

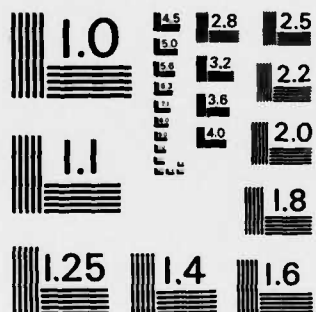
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LOGARITHMICALLY CONCAVE DENSITIES

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ABSTRACT

Let $\underline{X} = (X_1, \dots, X_n)$ be independent random variables with logarithmically concave symmetric densities. We show that for any logarithmically concave functions f and g on R^n that are invariant under sign changes,

$$\text{Cov}(f(\underline{X}), g(\underline{X})) \geq 0.$$

Bounds on the values of logarithmically concave densities on R^n evaluated at the mean vector are also given.

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I. INTRODUCTION AND SUMMARY.

The area of positive and negative dependence of multivariate distributions has attracted the attention of many authors over the past decade. (See Abdel-Hameed and Sampson (1978), Jogdeo (1977), Kanter (1975), Karlin and Rinott (1980), Dykstra (1980), and Prekopa (1973).) Pitt (1977) proves that if $n(x)$ is the standard normal density on R^2 and if A and B are balanced convex subsets of R^2 (i.e., $A=-A$ and $B=-B$) then

$$\int_{A \cap B} n(x) dx \geq \left(\int_A n(x) dx \right) \left(\int_B n(x) dx \right).$$

The question whether Pitt's result is true for any standard normal density on R^n , $n > 2$, remains unanswered.

In this paper we investigate covariance inequalities for a class of logarithmically concave densities. We show that if $X_1, \dots, X_n$ are independent random variables each having a symmetric logarithmically concave symmetric density, then the random variables $Y_1 = f(X_1, \dots, X_n)$ and $Y_2 = g(X_1, \dots, X_n)$ are positively correlated whenever f and g belong to a certain class of logarithmically concave functions on R^n . In particular it follows that if A and B are subsets of R^n that are symmetric along all the axes, then

$$\int_{A \cap B} h(x) dx \geq \left(\int_A h(x) dx \right) \left(\int_B h(x) dx \right),$$

where h is the joint density of the independent random variables $X_1, \dots, X_n$. We remark that a subset of R^n that is symmetric along all the axes is necessarily a balanced set and it follows that if

$(X_1, \dots, X_n)$ is a standard normal vector, with density n then

$$\int_{A \cap B} n(x) dx \geq \left(\int_A n(x) dx \right) \left(\int_B n(x) dx \right)$$

for all subsets A and B of R^n that are symmetric along all axes.

Throughout, the word "symmetric" will be used to mean "symmetric about the origin". For $n=1, 2, \dots$, let

$$H_n = \{f: R^n \rightarrow R_+, f \text{ is logarithmically concave and symmetric}\},$$

$$A_n = \{A: A \text{ is a } n \times n \text{ diagonal matrix with diagonal elements } \pm 1\},$$

$$G_n = \{f \in H_n: f(\underline{x}A) = f(\underline{x}) \text{ for all } \underline{x} \text{ in } R_n \text{ and all } A \in A_n\}, \text{ and}$$

$$L_n = \{K: K \text{ is a convex symmetric subset of } R^n\}.$$

Section 2. Positive Correlations of Functions of Multivariate Random Variables with Logarithmically Concave Densities.

In this section we will show that if $X_1, \dots, X_n$ are independent random variables each having a logarithmically concave symmetric density, then the random variables $Y_1 = f(X_1, \dots, X_n)$ and $Y_2 = g(X_1, \dots, X_n)$ are positively correlated whenever f and g belong to G_n .

2.1 Theorem. Let H be a convex subset of R^n . Then $f: H \rightarrow R_+$ is in H_n if and only if the set $H^+ = \{(\underline{x}, z): f(\underline{x}) \geq e^z\}$ is a convex subset of R^{n+1} and $\{\underline{x}: f(\underline{x}) \geq a\}$ is a symmetric subset of R^n for each $a \in R_+$.

