G into a complete graph $K = (N, F)$ on the same node set. The length function $l: F \rightarrow \mathbb{R}$ is defined as follows: for every $e \in F$, $l(e)$ is the length of the shortest path of G joining the endpoints of e . Solving the TSP on K instead of G clearly resolves the two difficulties just mentioned. Most of the existing literature on the TSP assumes an underlying complete graph.

However, the transformation of G into K has two drawbacks of its own. In most solution techniques a variable is associated with each edge of the graph. Therefore, the TSP on K requires $(|N|-1)|N|/2$ variables even when the original graph is sparse — which is often the case in applications (Many applications involve ^{planar} graphs or graphs with small thickness). The

second drawback is that the original problem on G may be easier to solve than the TSP on a complete graph. For example, Ratliff and Rosenthal [7] present a linear time algorithm for a version of the TSP that arises in the context of order picking in a rectangle warehouse. Their algorithm exploits the structure of the underlying graph G . We extend their results in Section 5. Another class of graphs for which the TSP can be solved in linear time is given in Cornuejols, Naddef and Pulleyblank [2]. For these reasons we prefer to avoid using the complete graph K . We propose a different way to overcome the deficiencies associated with the classical formulation of the TSP. Our approach is to introduce a new version of the TSP which we call the Graphical Traveling Salesman Problem. This formulation has also been used successfully by Fleischmann [4].

A tour of a connected graph G is a cycle going at least once through each node of G . (Here a cycle may use the same node or the same edge more

than once.) The length of a tour $T=(v_1, e_1, \dots, v_k, e_k, v_1)$ is $l(T) = \sum_{i=1}^k l(e_i)$.

The Graphical Traveling Salesman Problem, denoted by GTSP, consists in finding a tour of G whose length is minimum. Of course GTSP is NP-hard, since, given a graph G , the solution of GTSP with the length function $l(e) = 1$ for all $e \in E$, would show whether G is Hamiltonian, a known NP-complete problem.

A graph is Eulerian if it is connected and each of its nodes is incident with an even number of edges. It is well known and easy to prove that if a graph is Eulerian, then it contains a tour using each edge exactly once [1]. Conversely, given a cycle T , the graph H induced by the edges of T duplicated as many times as they are used in T , is an Eulerian graph. If T is a tour of G , then H spans all the nodes of G . In other

words, the tours of G correspond to the spanning Eulerian graphs obtained from the graph G by removing some edges and duplicating others.

If an edge of G has a negative length, then one can obtain tours of length as small as wanted by using this edge an indefinite number of times. In other words, there is no finite optimum solution. In the remainder we assume that all edge lengths are nonnegative. With this assumption it can be shown that there is an optimum solution using any edge at most twice. (Let T be some tour of G where some edge e is used three times or more. Consider the edge set obtained by taking the edges of T duplicated as many times as they are used in T and by removing two copies of e . The graph induced by this edge set is Eulerian and spanning. So it can be traversed by a tour T' . Clearly, $l(T') = l(T) - 2l(e) < l(T)$.)

To each tour of G we associate an integral vector $x = (x_e : e \in E)$ where x_e is the number of times that edge e occurs in the tour. In terms of x , the length of a tour is simply $\sum_{e \in E} l(e)x_e$. Note that there is a one-to-one correspondence between the vectors x associated with tours and the spanning Eulerian graphs defined two paragraphs earlier. In general, however, the same vector x can be associated with several tours. (For example, if G is a star and $x_e = 2$ for all $e \in E$, the tours associated with x can visit the branches of the star in any order.) Nonetheless, we will call the vector x associated with a tour, a tour itself.

In this paper, for $U \subseteq E$, $x(U)$ denotes $\sum_{e \in U} x_e$. For $S \subseteq N$, $\gamma(S)$ denotes the set of edges with both ends in S and $\delta(S)$ those with exactly one end in S . Also, $\delta(\{v\})$ is abbreviated by $\delta(v)$. Given a finite set J , R^J denotes the set of vectors $x = (x_j : j \in J)$ whose coordinates are real valued and indexed by the elements of J .

With this notation, the tours of G are those vectors $x \in \mathbb{R}^E$ which satisfy

- (1.1) $x_e \geq 0$ and integer for all $e \in E$,
- (1.2) $x(\delta(v))$ is a positive even integer for all $v \in N$,
- (1.3) the graph induced by the edges such that $x_e > 0$ is connected.

Conditions (1.2) and (1.3) follow from the fact that the graph obtained from G by making x_e copies of edge e must be Eulerian and spanning. An equivalent characterization of tours is given next.

An edge cutset $U \subseteq E$ of G is a set of edges such that $U = \delta(S) = \delta(N-S)$ for some nonempty $S \subset N$. The sets S and $N-S$ are called the shores of the edge cutset U . A vector $x \in \mathbb{R}^E$ is a tour if and only if it satisfies conditions (1.1), (1.2), and

- (1.4) $x(U) \geq 2$ for every edge cutset U of G .

This condition follows from the fact that a tour corresponds to a spanning cycle of G and cycles have even cardinality intersections with every edge cutset.

The convex hull of the tours of G will be denoted by $GTSP(G)$. Note that this polyhedron is not bounded and that the classical traveling salesman polytope $TSP(G)$ -- namely the convex hull ^{of the} incidence vectors of the Hamiltonian cycles of G -- is a face of the polyhedron $GTSP(G)$. In fact $TSP(G) = GTSP(G) \cap \{x \in \mathbb{R}^E : x(E) = |N|\}$.

In Section 2 we show that, when G is connected, the polyhedron $GTSP(G)$ is full-dimensional, i.e., it has dimension $|E|$. The inequalities (1.4) define facets (i.e., faces of dimension $|E|-1$) of this polyhedron if and only if the graphs induced by the shores of the edge cutset are both connected. These facets are in fact the subtour elimination inequalities which have become usual in the definition of $TSP(G)$.

The main result of Section 3 is a class of valid inequalities, called "path inequalities", for GTSP(G) which are shown to generalize the comb inequalities for TSP(G), see Grötschel and Padberg [6]. Other classes of facets are also presented in that section.

In Section 4 we introduce four polyhedra which are related to GTSP(G). In particular we give a full description of $P_1(G) = \text{conv}\{x \in \mathbb{R}^E : (1.1) \text{ and } (1.2) \text{ hold}\}$ and $P_2(G) = \text{conv}\{x \in \mathbb{R}^E : (1.1) \text{ and } (1.3) \text{ hold}\}$. The polyhedron $P_3(G) = \text{conv}\{x \in \mathbb{R}^E : (1.1) \text{ and } (1.4) \text{ hold}\}$ contains TSP(G) as a face and is also studied.

In Section 5 we introduce a generalization of the graphical traveling salesman problem. In addition to the graph G and the length function l, we are given a subset $V \subseteq N$ of the nodes. A Steiner tour is a cycle going at least once through each node of V. However, the cycle is not required to pass through the nodes of $N-V$. The Steiner Traveling Salesman Problem, denoted by STSP, consists in finding a Steiner tour of minimum length. Of course GTSP is the special case of STSP where $V=N$. We give an efficient algorithm to solve STSP in series-parallel graphs.

2. The Graphical Traveling Salesman Polyhedron

Full-dimensional polyhedra have the desirable property that a minimal set of defining inequalities is unique up to scaling any inequality by a positive constant. Moreover, in such a set, each inequality induces a facet. In other words, each facet has a unique description by a linear inequality, up to scaling by a constant. Of course, this is no longer true when the polyhedron is not full-dimensional. First, we show that the graphical traveling salesman polyhedron, GTSP(G), is full-dimensional if and only if G is a connected graph. Then we show that the convex hull of

the extreme points of $GTSP(G)$ is a full-dimensional polytope if and only if G is a connected bridgeless graph. (A bridge is an edge cutset of cardinality one.) Other results relating the connectivity of G to the TSP can be found in [5].

Theorem 2.1 If G is a connected graph, then $GTSP(G)$ is full-dimensional. Otherwise $GTSP(G)$ is empty.

Proof: If G is not connected, no tour exists. If G is connected, then consider any tour x and the $|E|$ tours $x + 2y^i$ where y^i is the unit vector such that $y_1^i = 1$ and $y_j^i = 0$ for $j \neq 1$. These $|E| + 1$ are affinely independent. ■

Theorem 2.2 Let G be a connected graph with k bridges. The convex hull of the extreme points of $GTSP(G)$ is a polytope of dimension $|E| - k$.

Proof: It follows from the remarks made in the introduction that every extreme point x of $GTSP(G)$ has components x_e which take the value 0, 1, or 2 for every $e \in E$. If e is a bridge, then $x_e = 2$ as a consequence of condition (1.4).

Denote by $P(G)$ the convex hull of the extreme points of $GTSP(G)$. We have just shown that, if G has k bridges, then $\dim P(G) \leq |E| - k$.

Conversely, we show that $\dim P(G) \geq |E| - k$ by induction on the number of edges of G . The property is true when $|E| = 1$. Now take a connected graph $G = (N, E)$. Let k be the number of bridges of G and let e be an edge such that the graph induced by the edge set $E - \{e\}$ is connected. We denote this graph by $G - \{e\}$.

If e is a bridge of G , then, by the choice of e , e is a pendent edge.

Therefore $G - \{e\}$ has $k-1$ bridges and, by the induction hypothesis, $P(G - \{e\})$ contains $|E| - k + 1$ affinely independent tours. All these tours can be extended to tours of $P(G)$ by adding the component $x_e = 2$. So $\dim P(G) \geq \dim P(G - \{e\}) = |E| - k$.

