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RESEARCH INTO TOPICS IN TRANSONIC FLOW THEORY(U)
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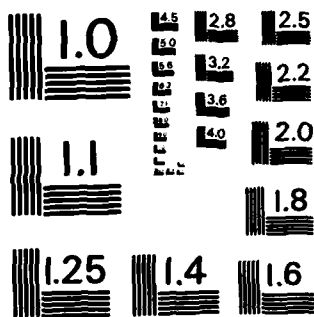
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by

David Nixon

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RESEARCH INTO TOPICS IN
TRANSONIC FLOW THEORY

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David Nixon

NEAR TR 331

Prepared under Contract No. N00014-83-C-0456

for

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Arlington, VA 22217

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The contract concerned certain research topics in transonic flow, mainly addressing the questions of why corrections for entropy and viscosity to transonic potential theory were large in certain cases. The physical implications arising from the accurate use of each of these corrections are incompatible. The investigation is due to the fact that the potential equation admits non-unique solution branches and that the corrections, in addition to modeling entropy and viscosity, allow the potential solution to		

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change to the correct branch. In view of this hypothesis, an analytic study into the causes of the nonunique potential solutions was initiated.

The study into the causes of the nonuniqueness was based on an integral equation formulation and the following results obtained.

a. The nonuniqueness only occur if there is a supersonic domain in the flow.

b. A nonunique solution does not have the same supersonic domain as a real solution.

c. The solution often can be made unique in steady flow by using a mixed conservative/nonconservative algorithm; this cannot be guaranteed however.

d. There is no way to differentiate between a solution that is physically correct and one that is not physically correct.

An additional study into the relative magnitudes of the effect of vorticity and entropy gradient on transonic solutions indicated that both are equal for most flow regions. However, the shock strength is controlled only by the entropy. This has relevance to the "strong shock" potential theories in which entropy but not vorticity is retained in the model.

Finally, an investigation into the stability of shock free supercritical flows was performed for three-dimensional and time-dependent cases. It is concluded that the results of the two-dimensional proof of Morawetz that such flows are unstable to small perturbations are also applicable to three-dimensional and unsteady flows.

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RESEARCH INTO TOPICS IN TRANSONIC FLOW THEORY

(ONR CONTRACT N00014-83-C-0456)

Summary

The contract concerned certain research topics in transonic flow, mainly addressing the questions of why corrections for entropy and viscosity to transonic potential theory were large in certain cases. The physical implications arising from the accurate use of each of these corrections are incompatible. The investigation concluded that the main cause of the large corrections is due to the fact that the potential equation admits nonunique solution branches and that the corrections, in addition to modeling entropy and viscosity, allow the potential solution to change to the correct branch. In view of this hypothesis, an analytic study into the causes of the nonunique potential solutions was initiated.

The study into the causes of the nonuniqueness was based on an integral equation formulation and the following results obtained.

- a. The nonuniqueness only occur if there is a supersonic domain in the flow.
- b. A nonunique solution does not have the same supersonic domain as a real solution.
- c. The solution often can be made unique in steady flow by using a mixed conservative/nonconservative algorithm; this cannot be guaranteed however.

- d. There is no way to differentiate between a solution that is physically correct and one that is not physically correct.

An additional study into the relative magnitudes of the effect of vorticity and entropy gradient on transonic solutions indicated that both are equal for most flow regions. However, the shock strength is controlled only by the entropy. This has relevance to the "strong shock" potential theories in which entropy but not vorticity is retained in the model.

Finally, an investigation into the stability of shock-free supercritical flows was performed for three-dimensional and time-dependent cases. It is concluded that the results of the two-dimensional proof by Morawetz are unstable to small perturbations are also applicable to three-dimensional and unsteady flows.

A detailed report on the work is attached as the Appendix.

1. INTRODUCTION

In the present range of computational fluid dynamics methods for transonic flow there is a hierarchy of equations that can be solved. For practical purposes the top of the range is the Reynolds averaged Navier-Stokes equations with some form of turbulence model. These equations will model transonic flows with viscous effects, strong shock waves and separated flow; the results of solving these equations agree quite well with experimental data. The next step in the simplification of the flow model is to solve the Euler equations which are essentially the Navier-Stokes equations with the viscous effects eliminated. The elimination of the diffusive effects leads to a loss of probable uniqueness in the problem in that a Kutta condition at the trailing edge of the wing may be necessary to close the problem. The Euler equations will model flows with strong shock waves and the vorticity caused by shocks. The final step in the simplification of the Navier-Stokes equations for transonic flows is the neglect of entropy and vorticity in the Euler equations which leads to the potential equation, or for small disturbances, the Transonic Small Disturbance (TSD) equations. These potential formulations will treat any attached flows with weak shock waves.

In certain cases the potential theory can be coupled with a boundary layer model to treat flows with mild viscous effects; for example see Ref. 1. This modification generally weakens the shock wave and moves it upstream. Another recent modification of potential theory are the strong shock theories pioneered by Nixon (Ref. 2) and Klopfer and Nixon (Ref. 3), which use a potential theory but with entropy introduced at the shock wave. The results of both these theories are considerably better than potential theory if the potential shock waves are strong. In fact the correction to the shock locations due to the inclusion of viscous and entropy effects is nearly the same, although by different mechanisms, and these results agree relatively well

with the solutions of the Navier-Stokes equations. The strong shock potential theories give results that are almost identical with the Euler Equations solutions. One question that now arises is why the potential theory plus boundary layer gives such good results when entropy and vorticity effects associated with the shock wave are neglected; this implies that vorticity and entropy effects are negligible relative to viscous effects. A second question is why does the strong shock theory give such good results if compared to the Euler equations results since vorticity is neglected; this implies that vorticity effects are negligible relative to entropy effects which would appear to be in contradiction to Crocco's Theorem. Finally a comparison of results of the potential, Euler and Navier-Stokes equations indicate that viscous effects are negligible compared with entropy and vorticity effects. The above noted implications are inconsistent and it is the purpose of the present work to resolve at least some of these inconsistencies.

In the investigation some computational results led to the hypothesis that the dominant factor in the large entropy and viscous corrections is the presence of the nonunique solutions first noted by Steinhoff and Jameson (Ref. 4) and further studied by Salas et. al. (Ref. 5) and Nixon (Ref. 6). It is suggested therefore, that the effects of viscosity, entropy and vorticity are relatively small compared with the effects of a nonunique solution. Accordingly a study into the causes of the nonuniqueness was initiated and the following conclusions reached.

First, it can be shown that the problem does not occur for subcritical flows, at least one shock wave is required in the flow. Secondly, the solution can be composed of a real lift component and a fictitious lift component due to the nonunique behavior; these two components appear as a pair and no way to decouple them could be found. Thirdly the appearance of the

nonunique component can be countered in steady flow by the use of a mixed nonconservative/conservative algorithm. This does not guarantee a unique solution for all cases however.

A second part of the study concerned the relative effects of entropy gradient and vorticity. It is found that the entropy gradient and vorticity effects are the same magnitude behind the shock but that there is an additional, perhaps dominant, effect of entropy in determining the shock strength and hence, position.

A separate study concerns the nonexistence proofs for shock free flow first outlined by Morawetz (Ref. 7) and alternately by Nixon and Kerlick (Ref. 8). The present work extends the theory of Ref. 8 to three dimensional steady and two dimensional unsteady flows.

2. VISCOUS EFFECTS

In order to get information for a study of the viscous effects the computer code GRUMFOIL (Ref. 1) was used to compute the flow around a NACA 0012 airfoil at $M_\infty = 0.8$, $\alpha = 0.75^\circ$. It proved difficult to obtain convergence but at least a partially converged result (residual of $\sim 10^{-3}$) was obtained. The grid sequence of iterations 600/400/500 was used; the fine grid is 160 X 32. The results are shown in Figure 1. The same problem was then solved with the boundary layer omitted and this result is also shown in Figure 1. In view of the considerable difference a third calculation was performed namely only ten iterations on the coarse and medium grids were used as a starting solution for 800 iterations on the fine grid. The result is also shown in Figure 1.

Because of the disparity in results the same case was calculated both viscously and inviscidly using the code TAIR (Refs. 9, 10). The inviscid results are compared with those of GRUMFOIL in Figure 2, the viscous calculations in Figure 3. The

TAIR results were converged.

Apart from the above calculations a series of inviscid calculations for $M_\infty = 0.8$ and different angles of attack were performed using GRUMFOIL and the results are shown in Figure 4. One set of results is for the usual grid sequencing algorithm while the other uses 10 iterations on the coarse and medium grids and 800 iterations on the fine grid.

Although the fact that most of the above results are not fully converged raises a warning flag the results are somewhat disquieting to the author since it appears that no great reliance can be placed on the computations. It is probable that the consistent multiplicity of results is due to the nonuniqueness of the potential formulation (Refs. 4-6) and that the viscous effects are in fact small, of the order of that shown in Figure 3. Consequently, the next part of the study was an investigation of the uniqueness of the potential formulation.

3. NONUNIQUE SOLUTIONS

In recent years multiple solutions to the numerical approximation to the full potential equations have appeared in the literature (Refs. 4 and 5). Initially the phenomena appeared in computations of the flow over a symmetric airfoil at zero angles of attack where two lifting solutions were present in addition to the expected nonlifting solution. In reference 4 some results, for a nonsymmetric airfoil, a RAE 2822 section, are also presented. Steinhoff and Jameson (Ref. 4) suggested that the change from one of the solutions to another is discontinuous and noted a hysteresis effect indicating that the lift coefficient (C_L) depended on whether the angle of attack (α) was increasing or decreasing. More recent work is by Salas (Ref. 5) who has extended the computations of the flows considered by Steinhoff and Jameson (Refs. 4) to show that it is possible to

construct a smooth C_L - α curve connecting the three solutions for a symmetric airfoil.

The investigations noted above are meticulously performed and are essentially numerical experiments. There is a limited amount of understanding that can be gained from such experiments and consequently a more analytic technique may yield additional information. Furthermore, although the numerical results are invaluable they do not exclude the possibility that the multiple solutions are due to the numerical approximation to the differential equation. The present investigation is based on the integral equation formulation (Ref. 11) which allows some degree of insight into the problem.

The transonic integral equation method of reference 11 is only applicable to the transonic small disturbance (TSD) equation rather than the full potential equation (FPE) that is used in the earlier work. Consequently the first step is to reproduce multiple solutions using the TSD equation. Once these solutions are obtained they can be analyzed using the ideas of the transonic integral equation theory.

3.1 Multiple Solutions for Small Disturbance Theory

Since it is easiest at present to use the integral equation theory to analyze small disturbance theory it is necessary to reproduce the multiple solutions using the TSD equation. This is achieved by using the computer code TSFOIL (Ref. 12) which solves the TSD equation using the conservative Murman-Cole algorithm and grid sequencing. The multiple solutions are found for a symmetric airfoil at zero angle of attack by imposing a 1° angle of attack on the coarse grid solution and then putting the angle of attack equal to zero in the medium and fine grid operations. It is found that such a device leads to multiple solutions over a small range of Mach numbers. Such solutions have been found for

a 11.8% Joukowski airfoil and a NACA 0012 airfoil. As a test for convergence, the solution for the Joukowski airfoil at $M_\infty = 0.85$ was converged to a residual of 10^{-7} . Krupp scaling was used in these results; the default Krupp grid is used, which has grid dimensions of 77 x 56. An example of a multiple solution is given in figure 5.