Proof. (If) Suppose that f is not in H_n . Then f is either not logarithmically concave or not symmetric. First assume that f is not logarithmically concave on H . Then there exists $\underline{x}_1, \underline{x}_2$ in H and a in $(0,1)$ such that

$$f(a\underline{x}_1 + (1-a)\underline{x}_2) < f^a(\underline{x}_1) f^{1-a}(\underline{x}_2).$$

Thus the point $(ax_1 + (1-a)x_2, a \ln f(x_1) + (1-a) \ln f(x_2))$ belongs to the line segment joining $(x_1, \ln f(x_1))$, $(x_2, \ln f(x_2))$ but not in H^+ . Since $(x_1, \ln f(x_1))$ and $(x_2, \ln f(x_2))$ are in H^+ , then H^+ is not convex.

If f is not symmetric on H , then there exists x_0 in H such that $f(x_0) \neq f(-x_0)$. Let $a_0 = (f(x_0) \vee f(-x_0))$ and $K_{a_0} = \{x: f(x) \geq a_0\}$. Then either x_0 or $-x_0$ is in K_{a_0} but not both, contradicting the assumption that K_{a_0} is symmetric.

(Only if). Let $f \in H_n$, (x_1, z_1) , (x_2, z_2) be any two points in H^+ and ℓ is the line joining them. Let $(\underline{x}, z_2)$ be any two points in H^+ and ℓ is the line joining them. Let $(\underline{x}, z)$ be any point on ℓ . Then there exists $0 \leq a \leq 1$ such that

$$\begin{aligned}\underline{x} &= ax_1 + (1-a)x_2, \\ \underline{z} &= az_1 + (1-a)z_2.\end{aligned}$$

Since $f(x_1) \geq e^{z_1}$ and $f(x_2) \geq e^{z_2}$, it follows that $f(\underline{x}) \geq e^{\underline{z}}$. Therefore, $(\underline{x}, z)$ is in H^+ . Thus H^+ is convex.

The fact that $\{x: f(x) > a\}$ is symmetric for each $a \in R_+$ is immediate since f is symmetric. ||

2.2 Corollary. Let H be a convex subset of R^n and assume that $f: H \rightarrow R_+$ is in H_n . Then, $\{x: f(x) \geq a\}$ is a convex and symmetric subset of R^n for each $a \in R_+$.

Proof: The symmetry of the set $\{x: f(x) \geq a\}$ is proved in Theorem 2.1. Now assume that x_1, x_2 are in $\{x: f(x) \geq a\}$. Then for a in $(0, 1)$ we have

$$f(ax_1 + (1-a)x_2) \geq f^a(x_1) f^{1-a}(x_2) \geq a.$$

Thus, $ax_1 + (1-a)x_2$ is in the set and hence it is convex; since a is arbitrary, the result follows. ||

The converse to Corollary 2.2 is not true:

Let f be a concave symmetric function on R which is not in H_1 . Then the sets $\{x: f(x) \geq a\}$ are convex symmetric subsets of R^1 . However f is not in H_1 .

The following lemma is due to Hoeffding and is a restatement of Lemma 2 of Lehmann [1966].

2.3 Lemma. Let X and Y be extended-valued random variables. Then

$$\text{Cov}(X, Y) = \int_{R^2} \text{Cov}\{I_{X^{-1}[x, \infty]}, I_{Y^{-1}[y, \infty]}\} dx dy, \text{ where the}$$

equality is valid even if one side is infinite.

2.4 Lemma. For any $K \in L_{n-1}$, $h \in H_{n-1}$, and $f \in H_n$, the function $g: R \rightarrow R_+$ defined by

$$g(x) = \int_K f(x_1, \dots, x_{n-1}, x) h(x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1}$$

is in H_1 .

Proof: The logconcavity of g follows from Theorem 6 of Prekopa [1973]. The symmetry of g follows from the symmetry of f , h , and k . Thus g is in H_1 , as desired. ||

2.5 Theorem. Let f and g be in H_1 . Suppose that (Ω, F, P) is a probability space and X is an extended-value random variable defined on (Ω, F, P) . Then for each f, g in H_1 we have

$$\text{Cov}(f(X), g(X)) \geq 0.$$

Proof: By Lemma 2.3 we have

$$\text{Cov}(f(X), g(X)) = \int_{R^2} \text{Cov}\{I_{[x, \infty]} \circ f(X), I_{[y, \infty]} \circ g(X)\} dx dy$$

since

$$I_{[x, \infty]} \circ f(X) = I_{f^{-1}[x, \infty]} \circ X.$$

From Corollary 2.2 we know that there exists a constant $a > 0$ such that $f^{-1}[x, \infty] = [-a, a]$. Thus there exists an $a > 0$ such that

