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STATISTIC FOR TESTING A S. (U) NORTH CAROLINA UNIV AT
CHAPEL HILL CENTER FOR STOCHASTIC PROC.
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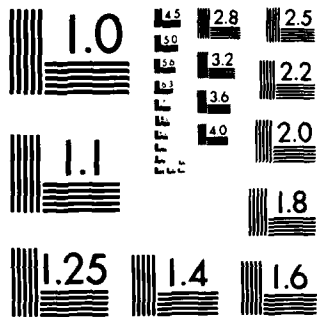
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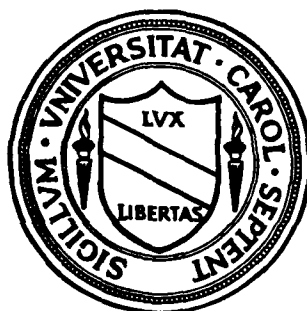
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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
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Chapel Hill, North Carolina

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The Asymptotic Behavior of the Likelihood Ratio Statistic for
Testing a Shift in Mean in a Sequence of Independent Normal Variates

by

Yi-Ching Yao

and

Richard A. Davis

Technical Report #80

September 1984

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 84 - 1152		
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR #80			7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research		
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina		6b. OFFICE SYMBOL (If applicable)		7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448	
6c. ADDRESS (City, State and ZIP Code) Statistics Department, Phillips Hall, 039-A, Chapel Hill NC 27514		8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620-82-C-0009	
8b. OFFICE SYMBOL (If applicable) NM		10. SOURCE OF FUNDING NOS.			
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448		PROGRAM ELEMENT NO. 61102F		PROJECT NO. 2304	TASK NO. A5
11. TITLE (Including Security Classification) THE ASYMPTOTIC BEHAVIOR OF THE LIKELIHOOD RATIO STATISTIC FOR TESTING A SHIFT IN MEAN IN A SEQUENCE OF INDEPENDENT NORMAL VARIATES					
12. PERSONAL AUTHOR(S) Yi-Ching Yao and Richard A. Davis					
13a. TYPE OF REPORT Technical		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Yr., Mo., Day) SEP 84	
				15. PAGE COUNT 15	
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB. GR.	Change point; normalized Brownian bridge; Ornstein-Uhlenbeck process; extreme value; asymptotics.		
19. ABSTRACT (Continue on reverse if necessary and identify by block number)					
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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL MAJ Brian W. Woodruff		22b. TELEPHONE NUMBER (Include Area Code) (202) 767- 5027		22c. OFFICE SYMBOL NM	

The Asymptotic Behavior of the Likelihood Ratio Statistic
for Testing a Shift in Mean in a Sequence of Independent Normal Variates

by

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ABSTRACT

Let $X_1, \dots, X_n$ be an independent sequence of random variables such that $X_1, \dots, X_r \sim \text{iid } N(\mu, \sigma^2)$ and $X_{r+1}, \dots, X_n \sim \text{iid } N(\mu + \theta, \sigma^2)$ where $\mu, \theta \neq 0, \sigma^2$ and r are unknown parameters. The asymptotic properties of the likelihood ratio in testing $H_0: r = n$ (no change point) vs. $H_1: r < n$ are derived. It is shown, using a result of Darling and Erdős, that the likelihood ratio, suitably normalized and under H_0 , converges in distribution to the double exponential extreme value distribution. The asymptotic operating characteristics of the likelihood ratio test are studied and comparisons are made between the likelihood ratio test and a Bayesian test.

Running head: Testing a shift in mean.

^{*}Partially supported by grants NSF DMS 8202335 and AFOSR F49620 82 C0009.

AMS 1980 subject classification. Primary 62F05; Secondary 62E20

Keywords and phrases. Change point, normalized Brownian bridge, Ornstein-Uhlenbeck process, extreme value, asymptotics.

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1. Introduction

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~~We~~ consider the problem of testing a sequence of independent and normally distributed random variables with common mean against alternatives involving a shift in the mean at an unknown time point. Specifically, let $X_1, \dots, X_n$ be an observed independent sequence such that $X_1, \dots, X_r \sim \text{iid } N(\mu, \sigma^2)$ and $X_{r+1}, \dots, X_n \sim \text{iid } N(\mu + \theta, \sigma^2)$ where $\mu, \theta \neq 0$ and $r(1 \leq r \leq n)$ are unknown and σ is either known or unknown. We want to test $H_0: r = n$ vs. $H_1: r < n$. Chernoff and Zacks (1964) derived, by assuming $\theta > 0$, a one-sided test statistic through a Bayesian argument. Their results were generalized to the one-parameter exponential family by Kander and Zacks (1966), and to two-sided test statistics by Gardner (1969) and MacNeill (1974). The asymptotic operating characteristics of these Bayesian tests were also studied by Gardner (1969) and MacNeill (1974). Sen and Srivastava (1975) derived various likelihood ratio tests and compared them to Bayesian procedures using computer simulations. Hawkins (1977) discussed heuristically the asymptotic null distribution of the likelihood ratio test. Unfortunately, his reasoning is not entirely correct (though the leading term is of the right order).

