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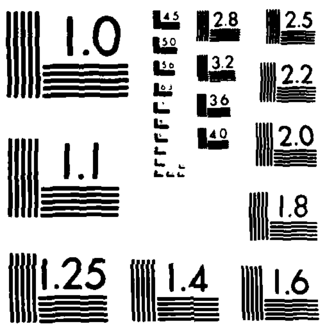

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ON THE SPANS OF POLYNOMIALS AND THE SPANS OF A LAGUERRE-POLYA-SCHUR SEQUENCE OF POLYNOMIALS

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ABSTRACT

If $P(x) = c(x-x_1) \cdots (x-x_n)$ is a real polynomial having only real zeros such that $x_n \leqq x_{n-1} \leqq \cdots \leqq x_1$, $x_n < x_1$ $(n \geqq 3)$, then following R. M. Robinson, A. Meir and A. Sharma, we call $S(P) = x_1 - x_n$ <u>the span of</u> $P(x)$. Meir and Sharma conjectured in 1969 that

$$S(P^{(k)}) \geqq S(P)\left(\frac{(n-k)(n-k-1)}{n(n-1)}\right)^{1/2} \text{ for } k = 1,\ldots,n-2 \quad ,$$

with equality only for polynomials of the form

$$P(x) = c(x-x_1)\left(x - \frac{x_1+x_n}{2}\right)^{n-2}(x-x_n) \quad .$$

We introduce the quantity $NS(P) \equiv S(P)/\sqrt{n(n-1)}$ and call it the normalized span of $P(x)$, using the notation $N_n^{(k)} \equiv NS(P^{(k)})$, $(k = 0,\ldots,n-2)$. We then prove in Theorem 1 the Meir-Sharma conjecture in the form

$$N_n^{(0)} \leqq N_n^{(1)} \leqq \cdots \leqq N_n^{(n-2)}, \quad (n \geqq 4) \quad .$$

We also show that these inequalities may assume only one of two forms described in Theorem 1.

In §§3 to 6 we apply these results to the infinite sequences of polynomials introduced by Polya and Schur in a famous paper from 1914.

AMS (MOS) Subject Classifications: 12D10, 30D15

Key Words: Polynomials having only real zeros, Polya-Schur sequences of polynomials

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SIGNIFICANCE AND EXPLANATION

If $P(x) = c(x-x_1) \cdots (x-x_n)$ is a real polynomial such that $x_n \leq x_{n-1} \leq \cdots \leq x_1$, $x_n < x_1$, $(n \geq 2)$, then following R. M. Robinson, A. Meir and A. Sharma, we call $S(P) = x_1 - x_n$ the span of $P(x)$. ~~We~~ prove a conjecture of Meir and Sharma from 1969 determining the least value of the span of ~~the~~ derivative $P'(x)$ for $n \geq 3$, if x_1 and x_n are kept fixed. Tools used are the Descartes rule of signs and the inequality $A \geq H$ between the arithmetic and harmonic mean. We also ~~apply our~~ results to the infinite sequences of polynomials introduced by G. Polya and I. Schur in a famous paper from 1914. A Polya-Schur sequence of polynomials is of the form

$$P_0(x) = 1,\ P_1(x), \ldots, P_n(x), \ldots ,$$

with $P_n(x) = x^n + \binom{n}{1} a_1 x^{n-1} + \binom{n}{1} a_2 x^{n-2} + \ldots + a_n$, such that all polynomials have only real zeros and satisfying

$$P_n'(x) = nP_{n-1}(x) \quad \text{for} \quad n = 1,2,\ldots .$$

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE SPANS OF POLYNOMIALS AND THE SPANS OF A LAGUERRE-POLYA-SCHUR SEQUENCE OF POLYNOMIALS

I. J. Schoenberg

1. On a conjecture of A. Meir and A. Sharma. If

$$P(x) = c(x-x_1) \cdots (x-x_n)$$

is a polynomial having only real zeros such that

(1.1) $$x_n \le x_{n-1} \le \cdots \le x_1, \; x_n < x_1, \; (n \ge 2) \quad ,$$

then, following R. M. Robinson, A. Meir and A. Sharma, we call

(1.2) $$S(P) \equiv x_1 - x_n$$

the span of $P(x)$. Clearly, the sequence of the spans of $P(x)$ and its derivatives

(1.3) $$S(P^{(k)}) \quad , \quad (k = 0,1,\ldots,n-2) \quad ,$$

form a non-increasing sequence.