3.2 Transonic Integral Equation Theory

Since the analysis of the multiple solutions is based on the integral equation theory it is helpful to outline the formulation.

For Krupp scaling the TSD equation can be written as

$$(1 - \phi_x) \phi_{xx} + \phi_{yy} = 0 \quad (1)$$

where if $\bar{\phi}$ is the perturbation velocity potential in a cartesian coordinate system $(\bar{x}, \bar{y})$ then

$$\begin{aligned} \phi &= k/\beta^2 \bar{\phi} \\ x &= \bar{x} \\ y &= \beta \bar{y} \end{aligned} \quad (2)$$

where

$$\begin{aligned} k &= (\gamma + 1) M_\infty^{1.75} \\ \beta^2 &= 1 - M_\infty^2 \end{aligned} \quad (3)$$

If $u = \phi_x$, $v = \phi_y$ then the physical perturbation velocity components $(\bar{u}, \bar{v})$ are given by

$$\begin{aligned} \bar{u} &= (\beta^2/k) u \\ \bar{v} &= (\beta^3/k) v \end{aligned} \quad (4)$$

In the formulation of Equation (1) the sonic point is given by

$$u = \phi_x = 1 \quad (5)$$

The boundary conditions for Equation (1) are that $\phi_x, \phi_y \rightarrow 0$ on the far field, that the tangency condition

$$\phi_y(x, \pm 0) = Y'_s(x, \pm 0) - \bar{A} \quad (6)$$

is satisfied and that the Kutta condition of zero velocity jump at the trailing edge and on the wake is satisfied. In Equation (6)

$$Y_s(x, \pm 0) = k/\beta^3 \bar{Y}_s(x, \pm 0) \quad (7)$$

where $y = \bar{Y}_s(x, \pm 0)$ denotes the geometry of the airfoil on its upper and lower surfaces, respectively. $\bar{A}$ is given by

$$\bar{A} = k/\beta^3 \alpha \quad (8)$$

where α is the angle of attack, although it should be noted that in the formulation used in TSFOIL

$$\bar{A} = k/\beta^3 M_\infty^{-1/4} \alpha \quad (9)$$

The basic idea of the integral equation method is to use Green's theorem to write the differential equation, Equation (1), and its associated boundary conditions in integral form. A detailed description of the method is given in reference 13.

For $y \neq 0$ the integral equation is given by

$$u - u^2/2 = u_{LS} + u_{LA} + I_T(u) \quad (10)$$

where

$$u_{LS}(x, y) = \frac{1}{2\pi} \int_0^1 [Y'_s(\xi, +0) - Y'_s(\xi, -0)] K_x(x, \xi; y, 0) d\xi \quad (11)$$

and is the solution of Equation (1) without the nonlinear terms,

$$u_{LA}(x, y) = \frac{1}{\pi} \int_0^1 \Delta u(\xi) K_\eta(x, \xi; y, 0) d\xi \quad (12)$$

where

$$\Delta u(\xi) = [u(\xi, +0) - u(\xi, -0)]/2 \quad (13)$$

The kernel function $K(x, \xi; y, \eta)$ is given by

$$K(x, \xi; y, \eta) = \frac{1}{2} \ln[(x - \xi)^2 + (y - \eta)^2] \quad (14)$$

The integral I_T is given by

$$I_T(u) = -\frac{1}{4\pi} \int_S \int K_{\xi x}(x, \xi; y, \eta) u^2(\xi, \eta) dS \quad (15)$$

The domain S is shown in figure 6.

If $y = \pm 0$, Equation (10) gives (see ref. 13) only the symmetric part of the solution and the antisymmetric part is given by

$$-\bar{A} + \frac{1}{2} [y'_S(x, +0) + y'_S(x, -0)] = -\frac{1}{\pi} \int_0^1 \frac{[\Delta u(\xi) - \Delta u^2(\xi)/2]}{x - \xi} d\xi + I_C(x) \quad (16)$$

where

$$\Delta u^2(\xi) = [u^2(\xi, +0) - u^2(\xi, -0)]/2 \quad (17)$$

and

$$I_C(x) = -\frac{1}{4\pi} \int_S \int K_{\xi y}(x, \xi; 0, \eta) [u^2(\xi, \eta) - \hat{u}^2(\xi)] dS \quad (18)$$

where

$$\hat{u}^2(\xi) = \begin{cases} u^2(\xi, +0), & \eta > 0 \\ u^2(\xi, -0), & \eta < 0 \end{cases} \quad (19)$$

This version of the theory is for normal shock waves and the integrals are continuous.

In the solution of the integral equations the circulation is given by Equation (16) although it should be emphasized that Equations (10) and (16) are not independent. The solution of Equation (16) is found by inverting the integral equation to give

$$\Delta u(x) - \frac{\Delta u^2(x)}{2} = u_{LA}(x,0) - \frac{1}{\pi} \left(\frac{1-x}{x}\right)^{1/2} \int_0^1 \frac{I_c(\xi)}{x-\xi} \cdot \left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \quad (20)$$

where u_{LA} is the antisymmetric solution of Equation (1) without the nonlinear terms and is given by

$$u_{LA}(x,0) = \frac{1}{\pi} \left(\frac{1-x}{x}\right)^{1/2} \int_0^1 \frac{-\bar{A} + \frac{1}{2} [Y'_S(\xi,+0) + Y'_S(\xi,-0)]}{(x-\xi)} \left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \quad (21)$$

The inversion procedure invokes the Kutta condition.

3.3 Application of the Integral Equations to Multiple Solutions

Consider for the moment the case of a symmetric airfoil at zero angle of attack in this case

$$\begin{aligned} \bar{A} &= 0 \\ Y_S(x,+0) &= -Y_S(x,-0) \end{aligned} \quad (22)$$

and the left hand side of Equation (16) is zero. A manipulation of the integrals in Equation (16) leads to the equation

$$\int_{S_1} \int K_{xy} [u^2(\xi,\eta) - u^2(\xi,\eta) - 2\Delta u(\xi)] d\xi d\eta = 0 \quad (23)$$

where S_1 covers half of the domain S . Equation (23) can be written in the more compact form.

$$\int_{S_1} \int K_{\xi\eta} f(\xi, \eta) d\xi d\eta = 0 \quad (24)$$

where

$$f(\xi, \eta) = [u^2(\xi, \eta) - u^2(\xi, -\eta) - 2\Delta u(\xi)]$$

One solution of this integral equation is

$$f(\xi, \eta) = 0 \quad (25)$$

which is the symmetric solution for the airfoil problem. A non-symmetric solution can be obtained if there is one or more functions $f(\xi, \eta) \neq 0$ that satisfy Equation (24). If there are such solutions then a multiple solution in transonic flow can exist if the equation

$$\frac{1}{2} [u^2(\xi, \eta) - u^2(\xi, -\eta) - 2\Delta u(\xi)] = f(\xi, \eta) \quad (26)$$

has a real solution for $u(\xi, \eta)$.

One nonzero form of $f(\xi, \eta)$ is

$$f(\xi, \eta) = \begin{cases} \frac{C}{[\xi(1-\xi)]^{1/2}} & ; 0 < \xi < 1 \\ 0 & ; \xi < 0, \xi > 1 \end{cases} \quad (27)$$

where C is an arbitrary constant. This is the form of $f(\xi, \eta)$ that is equivalent to the eigensolution that arises in the linear airfoil problem.

Consider now Equations (26) and (27) on $\eta = 0$; thus

$$\Delta u(\xi) = \frac{2C}{[\xi(1-\xi)]^{1/2} [u(\xi, +0) + u(\xi, -0) - 2]} \quad (28)$$

It can be seen that $\Delta u(\xi) \rightarrow \infty$ at the trailing edge, $\xi = 1$, unless the average velocity

$$[u(\xi, +0) + u(\xi, -0)]/2$$

behaves like

$$1 + (1 - \xi)^{-p/2}$$

as $\xi \rightarrow 1$, where $p > 1$. Hence, unless the constant C is zero, either $\Delta u(\xi)$ or the average velocity $\{u(\xi, +0) + u(\xi, -0)\}/2$ is infinite at the trailing edge. The condition that the velocity is finite at the trailing edge is thus sufficient to eliminate the eigensolution by putting $C = 0$. This is a slight variation on the classic use of the Kutta condition.

This integral equation is valid for shock waves normal to the freestream; if the shocks are not normal to the freestream a modified integral equation is used (ref. 14). Since the formulation changes are negligible the above set of equations will be used in the subsequent algorithm for clarity.

3.4 Number and Nature of Eigensolutions

It is desirable to estimate the number and the nature of the eigensolutions, $f(\xi, \eta)$ of equation (24). Consider equations (16) and (18). Using the inversion procedure that leads to equation (20), equations (16), (18) can be written in the form

$$\int_{S_1} \int K_{\xi\eta} [(u^2(\xi, \eta) - u^2(\xi, -\eta) - 2\Delta u(\xi) - 2I_{CL}(\xi))] d\xi d\eta = 0 \quad (29)$$

where

$$I_{CL}(x) = \frac{1}{\pi} \left(\frac{1-x}{x} \right)^{1/2} \int_0^1 \frac{-\bar{A} + \frac{1}{2} [Y'_S(\xi) + Y'_S(\xi)]}{(x-\xi)} \left(\frac{\xi}{1-\xi} \right)^{1/2} d\xi \quad (30)$$

and $u(\xi, \eta)$ is a transonic solution subject to the arbitrary boundary conditions in equation (6). Since the boundary condition is arbitrary it follows that an infinite number of solutions $u(\xi, \eta)$ exist.

Equation (29) is identical in form to equation (24) with $f(\xi, \eta)$ given by.

$$f(\xi, \eta) = u^2(\xi, \eta) - u^2(\xi, -\eta) - 2\Delta u(\xi) - 2I_{c_L}(\xi) \neq 0 \quad (31)$$

Since there are an infinite number of transonic solutions it follows that there are an infinite number of eigensolutions $f(\xi, \eta)$. A justification of a nonzero $f(\xi, \eta)$ is given in the Appendix.