The purpose of this paper is to study the asymptotic operating characteristics of the likelihood ratio test. In the next section, ^{the authors} ~~we~~ derive, by invoking a result of Darling and Erdős (1956), the asymptotic null distribution of the likelihood ratio test, which is related to the extreme value behavior of the Ornstein-Uhlenbeck process. In Section 3, ^{we} ~~we~~ study the asymptotic operating characteristics of the likelihood ratio test and make comparisons between likelihood ratio and Bayesian tests.

2. Asymptotic null distribution of the likelihood ratio test

Throughout this paper, we assume σ is known since the case of σ unknown

is asymptotically equivalent to that of σ known (see Remark 2.4 at the end of this section). So without loss of generality we set $\sigma = 1$.

Given a change in mean occurring between r and $r + 1$, the maximum likelihood is proportional to $\exp\{-\frac{1}{2}\left[\sum_{i=1}^r (X_i - \bar{X}_r)^2 + \sum_{i=r+1}^n (X_i - \bar{X}'_r)^2\right]\}$ where $\bar{X}_r$ denotes the sample mean of the first r observations and $\bar{X}'_r$ the sample mean of the last $n - r$ observations. So the (generalized) minus log likelihood ratio of H_0 vs. H_1 is proportional to (see for example Sen and Srivastava, 1975)

$$\begin{aligned} (2.1) \quad T_n^2 &\equiv \max_{1 \leq r \leq n-1} \left[\sum_{i=1}^r (X_i - \bar{X}_n)^2 - \sum_{i=1}^r (X_i - \bar{X}_r)^2 - \sum_{i=r+1}^n (X_i - \bar{X}'_r)^2 \right] \\ &= \max_{1 \leq r \leq n-1} \frac{\left(\frac{S_r}{\sqrt{n}} - \frac{r}{n} \frac{S_n}{\sqrt{n}} \right)^2}{\left(\frac{r}{n} \right) \left(1 - \frac{r}{n} \right)} \\ &= \left[\max_{1 \leq r \leq n-1} \frac{\left| \frac{S_r}{\sqrt{n}} - \frac{r}{n} \frac{S_n}{\sqrt{n}} \right|}{\left(\frac{r}{n} \right) \left(1 - \frac{r}{n} \right)^{\frac{1}{2}}} \right]^2 \end{aligned}$$

where $S_r = X_1 + \dots + X_r$. Suppose $\{W(t); 0 \leq t < \infty\}$ is a standard Brownian motion. Since under H_0 , $\{(S_r - r\mu)/\sqrt{n}; 1 \leq r \leq n\} \stackrel{D}{=} \{W(\frac{r}{n}); 1 \leq r \leq n\}$, we have

$$\begin{aligned} (2.2) \quad T_n &\equiv \max_{1 \leq r \leq n-1} \left| \frac{S_r}{\sqrt{n}} - \frac{r}{n} \frac{S_n}{\sqrt{n}} \right| / \left(\frac{r}{n} \right) \left(1 - \frac{r}{n} \right)^{\frac{1}{2}} \\ &\stackrel{D}{=} \max_{nt=1, \dots, n-1} |W(t) - tW(1)| / (t(1-t))^{\frac{1}{2}} \\ &= \max_{nt=1, \dots, n-1} |W_0(t)| / (t(1-t))^{\frac{1}{2}} \end{aligned}$$

where " $\stackrel{D}{=}$ " means "equal in distribution" and $W_0(t) = W(t) - tW(1)$, (i.e. Brownian bridge). Throughout, we shall use the convention that $\max_{nt=1, \dots, n-1} = \max_{1 \leq nt \leq n-1}$.

Theorem 2.1 Under H_0 ,

$$\lim_{n \rightarrow \infty} P(a_n^{-1} (T_n - b_n) \leq x) = \exp(-2\pi^{-1/2} e^{-x}), \quad -\infty < x < \infty$$

where $a_n = (2 \ln_2 n)^{-1/2}$, $b_n = a_n^{-1} + 2^{-1} a_n \ln_3 n$ and $\ln_k$ is the k -th iterated logarithm.

We need the following lemmas to prove the theorem.

