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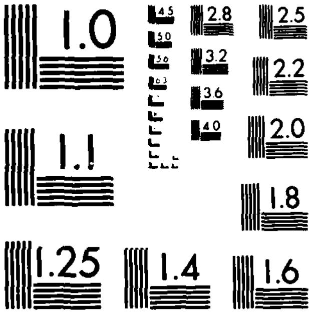

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SOME QUALITATIVE PROPERTIES OF BIVARIATE EULER-FROBENIUS POLYNOMIALS

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SOME QUALITATIVE PROPERTIES OF BIVARIATE EULER-FROBENIUS POLYNOMIALS

Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]

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ABSTRACT

Let M_n denote the bivariate box-spline corresponding to the directions (1,0), (0,1) (1,1), each occurring with multiplicity n. We determine all critical points of the polynomials

$$P_n(x) = \sum_{j \in \mathbb{Z}^2} M_n(j) e^{ijx}, \quad n \in \mathbb{Z}_+ .$$

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SIGNIFICANCE AND EXPLANATION

This is a further report in a series devoted to the study of box splines. Box splines ~~have been~~ introduced in MRC TSR #2320 and provide a natural generalization of univariate cardinal splines, i.e., splines with a uniform knot sequence.

The process of univariate spline interpolation becomes particularly simple in the cardinal case, and this report considers the corresponding bivariate process of interpolation at the integer points in the plane to a given function by a linear combination of integer translates of a box spline. In particular, the report shows that this process is well posed, i.e., any bounded continuous function f has exactly one such bounded interpolant If. The argument uses the Fourier transform to identify a certain trigonometric polynomial (in two variables) whose nonvanishing is equivalent to the asserted well-posedness. The minimum value of this polynomial yields a bound on the norm of the resulting interpolation projector I.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SOME QUALITATIVE PROPERTIES OF BIVARIATE EULER-FROBENIUS POLYNOMIALS

Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]

In a series of beautiful papers, I. J. Schoenberg developed the theory of univariate cardinal splines [6-8]. A basic result is the positivity of the Euler-Frobenius polynomials which implies the well posedness of cardinal interpolation.

Theorem 1 [6]. Let M_r denote the univariate cardinal B-spline with support centered at 0. The Euler-Frobenius polynomials

$$P_r(x) = \sum_{j\in Z} M_r(j)e^{ijx}, \quad r \in Z_+,$$

are strictly positive and attain their unique minimum (maximum) at $x = \pi \bmod 2\pi Z$ $(x = 0 \bmod 2\pi Z)$.

In this note we obtain the bivariate analogue of this result for box-splines. For a set of vectors $\Xi = \{\xi_1,\ldots,\xi_n\}$ with $\xi_\nu \in Z^m$, the box-spline M_Ξ is the functional on $C_0(R^m)$ defined by [1]

$$M_\Xi\phi := \int_{[-\frac{1}{2},\frac{1}{2}]^n} \phi\Big(\sum_{\nu=1}^{n} \lambda_\nu\xi_\nu\Big)d\lambda. \tag{1}$$

Equivalently, M_Ξ can be defined by its Fourier transform

$$\hat{M}_\Xi(y) = \prod_{\nu=1}^{n} S(\xi_\nu y) \tag{2}$$

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where $S(z) := (2/z)\sin(z/2)$. The latter definition stresses the similarity to the univariate case. We define the multivariate Euler-Frobenius polynomials by