3.5 Symmetric and Antisymmetric Integral Equations

Let $\bar{u}(\xi, \eta)$ and $\Delta u(\xi, \eta)$ be defined by

$$\bar{u}(\xi, \eta) = \frac{1}{2} [\bar{u}(\xi, \eta) + \bar{u}(\xi, -\eta)] \quad (32a)$$

$$\Delta u(\xi, \eta) = \frac{1}{2} [\bar{u}(\xi, \eta) - \bar{u}(\xi, -\eta)] \quad (32b)$$

Equation (10) can be manipulated to give the following symmetric and asymmetric parts.

$$\bar{u}(x, y) - \frac{1}{2} [\bar{u}^2(x, y) + \Delta u^2(x, y)] = u_{L_s} - \frac{1}{4\pi} \int_{s_1} \int_{\xi x} [\bar{u}^2(\xi, \eta) + \Delta u^2(\xi, \eta)] d\xi d\eta \quad (33)$$

$$\Delta u(x, y) - \Delta u(x, y) \bar{u}(x, y) = -\frac{1}{4\pi} \int_{s_1} \int_{\xi x} [u^2(\xi, \eta) - \bar{u}^2(\xi, \eta) - 2\Delta u(\xi, 0)] d\xi d\eta \quad (34)$$

These equations will be used in the following sections

3.6 First Compatibility Condition

If in equation (24), $f(\xi, \eta)$ is assumed known then equation (23) gives

$$\frac{1}{2} [u^2(\xi, \eta) - u^2(\xi, \eta) - 2\Delta u(\xi, 0)] = \Delta u(\xi, \eta) \bar{u}(\xi, \eta) - \Delta u(\xi, 0) = f(\xi, \eta) \quad (35)$$

If $\bar{u}(\xi, \eta)$ is known then this is an equation for $\Delta u(\xi, \eta)$ in the flow field and gives

$$\Delta u_0 = \frac{f_0}{\bar{u}_0 - 1} \quad (36)$$

where

$$f_0 = f(\xi, 0) \text{ etc.};$$

also

$$\Delta u = \frac{f + \Delta u_0}{\bar{u}} = \frac{1}{\bar{u}} \left[f - \frac{f_0}{1 - \bar{u}_0} \right] \quad (37)$$

In the normal transonic solution equation (16) only gives the value of $\Delta u(\xi, 0)$ or Δu_0 , the value of $\Delta u(x, y)$ being found from equation (34). Hence for a nonunique solution to exist Δu , as defined by equations (36) and (37) must be compatible with equation (34).

Substitution of equations (35), into equation (34) gives

$$\Delta u - \Delta u_0 = f - \frac{1}{4\pi} \int_{S_1} K_{\xi x} f \, d\xi d\eta \quad (38)$$

or, using equations (36), (37)

$$\frac{1}{\bar{u}} \left[f - f_0 \frac{(1 - \bar{u})}{(1 - \bar{u}_0)} \right] = f - \frac{1}{4\pi} \int_{S_1} K_{\xi x} f \, d\xi d\eta \quad (39)$$

An alternative form of equation (39) is the differential equation

$$F_{xx} + F_{yy} = f_x \quad (40)$$

$$F_y(x, \pm 0) = 0 \quad (41)$$

where

$$F(x, y) = \int^x \left\{ \frac{1}{\bar{u}(\xi, y)} f(\xi, y) - f(\xi, 0) \frac{[1 - \bar{u}(\xi, y)]}{[1 - \bar{u}(\xi, 0)]} \right\} d\xi \quad (42)$$

$$= \int^x [\Delta u(\xi, y) - \Delta u(\xi, 0)] d\xi \quad (43)$$

The far field boundary condition is that $F_x, F_y \rightarrow 0$.

The basic differential equation, (1) can be decoupled into symmetric and asymmetric parts. The asymmetric part is

$$\Delta \phi_{xx} + \Delta \phi_{yy} = (\Delta u \bar{u})_x \quad (44)$$

where

$$\Delta \phi(x, y) = \frac{1}{2} [\phi(x, y) - \phi(x, -y)] \quad (45)$$

the boundary condition is

$$\Delta \phi_y(x, 0) = -\bar{A} + \frac{1}{2} [Y'_s(x, +0) + Y'_s(x, -0)] \quad (46)$$

The symmetric part is

$$\bar{\phi}_{xx} + \bar{\phi}_{yy} = \frac{1}{2} [\bar{u}^2 + \Delta u^2]_x \quad (47)$$

with

$$\bar{\phi}_y(x, 0) = \frac{1}{2} [Y'_s(x, +0) - Y'_s(x, -0)] \quad (48)$$

Now equation (40) can be modified to give

$$G_{xx} + G_{yy} = g_x \quad (49)$$

where

$$G(x,y) = \int^x \frac{1}{\bar{u}(\xi,y)} g(\xi,y) d\xi \quad (50)$$

and

$$g(\xi,y) = f(\xi,y) - \frac{f(\xi,0)}{1-\bar{u}(\xi,0)} \quad (51)$$

with the boundary condition

$$G_y(x,\pm 0) = 0 \quad (52)$$

If $G(\xi,y)$ is identified with a lifting term $\Delta\phi$ then equations (47) and (49) can be added to give

$$(\bar{\phi} + G)_{xx} + (\bar{\phi} + G)_{yy} = \frac{1}{2} [(\bar{u} + \frac{g}{\bar{u}})^2]_x \quad (53)$$

$$\bar{\phi}_y(x,\pm 0) + G_y(x,\pm 0) = \frac{1}{2} [Y'_s(x,+0) - Y'_s(x,-0)] \quad (54)$$

where Δu on the right hand side of equation (47) has been identified as $g/\bar{u}$. Thus a fictitious value, G , can be added to a purely symmetric problem, denoted by $\bar{\phi}$, to give lift; the boundary condition is not affected. This is the mechanism of the appearance of the nonunique solutions. Equations (53) (54) are similar in form to equations (1) (6) if ϕ is identified with $\bar{\phi} + G$. The numerical algorithm will solve equations (53) in an identical manner to the real solution.

If there is real lift, that is a value that is calculated with no eigensolutions, then the lifting solution of equations (44) can be added to equation (53) to give

$$(\bar{\phi} + \Delta\phi^* + G)_{xx} + (\bar{\phi} + \Delta\phi^* + G)_{yy} = \frac{1}{2} [(\bar{u} + \frac{\Delta u^* + g}{\bar{u}})^2]_x \quad (55)$$

where "*" denotes a real lifting component. In this case Δu on the right hand side of equation (47) is identified with $[\Delta u^* + g/\bar{u}]$.

The boundary condition is

$$\bar{\phi}_y(x, \pm 0) + \Delta \phi_y^*(x, \pm 0) + G_y(x, \pm 0) = Y'_s(x, \pm 0) - \bar{A} \quad (56)$$

Again a fictitious lifting component $G(x, y)$ is added to the equation without a change in the boundary conditions. If G is present the numerical algorithm will solve equations (55) in an identical manner as the real solution. It is not possible to decouple the real and fictitious components.

3.7 Second Compatibility Condition

In the preceeding analysis it has been assumed known. For a solution to exist it also must be a solution of equation (33).

Equation (39) can be written as

$$\left(\frac{1-\bar{u}}{\bar{u}}\right) g = -\frac{1}{4\pi} \int_{S_1} \int K_{\xi x} f d\xi d\eta = I_f \quad (57)$$

where g is given by equation (51). It follows from equation (57) that

$$\bar{u} = \frac{g}{g + I_f} \quad (58)$$

it also follows from equation (58) that

$$\Delta u = g/\bar{u} = g + I_f \quad (59)$$

From equation (36)

$$\Delta u_o = \frac{f_o}{\bar{u}_o - 1}$$

and if the limit as $y \rightarrow 0$ is taken of equation (33) and if equations (58), (59) are used then

$$\bar{u}_0 - \frac{u_0^2}{2} = u_{LS_0} + \frac{1}{2} \left(\frac{f_0}{u_0 - 1} \right)^2 - \frac{1}{4\pi} \lim_{y \rightarrow 0} \int_{S_1} \int K_{\xi x} \left[\frac{g^2}{(g+I_f)^2} + (g+I_f)^2 \right] d\xi dn \quad (60)$$

where u_{LS_0} is the value of u_{LS} at $y = 0$.

Since f is assumed known and u_{LS_0} is known, equation (60) is an equation for $\bar{u}_0$. Hence g can be found.

Using equations (58), (59) to eliminate $\bar{u}$ and Δu in equation (33) it follows that a nonunique solution only exists if there is a solution f , to the equation

$$\frac{g}{g+I_f} - \frac{1}{2} \left(\frac{g}{g+I_f} \right)^2 = u_{LS} + \frac{1}{2} (g+I_f)^2 - \frac{1}{4\pi} \int_{S_1} \int K_{x\xi} \left[\frac{g^2}{(g+I_f)^2} + (g+I_f)^2 \right] d\xi dn \quad (61)$$

This solution must also satisfy equation (24).

If there is real lift due to the boundary condition of equation (6) then equation (24) gets replaced by

$$\int_{S_1} \int K_{\xi y} [f + \Delta u^* \bar{u}^* - \Delta u_0^* - I_{c_L}] d\xi dn \quad (62)$$

where I_{c_L} is given by equation (30) and $\Delta u^*, \bar{u}^*$ denote a lifting and a symmetric component of some solution u^* that satisfies the boundary condition represented by I_{c_L} . Since the shock waves in u^* cannot occur at the same location as in the real solution, u^* cannot be the real solution.

If f^* is defined by

$$f^* = \Delta u^* \bar{u}^* - \Delta u_0^* \quad (63)$$

then it follows that equation (26) is replaced by

$$\Delta u \bar{u} - \Delta u_0 = f + f^* \quad (64)$$

The analysis in section 3.6 and the present section is unchanged if f is replaced by $f + f^*$ since f always occurs as part of the term $f + f^*$; in this case $f + f^*$ must satisfy equation (62) rather than equation (24).

3.8 General Remarks on Integral Equations

Consider, for example, equation (10), which can be written as

$$u - u^2/2 = u_L + I_T(u^2) \quad (65)$$

The integral I_T can be discretized to give

$$u_i - u_i^2/2 = u_{L_i} + \sum_{j=1}^N A_{ij} u_j^2 \quad (66)$$

where A_{ij} is an influence coefficient and N is the number of discrete elements used in the evaluation of the integral.

The algebraic set of equation, (66) is quadratic and has, in general, 2^N solutions. As $N \rightarrow \infty$ there are an infinite number of solutions. However, if, during the solution of u_i say, only one root is taken then the number of solutions is 1^N or in other words a unique solution is obtained. Thus equation (66) will give a unique solution if one of the two choices of u_i is eliminated. A multiplicity of solutions occurs if a choice is allowed.

In the general transonic solution a change of sign occurs at the sonic line and the shock wave (Reference 13). The correct solution is that choice of root, or sign, that eliminates expansion shocks. A change in the location of the sign change indicates an alteration in the size of the supersonic domain.

Equation (61) can be written as

$$\left(\frac{g}{g+I_f} + I_f + g\right) - \frac{1}{2} \left(\frac{g}{g+I_f} + I_f + g\right)^2 = u_{LS} + u_{LA} - \frac{1}{4\pi} \int_{S_1} \int K_{x_S} \left[\frac{g}{I_f + g} + g + I_f\right]^2 d\xi d\eta \quad (67)$$

where

$$u_{LA} = \int_0^1 \frac{-f_0/(1-\bar{u}_0) y}{(x-\xi)^2 + y^2} d\xi$$

In equation (67) there is the possibility of two solutions for $\left[\frac{g}{I_f + g} + I_f + g\right]$; this is equivalent to u . If only the real solution is considered, that is, u^* , the value of u corresponding to f^* , then equation (61) will have a specified combination of sign changes. If the location of these sign changes does not alter, then, since u^* must be the only solution of the system, no fictitious term can arise. A fictitious term u can arise if there is an alteration of the sign changes, that is, the size of the supersonic domain changes. Thus in general, a nonunique solution will change the location of the shock wave.

In a subsonic flow only one root, the subsonic (negative) root is chosen and, since no sign change is possible, it follows that a nonunique solution cannot occur. Thus a nonunique solution can only occur if there is at least one shock wave in the flow. It should be noted that there is no information about the number of nonunique solutions since this depends on the number of possible shock locations.