$$I_{[x, \infty]} \circ f(X) = I_{[-a, a]} \circ X$$

Similarly we conclude that there is a $b > 0$ such that

$$I_{[y, \infty]} \circ g(X) = I_{[-b, b]} \circ X.$$

Therefore,

$$\text{Cov}(I_{[x, \infty]} \circ f(X), I_{[y, \infty]} \circ g(X)) \text{ equals}$$

$$P\{X \in [-a, b], [a, b]\} - P\{X \in [-a, a]\} \cdot P\{X \in [-b, b]\},$$

which is clearly nonnegative. From Lemma 2.3 it follows that $\text{Cov}(f(X), g(X))$ must be nonnegative. ||

2.6 Theorem. Let $X_1, \dots, X_n$ be independent random variables each having a density that belongs to H_1 . Then for all f and g in G_n we have $\text{Cov}(f(X), g(X)) \geq 0$.

Proof: We proceed by induction on n . By Theorem 2.5 the result is true for $n=1$. Now assume the result is true for some n_0 . For f and g in H_{n_0+1} and $\underline{X} = (X_1, \dots, X_{n_0+1})$, we write

$$\begin{aligned} \text{Cov}(f(\underline{X}), g(\underline{X})) &= E[\text{Cov}(f(\underline{X}), g(\underline{X}) | X_{n_0+1})] \\ &\quad + \text{Cov}[E f(\underline{X} | X_{n_0+1}), E g(\underline{X} | X_{n_0+1})]. \end{aligned}$$

For a fixed x_{n_0+1} , the function $f_{x_{n_0+1}} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is defined for any

$\underline{x} = (x_1, \dots, x_{n_0})$ by $f_{x_{n_0+1}}(x_1, \dots, x_{n_0}) = f(x_1, \dots, x_{n_0+1})$. For

$\underline{x}_1 = (x_{11}, \dots, x_{1n_0})$, $\underline{x}_2 = (x_{21}, \dots, x_{2n_0})$, we let $\underline{x}_1^* = (x_{11}, \dots, x_{1n_0},$

$x_{n_0+1})$ and $\underline{x}_2^* = (x_{21}, \dots, x_{2n_0}, x_{n_0+1})$. Then for a in $(0, 1)$,

$$\begin{aligned} f_{x_{n_0+1}}(a \underline{x}_1 + (1-a) \underline{x}_2) &= f(a \underline{x}_1^* + (1-a) \underline{x}_2^*) \geq f^a(\underline{x}_1^*) f^{1-a}(\underline{x}_2^*) \\ &= f_{x_{n_0+1}}^a(\underline{x}_1) f_{x_{n_0+1}}^{1-a}(\underline{x}_2) \end{aligned}$$

Moreover, for any matrix A in A_{n_0} and $\underline{x} = (x_1, \dots, x_{n_0})$ in R^{n_0} and $\underline{x}^* = (x_1, \dots, x_{n_0}, x_{n_0+1})$, we have $f_{x_{n_0+1}}(\underline{x}A) = f(\underline{x}^*A^*)$,

where $A^* =$ the $(n_0+1) \times (n_0+1)$ diagonal matrix defined by $A^* = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Since f is in G_{n_0+1} , then we have that

$$\begin{aligned} f_{x_{n_0+1}}(\underline{x}A) &= f(\underline{x}^*A^*) \\ &= f(\underline{x}^*) \\ &= f_{x_{n_0+1}}(\underline{x}). \end{aligned}$$

Therefore $f_{x_{n_0+1}}(x_1, \dots, x_{n_0})$ is in G_{n_0} . The induction hypothesis combined with the above argument gives

$$\begin{aligned} E[\text{Cov}\{f(\underline{X}), g(\underline{X})\} | X_{n_0+1} = x_{n_0+1}] &= \\ E[\text{Cov}\{f_{x_{n_0+1}}(\underline{X}^*), g_{x_{n_0+1}}(\underline{X}^*)\}] &\geq 0 \end{aligned}$$

where $\underline{x}^* = (x_1, \dots, x_{n_0})$ when $\underline{x} = (x_1, \dots, x_{n_0+1})$.

From Lemma 2.4 and the hypothesis, we deduce that

$Ef(\underline{X} | X_{n_0+1} = x_{n_0+1})$ as well as $Eg(\underline{X} | X_{n_0+1} = x_{n_0+1})$ are in H_1 .