In [1] Meir and Sharma consider the special case of the polynomials of the form

(1.4) $$P(x) = c(x-x_1)\left(x - \frac{x_1+x_n}{2}\right)^{n-2}(x-x_n) \quad , \quad (n \ge 2, \; x_n < x_1 \quad .$$

We call it the class of centered polynomials and denote it by the symbol π_n. We also say that $P(x)$ is centered at the point $(x_1+x_n)/2$.

The sequence of spans (1.3) for centered polynomials is easily determined. Without loss of generality we may assume that $x_1 = 1, x_n = -1$, and therefore

(1.5) $$P(x) = (x-1)x^{n-2}(x+1) = x^n - x^{n-2} \quad .$$

Their derivatives are

$$P^{(k)} = n(n-1)\ldots(n-k+1)x^{n-k} - (n-2)(n-3)\ldots(n-k-1)x^{n-k-2} \quad ,$$

hence

(1.6) $$P^{(k)}(x) = \frac{n!}{(n-k)!}\left(x^2 - \frac{(n-k)(n-k-1)}{n(n-1)}\right)x^{n-k-2}, \; (k = 0,\ldots,n-2) \quad .$$

This is a centered polynomial having the span

(1.7) $$S(P^{(k)}) = 2\sqrt{\frac{(n-k)(n-k-1)}{n(n-1)}} = S(P)\sqrt{\frac{(n-k)(n-k-1)}{n(n-1)}} \quad .$$

This equality for centered polynomials suggested to Meir and Sharma the conjecture that for

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any $P(x)$ of degree $n \geq 3$, having only real zeros, we have the inequality

(1.8) $$S(P^{(k)}) \geq S(P)\sqrt{\frac{(n-k)(n-k-1)}{n(n-1)}}\ , \quad \text{for } k = 1,\ldots,n-2\ ,$$

with the equality sign if and only if $P(x)$ is centered. This they prove in [1, page 530] for the largest value of $k = n-2$ only. The main point, however, is to establish (1.8) for the least value of $k = 1$, for then the result so obtained can be applied successively to P', P'', ..., $P^{(n-2)}$, proving the complete conjecture. This is what we are going to do.

The inequality (1.8) suggests normalization of the span $S(P)$: We call the quantity

(1.9) $$NS(P) = \frac{S(P)}{\sqrt{n(n-1)}}$$

the normalized span of $P(x)$, and to simplify notations we write

(1.10) $$N_n^{(0)} = NS(P),\ N_n^{(k)} = NS(P^{(k)})\ .$$

We may now state the following theorem whose second part goes beyond the Meir-Sharma conjecture.

Theorem 1. 1. If $P(x)$ is a polynomial of degree n whose zeros satisfy (1.1), then for the normalized spans of its derivatives we have that

(1.11) $$N_n^{(0)} \leq N_n^{(1)} \leq \ldots \leq N_n^{(n-3)} \leq N_n^{(n-2)}\ , \quad (n \geq 4)\ .$$

2. These inequalities can assume only one of the two forms: Either

(1.12) $$N_n^{(0)} < N_n^{(1)} < \ldots < N_n^{(n-3)} \leq N_n^{(n-2)}\ ,$$

or else

(1.13) $$N_n^{(0)} = N_n^{(1)} = \ldots = N_n^{(n-3)} = N_n^{(n-2)}\ .$$

The equations (1.13) hold only if $P(x)$ is centered, i.e. of the form (1.4). If $P(x)$ is not centered, then (1.12) holds. In (1.12) we have the equality sign in the last inequality, if and only if $P^{(n-3)}(x)$ is centered, hence is a cubic having its three zeros in an arithmetic progression.

The inequalities (1.11) are equivalent with the Meir-Sharma conjecture, while (1.12), (1.13), go beyond it; they will be derived below by Descartes rule of signs.

Proof of the inequalities (1.11). It suffices to prove the first one

(1.14) $$N_n^{(0)} \leq N_n^{(1)} \quad (n \geq 3)\ ,$$

as we simply iterate it to get (1.11).

We are also to show that if $n \geq 3$ then

(1.15) $$N_n^{(0)} = N_n^{(1)}$$

if and only if $P(x)$ is centered.

Let

(1.16) $y_{n-1} \leq y_{n-2} \leq \cdots \leq y_1$ be the zeros of $P'(x)$.