3.9 Non-Conservative Algorithms

A non-conservative algorithm, such as that of Murman and Cole. (ref. 15) adds source terms at the shock waves. Hence the algorithm is solving conservatively a differential equation of the form.

$$(1-\phi_x) \phi_{xx} + \phi_{yy} = [\sigma_i H(x - x_{s_i})]_x \quad (68)$$

where $\sigma_i(y)$ is a source term at the shock location x_{s_i} and $H(\)$ is the step function. The strength of the source term is not known explicitly in nonconservative algorithms.

If the asymmetric part of equation (68) is taken then

$$\Delta\phi_{xx} + \Delta\phi_{yy} = (\Delta u\bar{u})_x + \Delta[\sigma_i H(x - x_{s_i})]_x \quad (69)$$

If equation (69) is added to equation (49) the following result is obtained.

$$(\Delta\phi+G)_{xx} + (\Delta\phi+G)_{yy} = (\Delta u\bar{u})_x + \{g + \Delta[\sigma_i H(x - x_{s_i})]\}_x \quad (70)$$

If a fictitious positive component of lift starts to appear in a solution g will change from zero to a predominantly positive quantity, since g is equivalent to $(\Delta u\bar{u})$. This introduction of positive lift will change the location of the shock waves such that an upper surface shock moves aft while a lower surface shock moves forward. It follows that the source term introduced by the shock on the lower surface now acts at a further forward location while the countering source on the upper surface moves aft. Consequently, there is a considerable region between the shocks for which

$$\Delta[\sigma_i H(x - x_{s_i})] < 0$$

This counters the positive g and reduces the tendency of the nonunique solution to appear. However, it is necessary to point out that the nonconservative source error must be large enough to counter the g terms. The nonconservative algorithm only inhibits a nonunique solution from appearing; it will not necessarily remove an existing error. A similar analysis can be performed if the fictitious lift is negative. This hypothesis was tested by

using the computer code TSFOIL (Ref. 12) which has both conservative and nonconservative algorithms. A composite algorithm, given by

$$\text{Algorithm} = \lambda (\text{conservative}) + (1 - \lambda) (\text{nonconservative}) \quad (71)$$

was used. The parameter , λ , was taken to be given by

$$\lambda = -\epsilon | [C_L(t+\Delta t) - C_L(t)] | + 1 \quad (72)$$

Where t is the artificial iteration time and Δt is the iteration step. At convergence $\lambda = 1$ and the solution is conservative. It was found that for the nonunique solutions to be avoided in a computation of the example in Figure (5) $\epsilon [C_L(t+\Delta t) - C_L(t)]$ had to be of order unity during most of iteration. Smaller values did not inhibit nonuniqueness. Hence it is suggested that the "classic" degree of nonconservative algorithm is necessary to stop the appearance of nonunique solutions.

It is of interest to note that if a stabilizing term of the type used in TSFOIL namely $\epsilon \phi_{xt}$ is added to equation (1), where t is an artificial iteration time and ϵ is a parameter, then an analysis similar to that given in this section shows that this term will assist the formation of a nonunique solution. This is possibly the reason why the real solution cannot sometimes be computed.

3.10 Summary

The analysis given in this section can be summarized as follows.

The TSD equation can admit eigensolutions that satisfy all of the boundary conditions generally found in such problems; there are an infinite number of these eigensolutions. For a

nonunique solution to exist certain compatibility requirements must be met. These eigensolutions provide lift and act like an additional asymmetric source term in the flow field. If real lift is present the fictitious component provided by the nonuniqueness appears as a simple additive term to the real lifting component; there is no way to distinguish or uncouple the two components in a given numerical solution.

The nonuniqueness does not appear for subcritical flows and if, in a supercritical flow, it does appear the location of the shock waves changes from their real location. The nonuniqueness may be removed in some steady cases by using a nonconservative conservative algorithm which inhibits the appearance of such solutions. However, there is no guarantee that this will work in all cases. This partial cure only works for steady flows; it is possible that for unsteady flows that a "strong shock" theory of the type advocated in Ref. 3 could remove any nonuniqueness.

Since the behavior of the nonunique solutions is identical to that of the real solution the question arises as to whether these solutions are physically realizable. An analysis of the Navier-Stokes or, possibly, the Euler equations is necessary to determine this.

4.0 ENTROPY EFFECTS

The motivation for examining the magnitude of the entropy effects is the good agreement of the results of the strong shock theory of Klopfer and Nixon (Ref. 3) with results of the Euler equations. This good agreement implies that entropy effects are large compared to vorticity effects. This implication is studied in this section. The analysis technique will be based on an integral equation for the Euler equations developed by Nixon and Liu (Ref. 16).

4.1 An Integral Equation Formulation for the Euler Equations

For the present purpose only two-dimensional flow is considered. It is not necessary to consider the full set of the Euler equations at present and here only the conservation of mass equation

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0 \quad (73)$$

and the vorticity equation

$$\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = \omega \quad (74)$$

are considered, where ρ is density and ω is vorticity.

Equation (73) can be satisfied by a stream function where

$$\frac{\partial \psi}{\partial y} = \rho U; \quad \frac{\partial \psi}{\partial x} = -\rho V \quad (75)$$

Substitution of equation (75) into equation (74) gives

$$\psi_{xx} + \psi_{yy} = f \quad (76)$$

where

$$f = \rho \omega + \rho_y U - \rho_x V \quad (77)$$

Thin airfoil boundary conditions, namely

$$V(x, \pm 0) = -[\alpha + \gamma'_s(x, \pm 0)]U_\infty \quad (78)$$

are assumed to be applicable, and it is also assumed that freestream conditions apply ahead of and above and below the wing. At the downstream boundary constant (freestream) pressure is imposed.

Following the analysis of Nixon and Liu (Ref. 16) equation (76) can be put in integral form; thus

$$\psi = \frac{1}{2\pi} \int_0^\infty \{\Delta\psi_\eta K_O - K_{O\eta} \Delta\psi\} d\xi + \frac{1}{2\pi} \int_S K f dS \quad (79)$$

where

$$K = \frac{1}{2} \ln \{(x-\xi)^2 + (y-\eta)^2\} \quad (80)$$

and

$$K_O = \frac{1}{2} \ln \{(x-\xi)^2 + y^2\} \quad (81)$$

Differentiation of equation (79) with respect to y gives

$$\psi_y = \rho U = \frac{1}{2\pi} \int_0^\infty \{\Delta\psi_\eta K_{Oy} - K_{Oy\eta} \Delta\psi\} d\xi + \frac{1}{2\pi} \int_S K_y f dS \quad (82)$$

By integrating the second term in the line integral by parts with respect to ξ and the field integral by parts with respect to η , the following equation is obtained.

$$\begin{aligned} \rho U = \frac{1}{2\pi} \int_0^\infty \Delta(\rho V) K_{Ox} d\xi + \frac{1}{2\pi} \int_0^\infty \Delta(\rho U) K_{Oy} d\xi \\ - \frac{1}{2\pi} \int_S \int K_{\eta y} F dS \end{aligned} \quad (83)$$

where

$$F(x, y) = \int^y f(x, \eta) d\eta \quad (84)$$

Differentiation of equation (79) with respect to x gives

$$\psi_x = -\rho V = \frac{1}{2\pi} \int_0^\infty \{\Delta(\rho U) K_{Ox} - K_{O\eta x} \Delta\psi\} d\xi + \frac{1}{2\pi} \int_S K_x f dS \quad (85)$$

Performing the same integration by parts as above and letting $y \rightarrow 0$ gives

$$-\left[\rho V - \frac{\Delta(\rho V)}{2}\right]_0 = \frac{1}{2\pi} \int_0^\infty \frac{[\Delta(\rho U) - \Delta F]}{x-\xi} d\xi - \frac{1}{2\pi} \lim_{y \rightarrow 0} \int_S K_{\eta x} F dS \quad (86)$$

where the subscript "0" denotes a value at $y=0$. In equation (86) the left hand side is determined from the boundary condition, equation (78) ρ is known.

Consider the behavior of equation (83) as $x \rightarrow \infty$. The boundary condition for a flow variable P as $x \rightarrow \infty$ is

$$x \lim_{x \rightarrow \infty} \frac{\partial P}{\partial x} = 0 \quad (87)$$

Now, since the kernel K_{xy} vanishes as $|x - \xi| \rightarrow \infty$ it follows that as $x \rightarrow \infty$ only the value of $F(\xi, \eta)$ relatively close to x contributes to the integral. As $\frac{\partial F}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$ is a boundary condition $F(\xi, \eta)$ can be replaced by $F(x, \eta)$. Thus

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow \infty} \int_S K_{xy} F dS = \lim_{y \rightarrow 0} \left\{ \int_0^\infty F(x, \eta) [K_y|_{-\infty}^{x-\epsilon} + K_y|_{x+\epsilon}^\infty] d\eta + \int_{-\infty}^0 F(x, \eta) K_y|_{-\infty}^\infty d\eta \right\} = 0 \quad (88)$$

$$\lim_{x \rightarrow \infty} \int_0^\infty \frac{[\Delta(\rho U) - \Delta F]}{x - \xi} d\xi = [\Delta(\rho U) - \Delta F]_{x \rightarrow \infty} \{ \ln |x - \xi| \} \Big|_0^\infty$$

This integral is only finite if

$$\lim_{x \rightarrow \infty} [\Delta(\rho U) - \Delta F] = 0$$

In Reference (16) the use of the boundary condition of freestream pressure at $x \rightarrow \infty$ and equation (87) indicates that for symmetric flows

$$\lim_{x \rightarrow \infty} \rho U = \lim_{x \rightarrow \infty} F$$

Hence equation (87) is consistent with the result of Reference (16).

A final point is that, from equation (86),

$$\lim_{x \rightarrow \infty} \left[\rho V - \frac{\Delta(\rho V)}{2} \right]_0 = 0 \quad (91)$$

If the orthogonal coordinate system (x,y) is chosen such that the x axis aligns with the slit representing the airfoil and the wake it follows that equation (88) is satisfied, since there is no flow through the wake and hence

$$(\rho V)_0 = 0 \text{ on } y = 0 \quad (92)$$

4.2 Small Disturbance Versions of The Euler Equations in the Wake

It is convenient now to make a small disturbance assumption in the wake. Although the velocity over most of the airfoil may be large and these determine the entropy and vorticity distributions it is plausible to assume that at the trailing edge of the airfoil and on the wake a small disturbance form for the velocity can model the flow. Hence, in the region of the trailing edge and the wake it is assumed that

$$\begin{aligned} \rho &= \rho_\infty (1 + \rho') \\ U &= (1 + u) \\ V &= v \end{aligned} \quad (93)$$

and it is assumed that

$$|\rho'|, |u|, |v|, < < 1 \quad (94)$$

Furthermore, it is assumed that

$$\frac{S}{R} \sim \frac{S}{R}(u) \text{ in the wake} \quad (95)$$

Where S is entropy and R is the gas constant. Along the wake

$$\rho U - F = \int^Y [\rho U_Y - \rho \omega + \rho_x V] d\eta \quad (96)$$

From the energy equation (see Ref. 16)

$$\frac{\gamma}{(\gamma-1)} \cdot \left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} \frac{p_\infty}{\rho_\infty} e^{S/c_p} + \frac{1}{2} U^2 = \frac{1}{(\gamma-1)} a_\infty^2 + \frac{1}{2} U_\infty^2 \quad (97)$$

where C_p is the specific heat at constant pressure and a_∞ is the sonic speed at freestream conditions.