By Theorem 2.5 we have $\text{Cov}\{Ef(\underline{X} | X_{n_0+1}), Eg(\underline{X} | X_{n_0+1})\} \geq 0$. Thus we finally conclude that $\text{Cov}(f(\underline{X}), g(\underline{X})) \geq 0$. ||

The proof of the following theorem can be obtained by imitating the proof of Theorem 2.6 .

2.7 Theorem. Let $X = (X_1, \dots, X_n)$ be a standard normal vector with density n . Then

$$\int_{A \cap B} n(x) dx \geq \left(\int_A n(x) dx \right) \left(\int_B n(x) dx \right).$$

for all subsets A, B that are symmetric along all the axes.

Section 3. Bounds on Logarithmically Concave Densities.

In this section we derive some inequalities for strongly unimodal densities. First we define for $n=1,2,\dots$,

$$U_n = \{f: \mathbb{R}^n \rightarrow \mathbb{R}_+; f \text{ is a logarithmically concave density}\}.$$

3.1 Lemma. Let f and g be functions mapping $\mathbb{R}^n$ into $\mathbb{R}_+$. Then

$$\int f \ln(f/g) \geq (\int f) \ln(\int f / \int g).$$

In particular, if f is a density function on $\mathbb{R}^n$, then

$$\int f \ln(f/g) \geq -\ln \int g,$$

for any measurable function $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$.

Proof: First assume $\int f = \int g$. Then

$$\begin{aligned} \int f \ln(f/g) &= -\int f \ln(g/f) \geq -\int f \{(g/f) - 1\} \\ &[\text{since } \ln x \leq (x-1), x \geq 0] \\ &= \int f - \int g = 0. \end{aligned}$$

Thus the inequality is satisfied in this case.

Now, if $\int f \neq \int g$, then define $g^* = (\int f / \int g)g$, and note that $\int f = \int g^*$. Therefore, using the above inequality we have $\int f \ln(f/g^*) \geq 0$. Using the definition of g^* and simplifying we get $\int f \ln(f/g) \geq (\int f) \ln(\int f / \int g)$. ||

3.2 Theorem. Let X be a random vector with density f belonging to U_n with finite mean $\underline{\mu}$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $\int \exp(-g) = 1$, then $f(\underline{\mu}) \geq \exp(-\int g f)$.

Proof: Take $f_1 = f$, $f_2 = e^{-g}$. By Lemma 3.1, we have $\int f_1 \ln(\frac{f_1}{f_2}) \geq 0$. Take $f_1 = f$, $f_2 = e^{-g}$. Using Jensen's inequality and the fact that f

belongs to U_n , we deduce that

$$0 \leq \int f \{ \ln f + g \} \leq \ln f(\mu) + \int fg,$$

completing the proof. ||

3.3 Theorem. Let $\underline{X}$ be a nonnegative random vector with density f belonging to U_n with finite mean $\underline{\mu}$. Then $f(\underline{\mu}) \geq \frac{1}{\mu_1 x_1 \dots x_n} e^{-n}$. Equality is attained for $f(\underline{x}) = \prod_{i=1}^n \frac{1}{\mu_i} e^{-x_i/\mu_i}$ and therefore the bound is sharp.

Proof: Choose $g(\underline{x}) = \sum_1^n \log \frac{\mu_i}{a_i} + \sum_1^n a_i \frac{x_i}{\mu_i}$ for $\underline{x} \geq 0$. Then

$\int e^{-g} = \prod_{i=1}^n \int_0^\infty \frac{a_i}{\mu_i} e^{-a_i x_i/\mu_i} dx_i = 1$. Also $\int g f = \sum \log \frac{\mu_i}{a_i} + \sum a_i$. Thus by Theorem 3.2, $f(\underline{\mu}) \geq \prod_{i=1}^n (\frac{a_i}{\mu_i} e^{-a_i})$. The right hand side is maximized by choosing $a_i = 1, i=1, \dots, n$. Equality is attained for $f(\underline{x}) = \prod_{i=1}^n \frac{1}{\mu_i} e^{-x_i/\mu_i}$, as may be directly verified. ||

The following theorem gives a lower bound on the peak of density function belonging to U_n in terms of the determinant of its covariance matrix.

3.4 Theorem. Let $\underline{X}$ be a multivariate vector with mean $\underline{\mu}$, covariance matrix Σ , and density f belonging to U_n . Then

$$f(\underline{\mu}) \geq (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp(-n/2).$$

Proof: Let $g(\underline{x}) = \frac{1}{2} \ln |\Sigma| + (n/2) \ln (2\pi) + \frac{1}{2} (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})$.