The inequality (1.14) then amounts explicitly to

$$\frac{x_1-x_n}{\sqrt{n(n-1)}} \leq \frac{y_1-y_{n-1}}{\sqrt{(n-1)(n-2)}} ,$$

or

(1.17) $$y_1 - y_{n-1} \geq \sqrt{\frac{n-2}{n}}\,(x_1-x_n) .$$

To prove this we also need the new polynomial

(1.18) $$P_*(x) = (x-x_1)(x-\xi)^{n-2}(x-x_n)$$

where

(1.19) $$\xi = \frac{x_2+x_3+\ldots+x_{n-1}}{n-2} .$$

We will also show that in (1.17) we have the equality sign if and only if $P_*(x)$ is centered, hence when

(1.20) $$\xi = \frac{x_1+x_n}{2} .$$

We also need the derivative of $P_*(x)$, and let

(1.21) $\eta_{n-1} < \overbrace{\xi = \xi = \ldots = \xi}^{n-3} < \eta_1$ be the zeros of $P_*'(x)$.

The inequality (1.17) will be established as soon as we prove the two inequalities

(1.22) $$y_1 - y_{n-1} \geq \eta_1 - \eta_{n-1}$$

and

(1.23) $$\eta_1 - \eta_{n-1} \geq \sqrt{\frac{n-2}{n}}\,(x_1-x_n) .$$

We observe first that

(1.24) $$y_{n-1} < y_1 ,$$

for if $y_{n-1} = y_1$, their common value being $= 0$, say, then $P'(x) = c \cdot x^{n-1}$, and therefore $P(x) = c(x^n - x_1^n)/n$, which can not have only real zeros, as we assume that $n \geq 3$.

We also assume to start with that

$$x_2 < x_1 \quad , \tag{1.25}$$

which by Rolle's theorem implies that

$$y_1 - x_2 > 0 \quad . \tag{1.26}$$

Our next objective is to find an explicit expression for the quadratic polynomial $(x-\eta_1)(x-\eta_{n-1})$, which will turn out to be in terms of x_1, x_n, and ξ. Indeed, by differentiation of (1.18) we find

$$P_*'(x) = (x-\xi)^{n-3}\{(x-\xi)(x-x_n) + (n-2)(x-x_1)(x-x_n) + (x-x_1)(x-\xi)\}$$

showing that η_1 and η_{n-1} are the zeros of the second factor. This is the quadratic

$$\begin{aligned} Q(x) &= n(x-\eta_1)(x-\eta_{n-1}) \\ &= (x-\xi)(x-x_n) + (n-2)(x-x_1)(x-x_n) + (x-x_1)(x-\xi) \\ &= nx^2 - 2\left((n-1)\frac{x_1+x_n}{2} + \xi\right)x + (n-2)x_1x_n + (x_1+x_n)\xi \quad . \end{aligned} \tag{1.27}$$

The discriminant $b^2 - ac$ of $Q(x) = ax^2 + 2bx + c$ is easily evaluated and we find for difference $\eta_1 - \eta_{n-1} = \frac{2}{a}\sqrt{b^2-ac}$ the expression

$$\eta_1 - \eta_{n-1} = \sqrt{\frac{n-2}{n}(x_1-x_n)^2 + \frac{4}{n^2}\left(\frac{x_1+x_n}{2} - \xi\right)^2} \quad . \tag{1.28}$$

If we remove the second non-negative term under the square root sign we already prove the inequality (1.23), hence

$$\eta_1 - \eta_{n-1} \geqq \sqrt{\frac{n-2}{n}}\,(x_1-x_n) \quad . \tag{1.29}$$

<u>Proof of (1.22)</u>. We will first prove that

$$\eta_1 \leqq y_1 \quad . \tag{1.30}$$

From the identity

$$\frac{P'(x)}{P(x)} = \sum_1^n \frac{1}{x-x_i} \tag{1.31}$$

and $P'(y_1) = 0$, we obtain

$$\frac{1}{x_1-y_1} - \sum_{i=2}^{n-1} \frac{1}{y_1-x_i} - \frac{1}{y_1-x_n} = 0 \quad . \tag{1.32}$$

Here we should first observe that our assumption (1.25) implies (1.26) and therefore that all denominators in (1.32) are positive. For the special case of the polynomial $P_*(x)$, defined by (1.18) and (1.19), the equation (1.32) becomes

(1.33) $$\frac{1}{x_1-\eta_1} - \frac{n-2}{\eta_1-\xi} - \frac{1}{\eta_1-x_n} = 0 \quad .$$

We shall prove now if we replace in (1.33) the quantity η_1 by y_1, the left side of (1.33) becomes non-negative, hence

(1.34) $$\frac{1}{x_1-y_1} - \frac{n-2}{y_1-\xi} - \frac{1}{y_1-x_n} \geq 0 \quad .$$

Indeed, observe that the <u>arithmetic mean</u> A of the n - 2 positive quantities