Now assume that e is not a bridge of G . The graph $G - \{e\}$ contains the k bridges of G and possibly p new bridges, say $e_1, \dots, e_p$. By induction, $\dim P(G - \{e\}) = (|E| - 1) - (k + p)$. Consider $|E| - (k + p)$ affinely independent tours of $P(G - \{e\})$. We can extend them to tours of $P(G)$ by simply adding the component $x_e = 0$. In addition, note that one of these tours, say $\tilde{x}$, can be chosen so that $\tilde{x}_i = 2$ for every edge i of a spanning tree of $G - \{e\}$, otherwise. We will construct $p+1$ new tours from $\tilde{x}$. Let P be a path of $G - \{e\}$ joining the endpoints of e and such that $\tilde{x}_i > 0$ for every edge i of P . Note that the p bridges $e_1, \dots, e_p$ belong to P . Define the tour x^* by $x_i^* = \tilde{x}_i$ for $i \notin P$, $x_i^* = \tilde{x}_i - 1$ for $i \in P$ and $x_e^* = 1$. In addition define the tours x^j , $j=1, \dots, p$, by $x_i^j = \tilde{x}_i$ for $i \neq e, e_j$ and $x_{e_j}^j = 0$, $x_e^j = 2$. So we have a total of $|E| - k + 1$ tours in $P(G)$. To show that they are affinely independent, let us subtract $\tilde{x}$ from the $|E| - k$ others. We get the matrix

$ E - (k+p) - 1$		$x^* x^1 x^2 \dots x^p$
$\left. \begin{matrix} e \\ e_1 \\ \vdots \\ e_p \end{matrix} \right\} E - p - 1$		$\begin{matrix} 1 & 2 & \dots & 2 \\ -1 & 2 & & \\ & -2 & & \\ & & \ddots & \\ -1 & & & -2 \end{matrix}$
	linearly independent columns	$\begin{matrix} -1 & & & \\ -1 & & & \\ -1 & & & \\ 0 & & & \\ 0 & & & \end{matrix}$

It is easy to check that the rank of this matrix is $|E|-k$. So $\dim P(G) \geq |E|-k$. ■

The rest of this section is devoted to identifying valid inequalities and facets that define the polyhedron $GTSP(G)$.

Theorem 2.3 The inequality $x_e \geq 0$ defines a facet of $GTSP(G)$ if and only if the edge e is not a bridge of G .

Proof: If e is a bridge of G , then $x_e \geq 2$ in every tour of G . Conversely, if e is not a bridge, then let x be a tour such that $x_e = 0$. (Such a tour exists since $G-\{e\}$ is connected.) The tour x and the $|E|-1$ tours $x+2y^i$, $i \neq e$, are affinely independent, where y^i is the unit vector such that $y^i_1 = 1$ and $y^i_j = 0$ for $j \neq i$. ■

For any node set $S \subseteq N$, we denote by $G(S)$ the subgraph of G induced by S .

Theorem 2.4 Let U be an edge cutset with shores S and $\bar{S}$. The valid inequality $x(U) \geq 2$ is a facet of $GTSP(G)$ if and only if the graphs $G(S)$ and $G(\bar{S})$ are both connected.

Proof: If, say, $G(S)$ is not connected then let $S_1 \subset S$ be a node set which induces a connected component of $G(S)$. Then the cutset $\delta(S_1)$ is strictly contained in U . Therefore the valid inequality $x(\delta(S_1)) \geq 2$ is stronger than $x(U) \geq 2$, showing that the latter inequality cannot produce a facet.

Conversely, assume that $G(S)$ and $G(\bar{S})$ are both connected. For each $u \in U$, let x^u be a tour such that $x^u_u = 2$ and $x^u_e = 0$ for every $e \in U-\{u\}$.

Let x^* be one of the $|U|$ tours just defined. Consider the tours $x^* + 2y^i$, for $i \in E-U$, where y^i is the unit vector such that $y^i_1 = 1$ and $y^i_j = 0$ for $j \neq 1$. The $|E|$ tours $\{x^u : u \in U\} \cup \{x^* + 2y^i : i \in E-U\}$ are affinely independent and they all satisfy $x(U) = 2$. ■

It would be interesting to characterize the graphs G for which the facets of Theorem 2.3 and 2.4 completely define the polyhedron $GTSP(G)$. A small graph for which these facets are not sufficient is given in Figure 1.

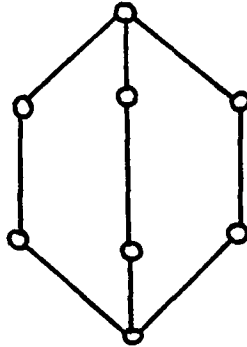


Figure 1

Note that the vector x such that $x_e = 1$ for every $e \in E$ satisfies the inequalities $x(U) \geq 2$ for every edge cutset U , yet it is not a tour. In fact this vector satisfies $x(E) = 9$ whereas it is easy to see that every tour satisfies the inequality $x(E) \geq 10$. Actually the inequality $x(E) \geq 10$ is a facet of the polyhedron $GTSP(G)$ for the graph G of Figure 1. We will prove this in the next section as a special case of a more general result. Paths with endnodes i and j are said to be internally node disjoint if they only meet in nodes i and j . When two distinct nodes of G are joined by an odd number of internally node disjoint paths of length 3 or more, we

will show in Section 3 how to generate a facet of $GTSP(G)$, called path inequality. These inequalities generalize the 3-star constraints defined by Fleischmann [4] and the comb inequalities of Grötschel and Padberg [6].

3. Facets of the Graphical Traveling Salesman Polyhedron

Let $G = (N, E)$ be a connected graph. Given two disjoint sets of nodes $N_1, N_2 \subset N$, we denote by (N_1, N_2) the set of edges of G with one end in N_1 and the other in N_2 . In the following, when we refer to two edges of (N_1, N_2) , these two edges may or may not be distinct.

A k-path configuration (see Figure 2) is defined by

- (3.1) an odd integer $k \geq 3$ and integers $n_i \geq 2$ for $i=1, \dots, k$,
- (3.2) a partition of the node set N into $\{A, Z, B_j^i\}$ for $i=1, \dots, k$ and $j=1, \dots, n_i$. For convenience, let $B_0^i \equiv A$ and $B_{n_i+1}^i \equiv Z$ for $i=1, \dots, k$.
- (3.3) The graphs $G(B_j^i)$ are connected for $i=1, \dots, k$ and $j=0, \dots, n_i+1$.
- (3.4) The edge set (B_j^i, B_{j+1}^i) is nonempty for every $i=1, \dots, k$ and $j=0, \dots, n_i$.

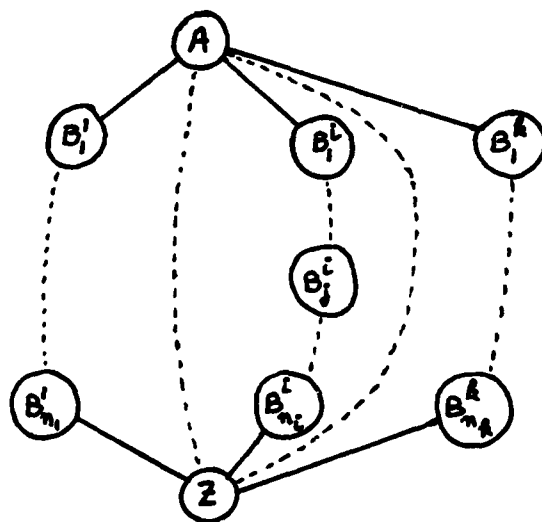


Figure 2. A k-path configuration.

The path inequality corresponding to this configuration is defined by

$$(3.5) \quad \sum_{e \in E} f_e x_e \geq 1 + \sum_{i=1}^k \frac{n_i+1}{n_i-1}$$

where

$$(3.6) \quad f_e = \begin{cases} 1 & \text{for } e \in (A, Z), \\ \frac{|j-p|}{n_i-1} & \text{for } e \in (B_j^i, B_p^i), i=1, \dots, k \text{ and } j \neq p \text{ such that} \\ & |j-p| \leq n_i, \\ \max\{\frac{p}{n_r-1} - \frac{j-2}{n_i-1}, \frac{j}{n_i-1} - \frac{p-2}{n_r-1}\} & \text{for } e \in (B_j^i, B_p^r), i \neq r, \\ & j=1, \dots, n_i \text{ and } p=1, \dots, n_r, \\ 0 & \text{otherwise.} \end{cases}$$

An interesting special case is obtained when $n_i = n \geq 2$ for all $i=1, \dots, k$. Such a configuration will be called n-regular. We can multiply the inequality (3.5) by $n-1$ to obtain integer coefficients. We get

$$(3.5') \quad \sum_{e \in E} g_e x_e \geq kn + n + k - 1$$

where

$$(3.6') \quad g_e = \begin{cases} n-1 & \text{for } e \in (A, Z), \\ |j-p| & \text{for } e \in (B_j^i, B_p^i), i=1, \dots, k \text{ and } j \neq p \text{ such that} \\ & |j-p| \leq n, \\ |j-p| + 2 & \text{for } e \in (B_j^i, B_p^r), i \neq r \text{ and } j, p=1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

When $k=3$ and $n=2$ this is exactly the 3-star constraint given by Fleischmann [4].

Let $a \equiv 1 + \sum_{i=1}^k (n_i+1)/(n_i-1)$. The skeleton of a k -path configuration

is the set of all edges in (B_j^i, B_{j+1}^i) for $i=1, \dots, k$ and $j=0, \dots, n_i$ and all edges with both ends in B_j^i for $i=1, \dots, k$ and $j=0, \dots, n_{i+1}$.

Theorem 3.1 Path inequalities are valid inequalities for GTSP(G).

Proof: Note that, since $f_e \geq 0$ for all $e \in E$, it suffices to prove the validity of the inequality $\sum_{e \in E} f_e x_e \geq a$ for a complete graph in order to prove it for all graphs. To show the validity of the inequality $\sum_{e \in E} f_e x_e \geq a$ we will show that, for the length function defined by the f_e 's, the length of any tour x is at least a .

First consider a minimum length tour x which only uses edges of the skeleton. Any such tour has the following form. For some $i^*=1, \dots, k$ and $j^*=0, \dots, n_{i^*}$, the tour x uses a single edge of $(B_j^{i^*}, B_{j+1}^{i^*})$ for $j \neq j^*$ and it uses two edges of $(B_j^{i^*}, B_{j+1}^{i^*})$ except for $j=j^*$ for which none is used. In other words, $k-1$ of the paths joining A to Z are used once whereas the last path is broken into two parts each used twice. The choice of the values of the f_e 's is such that, for any i^* and j^* , the length of x is a .

So, if a tour shorter than a exists, it must use some edges which are not in the skeleton. Let x be a tour shorter than a such that the number of edges not in the skeleton is minimum among all the tours which violate the inequality.