Then $\int e^{-g} = 1$, and $\int g(\underline{x}) f(\underline{x}) d\underline{x} = E g(\underline{X}) = \frac{1}{2} \ln |\Sigma| + (n/2) \ln (2\pi) + n/2$.

The desired conclusion then follows from Theorem 3.2. ||

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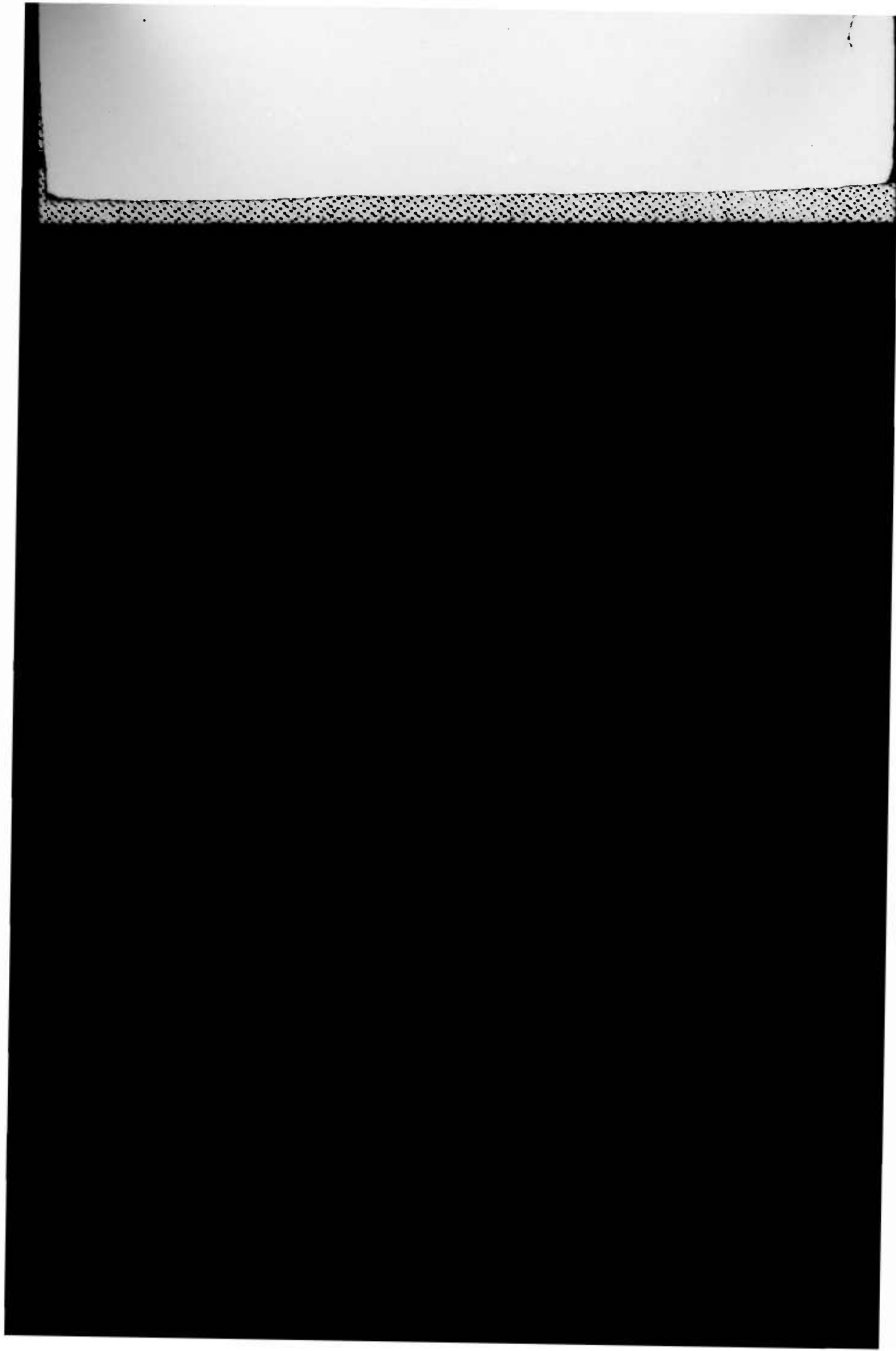
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