(1.35) $$y_1 - x_2,\ y_1 - x_3, \ldots, y_1 - x_{n-1}$$

is

(1.36) $$A = y_1 - \xi \quad ,$$

while their harmonic mean H is given by

(1.37) $$\sum_2^{n-1} \frac{1}{y_1-x_i} = \frac{n-2}{H} \quad .$$

However, by the classical inequality

(1.38) $$H \leq A \quad ,$$

and (1.36) we derive from (1.37) that

(1.39) $$\sum_2^{n-1} \frac{1}{y_1-x_i} \geq \frac{n-2}{y_1-\xi} \quad .$$

Therefore (1.32) shows that

(1.40) $$\frac{1}{x_1-y_1} - \frac{1}{y_1-x_n} \geq \frac{n-2}{y_1-\xi} \quad .$$

<u>I claim that</u> (1.40) <u>shows that</u> (1.30) <u>holds</u>: From (1.27) we have the identity in x

(1.41) $$\frac{1}{x_1-x} - \frac{n-2}{x-\xi} - \frac{1}{x-x_n} = \frac{Q(x)}{(x_1-x)(x-\xi)(x-x_n)} \quad .$$

If we set here $x = y_1$ we obtain from (1.40) that

$$\frac{1}{x_1-y_1} - \frac{n-2}{y_1-\xi} - \frac{1}{y_1-x_n} = \frac{Q(y_1)}{(x_1-y_1)(y_1-\xi)(y_1-x_n)} \geq 0 \quad ,$$

and therefore

(1.42) $$Q(y_1) \geq 0 \quad .$$

Since $Q(x) = c(x-\eta_1)(x-\eta_{n-1})$ and by Rolle $\xi \leq x_2 < y_1 < x_1$, we conclude that

(1.43) $$y_1 \geq \eta_1 \quad .$$

On replacing x by $-x$ we similarly find that

(1.44) $$y_{n-1} \leq \eta_{n-1} \quad .$$

Now (1.43) and (1.44) show that the inequality (1.22) holds.

Having established (1.22) and (1.23), it is clear the our main inequality (1.17) is also proved.

There remains the question: <u>When do we have the equality sign in (1.17)?</u>

Equality in (1.17) implies equality in both (1.22) and (1.23). However, we know that equality in (1.22) implies that the terms of the sequence (1.35) are all equal to each other, and so by (1.19) we have

(1.45) $$x_2 = x_3 = \ldots = x_{n-1} = \xi \quad .$$

Moreover, equality in (1.23) implies that the second term under the square root sign in (1.28) must vanish so that

(1.46) $$\frac{x_1+x_n}{2} = \xi \quad .$$

Now (1.45) shows that $P(x) = P_*(x)$, while (1.46) shows that $P(x)$ is a centered polynomial

$$P(x) = (x-x_1)\left(x - \frac{x_1+x_n}{2}\right)^{n-2}(x-x_n) \quad .$$

Can we eliminate the additional assumption (1.25)? The answer is affirmative: The $x_1, x_2, \ldots, x_n$ are independent, and all the other quantities, such as y_i, ξ, η_1, η_2 depend continuously on them. Therefore the string of inequalities (1.11), conjectured by Meir and Sharma, is established.

2. Proof of the inequalities (1.12) and (1.13).

We assume first that

(2.1) $$P(x) \in \pi_n \quad .$$

We then know from (1.5) implying (1.7), the equations (1.13) must hold.

<u>We now assume that</u>

(2.2) $$P(x) \notin \pi_n \quad .$$

<u>I claim that the inequalities (1.12) must hold.</u>

We distinguish two cases:

1. <u>If</u> $n = 4$ <u>we have</u>

(2.3) $$N_4^{(0)} < N_4^{(1)} \leq N_4^{(2)} ,$$

<u>with the first being "less than" sign,</u> because by (2.2) $P(x)$ is not centered.

2. <u>If</u> $n \geq 5$ <u>then none of the</u> $n - 4$ <u>polynomials</u>

(2.4) $$P'(x),\ P''(x),\ldots,P^{(n-4)}(x) ,$$

<u>can be centered, so that we have</u>

(2.5) $$N_n^{(0)} < N_n^{(1)} <\ldots< N_n^{(n-4)} < N_n^{(n-3)} \leq N_n^{(n-2)} .$$

<u>Proof</u>: Let us assume that one of the polynomials (2.4) is centered, e.g. $P^{(k)}(x) \in \pi_n$, where