From equation (97)

$$\rho U_Y = -\frac{\rho}{U} a_\infty^2 e^{S/c_p} \left\{ \frac{1}{\gamma} \left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} \frac{p_Y}{p_\infty} + \frac{1}{(\gamma-1)} \left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} \frac{S_Y}{c_p} \right\} \quad (98)$$

from Crocco's theorem

$$\rho \omega = -\frac{p_\infty}{RU} \left(\frac{p}{p_\infty}\right)^\gamma S_Y e^{S/c_v} \quad (99)$$

Where c_v is the specific heat at constant volume

Using the gas relation

$$\frac{p}{\rho^\gamma} = \left(\frac{p_\infty}{\rho_\infty^\gamma}\right) e^{S/c_v} \quad (100)$$

Equation (98) can be written as

$$\rho U_Y = -\rho_\infty \left\{ \frac{1}{U} \left(\frac{p}{p_\infty}\right)_Y + \frac{p_\infty}{RU} \cdot \left(\frac{p}{p_\infty}\right)^\gamma S_Y e^{S/c_v} \right\} \quad (101)$$

Using equations (99) and (101) in equation (97) gives

$$\rho U - F = - \int^Y \left\{ \frac{p_\infty}{U} \left(\frac{p}{p_\infty}\right) - \rho_x V \right\} d\eta \quad (102)$$

If the small disturbance assumption is valid in the wake and linear approximations are valid equation (102) is approximated by

$$\rho U - F \approx - \frac{1}{U_\infty} p \quad (103)$$

and hence

$$\Delta(\rho U) - \Delta F \approx -\frac{1}{U_\infty} \cdot (\Delta p) = 0 \quad (104)$$

since there is zero pressure jump across the wake. In this case the limits of integration in the line integral of equation (86) are replaced by 0, 1 and an explicit wake calculation is not necessary however, the wake geometry does appear through the field integral. This is a similar situation to the behavior of a solution of the full potential equation with nonlinear boundary conditions.

If the small disturbance assumption of equations (93) - (95) are used, then, in the wake

$$\rho U - F = -\frac{P_\infty}{U_\infty} \left(\frac{p}{p_\infty}\right) \approx -\frac{P_\infty}{U_\infty} \left(1 - \frac{S}{R} - \gamma M_\infty^2 u\right) \quad (105)$$

where the pressure relation

$$\frac{p}{p_\infty} = \left\{ e^{\frac{-S}{c}} \left[1 + \frac{\gamma-1}{2} \cdot M_\infty^2 (1 - U^2/U_\infty^2) \right] \right\}^{\frac{\gamma}{\gamma-1}} \quad (106)$$

has been expanded for small u and $\frac{S}{R}$. Using equation (104) in equation (86) yields

$$\begin{aligned} -\left[\rho V - \frac{\Delta(\rho V)}{2}\right]_0 = & \frac{1}{2\pi} \int_0^1 \frac{[\Delta(\rho U) - \Delta F]}{x-\xi} d\xi \\ & - \frac{1}{2\pi} \int_S \int K_{ox\eta} F dS \end{aligned} \quad (107)$$

This equation can be inverted to give

$$\Delta(\rho U) - \Delta F = -\frac{2}{\pi} \left(\frac{1-x}{x}\right)^{\frac{1}{2}} \int_0^\infty \frac{A}{(x-\xi)} \cdot \left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi \quad (108)$$

where

$$A = -\left[\rho V - \frac{\Delta(\rho V)}{2}\right]_0 + \frac{1}{2\pi} \lim_{y \rightarrow 0} \int_S \int K_{x\eta} F dS \quad (109)$$

In the inversion use has been made of the fact that at the trailing edge

$$\Delta(\rho U) - \Delta F \sim -\frac{1}{U_\infty} \Delta p = 0 \quad (110)$$

This is a Kutta condition and is necessary to ensure a finite value of $[\Delta(\rho V) - \Delta F]$ at the trailing edge. However, if the solution algorithm for the Euler equations automatically captures a contact discontinuity at the trailing edge across which Δp is zero then such a condition is superfluous. The necessity of a Kutta condition is determined by whether or not the algorithm captures a contact discontinuity at the trailing edge.

It is instructive to examine the wake problem if a "strong shock" potential theory (e.g. Ref. 3) is used.

The theory of Ref. 3 neglects vorticity while retaining entropy. If the small disturbance approximation is used in the wake, equation (103) is replaced by

$$\rho U - F = -\frac{1}{U_\infty} \left[p + \frac{p_\infty}{R} S \right] \quad (111)$$

and

$$\Delta(\rho U) - \Delta F = -\frac{p_\infty}{U_\infty R} \Delta S \text{ on the wake} \quad (112)$$

Substitution of equation (112) into equation (107) gives

$$-\left[\rho V - \frac{\Delta(\rho V)}{2} \right]_0 = -\frac{1}{2\pi} \int_0^1 \frac{[\Delta(\rho U) - \Delta F]}{x - \xi} d\xi + \frac{1}{2\pi} \frac{p_\infty}{U_\infty R} \int_1^\infty \frac{\Delta S}{x - \xi} d\xi - \frac{1}{2\pi} \int_S \int K_{ox\eta} F dS \quad (113)$$

Since the entropy jump is constant along the wake and is determined by the shock strength, the second term on the right hand side acts like a boundary condition. Since ΔS is constant

this integral is infinite and is thus inappropriate. In the numerical examples in Ref. 3 the downstream boundary is at a finite distance from the airfoil. In such cases the entropy integral is finite but is dependent on the location of the boundary. It appears that a consistent strong shock theory is not possible; for results that are independent of the computational boundary ΔS should be zero across the wake.

It should be noted that if ΔS is positive then the entropy integral acts to produce a negative camber and thus reduces the lift. This accounts, perhaps, for the excellent agreement of the method of Ref. 3 with the results of the Euler equations.

Equation (113) is similar to one used in aeroelasticity and can be inverted to give

$$\begin{aligned} \Delta(\rho U) - \Delta F = & \frac{2}{\pi} \left(\frac{1-x}{x}\right)^{\frac{1}{2}} \int_0^1 \left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} \frac{\tilde{A}}{x-\xi} d\xi \\ & + \frac{4}{\pi} \left(\frac{1-x}{x}\right)^{\frac{1}{2}} \int_1^\infty \frac{\Delta S}{(x-\xi)} d\xi \end{aligned} \quad (114)$$

where

$$\tilde{A} = A - \frac{1}{2\pi} \int_1^\infty \frac{\Delta S}{x-\xi} d\xi \quad (115)$$

and A is given by equation (109). In the inversion a Kutta condition ensuring that

$$[\Delta(\rho U) - \Delta F]_{x=1} = 0 \quad (116)$$

must be enforced. If the small disturbance formulation is applicable at the trailing edge, then, neglecting vorticity,

$$\Delta(\rho U) - \Delta F = \gamma M_\infty^2 p_\infty \Delta u \quad (117)$$

and thus the Kutta condition enforces a zero velocity jump at the trailing edge for a finite solution rather than a zero pressure jump as is the case with the Euler equations. In other words, the strong shock theory requires a Kutta condition to make the velocity jump finite at the trailing edge and this condition disallows the possibility of making the pressure jump zero unless $\Delta S = 0$.

4.3 Shock Location

The most noticeable difference between potential and Euler equation solutions is the shock location consequently it is of interest to investigate the mechanism that fixes the shock location. In the following analysis only non-lifting flows are considered.

The integral equation for non-lifting flows can be derived from equation (83) and gives

$$\rho U = \frac{1}{2\pi} \int_0^1 \Delta(\rho V) K_{Ox} d\xi - \frac{1}{2\pi} \int_S \int K_{ny} F dS \quad (118)$$

The shock location appears only through the field integral over S so that the location of the shock depends on the magnitude of F . The analysis can be simplified if the field integral is integrated by parts to give

$$\int_S \int K_{\xi y} F dS = - \int_S \int K_y f dS \quad (119)$$

and f is given by equation (77)

The vorticity on the airfoil surface is given by Crocco's theorem

$$\omega = - \frac{1}{R} \frac{\rho}{U} e^{\gamma-1} e^{S/c_v} \frac{\partial S}{\partial y} \quad (120)$$

and hence

$$\rho \omega = - \frac{1}{R} \frac{\rho}{U} e^{S/c_v} \frac{\partial S}{\partial y} \quad (121)$$

The density is given by

$$\rho = \rho_{\infty} e^{-S/R} \left\{ 1 + \frac{\gamma-1}{2} M_{\infty}^2 [1 - (U^2 + V^2)/U_{\infty}^2] \right\}^{1/(\gamma-1)} \quad (122)$$

If the function f is expanded about the freestream conditions then

$$f = \frac{-P_{\infty}}{RU_{\infty}} \frac{\partial S}{\partial y} - \frac{\rho_{\infty} U_{\infty}}{R} \frac{\partial S}{\partial y} - \rho_{\infty} M_{\infty}^2 U_{\infty} \frac{\partial u}{\partial y} + \text{second order terms} \quad (123)$$

The first term is an approximation to the vorticity term while the remaining first order terms are an approximation to $\rho_y U$. It can be seen that the vorticity term is small compared to the entropy effect in the density only if

$$\frac{\rho_{\infty} U_{\infty}^2}{\rho_{\infty}} = \gamma M_{\infty}^2 \ll 1 \quad (124)$$

Note that both the vorticity and entropy contributions have the same sign. Hence the inclusion of only one component is probably an improvement on the potential theory.

This condition is not satisfied generally in transonic flows. Hence, in this case vorticity effects and entropy gradient effects are the same order of magnitude.

There is another effect of entropy on a transonic solution, namely the effect on the shock jump condition. This weakens the shock wave and moves the location forward. The magnitude of this effect is difficult to estimate theoretically. As a general statement, vorticity cannot be neglected relative to entropy; if this is done only a part of the necessary overall correction is retained. However, the entropy correction alone is probably better than no correction.

4.4 Summary

In this section it has been shown that vorticity cannot be neglected if entropy is retained in the flow equations. However, if only the entropy effect is retained this is probably an improvement on potential theory. It has also been shown that if the entropy jump across the wake is retained in a strong shock theory the solution will be dependent on the size of the computational boundary; it is also impossible to have both a finite value of ΔU and a zero pressure jump at the trailing edge.

5.0 PERTUBATIONS OF A SHOCK FREE SUPERCRITICAL FLOW

In the early days of transonic flow research the question of the existence of a shock free supercritical flow was posed. The question was answered in a series of papers by Morawetz (7,17) who used the hodograph transformed equations to show that a shock free supercritical flow is unstable to arbitrary small perturbations. This instability was regarded as proof that such flows could not exist. However, several years later the work of Nieuwland (18), also using the hodograph transformation, indicated that shock free supercritical flows could exist. These two apparently conflicting results were reconciled by realizing that shock free supercritical flows could exist but were unstable to a small perturbation.

The hodograph transformation used in the analysis of Morawetz cannot be used for either three dimensional or unsteady flows. In the present note the transonic integral equation method (11,13) is used to examine the perturbation of a shock free supercritical flow. The analysis is a minor extension of that given in Reference (8).