It is easy to see that x does not contain any edge from (B_j^i, B_p^i) for $i=1, \dots, k$ and $2 \leq p-j \leq n_i$, since one can replace any such edge by a path using one edge from each (B_t^i, B_{t+1}^i) , $t=j, \dots, p-1$. The resulting tour has the same length but uses less edges outside the skeleton.

Next we will show that x cannot contain an edge of (A, Z) . Let $e \in (B_j^1, B_p^r)$, $r \neq 1$. Without loss of generality $f_e = p/(n_r-1) - (j-2)/(n_1-1) \geq j/(n_1-1) - (p-2)/(n_r-1)$. Therefore, $(p-1)/(n_r-1) \geq (j-1)/(n_1-1)$ and $f_e \geq 1/(n_r-1) + 1/(n_1-1)$. This shows that the shortest edges with one node in the set B_j^1 belong to the skeleton. As a consequence, the shortest completion of an edge of (A, Z) into a tour of G uses a single edge of each (B_j^1, B_{j+1}^1) . But such a tour satisfies (3.5). So x does not contain an edge of (A, Z) .

Now we use the fact that k is odd. Since x does not contain any edge from (A, B_j^1) for $i=1, \dots, k$ and $j=2, \dots, n_1+1$, there must exist some $s=1, \dots, k$ such that either 0 or 2 edges of (A, B_1^s) belong to the tour x .

First assume that x does not contain any edge from (B_j^1, B_p^s) where s is the index defined above, $1 \leq i \leq k$, $j=1, \dots, n_1$ and $p=1, \dots, n_s$. Then x contains two edges of each (B_t^s, B_{t+1}^s) , $t=0, \dots, n_s$, except perhaps for one value of t . But then the tour obtained from x by taking the same edges outside the s^{th} path, taking one edge of each (B_t^s, B_{t+1}^s) , $t=0, \dots, n_s$, and one edge from (A, Z) yields a tour of length at most that of x , which we just saw is impossible.

So x must contain an edge from (B_j^1, B_p^s) for some $i \neq s$, $1 \leq i \leq k$, $j=1, \dots, n_1$ and $p=1, \dots, n_s$. Among all such edges, let e be one corresponding to the smallest value of p . By the choice of p and s , the tour contains two edges of (B_t^s, B_{t+1}^s) for every $t=0, \dots, p-1$ except perhaps for one value of t . Since $f_e \geq j/(n_1-1) - (p-2)/(n_s-1)$, the tour obtained from x by deleting e , replacing the edges of (B_t^s, B_{t+1}^s) , $t=0, \dots, p-1$, by one edge from each (B_t^s, B_{t+1}^s) , $t=0, \dots, p-1$, and finally adding one edge from each (B_t^1, B_{t+1}^1) , $t=0, \dots, j-1$, yields a tour of length at most that of x and with one less edge outside the skeleton, which contradicts the minimality of x .

This completes the proof. ■

Theorem 3.2 Path inequalities define facets of GTSP(G).

Proof: Let $\sum_{e \in E} f_e x_e \geq a$ be a path inequality (which we will refer to as inequality 1 in this proof). Assume it does not define a facet. Then there exists an inequality, say $\sum_{e \in E} c_e x_e \geq a$ (inequality 2), such that the face defined by the first inequality is contained in the facet defined by the second. Our aim is to show that $c_e = f_e$ for all $e \in E$.

First we show that $c_e = 0$ if e has both ends in the same set B_j^i for $i=1, \dots, k$ and $j=0, \dots, n_{i+1}$. Let x be a tour which satisfies inequality 1 with equality. (Such tours exist as we have seen in the proof of Theorem 3.1.) The tour x must also satisfy inequality 2 with equality. Now consider $x + 2y^e$ where y^e is the unit vector such that $y_t^e = 1$ and $y_t^e = 0$ for $t \neq e$. Since $f_e = 0$, this tour satisfies inequality 1 with equality. Therefore it must also satisfy inequality 2 with equality. Since both x and $x + 2y^e$ satisfy inequality 2 with equality, we must have $c_e = 0$.

Next we show that c_e has the same value for all edges in (B_j^i, B_{j+1}^i) and then that this value does not depend on j . Let x be a tour which satisfies inequality 1 with equality and which only uses edges of the skeleton. As pointed out in the proof of Theorem 3.1, there exists such an x which only uses one edge of (B_j^i, B_{j+1}^i) , say edge e_1 . Modify x so that it uses another edge of (B_j^i, B_{j+1}^i) , say edge e_2 , instead of e_1 . In order to still have a tour we may have to change x within B_j^i and B_{j+1}^i , but we can keep x unchanged anywhere else. Since both tours satisfy inequality 1 (and therefore inequality 2) with equality we must have $c_{e_1} = c_{e_2}$. Now

let x be a tour of the skeleton such that one edge of (B_p^r, B_{p+1}^r) is used

for all $r \neq i$ and $p=0, \dots, n_r$, and two edges of every (B_j^i, B_{j+1}^i) are used except for one index $j=j^*$. Any tour obtained from x by just changing the value of j^* satisfies both inequalities with equality which proves that there is a constant c^i such that $c_e = c^i$ for all $e \in (B_j^i, B_{j+1}^i)$, $j=0, \dots, n_i$. The tours just defined satisfy both inequalities 1 and 2 with equality for any value of $i=1, \dots, k$. Hence, the constants c^i must satisfy a system of k equalities whose unique solution is $c^i = 1/(n_i-1)$ for $i=1, \dots, k$.

Now let e^* be an edge not in the skeleton, and let x be a tour using e^* and edges of the skeleton. In addition, assume that x satisfies inequality 1, and hence inequality 2, with equality. Such a tour exists by our choice of the coefficients f_e . Since $c_e = f_e$ for all edges in the skeleton, we must also have $c_{e^*} = f_{e^*}$, which completes the proof. ■

We have pointed out earlier that $TSP(G)$ — the convex hull of the incidence vectors of the Hamilton cycles of G — is a face of the polyhedron $GTSP(G)$. We will show how the path inequalities relate to well-known facets of $TSP(G)$.

A comb is defined by a set $W_0 \subset N$ called the handle and k disjoint subsets $W_1, \dots, W_k \subset N$, k odd, called the teeth, each of these subsets having a nonempty intersection with W_0 . The comb inequality associated with the comb $(W_0, W_1, \dots, W_k)$ is

$$\sum_{i=0}^k x(\gamma(W_i)) \leq \sum_{i=0}^k |W_i| - (3k+1)/2.$$

Grötschel and Padberg [6] showed that these inequalities are facets of $TSP(G)$ when G is a complete graph.

Theorem 3.3 The path inequalities generalize the comb inequalities such

that $W_0 = \bigcup_{i=1}^k W_i \neq \emptyset$ and $\bigcup_{i=0}^k W_i \neq N$. More precisely, these comb inequalities define the same faces of $TSP(G)$ as the 2-regular path inequalities.

Proof: Recall that a 2-regular path inequality $\sum_{e \in E} f_e x_e \geq a$ is one where $n_i = 2$ for all $i = 1, \dots, k$. Let $\{A, Z, B_1^1, B_2^1, i = 1, \dots, k\}$ be the node partition which defines the corresponding path configuration. Let $W_0 = A \cup (\bigcup_{i=1}^k B_1^1)$ and $W_i = B_1^i \cup B_2^i$ for $i = 1, \dots, k$. We will show that the comb inequality associated with $(W_0, W_1, \dots, W_k)$ can be derived from the 2-regular path inequality and the equations $x(\delta(v)) = 2$ for all $v \in N$.

These equations imply that, for any $W \subset N$, $x(\gamma(W)) = |W| - x(W, \bar{W})/2$.

Therefore,

$$\begin{aligned} \sum_{i=0}^k x(\gamma(W_i)) &= \sum_{i=0}^k (|W_i| - x(W_i, \bar{W}_i)/2) \\ &= \sum_{i=0}^k |W_i| - (\sum_{e \in E} g_e x_e)/2, \text{ as defined in (3.6')} \\ &\leq \sum_{i=0}^k |W_i| - (3k+1)/2, \text{ by (3.5')}. \blacksquare \end{aligned}$$

An obvious question at this point is whether the comb inequalities

with either $W_0 = \bigcup_{i=1}^k W_i = \emptyset$ or $\bigcup_{i=0}^k W_i = N$ can also be generalized to facets

of GTSP(G). We will show that this is indeed the case using the concepts of wheelbarrow configurations and bicycle configurations. These configurations are very similar to k-path configurations, the main differences being that $Z=\emptyset$ and $A=Z=\emptyset$ respectively.

More specifically, a wheelbarrow configuration (see Figure 3) is defined by

(3.7) an odd integer $k \geq 3$ and integers $n_i \geq 2$ for $i=1, \dots, k$,

(3.8) a partition of the node set N into $\{A, B_j^i$ for $i=1, \dots, k$ and $j=1, \dots, n_i\}$. For convenience, let $B_0^1 = A$.

(3.9) The graphs $G(B_j^i)$ are connected for $i=1, \dots, k$ and $j=0, \dots, n_i$.

(3.10) The edge set (B_j^i, B_{j+1}^i) is nonempty for every $i=1, \dots, k$ and $j=0, \dots, n_i-1$.

(3.11) The edge set $(B_{n_i}^i, B_{n_i+1}^{i+1})$ is nonempty for every $i=1, \dots, k$.

In (3.11) and in the remainder of this section, the index $i+1$ is defined to be equal to 1 when $i=k$. In general indices will be defined modulo the largest value that they can assume.

A bicycle configuration (see Figure 3) is defined by

(3.12) an odd integer $k \geq 3$ and integers $n_i \geq 2$ for $i=1, \dots, k$,

(3.13) a partition of the node set N into $\{B_j^i$ for $i=1, \dots, k$ and $j=1, \dots, n_i\}$.

(3.14) The graphs $G(B_j^i)$ are connected for $i=1, \dots, k$ and $j=1, \dots, n_i$.

(3.15) The edge set (B_j^i, B_{j+1}^i) is nonempty for every $i=1, \dots, k$ and $j=1, \dots, n_i-1$.

(3.16) The edge sets (B_1^i, B_1^{i+1}) and $(B_{n_i}^i, B_{n_i+1}^{i+1})$ are nonempty for every $i=1, \dots, k$.