(2.6) $$1 \leq k \leq n-4 ,$$

and let us reach a contradiction to our assumption (2.2). We may assume by a change of scale that

$$P^{(k)}(x) = (x-1)x^{n-k-2}(x+1) = x^{n-k} - x^{n-k-2}$$

and therefore, by integration, that

(2.7) $$P^{(k-1)}(x) = \frac{1}{n-k+1} x^{n-k+1} - \frac{1}{n-k-1} x^{n-k-1} + C ,$$

and <u>let us show that necessarily</u>

(2.8) $$C = 0 .$$

Indeed, let us assume that $C > 0$, when (2.7) shows, by Descartes rule of signs, that for the number of zeros in $(0,\infty)$ of $P^{(k-1)}$ we have

(2.9) $$Z(0,\infty) \leq 2 .$$

Moreover, from

(2.10) $$P^{(k-1)}(-x) = \frac{(-1)^{n-k+1}}{n-k+1} x^{n-k+1} - \frac{(-1)^{n-k-1}}{n-k-1} x^{n-k-1} + C$$

we conclude that

(2.11) $$Z(-\infty,0) \leq \begin{cases} 2 & \text{if } n-k \text{ is odd} , \\ 1 & \text{if } n-k \text{ is even} . \end{cases}$$

In any case we have that

(2.12) $$Z(-\infty,\infty) \leq 4 ,$$

while (2.7), having only real zeros, should have at least, by

$$n-k+1 \geq n-(n-4)+1 = 5 ,$$

five zeros, in contradiction with (2.12).

Likewise, if $C < 0$ we have

$$Z(0,\infty) \leq 1$$

and from (2.10)

$$Z(-\infty,0) \leq \begin{cases} 1 & \text{if } n-k \text{ is odd} \\ 2 & \text{if } n-k \text{ is even} \end{cases} ,$$

hence again $Z(-\infty,\infty) \leq 4$, leading to the same contradiction. The conclusion, that $P^{(k)}(x) \in \pi_{n-k}$ implies that $P^{(k-1)}(x) \in \pi_{n-k+1}$, may be iterated until we reach the conclusion that $P(x) \in \pi_n$, in contradiction to our assumption (2.2).

3. On the spans of a Laguerre-Polya-Schur sequence of polynomials.

Let

(3.1) $$F(z) = \sum_0^\infty \frac{a_n}{n!} z^n, \ (a_0 = 1) \quad ,$$

be a formal real power series. Multiplying by e^{xz} we obtain

(3.2) $$F(z)e^{xz} = \sum_0^\infty \frac{P_n(x)}{n!} z^n \quad ,$$

which generates a so-called Appell sequence of polynomials

(3.3) $$P_n(x) = x^n + \binom{n}{1}a_1x^{n-1} + \ldots + a_n \ (P_0(x) = 1, \ n = 1,2,\ldots\ ) \quad ,$$

having the property that

(3.4) $$P_n'(x) = n\,P_{n-1}(x) \quad .$$

A fundamental theorem of Polya and Schur from 1914 [2] states that

(3.5) all $P_n(x)$ have only real zeros

if and only if the function $F(z)$ is entire and belongs to the class $\mathcal{E}_2^*$ of entire functions

(3.6) $$F(z) = e^{-\gamma z^2+\delta z} \prod_{k=1}^\infty (1 + \delta_k z)e^{-\delta_k z}$$

such that

(3.7) $$\gamma \geq 0, \ 0 < \gamma + \sum_1^\infty \delta_k^2 < \infty \quad .$$

Here we have excluded the trivial case when $\gamma = 0$ and all $\delta_k = 0$, when $P_n(x) = (x+\delta)^n$ for all n.

What can we say about the normalized spans of the polynomials $P_n(x)$? We recall from (1.4) that we denote by π_n the class of centered polynomials

$$P(x) = c(x-x_1)\left(x - \frac{x_1+x_n}{2}\right)^{n-2}(x-x_n) \quad (n \geq 3,\ x_n < x_1) \quad . \tag{3.8}$$

From our Theorem 1 we can easily derive our

Theorem 2. *Let* $F(z) \in \mathcal{E}_2^*$. *For the sequence of normalized spans*

$$NS(P_n) \quad (n = 2,3,\dots) \tag{3.9}$$

we have one of the following two alternatives: Either

$$NS(P_2) \geq NS(P_3) > NS(P_4) > \dots > NS(P_n) > \dots \tag{3.10}$$

or else

$$NS(P_2) = NS(P_3) = \dots = NS(P_n) = \dots \quad . \tag{3.11}$$

The second alternative (3.11) *holds only if* $P_4(x) \in \pi_4$, *which implies that*

$$P_n(x) \in \pi_n \quad \text{for} \quad n = 3,4,\dots \quad . \tag{3.12}$$

In (3.10) *we have the equality sign among the first two terms if and only if* $P_3(x) \in \pi_3$, *which means that* $P_3(x)$ *is a cubic having its zeros in arithmetic progression.*