5.1 Some Comments on Integral Equations

Since the analysis is to be based on the transonic integral equation method some basic points are outlined below. Full details of the particular formulation can be found in Reference 13.

The three-dimensional, low frequency transonic small disturbance equation is

$$(1-M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} - 2 \frac{M_\infty^2}{U_\infty} \phi_{xt} = (\gamma+1) M_\infty^2 \phi_x \phi_{xx} \quad (125)$$

z is normal to the wing.

This can be written in integral form to give (Ref. 14, 19).

$$u - \frac{m^2 u^2}{2} = I_L + I_S = F \quad (126)$$

where

$$I_L = \frac{1}{4\pi} \int_A \int \Delta \phi_z \bar{K}_{Ox} dA - \frac{1}{4\pi} \int_A \int \Delta u K_{Oz} dA \quad (127)$$

the kernel K is given by

$$K(x, \xi; y, \eta; z, \zeta) = 1 / [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{\frac{1}{2}} \quad (129)$$

A is the planform of the wing and wake, S is the flow domain after removing the wing, wake, shock waves and the field point (x, y) .

Also

$$u = \frac{(\gamma+1) M_\infty^2}{(1-M_\infty^2)} \phi_x; \quad B = 2 \frac{M_\infty^2 c}{U_\infty (1-M_\infty^2)} \quad (130)$$

For a shock normal to the freestream I_S is a smooth integral and m is unity. In general m is a function of the shock angle for I_S

to be a smooth function. I_L involves only the geometry of the wing. In the formulation sonic conditions exist when $u = 1$. Equation (126) can be rearranged to give

$$u = \frac{1}{m^2} [1 \pm (1 - 2 m^2 F)^{\frac{1}{2}}] \quad (131)$$

and the question arises as to which root to take. If the flow is initially subsonic then $u < 1$ and the negative root must be taken. As the air flows over the airfoil the velocity increases until $u = \frac{1}{m^2}$. This value is the maximum that the negative root in Equation^m(131) will permit and occurs when

$$F = \frac{1}{2m^2}$$

If the velocity is to increase beyond $1/m^2$ then the positive root must be taken. However, it is possible that the velocity could jump from a value less than $1/m^2$ by simply switching signs in Equation (138). This would give an expansion shock which is not permitted since this implies that entropy decreases. Hence at an "accelerating" switch, that is, a switch in sign when the flow is accelerating at that point, Equation (132) must be satisfied. A compression jump is possible and consequently Equation (132) need not be satisfied when the flow is decelerating. Finally, for the solution to be real throughout the domain, it is necessary that

$$F < \frac{1}{2m^2} \quad (133)$$

Equations (129) and (130) lead to the following regularity conditions which eliminate expansion shocks:

$$\left(\frac{\partial F}{\partial x}\right)_{x_0, y_0} = 0 \quad (134 \text{ a})$$

$$\left(\frac{\partial F}{\partial y}\right)_{x_0, y_0} = 0 \quad (134 \text{ b})$$

$$(F)_{x_0} = \frac{1}{2m^2} \quad (134 \text{ c})$$

Here, $x = x_0$, $y = y_0$ is the switching point for a zero strength jump. If Equations (134) are satisfied at a decelerating point in the flow then there is no compression shock and the flow is shock free.

The transonic integral equation is given by Equation (126) and if the flow has a supersonic zone then the regularity conditions of Equation (134) must be satisfied. These regularity conditions in general ensure that no discontinuities are present at given switching lines x_{oi} , y_{oi} ($i = 1, 2$). In addition to the $u(x, y, z, t)$ variable in equation (126) there are only four unknowns for the regularity conditions, namely the value of the x_{oi} , y_{oi} . Since six equations, (equations 134) must be satisfied to ensure that no discontinuities occur it is apparent that a transonic flow must in general have a discontinuity because there is not a sufficient number of unknowns to eliminate both shockwaves. This discontinuity is a compression shock if expansion shocks are excluded. However, it is possible that the necessary six regularity equations are satisfied by the four unknown x_{oi} , y_{oi} . In this case the flow has a supersonic zone but no discontinuities. An airfoil with this type of flow is a supercritical shock free airfoil.

Since $u(x, y)$ is a function of the forcing term I_L , Equations (134) can be written as

$$I_{Lx}(x_{oi}, y_{oi}, t) + I_{sx}(I_L, x_{oi}, y_{oi}, t) = 0 \quad (135 \text{ a})$$

$$I_{Ly}(x_{oi}, y_{oi}, t) + I_{sy}(I_L, x_{oi}, y_{oi}, t) = 0 \quad (135 \text{ b})$$

$$I_L(x_{oi}) + I_s(I_L, x_{oi}) = \frac{1}{2m^2} \quad (135 \text{ c})$$

For x_{oi}, y_{oi} ($i = 1, 2$) these are the six regularity equations for the four switching points x_{oi}, y_{oi} and in general the equation set is inconsistent. However, as noted above, it is possible that the flow is such that Equations (135) are satisfied at both x_{oi}, y_{oi} . It is of interest to see what happens if such a shock free supercritical flow is perturbed in some way. A perturbation, either in Mach number or geometry, is transmitted through the forcing term I_L and hence the effect of an infinitesimal change in I_L , ΔI_L , on the solution is investigated. Since the I_L and I_s are functions that are differentiable Equations (135) can be expanded to give

$$[I_{Lxy}(x_{oi}, y_{oi}, t) + I_{sxy}(I_L, x_{oi}, y_{oi}, t)] \Delta x_{oi} + \Delta y_{oi} [I_{Lyy}(x_{oi}, y_{oi}, t) + I_{syy}(I_L, x_{oi}, y_{oi}, t)] + \Delta I_{Ly}(x_{oi}, y_{oi}, t) + \Delta I_{sy}[I_L, \Delta I_L, x_{oi}, y_{oi}, t] = 0 \quad (136 a)$$

$$[I_{Lxx}(x_{oi}, y_{oi}, t) + I_{sxx}(I_L, x_{oi}, y_{oi}, t)] \Delta x_{oi} + \Delta I_{Lx}(x_{oi}, y_{oi}, t) + \Delta I_{sx}(I_L, \Delta I_L, x_{oi}, y_{oi}, t) + \Delta y_{oi} [I_{sxy}(I_L, x_{oi}, y_{oi}, t) + I_{Lxy}(x_{oi}, y_{oi}, t)] = 0 \quad (136 b)$$

$$\Delta I_{Li}(x_{oi}, y_{oi}, t) + \Delta I_s(I_L, \Delta I_L, x_{oi}, y_{oi}, t) = 0 \quad (136 c)$$

where ΔI_s is the change in I_s due to the change in I_L of ΔI_L . In the derivation of Equation (136 c), equation (135) is used. Equations (136) give six regularity conditions that must be satisfied to retain the shock free flow, and again only four parameters, $\Delta x_{oi}, \Delta y_{oi}$ ($i = 1, 2$) that can be chosen to effect these conditions. Equation (136 c) does not contain Δx_{oi} or Δy_{oi} and hence it is impossible to satisfy Equation (136 c) unless ΔI_L is considered an undetermined function. Thus a shock free flow cannot in general be perturbed, either by Mach

number or geometry, and still remain shock free. However, if ΔI_L can be chosen such that equations (136 c) satisfied for all $z = \text{constant}$ lines covering the supersonic zone then Equation (136 a, 136 b) can be satisfied by a judicious choice of $\Delta x_0, \Delta y_0$, thus it is possible in certain circumstances to perturb a shock-free flow and retain the shock-free characteristic if the geometry perturbation is suitably chosen. Since ΔI_L is a single parameter function of Mach number for angle of attack (see Reference 13), ΔI_L is determined for all (x, y, z, t) by a single parameter family. Thus it is not possible for a shock free airfoil to remain shock free under a simple Mach number or angle of attack change since Equations (136) cannot be satisfied on all $z = \text{constant}$ lines where $u > 1$.

5.2 Summary

A theory for the existence of shock free steady flow for airfoils (Ref. 8) has been extended to treat three dimensional low frequency unsteady flow. The general well known result, that shock free flows are singular points, unstable to small perturbations, is unaffected by the extension.

6.0 CONCLUDING REMARKS

The main objective of the research reported here is to determine the relative magnitude of viscosity, vorticity and entropy in a transonic calculation. The first conclusion is that, for attached flow, the viscous effect is relatively small; the large corrections reported in the literature being due to the presence of nonunique solutions. Second, the vorticity and entropy effects are the same order of magnitude, with the added effect of entropy alone on the shock strength and hence location. The report also considers the existence of shock free flows in three dimensional, low frequency unsteady transonic flows and the stability of such flows to small perturbations.

As a result of the investigation some insight into the appearance of nonunique solutions in transonic potential flow is given. Also the effect of entropy and vorticity on Euler equation solutions is studied.

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REFERENCES

1. Melnik, P. E., Chow, R., and Mead, H.: Theory of Viscous Transonic Flow Over Airfoils at High Reynolds Number. AIAA Paper 77-680, 1977.
2. Nixon, D.: Transonic Small Disturbance Theory with Strong Shock Waves. AIAA Journal Vol. 18, No. 6 1980.
3. Klopfer, G. H. and Nixon, D.: Non-Isentropic Potential Formulation for Transonic Flows. AIAA Journal Vol. 22 No. 6 1984.
4. Steinhoff, J. S. and Jameson, A.: Multiple Solutions of the Transonic Potential Flow Equation. AIAA Journal Vol. 20, No. 11, 1982.
5. Salas, M. D., Gumbert, C. R., and Turkel, E.: Nonunique Solutions to the Transonic Potential Flow Equation. AIAA Journal Vol, 22 No. 1, 1984.
6. Nixon, D.: Observations on the Occurrence of Multiple Solutions in Transonic Flow. Nielsen Engineering & Research Paper No. 179, 1983.
7. Morawetz, C. S.,: "On the Non-Existence of Continuous Transonic Flow Past Profiles, I," Comm. on P. & App. Math, Vol. 9, 1956.
8. Nixon, D. and Kerlick, G. D.: Perturbation of a Transonic Flow with Vanishing Shock Waves. NEAR Paper 153, 1982.
9. Holst, T.: Fast Conservative Algorithms for Solving the Transonic Full Potential Equation. AIAA Journal Vol. 18 No. 12. 1980.

REFERENCES Cont.

10. Van Dalsem, R.: Private Communication 1984.
11. Nixon, D.: Calculation of Transonic Flows Using an Extended Integral Equation Method. AIAA Journal, Vol. 15, No. 3, 1977.
12. Stahara, S. S.: Operational Manual for Two Dimensional Transonic Code TSFOIL. NASA CR 3064, 1978.
13. Nixon, D.: Calculation of Transonic Flows Using the Integral Equation Method. Ph.D. Thesis, University of London, England, 1976.
14. Nixon, D.: The Transonic Integral Equation Method with Curved Shock Waves. Acta Mechanica Vol. 32/1-3. 1979.
15. Murman, E. M. and Cole J. D.: Calculation of Plane Steady Transonic Flow. AIAA Journal, Vol. 9 No. 1 1971.
16. Nixon, D. and Liu Y.: Concepts for Radically Increasing The Numerical Convergence Rate of the Euler Equations. Nielsen Engineering & Research Report. TR 326 1974.
17. Morawetz, C.S.: On the Non-Existence of Continuous Transonic Flow Past Profiles, II, Comm. on P. & App. Math, Vol. 10, 1956.
18. Nieuwland, G. Y.: Theoretical Design of Shock Free Transonic Flow Around Airfoils, Aerospace Proceedings, Vol. 1, 1966.