Lemma 2.2 (Theorem 2 of Darling and Erdős, 1956)

$$\lim_{n \rightarrow \infty} P(a_n^{-1} \left(\max_{k=1, \dots, n} \left| \frac{W(k)}{\sqrt{k}} \right| - b_n \right) \leq x) = \exp(-\pi^{-1/2} e^{-x}), \quad -\infty < x < \infty.$$

Lemma 2.3

$$\max_{1 \leq nt \leq [n/\ln n]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} - \max_{1 \leq nt \leq [n/\ln n]} \frac{|W(t)|}{\sqrt{t}} = o_p(a_n)$$

Proof. For n large and $t \leq 1/\ln n$, we have

$$\begin{aligned} \left| \frac{W(t) - tW(1)}{\sqrt{t(1-t)}} - \frac{W(t)}{\sqrt{t}} \right| &\leq \left| \frac{W(t) - tW(1)}{\sqrt{t(1-t)}} - \frac{W(t)}{\sqrt{t}} \right| \\ &\leq \left| \frac{W(t)}{\sqrt{t}} \left(\frac{1}{\sqrt{1-t}} - 1 \right) \right| + \left| \sqrt{\frac{t}{1-t}} W(1) \right| \\ &\leq \sqrt{t} |W(t)| + 2\sqrt{t} |W(1)| \\ &\leq (\ln n)^{-1/2} (|W(t)| + 2|W(1)|). \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \max_{1 \leq nt \leq [n/\ln n]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} - \max_{1 \leq nt \leq [n/\ln n]} \frac{|W(t)|}{\sqrt{t}} \right| &\leq \max_{1 \leq nt \leq [n/\ln n]} \left| \frac{|W_0(t)|}{\sqrt{t(1-t)}} - \frac{|W(t)|}{\sqrt{t}} \right| \\ &\leq (\ln n)^{-1/2} \max_{1 \leq nt \leq [n/\ln n]} (|W(t)| + 2|W(1)|) \\ &= o_p((\ln n)^{-1/2}). \end{aligned}$$



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Lemma 2.4

$$\max_{[n/\ln n] \leq nt \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} = o_p((\ln_3 n)^{1/2})$$

Proof. By the law of the iterated logarithm, for t small enough,

$$|W(t)| < 2\sqrt{2t \ln_2 t^{-1}}, \text{ W.P.1.}$$

It follows that as $s \rightarrow 0+$,

$$\max_{t \in [s, 1/2]} \frac{|W(t)|}{\sqrt{t}} = o_p((\ln_2 s^{-1})^{1/2})$$

so that

$$\begin{aligned} \max_{[n/\ln n] \leq nt \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} &\leq \max_{[n/\ln n] \leq nt \leq [n/2]} \frac{|W(t)|}{\sqrt{t}} + \max_{[n/\ln n] \leq nt \leq [n/2]} \sqrt{\frac{t}{1-t}} |W(1)| \\ &= o_p((\ln_3 n)^{1/2}) + o_p(1). \end{aligned} \quad \square$$

Lemma 2.5

$$\lim_{n \rightarrow \infty} P(a_n^{-1} \left(\max_{1 \leq nt \leq [n/\ln n]} \frac{|W(t)|}{\sqrt{t}} - b_n \right) \leq x) = \exp(-\pi^{-1/2} e^{-x}).$$

Proof. Since $\{\frac{W(t)}{\sqrt{t}}; t = \frac{1}{n}, \dots, \frac{[n/\ln n]}{n}\} \stackrel{D}{=} \{\frac{W(t)}{\sqrt{t}}; t = 1, \dots, [n/\ln n]\}$,

it follows from Lemma 2.2 that

$$P\left(\max_{1 \leq t \leq [n/\ln n]} \frac{|W(t)|}{\sqrt{t}} \leq a_{[n/\ln n]} x + b_{[n/\ln n]}\right) \rightarrow \exp(-\pi^{-1/2} e^{-x}).$$

But,

$$a_{[n/\ln n]} = a_n + o(a_n) \text{ and } b_{[n/\ln n]} = b_n + o(a_n)$$

which establishes the lemma by the convergence to types result. □

Lemma 2.6

$$\max_{1 \leq nt \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} - \max_{1 \leq nt \leq [n/\ln n]} \frac{|W(t)|}{\sqrt{t}} = o_p(a_n).$$

Proof. From Lemmas 2.3 and 2.5, we have

$$(2 \ln n)^{-1/2} \max_{1 \leq nt \leq [n/\ln n]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} \xrightarrow{n \rightarrow \infty} 1 \text{ in probability.}$$

This property and Lemma 2.4 together imply

$$P\left(\max_{[n/\ln n] \leq nt \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} \geq \max_{1 \leq nt \leq [n/\ln n]} \frac{|W_0(t)|}{\sqrt{t(1-t)}}\right) \xrightarrow{n \rightarrow \infty} 0.$$

Now, applying Lemma 2.3 completes the proof. $\square$

Lemma 2.7

$$\max_{1 \leq n(1-t) \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} - \max_{1 \leq n(1-t) \leq [n/\ln n]} \frac{|W(t) - W(1)|}{\sqrt{1-t}} = o_p(a_n).$$

Proof. This follows from Lemma 2.6 and the symmetry of the Brownian bridge

$W_0(t)$ with respect to $t = \frac{1}{2}$; i.e. $\{W_0(t); 0 \leq t \leq 1\} \stackrel{D}{=} \{W_0(1-t); 0 \leq t \leq 1\}$. $\square$