Proof: 1. Let us consider $P_n(x)$ for some $n \geq 4$. If $P_n(x) \in \pi_n$, then we know, by (3.4), and Theorem 1 that

$$NS(P_2) = NS(P_3) = \dots = NS(P_n) \quad .$$

On the other hand we know from (2.6), (2.7) and (2.8), that all integrals of $P_4(x)$, having only real zeros, are centered. This gives the remaining equations $NS(P_n) = NS(P_{n+1}) = \dots$.

2. If

$$P_4(x) \notin \pi_4 \quad , \tag{3.13}$$

then, by Theorem 1, we have $NS(P_3) > NS(P_4)$. We also know that no integral of $P_4(x)$ having only real zeros, can be centered. Hence (3.13) implies that $P_n(x) \notin \pi_n$ if $n > 4$. This shows that $NS(P_4) > NS(P_5) > \dots$. Finally, we have $NS(P_2) = NS(P_3)$ or $NS(P_2) > NS(P_3)$, depending on whether $P_3(x)$ is centered or not.

An example. Let $H_n(x)$ be the Hermite polynomials as defined by Szegö [3, page 105]:

(3.14) $$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \quad \text{for } n = 2,3,\ldots$$

with $H_0(x) = 1$, $H_1(x) = 2x$. Also $H_2(x) = 4x^2 - 2$ and $H_3(x) = 8x^3 - 12x$. We see that $H_3(x)$ is centered, while $H_n(k)$ can not be centered if $n \geq 4$, because $\{H_n(x)\}$ form an orthogonal system of polynomials and can therefore not have multiple zeros. It follows that we have

(3.15) $$NS(H_2) = NS(H_3) > HS(H_4) > \ldots > NS(H_n) > \ldots \quad .$$

What is the limit of the decreasing sequence (3.15) of positive numbers? We can easily show that

(3.16) $$\lim_{n\to\infty} NS(H_n) = 0$$

using an estimate of the largest zero of $H_n(x)$ found in Szegö [3, page 119, (6.2.18)]. Szegö shows that the largest zero of $H_n(x)$ does not exceed $d_n = \sqrt{2}\,(n-1)/\sqrt{n+2}$, which implies that

$$NS(H_n) \leq \frac{2}{\sqrt{n(n-1)}} d_n \to 0 \quad .$$

4. On the vanishing of consecutive coefficients of $F(z)$.

We wish to prove the

Theorem 3. *If in the expansion* (3.1) *of the function* (3.6) *we have*

(4.1) $$a_m \neq 0 \ (m \geq 1) \quad ,$$

and

(4.2) $$a_{m+1} = 0, \ a_{m+2} = 0 \quad ,$$

then

(4.3) $$a_n = 0 \ \text{ for all } \ n > m \quad .$$

Proof: We shall use the property of the Descartes rule of signs to the effect that for a polynomial having only real zeros the number w of variations of sign of its coefficients is equal to the number of its positive zeros. We consider the three polynomials

$$P_m(x) = x^m + \binom{m}{1}a_1x^{m-1} + \ldots + a_m$$

(4.4) $$P_{m+1}(x) = x^{m+1} + \binom{m+1}{1}a_1x^m + \ldots + \binom{m+1}{m}a_mx$$

$$P_{m+2}(x) = x^{m+2} + \binom{m+2}{1}a_1x^{m+1} + \ldots + \binom{m+2}{m}a_mx^2 + a_{m+3} \quad .$$

having only real zero. The next polynomial is

(4.5) $$P_{m+3}(x) = x^{m+3} + \binom{m+3}{1}a_1x^{m+2} + \ldots + \binom{m+3}{m}a_mx^3 + a_{m+3} \quad .$$

We consider the numbers of variations of sign

(4.6) $$w = v(1,a_1,\ldots,a_m),\ w' = v((-1)^m,(-1)^{m-1}a_1,\ldots,a_m) \quad .$$

By our introductory remark we have that for $P_m(x)$

(4.7) $$Z_m(0,\infty) = w,\ Z_m(-\infty,0) = w', \text{ and therefore}$$

(4.8) $$w + w' = m \quad .$$

We now consider $P_{m+3}(x)$. If we assume for the moment that

(4.9) $$a_{m+3} > 0 \quad ,$$

then

$$Z_{m+3}(0,\infty) = w,\ Z_{m+3}(-\infty,0) = w' + 1, \text{ if } a_m > 0$$

and

$$Z_{m+3}(0,\infty) = w + 1,\ Z_{m+3}(-\infty,0) = w', \text{ if } a_m < 0 \quad .$$

From (4.9) and (4.8), it follows that

(4.10) $$Z_{m+3}(-\infty,\infty) = w + w' + 1 = m + 1 \quad ,$$

in contradiction with the reality of all zeros of P_{m+3}.