REFERENCES Concluded

19. Nixon D.: High Subsonic Flow Past a Steady Finite Wing.
London, England Queen Mary College, London, England Report
OMC EP 1015. 1974.
20. Nixon, D. and Hancock, G. J.: Integral Equation Methods - A
Reappraisal, Symposium Transsonicum, Springer Verlag, 1976.

APPENDIX

Further Comments On The Number Of Eigensolutions

It is possible that the only solution to equation (31) is

$$f(\xi, \eta) = 0 \quad (A1)$$

In this case

$$\Delta u \bar{u} - \Delta u_o - I_{cL} = 0 \quad (A2)$$

and

$$\Delta u_o = \frac{I_{cL}}{\bar{u}_o - 1} \quad (A3)$$

$$\Delta u = - I_{cL} \left[\frac{\bar{u}_o}{\bar{u}_o - 1} \right] / \bar{u} \quad (A4)$$

If equation (A2) is substituted into equation (34) it follows that

$$\Delta u - \Delta u \bar{u} = \frac{1}{\pi} \int_0^1 \frac{I_{cL}(\xi) y}{(x-\xi)^2 + y^2} d\xi \quad (A5)$$

The right-hand side of equation (A5) can be recognized as Δu_L , the linear value of Δu in the flow field. Hence equation (A5) becomes

$$\Delta u (1 - \bar{u}) = \Delta u_L \quad (A6)$$

It can be seen that as $\bar{u} \rightarrow 1$ Δu becomes infinite which will give an expansion shock. Consequently, equation (A1) cannot be the correct solution of equation (34); a nonzero value of $f(\xi, \eta)$ must exist.

ITERATION
SEQUENCE AND

CODE	REMARKS	GRID	CONVERGENCE	C_L
GRUMFOIL	Viscous	600/400/500	0.3×10^{-3}	0.237
	Inviscid	600/400/500	0.3×10^{-3}	0.976
	Inviscid	10/10/800	0.5×10^{-3}	0.255
	Inviscid	Viscous (600-400) Inviscid fine grid, (500)	0.9×10^{-3}	0.205
TAIR	Inviscid	"C" grid	default	0.511
	Viscous	"C" grid	default	0.225
	Inviscid	"c" grid	default	0.418