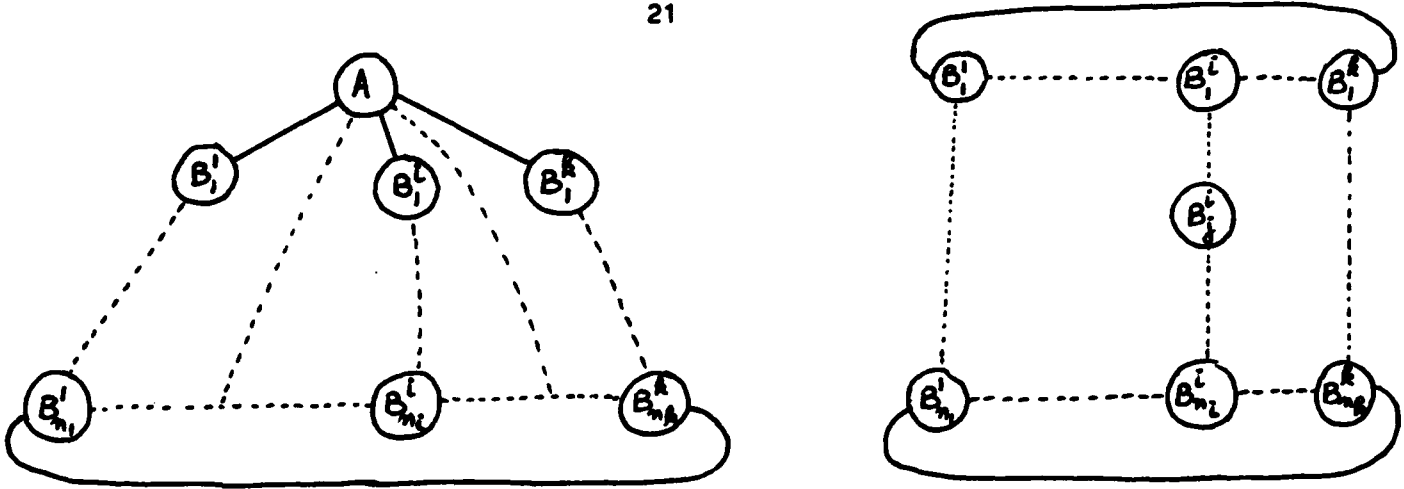


Figure 3. A wheelbarrow and a bicycle

The wheelbarrow inequality associated with a wheelbarrow configuration and the bicycle inequality associated with a bicycle configuration are both defined by

$$(3.17) \quad \sum_{e \in E} f_e x_e \geq a$$

where a is as defined earlier and

$$(3.18) \quad f_e = \begin{cases} \frac{|j-p|}{n_1-1} & \text{for } e \in (B_j^i, B_p^i), i=1, \dots, k \text{ and } j \neq p \text{ such that } |j-p| \leq n_1, \\ \max\left\{\frac{p}{n_r-1} - \frac{j-2}{n_1-1}, \frac{j}{n_1-1} - \frac{p-2}{n_r-1}\right\} & \text{for } e \in (B_j^i, B_p^r), i \neq r, \\ & j=1, \dots, n_1 \text{ and } p=1, \dots, n_r, \\ 0 & \text{otherwise.} \end{cases}$$

As earlier, a n -regular configuration is obtained when $n_1 = n \geq 2$ for all $i=1, \dots, k$. The inequality (3.17) becomes

$$(3.17') \quad \sum_{e \in E} g_e x_e \geq kn + k + n - 1$$

where

$$(3.18') \quad g_e = \begin{cases} |j-p| & \text{for } e \in (B_j^i, B_p^i), i=1, \dots, k \text{ and } j \neq p \text{ such that } |j-p| \leq n, \\ |j-p|+2 & \text{for } e \in (B_j^i, B_p^r), i \neq r \text{ and } j, p=1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.4 Wheelbarrow and bicycle inequalities are valid inequalities for GTSP(G).

Proof: First consider a wheelbarrow configuration $\{A, B_j^i \text{ for } i=1, \dots, k \text{ and } j=1, \dots, n_i\}$ and the associated inequality. Let G' be the graph constructed from G by adding a node z joined to at least one node in each $B_{n_i}^i$ for $i=1, \dots, k$. This defines a k -path configuration in G' , namely

$\{A, \{z\}, B_j^i \text{ for } i=1, \dots, k \text{ and } j=1, \dots, n_i\}$. Extend the definition of f_e to every edge in G' using (3.6). We know that, for the length function defined by the f_e 's, every tour of G' has length at least a .

Now let x be a minimum length tour of G . We can assume that x does not contain any edge from (B_j^i, B_p^i) for $p \geq j+2$ since this edge could be replaced by a path of the same length using one edge of each (B_t^i, B_{t+1}^i) , $t=j, \dots, p-1$.

If x does not use any edge from (B_j^i, B_p^r) for $i \neq r$ and $j=1, \dots, n_i$,

$p=1, \dots, n_r$ then its length is $\sum_{i=1}^k 2n_i/(n_i-1) > a$. So assume that x uses

such an edge.

Among all such edges, let e be one corresponding to the largest possible value of p . Say $e \in (B_j^i, B_p^r)$. Then by our choice of p , the tour x contains two edges of (B_t^r, B_{t+1}^r) for $t=p, \dots, n_r-1$, if $p < n_r$. Let x' be

the tour of G' defined as follows. Remove the edge e from x and, if $p < n_r$, remove also one edge from each (B_t^r, B_{t+1}^r) for $t=p, \dots, n_r-1$. Then add one edge from $(B_{n_r}^r, \{z\})$ as well as one from $(\{z\}, B_{n_1}^1)$ and one from each

(B_t^1, B_{t+1}^1) , $t=j, \dots, n_1-1$. Since $f_e \geq \frac{p}{n_r-1} - \frac{j-2}{n_1-1}$, the tour x is at least as long as the tour x' . So x' has length at least a .

Now consider the case of a bicycle configuration. We construct a graph G'' by adding to G a node α joined to each B_i^1 for $i=1, \dots, k$. We will use the fact that the wheelbarrow inequality associated with $(\{\alpha\}, B_j^1)$ for $i=1, \dots, k$ and $j=1, \dots, n_i$ is valid in G'' . Given a minimum length tour x of G with no edge from (B_j^1, B_p^1) for $p \geq j+2$, let e be an edge of (B_j^1, B_p^r) , $i \neq r$, corresponding to the smallest value of p . Remove from x the edge e and, if necessary, one edge from each (B_t^r, B_{t+1}^r) for $t=1, \dots, p-1$. Then add one edge from $(B_1^r, \{\alpha\})$, one from $(\{\alpha\}, B_1^1)$ and one from each (B_t^1, B_{t+1}^1) , $t=1, \dots, j-1$. Since $f_e \geq \frac{j}{n_i-1} - \frac{p-2}{n_r-1}$, the tour x'' of G'' just constructed is at most as long as the tour x . Therefore $\sum_{e \in E} f_e x_e \geq$

$$\sum_{e \in E} f_e x_e'' \geq a. \quad \blacksquare$$

Theorem 3.5 Wheelbarrow inequalities define facets of GTSP(G). Bicycle inequalities such that the edge sets (B_1^i, B_1^{i+2}) and $(B_{n_i}^i, B_{n_i}^{i+2})$ are nonempty for every $i=1, \dots, k$, define facets of GTSP(G).

Proof: The proof is very similar to that of Theorem 3.2. Let $\sum_{e \in E} f_e x_e \geq a$

be an inequality associated with a wheelbarrow or a bicycle configuration. (We will refer to this inequality as inequality 1 in this proof.) Since it is satisfied with equality by some tours (see Figures 4 and 5), it defines a face of GTSP(G). Let $\sum_{e \in E} c_e x_e \geq a$ (inequality 2) define a facet

which contains the face defined by inequality 1. We will show that $c_e = f_e$ for all $e \in E$. As in the proof of Theorem 3.2, we must have $c_e = 0$ for every edge e with both ends in B_j^i , $i=1, \dots, k$ and $j=(0, 1, 2, \dots, n_i)$, and c_e must assume the same value for all the edges in the edge set (B_j^i, B_{j+1}^i) .

Now consider the wheelbarrow inequality and the tours of Figure 4(a) where we vary the index j of the set (B_j^i, B_{j+1}^i) whose edges are not used in the tour. Since all these tours satisfy inequality 1 (and therefore inequality 2) with equality, there must be a constant c^i such that $c_e = c^i$ for every edge $e \in (B_j^i, B_{j+1}^i)$ and every $j=0, \dots, n_i-1$. Furthermore, comparing the tours of Figures 4(a), (b) and (c) we get the following equations, for $i=1, \dots, k$, where b_i is the value of c_e for $e \in (B_{n_i}^i, B_{n_i+1}^{i+1})$.

$$b_{i-1} + b_i + (n_i - 2)c^i = b_{i-1} + n_{i+1}c^{i+1} = b_i + n_{i-1}c^{i-1}.$$

The solution of this system is $c^i = \frac{a}{n_i - 1}$ and $b_i = \frac{a}{n_i - 1} + \frac{a}{n_{i+1} - 1}$ for

$i=1, \dots, k$. Writing once more that the tours of Figure 4 satisfy inequality 2 with equality, we get $a=1$. So $c_e = f_e$ for all the edges of the skeleton.