We get the same contradiction (4.10) if $a_{m+3} < 0$. Our conclusion is that $a_{m+3} = 0$.

This proves Theorem 3 if we apply the above reasoning to P_{m+k}, if a_{m+k} should be the first non-vanishing coefficient.

This shows that the assumptions (4.1) and (4.2) imply that $F(z)$ reduces to a polynomial. This is what we assume in our next section.

5. The case when $F(z)$ is a polynomial.

(5.1) $$F(z) = 1 + \frac{a_1}{1!} z + \dots + \frac{a_m}{m!} z^m \quad (a_m \neq 0)$$

and therefore

(5.2) $$P_n(x) = x^n + \binom{n}{1} a_1 x^{n-1} + \dots + \binom{n}{m} a_m x^{n-m} \quad \text{if} \quad n \geq m \quad .$$

We raise the question: Can we decide which case of the dichotomy (3.10) or (3.11) of Theorem 2 applies to the normalized spans of the polynomials (5.2)?

We first observe that without loss of generality we may assume that $a_1 = 0$. From (3.6) and (3.1) we find that

(5.3) $$a_1 = \delta, \quad a_2 = \delta^2 - 2\gamma - \sum_1^\infty \delta_k^2 \quad .$$

However, the effect of the value of δ is trivial: If we multiply (3.2) by $e^{-\delta z}$ we obtain

$$F(z) e^{(x-\delta)z} = \sum_0^\infty \frac{P_n(x-\delta)}{n!} z^n \quad ,$$

showing that the factor $e^{\delta z}$ in (3.6) merely translates the x-axis by the amount δ. However, the property of a polynomial being centered is invariant if we replace x by $x - \delta$. for this reason we shall assume that

(5.4) $$\delta = 0 \quad \text{and by (5.3), that} \quad a_1 = 0 \quad ,$$

when

(5.5) $$a_2 = -(2\gamma + \sum_1^\infty \delta_k^2) < 0 \quad .$$

We may now easily answer the question concerning the dichotomy between (1.10) and (1.11) by proving

Theorem 4. For the case when (5.1) holds we have for the normalized spans of the polynomials (5.2), that (1.11) holds, hence

(5.6) $$NS(P_2) = NS(P_3) = \dots = NS(P_n) = \dots \quad ,$$

if and only if in (5.1) we have $m = 2$, hence

(5.7) $$F(z) = 1 + \frac{a_2}{2!} z^2 \quad (a_2 < 0) \quad .$$

If $m \geq 3$ *we have the inequalities* (3.10), *hence*

$$NS(P_2) \geq NS(P_3) > NS(P_4) > \dots > NS(P_n) > \dots \quad . \tag{5.8}$$

Proof: By Theorem 2 we know that (3.10) holds, hence (5.8), if and only if

$$P_4(x) = x^4 + 6a_2x^2 + 4a_3x + a_4 \tag{5.9}$$

is *not* centered. From $a_1 = 0$ it is clear that if P_4 is centered, it can only be centered at point $x = 0$.

If $a_4 \neq 0$, the $P_4(x)$ has no vanishing zero, and can therefore not be centered at 0. If $a_4 = 0$, hence $a_3 \neq 0$, then

$$P_4(x) = (x^3 + 6a_2x + a_3)x \quad .$$

The cubic factor has three real zeros all $\neq 0$, and so $P_4(x)$ can not be centered at 0, where it should have a double zero. Therefore $P_4(x)$ if not centered, and Theorem 4 is established.

The last question to be discussed is as follows.

6. *What is* $\lim_{n\to\infty} NS(P_n)$ *in (5.8)*?