TABLE 1

Various value of C_L for different codes

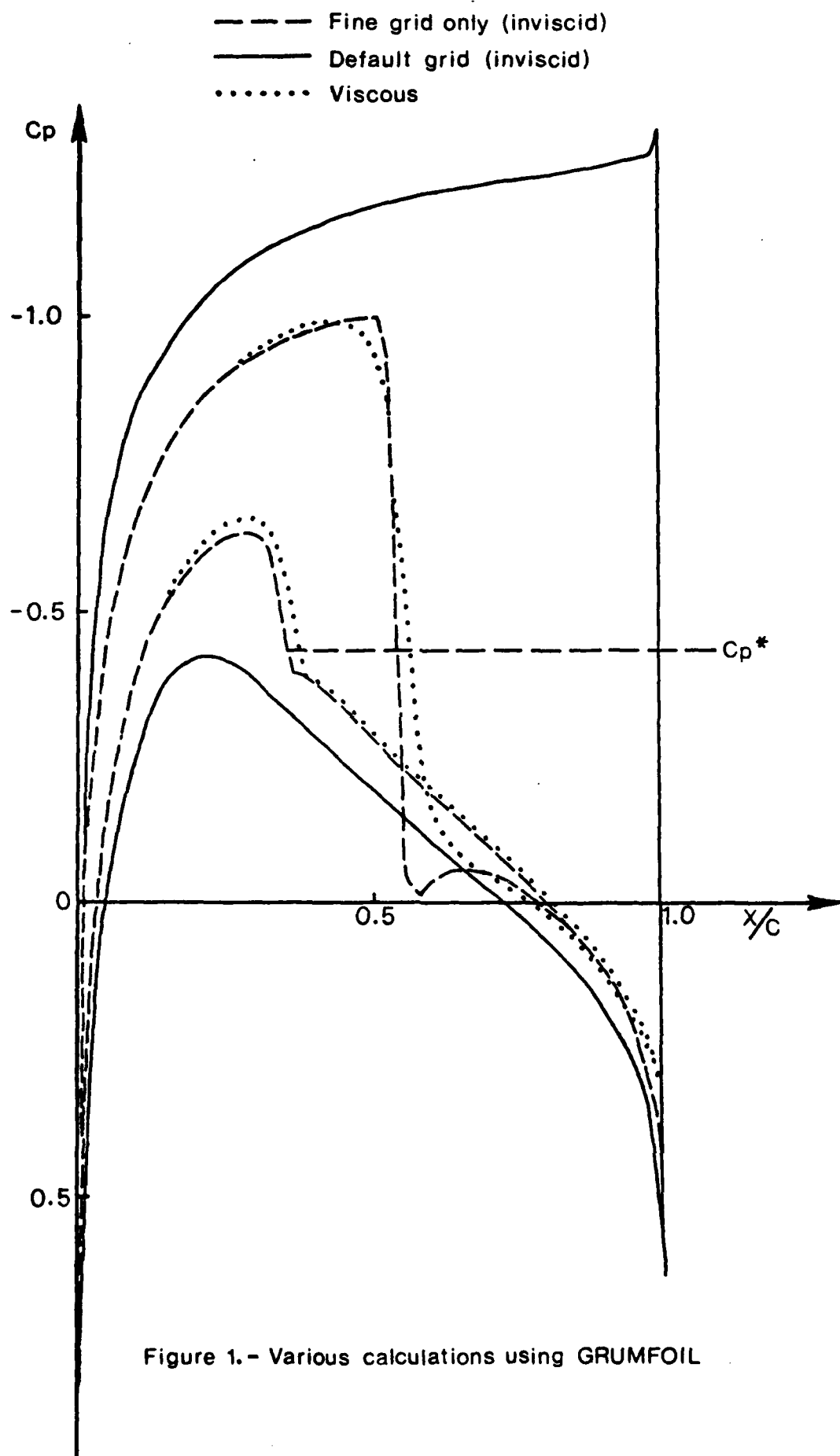


Figure 1.- Various calculations using GRUMFOIL

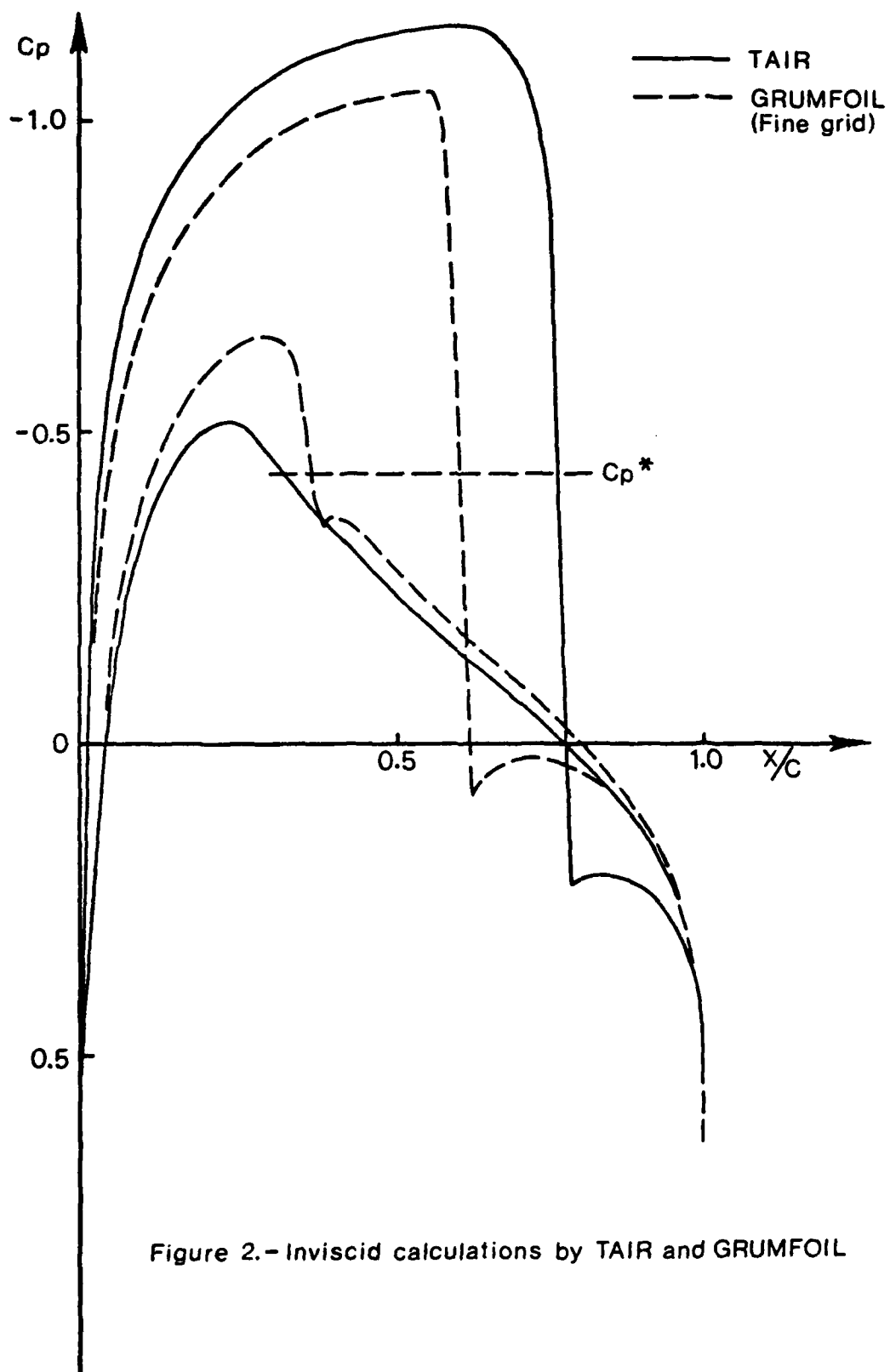


Figure 2.- Inviscid calculations by TAIR and GRUMFOIL

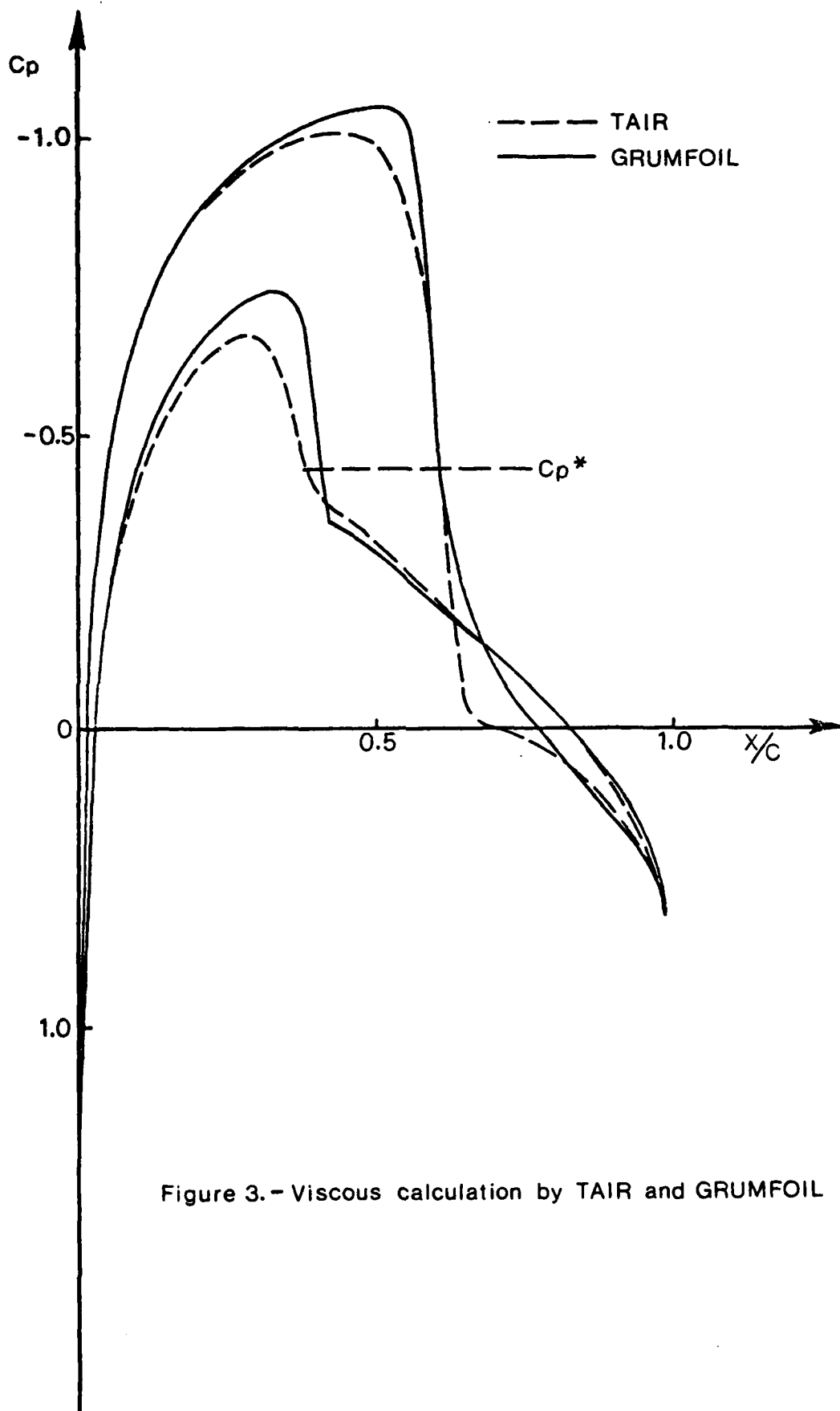


Figure 3.- Viscous calculation by TAIR and GRUMFOIL

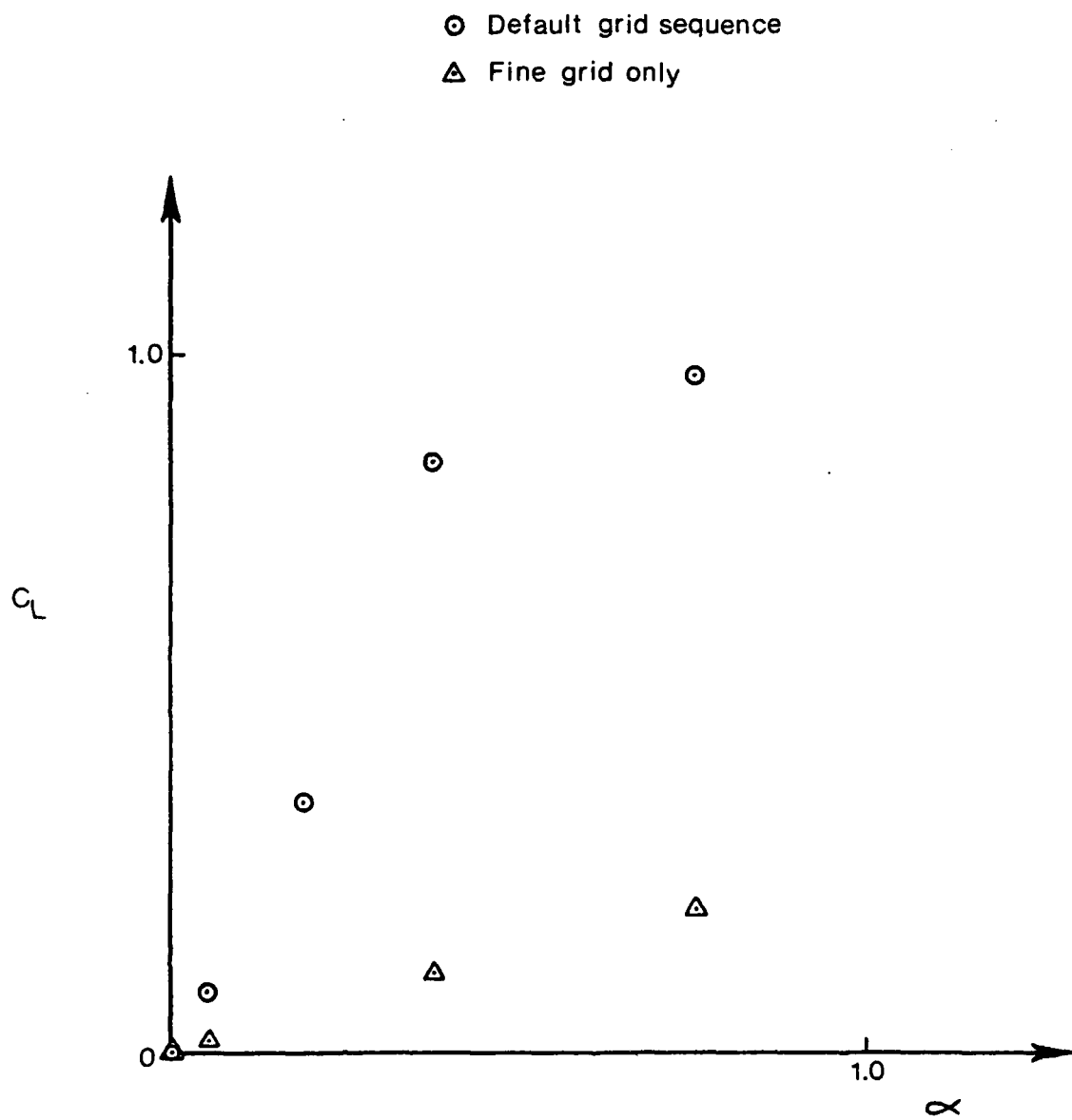


Figure 4.- Variation of C_L with α ; NACA 0012 Airfoil, $M_\infty = 0.8$.

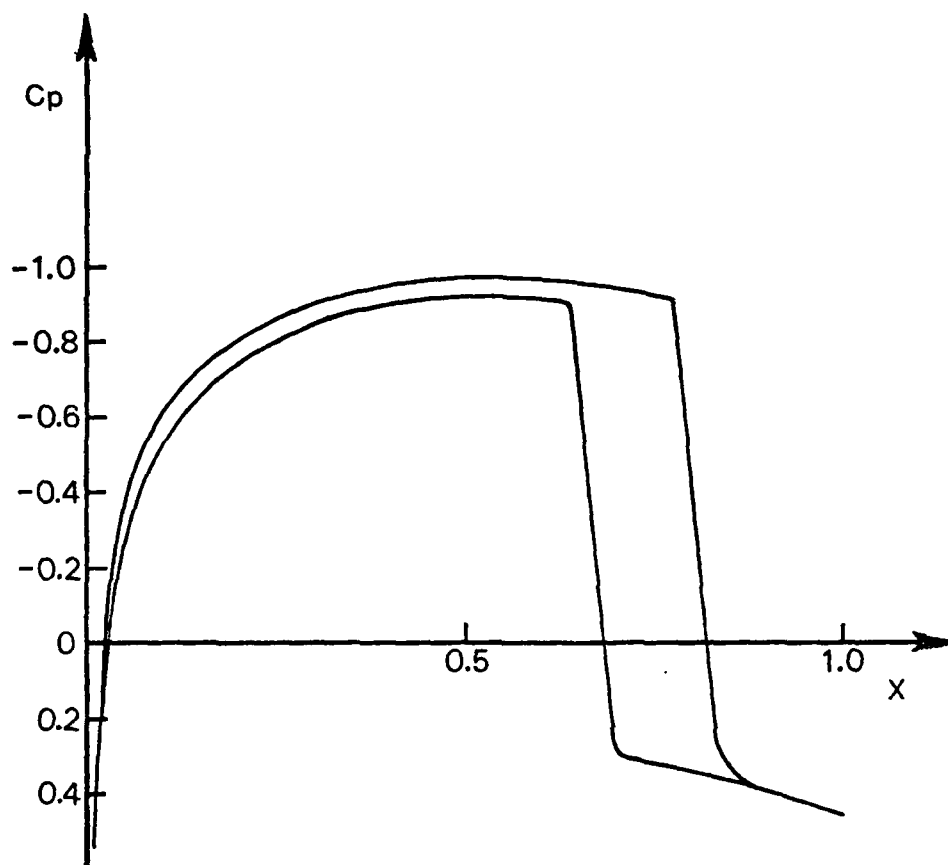


Figure 5.— Pressure distribution around a 11.8% Joukowski airfoil; $M_\infty = 0.85$, $\alpha = 0.0^\circ$.

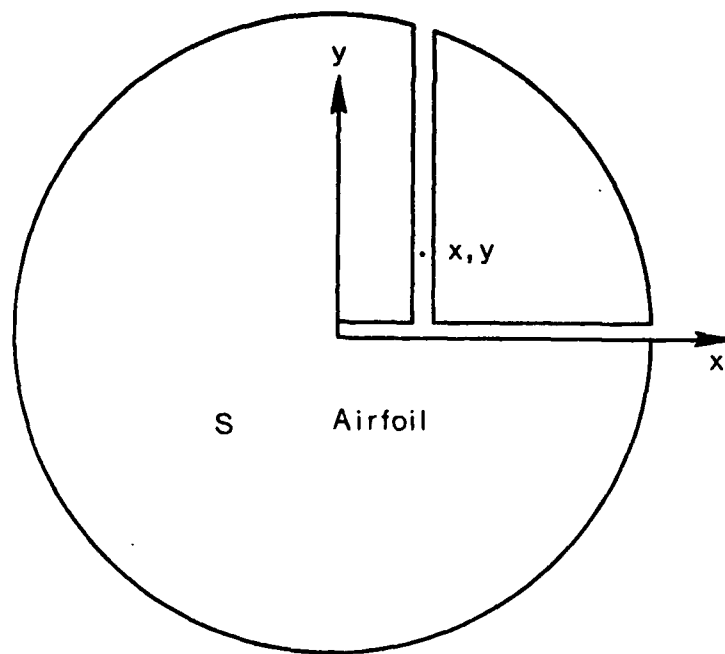


Figure 6.- Domain of integration S .

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