Proof of Theorem 2.1. We have

$$\begin{aligned} P(T_n \leq a_n x + b_n | H_0) &= P\left(\max_{1 \leq nt \leq n-1} \frac{|W_0(t)|}{\sqrt{t(1-t)}} \leq a_n x + b_n\right) \\ &= P\left(\max_{1 \leq nt \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} \leq a_n x + b_n, \max_{1 \leq n(1-t) \leq [n/2]} \frac{|W_0(t)|}{\sqrt{t(1-t)}} \leq a_n x + b_n\right) \end{aligned}$$

which by Lemmas 2.6 and 2.7 is equal to

$$\begin{aligned} P\left(\max_{1 \leq nt \leq [n/\ln n]} \frac{|W(t)|}{\sqrt{t}} \leq a_n x + b_n, \max_{1 \leq n(1-t) \leq [n/\ln n]} \frac{|W(t) - W(1)|}{\sqrt{1-t}} \leq a_n x + b_n\right) + o(1) \\ \rightarrow \exp(-2\pi^{-1/2} e^{-x}). \end{aligned}$$

by Lemma 2.5 and the independence of the increments of Brownian motion. $\square$

Remark 2.1. From extreme value theory, it is known that the extreme value of a Gaussian sequence converges to its limit distribution very slowly. Therefore, one must be careful in using the limit distribution as an approximation for finite n .

Remark 2.2. Hawkins (1977) approximated the asymptotic null distribution by that of the maximum of $\ln n$ consecutive observations from a weakly dependent stationary Gaussian sequence. Appealing to a result of Berman (1964), this

statistic has the same limit distribution as the maximum of $\ln n$ observations from an iid $N(0, 1)$ sequence. However, since $W(t)/\sqrt{t} = X(\frac{1}{2} \ln t)$ where $X(t)$ is the Ornstein-Uhlenbeck process, the above argument shows that the null distribution is approximated by the extreme value (in continuous time) of the Ornstein-Uhlenbeck process. The normalizing constants for the maximum of this process differ from those of the maximum from an iid $N(0, 1)$ sequence (cf. Leadbetter, Lindgren and Rootzen, 1983) which accounts for the discrepancy between the normalizing constants in Theorem 2.1 and those suggested by Hawkins.

Remark 2.3. The statistic T_n was derived from the two-sided likelihood ratio (i.e. θ could be either positive or negative). If it is given that $\theta \leq 0$ (i.e. downward shift in mean), a statistic derived from the one-sided likelihood ratio is (cf. (2.5) of Sen and Srivastava, 1975)

$$(2.3) \quad T'_n = \max \left(0, \max_{1 \leq r \leq n-1} \frac{\frac{S_r}{\sqrt{n}} - \frac{r}{n} \frac{S_n}{\sqrt{n}}}{\left(\frac{r}{n} (1 - \frac{r}{n}) \right)^{1/2}} \right).$$

Applying Theorem 1 of Darling and Erdős (1956) and following the pattern of the proof of Theorem 2.1, it can be shown that under H_0 ,

$$(2.4) \quad \lim_{n \rightarrow \infty} P(a_n^{-1}(T'_n - b_n) \leq x) = \exp(-\pi^{-1/2} e^{-x}), \quad -\infty < x < \infty.$$

Remark 2.4. When the common variance σ^2 is unknown, the likelihood ratio statistic of H_0 vs. H_1 becomes

$$\begin{aligned} \min_{1 \leq r \leq n-1} \left(\frac{\hat{\sigma}_r^2}{\hat{\sigma}_n^2} \right)^{n/2} &= \left(\min_{1 \leq r \leq n-1} \frac{n \hat{\sigma}_r^2}{n \hat{\sigma}_n^2} \right)^{n/2} \\ &= \left(1 - \frac{1}{n} \left[\max_{1 \leq r \leq n-1} \hat{\sigma}_n^{-2} \left\{ \sum_{i=1}^n (X_i - \bar{X}_n)^2 \right. \right. \right. \\ &\quad \left. \left. \left. - \sum_{i=1}^r (X_i - \bar{X}_r)^2 - \sum_{i=r+1}^n (X_i - \bar{X}_r')^2 \right\} \right] \right)^{n/2} \end{aligned}$$

where $\hat{\sigma}_r^2 = n^{-1} \left[\sum_{i=1}^r (X_i - \bar{X}_r)^2 + \sum_{i=r+1}^n (X_i - \bar{X}_r')^2 \right]$. So a test statistic is

$$T_n'' = \frac{\sigma^2}{\hat{\sigma}_n^2} \max_{1 \leq r \leq n-1} \sigma^{-2} \left[\sum_{i=1}^r (X_i - \bar{X}_n)^2 - \sum_{i=1}^r (X_i - \bar{X}_r)^2 - \sum_{i=r+1}^n (X_i - \bar{X}_r')^2 \right]$$

where a large value of T_n'' indicates significance. The random variable

$\max_{1 \leq r \leq n-1} \sigma^{-2} \left[\sum_{i=1}^r (X_i - \bar{X}_n)^2 - \sum_{i=1}^r (X_i - \bar{X}_r)^2 - \sum_{i=r+1}^n (X_i - \bar{X}_r')^2 \right]$ has the same null

distribution as the likelihood ratio statistic under the condition of σ being known equal to 1. We also have, under H_0 ,

$$\frac{\sigma^2}{\hat{\sigma}_n^2} = 1 + O_p(n^{-1/2})$$

Therefore, the likelihood ratio statistic for both cases of σ known and unknown has the same asymptotic null distribution.