We may and do assume that

$$a_1 = 0 \quad . \tag{6.1}$$

This implies that the zeros of

$$P_n(x) = x^{n-m}\{x^m + \binom{n}{2} a_2x^{m-2} + \dots + \binom{n}{m} a_m\} \quad (m \geq n) \tag{6.2}$$

have the sum 0; it follows that the point 0 is between the extreme zeros of $P_n(x)$. Therefore the span of $P_n(x)$ does not change if we omit the factor x^{n-m}. We therefore have

$$S(P_n(x)) = S(x^m + \binom{n}{2} a_2x^{m-2} + \dots + \binom{n}{m} a_m) \quad .$$

If we now replace x by nx in the polynomial on the right side, then its zeros shrink by the factor $1/n$, and its span is similarly contracted. Therefore we restore the equation by multiplying by n, obtaining

$$S(P_n(x)) = nS(n^mx^m + n^{m-2}\binom{n}{2} a_2x^{m-2} + \dots + n^mn^{-m}\binom{n}{m} a_m) \quad .$$

Dividing the right side polynomial by n^m, we obtain

$$S(P_n(x)) = nS(x^m + \frac{1\cdot(1-\frac{1}{n})}{2!} a_2 x^{m-2} + \ldots + \frac{1\cdot(1-\frac{1}{n})\ldots(1-\frac{n-m+1}{n})}{m!} a_m) \quad .$$

Dividing this equation by $\sqrt{n(n-1)}$ and letting $n \to \infty$ we obtain

Theorem 5. <u>For the generating function</u> (5.1), <u>satisfying</u> (6.1) <u>we have that</u>

$$\lim_{n\to\infty} NS(P_n(x)) = S(x^m + \frac{a_2}{2!} x^{m-2} + \ldots + \frac{a_m}{m!}) \quad . \tag{6.3}$$

By (5.1) this limit can also be expressed as

$$\lim_{n\to\infty} NS(P_n(x)) = S(x^m F(\frac{1}{x})) \quad . \tag{6.4}$$

<u>A numerical example</u>. For (5.1) we choose $m = 4$ and

$$F(z) = (1 - \frac{z^2}{1^2})(1 - \frac{z^2}{2^2}) = 1 - \frac{5}{4} z^2 + \frac{1}{4} z^4 \quad . \tag{6.5}$$

On combining with (5.1) we find $a_1 = 0$, $a_2 = -5/2$, $a_3 = 0$, $a_4 = 6$. For the right side of (6.4) we therefore have

$$S(x^4 F(1/x)) = S\{(x^2 - \frac{1}{1^2})(x^2 - \frac{1}{2^2})\} = 2 \quad , \tag{6.6}$$

as the extreme zeros are -1 and $+1$.

Let us find the largest zero $x_1^{(n)}$ of (6.2), hence of

$$P_n(x) = (x^4 - \binom{n}{2}\frac{5}{2}x^2 + \binom{n}{4} 6)x^{n-4} \quad . \tag{6.7}$$

From (6.4) and (6.6) we obtain after canceling the factor 2 on both sides

$$\lim_{n\to\infty} x_1^{(n)}/\sqrt{n(n-1)} = 1 \quad ,$$

hence

$$x_1^{(n)} \sim n \quad . \tag{6.8}$$

<u>Finding the largest zero of</u> (6.7) <u>for</u> $n = 7,8,9,10$ <u>we obtain the table</u>

n	7	8	9	10
x_1^n	6.93812	7.96069	8.98372	9.99771

which illustrates well the relation (6.8), or $x_1^{(n)}/n \to 1$.

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If $P(x) = c(x-x_1) \cdots (x-x_n)$ is a real polynomial having only real zeros such that $x_n \leqq x_{n-1} \leqq \cdots \leqq x_1$, $x_n < x_1$ $(n \geqq 3)$, then following R. M. Robinson, A. Meir and A. Sharma, we call $S(P) = x_1 - x_n$ the span of $P(x)$. Meir and Sharma conjectured in 1969 that

$$S(P^{(k)}) \geqq S(P)\left(\frac{(n-k)(n-k-1)}{n(n-1)}\right)^{1/2} \quad \text{for} \quad k = 1,\ldots,n-2 \quad ,$$

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ABSTRACT (continued)

with equality only for polynomials of the form

$$P(x) = c(x-x_1)\left(x - \frac{x_1+x_n}{2}\right)^{n-2}(x-x_n) \quad .$$

We introduce the quantity $NS(P) \equiv S(P)/\sqrt{n(n-1)}$ and call it the normalized span of $P(x)$, using the notation $N_n^{(k)} \equiv NS(P^{(k)})$, $(k = 0,\ldots,n-2)$. We then prove in Theorem 1 the Meir-Sharma conjecture in the form

$$N_n^{(0)} \leqq N_n^{(1)} \leqq \cdots \leqq N_n^{(n-2)}, \quad (n \geqq 4) \quad .$$

We also show that these inequalities may assume only one of two forms described in Theorem 1.

In §§3 to 6 we apply these results to the infinite sequences of polynomials introduced by Polya and Schur in a famous paper from 1914.

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