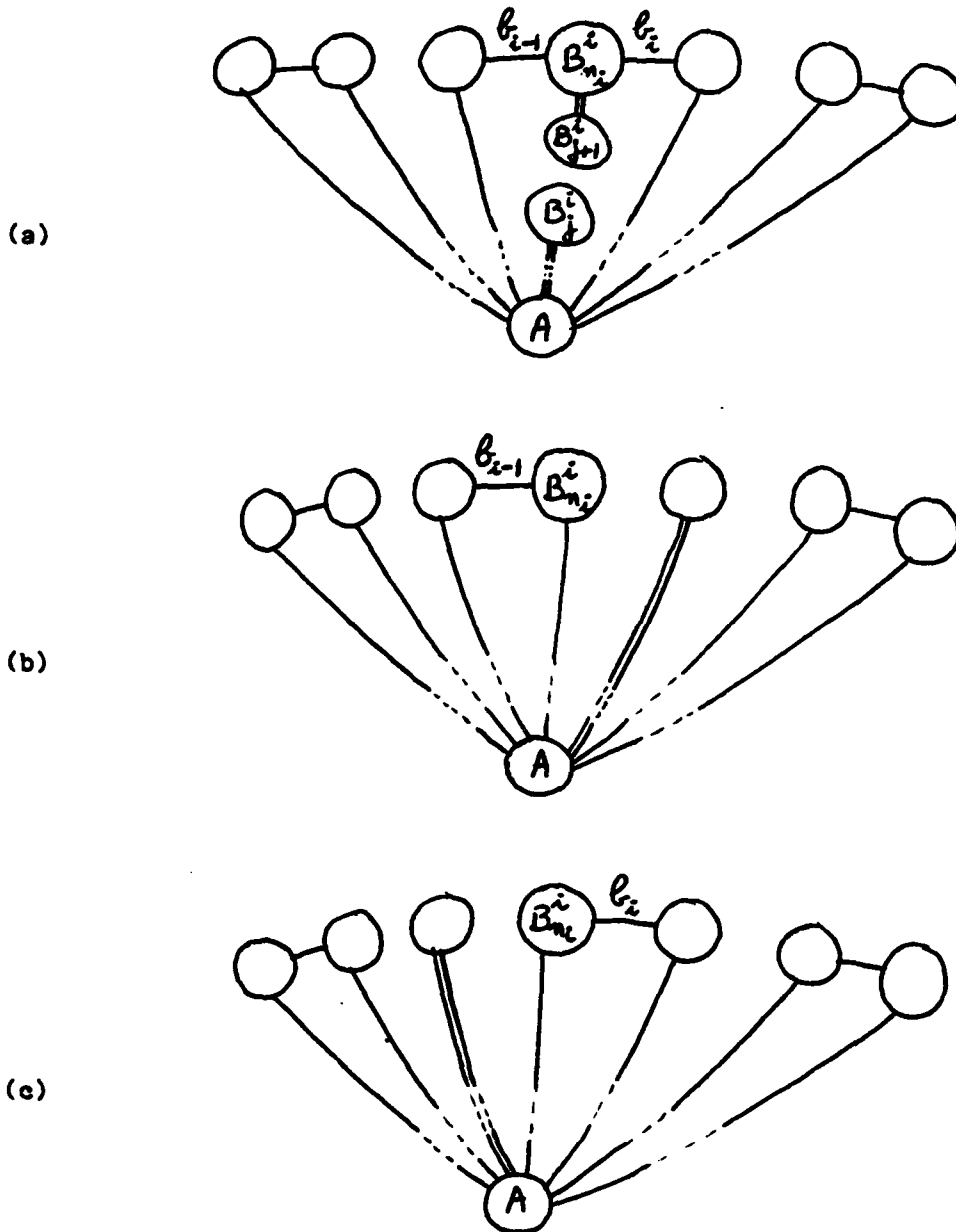


Figure 4. Tours satisfying the wheelbarrow inequality with equality.

Now consider the bicycle inequality and the tours of Figure 5(a) where we vary the index j of the set (B_j^1, B_{j+1}^1) which does not have any edge in the tour. This shows that there is a constant c^1 such that $c_e = c^1$ for every $e \in (B_j^1, B_{j+1}^1)$ and every $j=1, \dots, n_1-1$. Let b_1 be the value of c_e

for $e \in (B_{n_i}^i, B_{n_{i+1}}^{i+1})$ and d_i the value of c_e for $e \in (B_1^i, B_1^{i+1})$. By comparing

the tours of Figures 5(c) and 5(d), we get

$$b_i + (n_i - 1)c^i = d_i + (n_{i+1} - 1)c^{i+1}.$$

By comparing the tours of Figures 5(e) and 5(f), we get

$$d_i + (n_i - 1)c^i = b_i + (n_{i+1} - 1)c^{i+1}.$$

This implies that $b_i = d_i$ and that $(n_i - 1)c^i = (n_{i+1} - 1)c^{i+1}$. Therefore, there exists a constant α such that $c^i = \frac{\alpha}{n_i - 1}$, for $i=1, \dots, k$. Finally,

comparing the tours of Figures 5(a) and 5(b) we get

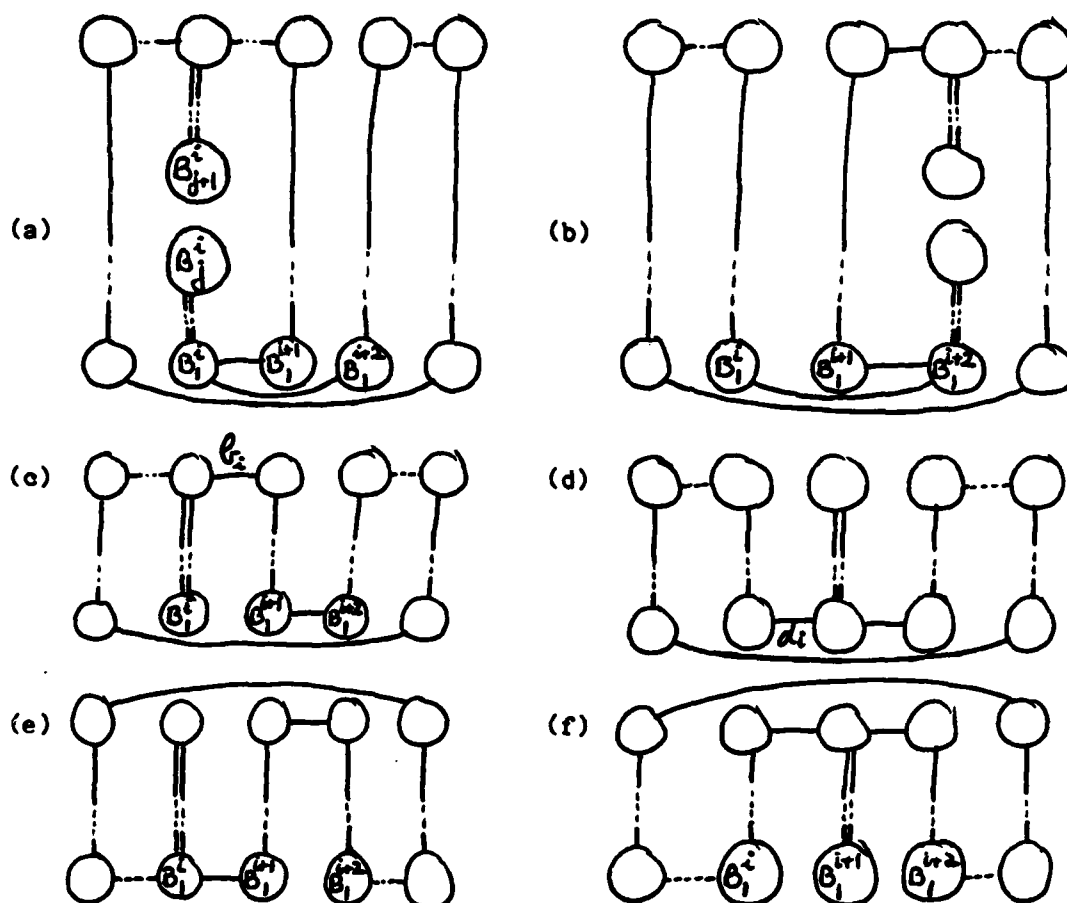
$$(n_i - 2)c^i + b_i + d_i = (n_{i+2} - 2)c^{i+2} + b_{i+1} + d_{i+1}.$$

This implies $b_i = d_i = \frac{\alpha}{n_i - 1} + \frac{\alpha}{n_{i+1} - 1}$. Writing that the tour of Figure 5(a)

satisfies inequality 2 with equality, we get $\alpha=1$. So $c_e = f_e$ for the edges of the skeleton.

Finally, we have to determine c_e for the edges which are not in the skeleton of the wheelbarrow or bicycle configurations. For each such edge e^* we will show that there is a tour which uses the edge e^* and only edges of the skeleton, and which satisfies inequality 1 with equality. The fact that this tour also satisfies inequality 2 with equality will imply that $c_{e^*} = f_{e^*}$.

When $e^* \in (B_j^{i+1}, B_p^{i+1})$ $i=0, \dots, k-1$, and $p-j \geq 2$, it is easy to obtain the required tour by modifying the tours



configurations requires edges in some of the sets (B_1^{h-2}, B_1^h) and $(B_{n_h-2}^{h-2}, B_{n_h}^h)$

for h in the odd section.

This completes the proof of Theorem 3.5. ■

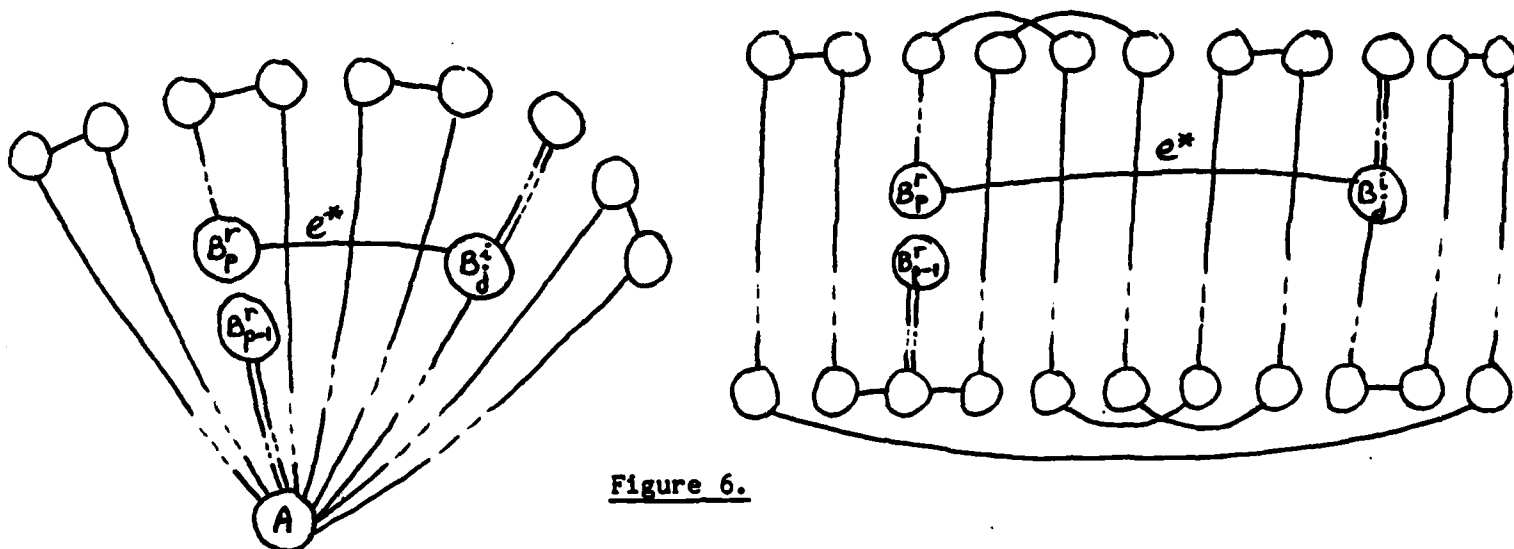


Figure 6.