3. Operating characteristics of the likelihood ratio test $\{T_n\}$

We shall compute the local powers of T_n and compare the performance of T_n

and the Bayesian statistic $U_n \equiv \frac{1}{n^2} \sum_{r=1}^{n-1} \left[\sum_{i=r}^{n-1} (X_{i+1} - \bar{X}_n) \right]^2$, (see (1.3) of Gardner

(1969), assuming the uniform prior distribution on the location of the change point).

Let α be a given significance level. From Theorem 2.1, $\lim_{n \rightarrow \infty} P(T_n > c_n) = \alpha$ where

$$(3.1) \quad c_n = b_n - a_n \ln[2^{-1} \pi^{1/2} \ln(1 - \alpha)^{-1}].$$

We consider the alternative $H(t_0, \eta, n)$ that there is a change between $[n t_0]$ and $[n t_0] + 1$ and the amount of (positive) shift in mean is

$$(3.2) \quad \theta = (n t_0 (1 - t_0))^{-1/2} ((2 \ln_2 n)^{1/2} + \eta + o(1))$$

where t_0 and η are two constants with $0 < t_0 < 1$. (The results below are exactly the same for negative shifts.)

Theorem 3.1.

$$\lim_{n \rightarrow \infty} P(T_n > c_n | H(t_0, \eta, n)) = \alpha + \Phi(\eta)(1 - \alpha)$$

where $\Phi(\cdot)$ is the standard normal cdf.

Proof of Theorem 3.1.

Again, let $W_0(t) = W(t) - tW(1)$ be a Brownian bridge. Under $H(t_0, \eta, n)$,

$$(3.3) \quad T_n \equiv \max_{1 \leq r \leq n-1} \left| \frac{S_r}{\sqrt{n}} - \frac{r}{n} \frac{S_n}{\sqrt{n}} \right| / \left(\frac{r}{n} (1 - \frac{r}{n}) \right)^{1/2}$$

$$\stackrel{D}{=} \max_{1 \leq nt \leq n-1} |W_0(t) - f_n(t)| / (t(1-t))^{1/2}$$

where

$$(3.4) \quad f_n(t) = n^{1/2} \theta (1 - [n t_0]/n)t \quad \text{if } nt \leq [n t_0],$$

$$= n^{1/2} \theta \frac{[n t_0]}{n} (1-t) \quad \text{if } nt > [n t_0].$$

Denote by $E_{n,\ell}$ ($1 \leq \ell \leq 6$) the event that

$$\left\{ \max_{t \in D_{n,\ell}} |W_0(t) - f_n(t)| / (t(1-t))^{1/2} > c_n \right\}$$

where

$$D_{n,1} = \{k/n: 1 \leq k \leq [n/\ell n n]\}$$

$$D_{n,2} = \{k/n: n - [n/\ell n n] \leq k \leq n - 1\}$$

$$D_{n,3} = \{k/n: [n/\ell n n] < k < [n t_0] - [n(\ell n_3 n)/(\ell n_2 n)^{1/2}]\}$$

$$D_{n,4} = \{k/n: [n t_0] + [n(\ell n_3 n)/(\ell n_2 n)^{1/2}] < k < n - [n/\ell n n]\}$$

$$D_{n,5} = \{k/n: [n t_0] - [n(\ell n_3 n)/(\ell n_2 n)^{1/2}] \leq k \leq [n t_0]\}$$

$$D_{n,6} = \{k/n: [n t_0] < k \leq [n t_0] + [n(\ell n_3 n)/(\ell n_2 n)^{1/2}]\}.$$

Obviously,

$$P(T_n > c_n | H(t_0, \eta, n)) = P\left(\max_{1 \leq nt \leq n-1} |W_0(t) - f_n(t)| / (t(1-t))^{1/2} > c_n \right)$$

$$= P\left(\bigcup_{\ell=1}^6 E_{n,\ell} \right).$$

We now break up the proof into a few lemmas.

Lemma 3.2

$$P(E_{n,1} \Delta \left\{ \max_{t \in D_{n,1}} \frac{|W(t)|}{\sqrt{t}} > c_n \right\}) = o(1)$$

$$P(E_{n,2} \Delta \left\{ \max_{t \in D_{n,2}} \frac{|W(t) - W(1)|}{\sqrt{1-t}} > c_n \right\}) = o(1).$$

(Δ denotes symmetric difference.)