Theorem 3.6 The path, wheelbarrow and bicycle inequalities generalize all the comb inequalities.

Proof: The proof of Theorem 3.3 is still valid when $Z=\emptyset$ and/or when $A=\emptyset$. ■

We conclude this section with another class of facets of $GTSP(G)$. A hypohamiltonian graph is a nonhamiltonian graph such that the deletion of any node yields a hamiltonian graph. The Petersen graph is a classical example of a hypohamiltonian graph. Let $A_1, \dots, A_k$ be a partition of the node set of $G=(N,E)$ with the following properties:

(3.19) each graph $G(A_i)$ is connected, $i=1, \dots, k$,

(3.20) The graph G^* obtained by shrinking each set A_i to a single node a_i contains a hypohamiltonian subgraph which spans all the nodes a_i , $i=1, \dots, k$.

Let H be an edge maximal hypohamiltonian subgraph of G^* , i.e., the addition of any edge of G^* to H would make H hamiltonian. Denote by F the edge set of H .

Theorem 3.7 Let $A_1, \dots, A_k$ be a partition of N satisfying the conditions (3.19) and (3.20), and let F be the edge set of a maximal hypohamiltonian subgraph of G^* . Then the following inequality is valid and defines a facet of $GTSP(G)$.

$$\sum_{e \in E} f_e x_e \geq |N| + 1$$

$$\text{where } f_e = \begin{cases} 1 & \text{if } e \in (A_i, A_j) \text{ and } a_i a_j \in F, \\ 2 & \text{if } e \in (A_i, A_j) \text{ and } a_i a_j \notin F, \\ 0 & \text{otherwise} \end{cases}$$

Proof: The fact that the inequality is valid is obvious. Assume that it defines a face of $GTSP(G)$ which is not a facet, and let $\sum_{e \in E} c_e x_e \geq |N| + 1$

be an inequality defining a facet of $GTSP(G)$ which contains that face.

First note that $c_e = 0$ for every edge e with both ends in the same set A_i . Next we show that, for any given $i=1, \dots, k$, the value of c_e is the same, say $c_e = c^i$, for all the edges $e \in (A_i, A_j)$ such that $a_i a_j \in F$. Since $H - \{a_i\}$ is hamiltonian, there is a tour of H using a Hamilton cycle of $H - \{a_i\}$ and some edge $a_i a_j \in F$ twice. This tour satisfies $\sum f_e x_e = |N| + 1$ and therefore also $\sum c_e x_e = |N| + 1$. By changing the edge $e \in (A_i, A_j)$ such that $a_i a_j \notin F$, we get that c_e is identical for all these edges, say $c_e = c^i$. Since this argument holds for any $i=1, \dots, k$, and since G is connected, the constant c^i does not depend on i and $c^i = 1$.

Finally, let $e \in (A_i, A_j)$ such that $a_i a_j \notin F$. There exists a Hamilton cycle which uses e and only edges of F . Since this Hamilton cycle is on the face defined by $\sum f_e x_e \geq |N|+1$, we must have $c_e=2$. ■

4. Some Related Integer Polyhedra

First consider the polyhedra $P_1(G)$ and $P_2(G)$ defined in the introduction. Recall that $P_1(G)$ is the convex hull of the nonnegative integer vectors $x \in \mathbb{R}^E$ satisfying

- (1.2) for each node v of G , the sum of the values x_e over the edges e incident with v is even and at least 2.

The polyhedron $P_2(G)$ is the convex hull of the nonnegative integer vectors $x \in \mathbb{R}^E$ satisfying

- (1.3) the graph induced by the edges of G such that $x_e > 0$ is connected.

Theorem 4.1 The following system of linear inequalities are sufficient to define $P_1(G)$ and $P_2(G)$.

$$P_1(G) = \{x \in \mathbb{R}^E : x_e \geq 0 \text{ for all } e \in E \text{ and } x(\delta(v)) \geq 2 \text{ for all } v \in N\},$$

$$P_2(G) = \{x \in \mathbb{R}^E : x(F) \geq p-1 \text{ for all } F \subseteq E, \text{ where } p \text{ is the number of connected components of } G \setminus F \equiv (N, E-F)\}.$$

Proof: To impose condition (1.2), add a triangle to each node $v \in N$. See Figure 7. Consider the b -matching problem with equality requirements at

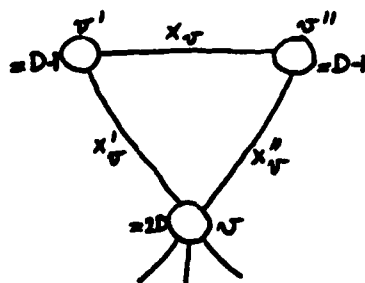


Figure 7.

each node ($=2D$ at node v and $D-1$ at each of the additional nodes v' and v''). Then the solutions of the b -matching problem are exactly the nonnegative integer solutions satisfying (1.2), assuming that D is large.

Take D to be odd and large. Then the requirement at each node i of the expanded graph is an even b_i , so there is no blossom constraint in the b -matching problem. In other words, the b -matching polyhedron is given by

$$\begin{aligned} x_e &\geq 0 && \text{for all } e \in E, \\ x'_v + x''_v + x(\delta(v)) &= 2D && \text{for all } v \in N \text{ [here } \delta \text{ is defined in G]}, \\ x'_v + x_v &= D-1 && \text{for all } v \in N, \\ x''_v + x_v &= D-1 && \text{for all } v \in N, \\ x_v, x'_v, x''_v &\geq 0 && \text{for all } v \in N. \end{aligned}$$

The variables x'_v and x''_v are easily eliminated from this system. The remaining system in terms of x_e , $e \in E$, and x_v , $v \in N$, is

$$\begin{aligned} x_e &\geq 0 && \text{for all } e \in E, \\ x_v &= x(\delta(v)) - 2 && \text{for all } v \in N, \\ x_v &\geq 0 && \text{for all } v \in N. \end{aligned}$$

Again it is easy to eliminate x_v . So we get

$$P_1(G) = \{x \in \mathbb{R}^E : x_e \geq 0 \text{ for all } e \in E \text{ and } x(\delta(v)) \geq 2 \text{ for all } v \in N\}.$$

Now consider $P_2(G)$. The condition (1.3) can be stated using matroid terminology. It says that $x \geq y$ for any vector y which is the $(0,1)$ -

incidence vector of a spanning set of the graphic matroid associated with G . Equivalently $y^* = 1-y$ is an independent set of the dual matroid M^* . The system of inequalities describing the convex hull of the y^* 's is

$$\begin{aligned} y^* &\geq 0, \\ y^*(F) &\leq r^*(F) \text{ for all } F \subseteq E, \end{aligned}$$

where r^* is the rank function of M^* [3]. r^* is related to the rank function r of M by $r^*(F) = |F| + r(E-F) - r(E)$. So the convex hull of the vectors y is

$$\begin{aligned} y &\leq 1, \\ y(F) &\geq r(E) - r(E-F) \text{ for all } F \subseteq E. \end{aligned}$$

Therefore the polyhedron $P_2(G)$ is

$$P_2(G) = \{x \in \mathbb{R}^E : x(F) \geq r(E) - r(E-F) \text{ for all } F \subseteq E\}.$$

For a graphic matroid, $r(E) - r(E-F)$ is one less than the number of connected components of $G \setminus F = (N, E-F)$. ■

Define $P_3(G)$ as the convex hull of the nonnegative integer vectors $x \in \mathbb{R}^E$ satisfying

(1.4) the sum of the values x_e on any edge cutset is at least 2.

Remark 4.2 $P_3(G) \subseteq P_1(G)$ since all the constraints $x(\delta(v)) \geq 2$ which define $P_1(G)$ are also valid for $P_3(G)$. In general the inclusion is strict.

Remark 4.3 $P_3(G) \subset P_2(G)$. In fact none of the inequalities $x(F) \geq p-1$ which define $P_2(G)$ are tight for $P_3(G)$. It is easy to show that the system $x(F) \geq p$ for all $F \subseteq E$, where p is the number of connected components of $G \setminus F$, is valid for $P_3(G)$. This follows by taking linear combinations of the valid inequalities $x_e \geq 0$ and $x(F) \geq 2$ for any edge cutset F .

Remark 4.4 $GTSP(G) \subset P_3$ and in general the inclusion is strict as shown by the example of Figure 1. In that example, the point $x_e = 1$ for all $e \in E$ is an extreme point of $P_3(G)$ which does not belong to $GTSP(G)$.

Remark 4.5 The extreme points of $GTSP(G)$ are those extreme points of $P_3(G)$ which are also extreme in $P_1(G)$.

Remark 4.6 $GTSP(G)$ and $P_3(G)$ are both full-dimensional polyhedra and they both have $TSP(G)$ as a face.

Define the polyhedron $P_4(G)$ as

$$P_4(G) = \{x \in \mathbb{R}^E : x_e \geq 0 \text{ for all } e \in E \text{ and } x(F) \geq 2 \text{ for all edge cutsets } F \subseteq E\}.$$

In general $P_4(G)$ has fractional extreme points. An interesting question is to characterize those graphs G for which $P_4(G)$ has only integral extreme points, i.e., when $P_4(G) = P_3(G)$. We do not have a complete characterization of these graphs but only a sufficient condition.

A series-parallel graph is any graph that can be obtained by a recursive application of the following operations, starting from the graph consisting of two nodes joined by an edge.