Proof. We only prove the first equation. The second one can be done similarly.

For $1 \leq nt \leq [n/\ln n]$ and n large, we have

$$\frac{|f_n(t)|}{\sqrt{t(1-t)}} \leq 2 \frac{|f_n(t)|}{\sqrt{t}} = 2(n^{\frac{1}{2}} \theta) (1 - \frac{[nt_0]}{n}) \sqrt{t}$$

and using (3.2), this bound is $O((\ln_2 n / \ln n)^{\frac{1}{2}}) = o(a_n)$ uniformly in $t \in D_{n,1}$.

So,

$$\max_{t \in D_{n,1}} \frac{|W_o(t)|}{\sqrt{t(1-t)}} - \max_{t \in D_{n,1}} \frac{|W_o(t) - f_n(t)|}{\sqrt{t(1-t)}} = o_p(a_n)$$

which together with Lemmas 2.3 and 2.5 completes the proof. $\square$

Lemma 3.3

$$P(E_{n,\ell}) = o(1), \quad \ell = 3, 4.$$

Proof. We only consider the case $\ell = 3$. From Lemma 2.4,

$$\max_{t \in D_{n,3}} \frac{|W_o(t)|}{\sqrt{t(1-t)}} = O_p((\ln_3 n)^{\frac{1}{2}}).$$

Also for n large,

$$\max_{t \in D_{n,3}} \frac{|f_n(t)|}{\sqrt{t(1-t)}} = \max_{t \in D_{n,3}} n^{\frac{1}{2}} \theta (1 - [nt_0]/n) (t/(1-t))^{\frac{1}{2}}$$

$$= (n^{\frac{1}{2}} \theta) (1 - [nt_0]/n) (t/(1-t))^{\frac{1}{2}} \text{ at } t = t^* = n^{-1} \left([nt_0] - \left[\frac{n \ln_3 n}{(\ln_2 n)^{\frac{1}{2}}} \right] \right)$$

$$= t_0 - (\ln_3 n) / (\ln_2 n)^{\frac{1}{2}} + O(n^{-1})$$

$$\begin{aligned}
 &= [t_0(1-t_0)]^{-\frac{1}{2}}((2 \ln_2 n)^{\frac{1}{2}} + n + o(1))(1 - t_0 + o(n^{-1}))(t^*/(1-t^*))^{\frac{1}{2}} \\
 &= (2 \ln_2 n)^{\frac{1}{2}} - 2^{-\frac{1}{2}}(1 + o(1))(\ln_3 n)/(t_0(1-t_0)).
 \end{aligned}$$

Therefore, the lemma follows from the inequality

$$\begin{aligned}
 \max_{t \in D_{n,3}} \frac{|W_0(t) - f_n(t)|}{\sqrt{t(1-t)}} &\leq O_p((\ln_3 n)^{\frac{1}{2}}) + (2 \ln_2 n)^{\frac{1}{2}} \\
 &\quad - 2^{-\frac{1}{2}}(1 + o(1))(\ln_3 n)/(t_0(1-t_0)). \quad \square
 \end{aligned}$$

Lemma 3.4

$$P(E_{n,\ell} \Delta \{(t_0(1-t_0))^{\frac{1}{2}} n^{\frac{1}{2}} \theta - \frac{W_0(t_0)}{\sqrt{t_0(1-t_0)}} > c_n\}) = o(1), \ell = 5, 6.$$

Proof. Once again we only supply the proof for one case $\ell = 5$. Observe that

$$(3.5) \quad \max_{t \in D_{n,5}} \frac{f_n(t)}{\sqrt{t(1-t)}} = n^{\frac{1}{2}} \theta ((1-t_0)t_0)^{\frac{1}{2}} + O((\ln_2 n)^{\frac{1}{2}}/n) \sim (2 \ln_2 n)^{\frac{1}{2}}$$

and since $W_0(t)$ is W.P.1 continuous

$$(3.6) \quad \max_{t \in D_{n,5}} \left| \frac{W_0(t)}{\sqrt{t(1-t)}} - \frac{W_0(t_0)}{\sqrt{t_0(1-t_0)}} \right| = o_p(1).$$

It follows that $R_n \equiv \max_{t \in D_{n,5}} \frac{f_n(t) - W_0(t)}{\sqrt{t(1-t)}}$ is eventually equal to

$$\max_{t \in D_{n,5}} \frac{|W_0(t) - f_n(t)|}{\sqrt{t(1-t)}}. \quad \text{Set } Z_n = (t_0(1-t_0))^{\frac{1}{2}} n^{\frac{1}{2}} \theta - W_0(t_0)(t_0(1-t_0))^{-\frac{1}{2}}. \quad \text{It thus}$$

suffices to show that for every $\epsilon > 0$,

$$(3.7) \quad P(|Z_n - R_n| > \epsilon) \rightarrow 0$$

and

$$(3.8) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P(c_n - \epsilon \leq Z_n \leq c_n + \epsilon) = 0.$$

We have

$$\begin{aligned} \max_{t \in D_{n,5}} \frac{f_n(t)}{\sqrt{t(1-t)}} - \max_{t \in D_{n,5}} \frac{W_0(t)}{\sqrt{t(1-t)}} - Z_n &\leq R_n - Z_n \\ &\leq \max_{t \in D_{n,5}} \frac{f_n(t)}{\sqrt{t(1-t)}} - \min_{t \in D_{n,5}} \frac{W_0(t)}{\sqrt{t(1-t)}} - Z_n. \end{aligned}$$

By (3.5) and (3.6) the two outside terms of this inequality are both $o_p(1) + O((\ln n)^{1/2}/n) = o_p(1)$ which proves (3.7). As for (3.8), we have $c_n - (t_0(1-t_0))^{1/2} n^{1/2} \theta = -\eta + o(1)$ so that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P(c_n - \epsilon \leq Z_n \leq c_n + \epsilon) &= \lim_{\epsilon \rightarrow 0} P\left(\eta - \epsilon \leq \frac{W_0(t_0)}{\sqrt{t_0(1-t_0)}} \leq \eta + \epsilon\right) \\ &= 0. \quad \square \end{aligned}$$

Continuation of the proof of Theorem 3.1: First it is a straightforward calculation to show

$$(3.9) \quad P(\{(t_0(1-t_0))^{1/2} n^{1/2} \theta - W_0(t_0)/(t_0(1-t_0))^{1/2} > c_n\} \Delta A_n) = o(1)$$

where

$$A_n = \{(t_0(1-t_0))^{1/2} n^{1/2} \theta - \frac{[W(t_0) - W((\ln n)^{-1/2}) - t_0(W(1-(\ln n)^{-1/2}) - W((\ln n)^{-1/2}))]}{(t_0(1-t_0))^{1/2}} > c_n\}.$$

By (3.9), Lemmas 2.5, 3.2-3.4, and the independence of the Brownian increments, we have

$$\begin{aligned} P(T_n > c_n | H(t_0, n, n)) &= P\left(\bigcup_{\ell=1}^6 E_{n,\ell}\right) \\ &= P(\{\max_{t \in D_{n,1}} \frac{W(t)}{\sqrt{t}} > c_n\} \cup \{\max_{t \in D_{n,2}} \frac{|W(t) - W(1)|}{\sqrt{1-t}} > c_n\} \cup A_n) + o(1) \\ &\rightarrow \alpha + \phi(\eta)(1-\alpha). \quad \square \end{aligned}$$

MacNeill (1971) studied the asymptotic behavior of $U_n \equiv n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} (X_{i+1} - \bar{X}_n) \right)^2$.

We state his results under H_0 and under the alternative $H'(\psi, n)$ that there is a change between $[n t_0]$ and $[n t_0] + 1$ and the amount of shift in mean is $\psi n^{-1/2}$.

Lemma 3.5

(i) Under H_0 , $U_n \xrightarrow{D} \int_0^1 W_0^2(t) dt$

(ii) Under $H'(t_0, \psi, n)$,

$$U_n \xrightarrow{D} \int_0^1 (W_0(t) + h(t))^2 dt$$

$$\begin{aligned} \text{where } h(t) &= \psi t_0 t & 0 \leq t \leq 1 - t_0 \\ &= \psi (1-t_0)(1-t), & 1 - t_0 \leq t \leq 1 \end{aligned}$$

We now compare the two tests relative to two sequences of alternatives both of which are approaching H_0 (in some sense). For the first comparison, t_0 is fixed and a sequence of shifts $\psi(t_0, \alpha, \beta)n^{-1/2}$ converging to 0 is specified so that the sequence of tests $\{U_n\}$ has asymptotic significance α and power $\beta > \alpha$. In other words, if $c(\alpha)$ is such that $P(\int_0^1 W_0^2(t) dt > c(\alpha)) = \alpha$, then $\psi = \psi(t_0, \alpha, \beta)$ is chosen so that $P(\int_0^1 (W_0(t) + h(t))^2 dt > c(\alpha)) = \beta$. Then the sample size $n' = n'(n)$ required in order for the likelihood ratio test to have asymptotic significance α and power β w.r.t. the sequence of shifts $\psi(t_0, \alpha, \beta)n^{-1/2}$ is found by solving $(n' t_0 (1-t_0))^{-1/2} ((2 \ln_2 n')^{1/2} + \eta + o(1)) = \psi(t_0, \alpha, \beta)n^{-1/2}$ (cf. (3.2)) where $\eta = \eta(\alpha, \beta)$ satisfies $\alpha + \Phi(\eta)(1-\alpha) = \beta$. It is easily seen that n' should grow like $(2n \ln_2 n) / (\psi^2(t_0, \alpha, \beta) t_0 (1-t_0))$. This suggests defining the analogue of Pitman efficiency (depending on t_0, α, β) of the sequence of tests $\{T_n\}$ w.r.t. $\{U_n\}$ by $n/n' \sim \psi^2(t_0, \alpha, \beta) t_0 (1-t_0) / (2 \ln_2 n)$. Thus for the sequence of alternatives specified above, $\{U_n\}$ is more efficient than $\{T_n\}$ when n is very large (although $\ln \ln n$ grows rather slowly).