(4.1) duplicate an edge (i.e., add an edge joining the same end nodes),

(4.2) replace an edge uv by two edges uw and wv where w is a new node.

Series-parallel graphs will be discussed at greater length in Section 5. Here we just prove the following theorem.

Theorem 4.7 If G is a series-parallel graph, then $P_3(G) = P_4(G)$.

Proof: The theorem is true for the graph with two nodes joined by an edge. Assume it is true for all series-parallel graphs with m edges. We will prove the theorem for series-parallel graphs with $m+1$ edges. Such a graph is obtained from a series-parallel graph with m edges by operation (4.1) (Edge duplication) or operation (4.2) (Node insertion).

If an edge is duplicated, then every edge cutset contains either both the edge and its duplicate or neither of them. Therefore, if x is an extreme point of $P_4(G)$, at most one of the two variables x_1 and x_2 associated with the edge and its duplicate takes a positive value and, by the induction hypothesis, x must be integral.

If a node is inserted on an edge, then let x_1 and x_2 be the two new edge variables. The new system of inequalities defining $P_4(G)$ is made of two copies of the former system, one with x_1 , the other with x_2 , instead of the variable associated with the divided edge, to which is added the constraint $x_1 + x_2 \geq 2$. We want to show that every extreme point of the new system is integral.

If $x_1 = x_2$ for some extreme point of $P_n(G)$, then this extreme point must be integral by the induction hypothesis. So assume $x_1 > x_2$ for some extreme point. Note that no constraint involving x_1 can be tight except possibly $x_1 + x_2 \geq 2$. If even this constraint is not tight, then x_1 could be decreased and therefore x was not an extreme point. Now assume $x_1 + x_2 = 2$. This equation determines the value of x_2 whereas the rest of the system defining x_1 has an integral solution by the induction hypothesis. So x_2 is integral too. Again the extreme point x is integral. ■

We conclude this section with a property of the extreme points of $P_n(G)$. Our theorem can be proved using the next lemma.

A family of subsets $\{S_i \subset N: i=1, \dots, k\}$ is said to be crossing if it does not contain the empty set, it contains the complement of each of its members and, finally, given any S_1 and S_2 in the family such that $S_1 \not\subset S_2$, $S_2 \not\subset S_1$, $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cup S_2 \neq N$, then both $S_1 \cap S_2$ and $S_1 \cup S_2$ belong to the family. (The sets S_1 and S_2 are said to cross.)

Lemma 4.8 The family $\{S \subset N: x(S, \bar{S}) = 2\}$ is a crossing family. Furthermore, if S_1 and S_2 are two sets of the family which cross, then $x(S_1 - S_2, S_2 - S_1) = x(S_1 \cap S_2, \overline{S_1 \cup S_2}) = 0$, $x(S_1 - S_2, \overline{S_1 - S_2}) = x(S_2 - S_1, \overline{S_2 - S_1}) = 2$ and $x(S_1 \cap S_2, S_1 - S_2) = x(S_1 \cap S_2, S_2 - S_1) = x(\overline{S_1 \cup S_2}, S_1 - S_2) = x(\overline{S_1 \cup S_2}, S_2 - S_1) = 1$.

Proof: Let S_1 and S_2 be two node sets in the family such that $S_1 \not\subset S_2$, $S_2 \not\subset S_1$, $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cup S_2 \neq N$ (i.e., the sets S_1 and S_2 cross). Let $S_3 = S_1 \cap S_2$ and $S_4 = S_1 \cup S_2$.

$$x(S_3, \bar{S}_3) = x(S_3, S_1 - S_2) + x(S_3, S_2 - S_1) + x(S_3, \bar{S}_4)$$

$$x(S_4, \bar{S}_4) = x(S_1 - S_2, \bar{S}_4) + x(S_2 - S_1, \bar{S}_4) + x(S_3, \bar{S}_4)$$

$$x(S_1, \bar{S}_1) = x(S_3, S_2 - S_1) + x(S_3, \bar{S}_4) + x(S_1 - S_2, S_2 - S_1) + x(S_1 - S_2, \bar{S}_4)$$

$$x(S_2, \bar{S}_2) = x(S_3, S_1 - S_2) + x(S_3, \bar{S}_4) + x(S_2 - S_1, S_1 - S_2) + x(S_2 - S_1, \bar{S}_4).$$

Therefore

$$x(S_1, \bar{S}_1) + x(S_2, \bar{S}_2) = x(S_3, \bar{S}_3) + x(S_4, \bar{S}_4) + 2x(S_1 - S_2, S_2 - S_1)$$

Since $x(S_1, \bar{S}_1) = x(S_2, \bar{S}_2) = 2$ and $x(S_3, \bar{S}_3) \geq 2$, $x(S_4, \bar{S}_4) \geq 2$, we deduce that $x(S_3, \bar{S}_3) = x(S_4, \bar{S}_4) = 2$ and therefore the family is crossing. In addition we must have $x(S_1 - S_2, S_2 - S_1) = 0$.

Again summing the equations defining $x(S_1, \bar{S}_1)$ and $x(S_2, \bar{S}_2)$, but grouping the terms differently, we get $x(S_1, \bar{S}_1) + x(S_2, \bar{S}_2) = x(S_2 - S_1, \bar{S}_2 - \bar{S}_1) + x(S_1 - S_2, \bar{S}_1 - \bar{S}_2) + 2x(S_3, \bar{S}_4)$. Since $x(S_1, \bar{S}_1) = x(S_2, \bar{S}_2) = 2$ and $x(S_2 - S_1, \bar{S}_2 - \bar{S}_1) \geq 2$, $x(S_1 - S_2, \bar{S}_1 - \bar{S}_2) \geq 2$, we deduce that $x(S_2 - S_1, \bar{S}_2 - \bar{S}_1) = x(S_1 - S_2, \bar{S}_1 - \bar{S}_2) = 2$ and $x(S_3, \bar{S}_4) = 0$.

Now substitute $x(S_3, \bar{S}_3)$, $x(S_4, \bar{S}_4)$, $x(S_1, \bar{S}_1)$, $x(S_2, \bar{S}_2)$, $x(S_3, \bar{S}_4)$ and $x(S_1 - S_2, S_2 - S_1)$ by their known value in the equations defining $x(S_3, \bar{S}_3)$, $x(S_4, \bar{S}_4)$, $x(S_1, \bar{S}_1)$ and $x(S_2, \bar{S}_2)$. We obtain a system of four equations with four unknowns whose unique solution is

$$x(S_3, S_1 - S_2) = x(S_3, S_2 - S_1) = x(S_1 - S_2, \bar{S}_4) = x(S_2 - S_1, \bar{S}_4) = 1. \quad \blacksquare$$

A family of subsets $\{S_i : i=1, \dots, k\}$ is said to be nested if, for any $i \neq j$, either $S_i \subset S_j$ or $S_j \subset S_i$ or $S_i \cap S_j = \emptyset$. A family of edge cutsets $\{(S_i, \bar{S}_i) : i=1, \dots, k\}$ is said to be laminar if the family $\{S_i : i=1, \dots, k\}$ is nested.

Any extreme point of the polyhedron $P_4(G)$ can be defined as the unique solution of a system of $|E|$ equations of the form

$$(4.3) \quad x_e = 0, \quad e \in E_0, \text{ and}$$

$$(4.4) \quad x(S, \bar{S}) = 2 \text{ for } (S, \bar{S}) \in B$$

where $E_0 \subseteq E$, B is a family of edge cutsets and $|B| + |E_0| = |E|$.

Theorem 4.9 Let x be an extreme point of $P_4(G)$. The system (4.3), (4.4) of equations defining x can be chosen so that the edge cutsets in (4.4) form a laminar family.

Proof: Assume that x is an extreme point of $P_4(G)$ for which the theorem does not hold. Consider a system of the form (4.3), (4.4) where the family B of cutsets is such that $a(B) = \sum (|S| : (S, \bar{S}) \in B)$ is minimum. We denote this system by (E_0, B) .

Since B is not laminar, there exist $(S_1, \bar{S}_1)$ and $(S_2, \bar{S}_2)$ in B such that $S_1 \not\subseteq S_2$, $S_2 \not\subseteq S_1$ and $S_1 \cap S_2 \neq \emptyset$.

First assume that $S_1 \cup S_2 \neq N$. Since the system defining x has rank $|E|$, any valid equation can be obtained as a linear combination of equations in this system. In particular, by lemma 4.8, we have the valid equalities $x(S_1 - S_2, \bar{S}_1 - \bar{S}_2) = 2$ and $x(S_2 - S_1, \bar{S}_2 - \bar{S}_1) = 2$. So

$$x(S_1 - S_2, \bar{S}_1 - \bar{S}_2) = \sum (\alpha_i x(S_1, \bar{S}_1) : (S_1, \bar{S}_1) \in B) + \sum (\gamma_e x_e : e \in E_0)$$

$$x(S_2 - S_1, \bar{S}_2 - \bar{S}_1) = \sum (\beta_i x(S_1, \bar{S}_1) : (S_1, \bar{S}_1) \in B) + \sum (\delta_e x_e : e \in E_0)$$

$$\text{where } \sum (\alpha_i : (S_1, \bar{S}_1) \in B) = \sum (\beta_i : (S_1, \bar{S}_1) \in B) = 1.$$

If $\alpha_1 \neq 0$, we can replace $x(S_1, \bar{S}_1) = 2$ in the system (E_0, B) by $x(S_1 - S_2, \bar{S}_1 - \bar{S}_2) = 2$ and still have a system whose unique solution is x . This contradicts the minimality of $a(B)$.

Similarly $\beta_2 \neq 0$ would allow us to replace $x(S_2, \bar{S}_2) = 2$ in the system (E_0, B) by $x(S_2 - S_1, \bar{S}_2 - \bar{S}_1) = 2$ contradicting the minimality of $a(B)$.