For the second comparison, we consider the sequence of alternatives $\{H_{(n)}\}$ where the amount of shift θ is now held constant but $t_0 = n^{-1}[n - n^{1-\gamma}] \approx 1 - n^{-\gamma}$, $1/2 < \gamma < 1$, is approaching 1. Then for the alternatives $\{H_{(n)}\}$ the sequence of tests $\{T_n\}$ is more powerful than $\{U_n\}$ since

$$(3.10) \quad P(U_n > c(\alpha) | H_{(n)}) \rightarrow \alpha$$

while

$$(3.11) \quad P(T_n > c_n | H_{(n)}) \rightarrow 1,$$

where c_n is given by (3.1).

The proof of these two results is as follows: Let Y_i be iid $N(0, 1)$. Then

$$\begin{aligned} U_n &= n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} (X_{i+1} - \bar{X}) \right)^2 \stackrel{D}{=} n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} [(Y_{i+1} - \bar{Y}) - g_n(i+1)] \right)^2 \\ &= n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} (Y_{i+1} - \bar{Y}) \right)^2 - 2n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} (Y_{i+1} - \bar{Y}) \right) \left(\sum_{i=r}^{n-1} g_n(i+1) \right) \\ &\quad + n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} g_n(i+1) \right)^2 \end{aligned}$$

where

$$g_n(i) = \begin{cases} -(1 - [n - n^{1-\gamma}]/n)\theta & \text{if } i \leq [n - n^{1-\gamma}] \\ ([n - n^{1-\gamma}]/n)\theta & \text{if } i > [n - n^{1-\gamma}] \end{cases}$$

Since $n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} (Y_{i+1} - \bar{Y}) \right)^2 \xrightarrow{D} \int_0^1 W_0^2(t) dt$, it suffices to show

$$(3.12) \quad n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} g_n(i+1) \right)^2 = o(1).$$

We have for $\frac{1}{2} < \gamma < 1$,

$$\begin{aligned} n^{-2} \sum_{r=1}^{n-1} \left(\sum_{i=r}^{n-1} g_n(i+1) \right)^2 &\leq n^{-2} n \{ n^{-1} (n - [n - n^{1-\gamma}]) n |\theta| \\ &\quad + (n - [n - n^{1-\gamma}]) |\theta| \}^2 \\ &\leq n^{-1} (2n^{1-\gamma} |\theta| + o(1))^2 \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

which establishes (3.12) and thus proves (3.10).

As for (3.11), we have

$$T_n \geq \frac{|W_o(t_o) - f_n(t_o)|}{(t_o(1-t_o))^{\frac{1}{2}}} = \left| \frac{W_o(t_o)}{(t_o(1-t_o))^{\frac{1}{2}}} - \theta n^{\frac{1}{2}(1-\gamma)} + o(n^{\frac{1}{2}(1-\gamma)}) \right|$$

$$= |\theta| n^{\frac{1}{2}(1-\gamma)} + o_p(n^{\frac{1}{2}(1-\gamma)}),$$

whence

$$P(T_n > c_n | H_{(n)}) \rightarrow 1.$$

Note that (3.11) remains true if $t_o = 1 - n^{-\gamma}$ for any γ , $0 < \gamma < 1$.

Remark 3.1. The simulation results of Sen and Srivastava (1975) tend to support the above conclusions even for moderate sample sizes. They show that for $20 \leq n \leq 100$, $\{T_n\}$ is less powerful than $\{U_n\}$ when t_o is close to $\frac{1}{2}$ and more powerful when t_o is close to 0 or 1. When n is moderately large, the factor $\psi^2(t_o, \alpha, \beta)t_o(1-t_o)$ in the above efficiency calculation may dominate the $2 \ln_2 n$ piece. Based on Sen and Srivastava's simulation results, it seems likely that for fixed α and β , $\psi^2(t_o, \alpha, \beta)t_o(1-t_o)$ increases as $|t_o - \frac{1}{2}| \rightarrow \frac{1}{2}$. Actually when $t_o \approx 1$, then $h(t) = 0(1 - t_o)$ and $\psi(t_o, \alpha, \beta)$ is of the order $(1 - t_o)^{-1}$ so that the factor $\psi(t_o, \alpha, \beta)t_o(1 - t_o)$ is of the order $(1 - t_o)^{-1}$. This argument gives a heuristic explanation as to why the sequence of tests $\{T_n\}$ is more powerful than $\{U_n\}$ when $(1 - t_o)$ (or t_o) converges to zero sufficiently fast.

Acknowledgments. The authors wish to thank Professor Sidney Resnick for bringing to their attention the result of Darling and Erdős.

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