So we must have $\alpha_1 = \beta_2 = 0$. Now we use the following fact, proved in Lemma 4.8.

$$x(S_1, \bar{S}_1) + x(S_2, \bar{S}_2) = x(S_1 - S_2, \overline{S_1 - S_2}) + x(S_2 - S_1, \overline{S_2 - S_1}) + 2x(S_1 \cap S_2, \overline{S_1 \cup S_2}).$$

Since $x(S_1, \bar{S}_1) = 2$ and $x(S_2, \bar{S}_2) = 2$ are independent of the $|E| - 2$ other equations in the system (E_0, B) , we must have $\alpha_2 = 1$ and $\beta_1 = 1$. Therefore a new system whose unique solution is x can be obtained from the system (E_0, B) by replacing the equations $x(S_1, \bar{S}_1) = 2$ and $x(S_2, \bar{S}_2) = 2$ by the two equations $x(S_1 - S_2, \overline{S_1 - S_2}) = 2$ and $x(S_2 - S_1, \overline{S_2 - S_1}) = 2$. This new system contradicts the minimality of $a(B)$.

Now assume that $S_1 \cup S_2 = N$. Then $S_1 = \overline{S_2 - S_1}$ and $S_1 = S_2 - S_1$, so $x(S_1, \bar{S}_1) = x(S_2 - S_1, \overline{S_2 - S_1})$. Similarly $x(S_2, \bar{S}_2) = x(S_1 - S_2, \overline{S_1 - S_2})$. By replacing $x(S_1, \bar{S}_1) = 2$ and $x(S_2, \bar{S}_2) = 2$ in the system (E_0, B) by $x(S_1 - S_2, \overline{S_1 - S_2}) = 2$ and $x(S_2 - S_1, \overline{S_2 - S_1}) = 2$, we obtain a contradiction to the minimality of $a(B)$. This completes the proof of the theorem. ■

This result ~~seems to have~~^{has} been proven independently by W. Cunningham in unpublished work on the polyhedron $P_4(G)$.

5. The Steiner Traveling Salesman Problem in Series-Parallel Graphs.

Let $G = (N, E)$ be a graph, $l: E \rightarrow \mathbb{R}$ a nonnegative length function and $\tilde{N} \subseteq N$ a set of nodes called Steiner points. The Steiner tree problem is to find a minimum length ^{tree} in G which spans the nodes of $\tilde{N}$. This problem is known to be NP-hard. Similarly one can define the Steiner Traveling Salesman Problem as the problem of finding a minimum length cycle of G which goes at least once through each node of $\tilde{N}$. We call such a cycle a Steiner tour. Recall that, in this paper, cycles may contain the same node or the same edge more than once. In both Steiner problems the nodes of $N - \tilde{N}$ may or may not be on the tree or the cycle.

Many problems which are NP-hard in general graphs have been solved in polynomial time in Series-Parallel graphs. Series-Parallel graphs

(SP-graphs) were defined in Section 4. Takamizawa, Nishizeki and Saito [8] and Wald and Colbourn [9] give linear time algorithms for various Steiner tree problem. Ratliff and Rosenthal [7] solve the problems on SP-graphs, including the Steiner Traveling Salesman problem in graphs that model a rectangular warehouse, where $\tilde{N}$ represents a set of points along the aisles. These graphs are series-parallel. In this section we show that the algorithm of Ratliff and Rosenthal can be extended to all SP-graphs. The resulting algorithm runs in linear time. We will also discuss briefly how the same technique can be used to solve the Steiner tree problem in SP-graphs.

Consider a connected graph G with a two-node cutset, say $\{u, v\}$. Let S_1 be the node set of a connected component of $G - \{u, v\}$. Define G_1 as the graph induced by $S_1 \cup \{u, v\}$ and G_2 as the graph induced by the edges of G which are not in G_1 . Let T_1 and T_2 be the restrictions of a Steiner tour of G to G_1 and G_2 respectively. We will characterize a partial Steiner tour such as T_1 by (a, b, c) where a is the parity of the number of edges of T_1 which are incident with the node u , b is the parity of the number of edges of T_1 incident with v , and c is the number of connected components of T_1 . The elements a and b can take the values E (for even), U (for uneven or odd) and O (for zero) whereas c can take the values 0, 1, or 2. It is easy to enumerate all the possible combinations of (a, b, c) which arise from partial Steiner tours.

Proposition 5.1 The partial Steiner tours can be partitioned into the following seven classes:

$(E, E, 1), (E, E, 2), (E, O, 1), (O, E, 1), (U, U, 1), (O, O, 1)$ and $(O, O, 0)$.

Note that $(O, O, 1)$ means that $\tilde{N} \cap S_1 = \emptyset$, i.e. there are no Steiner points in $N - S_1$. Similarly $(O, O, 0)$ means that there are no Steiner points in $S_1 \cup \{u, v\}$.

Let T_1 and T_2 be defined as above. Now let T'_1 be a partial Steiner tour of G_1 which belongs to the same class as T_1 . The partial tour T'_1 is also a valid completion of T_2 into a Steiner tour of G . So, in each class, it suffices to keep a minimum length solution. Let $l(a,b,c)(G_1)$ be the minimum length of a partial Steiner tour of G_1 in the class (a,b,c) . (Set $l(a,b,c)(G_1) = \infty$ if the class is empty). A minimum length Steiner tour of G can be obtained by combining compatible partial tours and taking the overall minimum length solution. (This is nothing but the optimality principle of dynamic programming.) More precisely, the optimum solution is obtained as the minimum of

$$l(U,U,1)(G_1) + l(U,U,1)(G_2),$$

$$l(E,E,1)(G_1) + l(E,E,1)(G_2),$$

$$l(E,E,1)(G_1) + l(E,E,2)(G_2),$$

$$l(E,E,1)(G_1) + l(E,0,1)(G_2),$$

$$l(E,E,1)(G_1) + l(0,E,1)(G_2),$$

$$l(E,E,1)(G_1) + l(0,0,0)(G_2),$$

$$l(E,E,2)(G_1) + l(E,E,1)(G_2),$$

$$l(E,0,1)(G_1) + l(E,E,1)(G_2),$$

$$l(0,E,1)(G_1) + l(E,E,1)(G_2),$$

$$l(0,0,0)(G_1) + l(E,E,1)(G_2),$$

$$l(0,E,1)(G_1) + l(0,E,1)(G_2),$$

$$l(E,0,1)(G_1) + l(E,0,1)(G_2),$$

$$l(E,0,1)(G_1) + l(0,0,0)(G_2),$$

$$l(0,0,0)(G_1) + l(E,0,1)(G_2),$$

$$l(0,E,1)(G_1) + l(0,0,0)(G_2),$$

$$l(0,0,0)(G_1) + l(0,E,1)(G_2),$$

$$l(0,0,1)(G_1) + l(0,0,0)(G_2),$$

$$l(0,0,0)(G_1) + l(0,0,1)(G_2).$$

In order to apply this procedure, the graph G must have a two-node cutset. In SP-graphs, two-node cutsets can be found recursively as follows.

If every node of G has degree at most two, stop. Otherwise the graph G must contain some parallel paths joining the same endpoints, say u and v . [The existence of such paths follows by considering the sequence of edge duplications and node insertions that led to the graph G and by considering the last edge duplications performed in this sequence.] To find the nodes u and v it suffices to ignore the nodes of G which have degree two and to look for parallel edges in the resulting graph.

Now let G_2 be ^{the} graph induced by one of the paths joining u and v and let G_1 be the graph induced by the other edges. The graph G_1 is an SP-graph with fewer ^{edges} than G , and therefore the recursion can be applied to G_1 .

To complete the description of the algorithm it suffices to show how to find a minimum cost partial Steiner tour of G_2 , for each of the seven classes. See Figure 8.

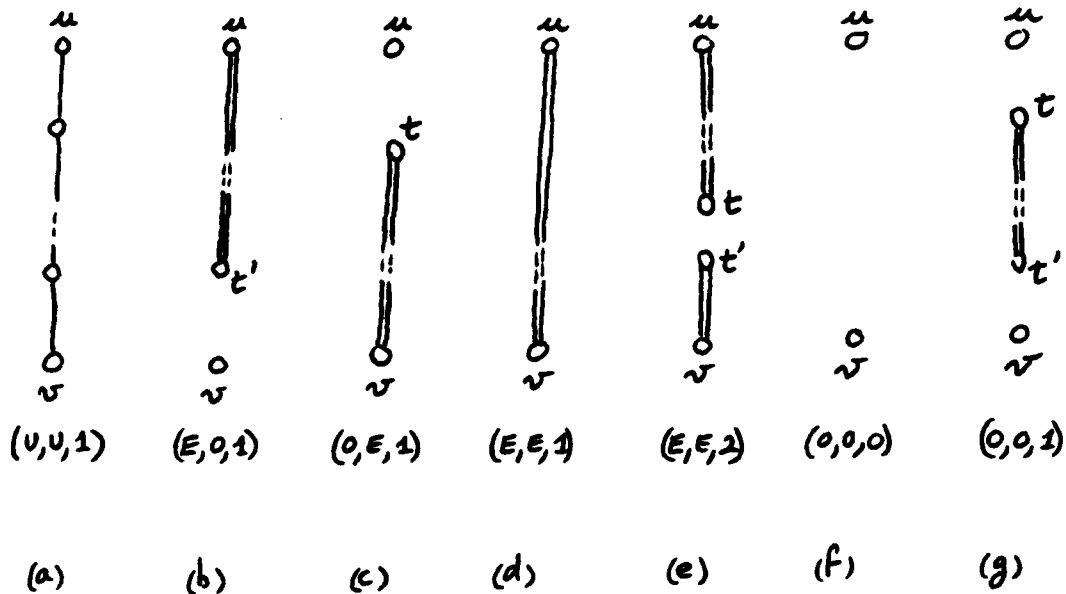


Figure 8

In cases (b), (c) and (g) there are no Steiner points between u and t and between t' and v . In case (e), the nodes t and t' are two Steiner points such that there is no other Steiner point between them and the distance from t to t' is the largest among all pairs of nodes with this property.

Let us turn now to the Steiner Tree problem. Since there are no parity requirements an edge is never taken more than once and there are only six classes to consider, namely $(\neq 0, \neq 0, 2), (\neq 0, \neq 0, 1), (\neq 0, 0, 1), (0, \neq 0, 1), (0, 0, 1)$ and $(0, 0, 0)$. The rest of the algorithm is almost identical to the algorithm for the Steiner Traveling Salesman problem, so the details are left to the reader.

We conclude this section by noting that, although we have a polynomial algorithm for the Steiner Tree and Steiner Traveling Salesman problems in SP-graph, we do not have a description of the corresponding integer polyhedra.

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