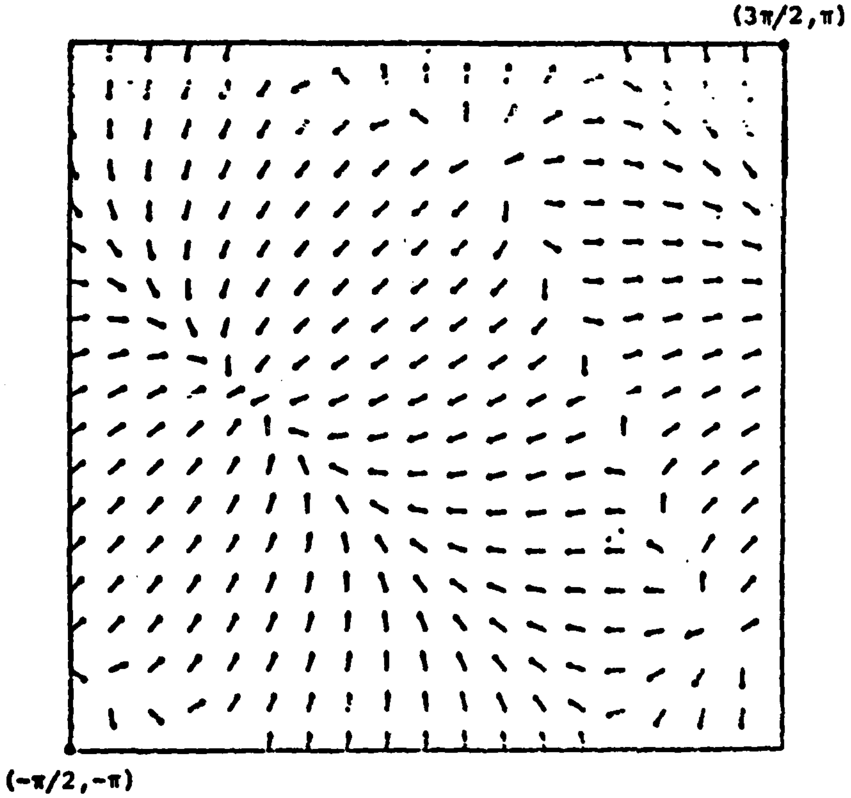
$$P_{\Xi}(x) := \sum_{j\in Z^m} M_{\Xi}(j)e^{ijx}. \tag{3}$$

In the bivariate case $(m = 2)$ we proved [3] the following conjecture. The polynomials P_{Ξ} are strictly positive iff the box-splines $M_{\Xi}(\cdot - j)$, $j \in Z^m$, are linearly independent.

If valid in general $(m > 2)$ the conjecture would imply that cardinal interpolation is well posed if the obvious necessary condition of linear independence is satisfied. For two variables it was shown in [2] that the box-splines are linearly independent only on the "standard" three direction mesh, up to symmetry the vectors in Ξ have to be chosen from the set $\{(1,0), (0,1), (1,1)\}$. While the corresponding grid is very regular, the analysis of the interpolation problem is complicated. Our results [3,4] are not as complete as in I. J. Schoenberg's univariate theory. E.g. we were not able to determine the location of the minimum for P_{Ξ} which in general depends on Ξ. We conjectured that in the symmetric case, when each of the three vectors in Ξ occurs with multiplicity n, the polynomial $P_n = P_{\Xi}$ attains its minimum at the point $(\frac{2\pi}{3},\frac{2\pi}{3})$. In this note we prove this conjecture and determine all critical points of P_n.

Theorem 2. The polynomials P_n, $n \in Z_+$, attain their minima at $\pm(\frac{2\pi}{3},\frac{2\pi}{3}) \bmod 2\pi\, Z^2$, their maxima at the points $2\pi\, Z^2$ and have saddle points at $\pi\, Z^2 \setminus 2\pi\, Z^2$. These are the only critical points of P_n.

Figure 1 below shows the gradient field of P_3 on $[\pi/2,3\pi/2] \times [-\pi,\pi]$ which illustrates the general situation.



< Figure 1 >

The proof of Theorem 2 relies heavily on the symmetries of P_n. Let $\mathcal{A}$ denote the group of 12 linear transformations which leave the mesh generated by the three directions (1,0), (0,1), (1,1) invariant. This group is generated by the matrices

$$(4) \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

which correspond to reflection at the origin and permutation of the directions. The symmetric box-spline M_n is invariant under composition with $\mathcal{A}$, i.e.

$$(5) \qquad M_n(Ax) = M_n(x), \quad A \in \mathcal{A}.$$

Therefore, the corresponding Euler-Frobenius polynomials satisfy

$$(6) \qquad P_n(A^*x + 2\pi j) = P_n(x), \quad A \in \mathcal{A}, \quad j \in \mathbb{Z}^2,$$

where A^* denotes the transpose of A. These relations give much information about the structure of P_n. Denote by $\nabla f(u,v) :=$ $(D_u f(u,v), D_v f(u,v))$ the gradient of a function f. Differentiating the identity (6) we obtain

$$(7) \qquad (\nabla P_n(A^*x + 2\pi j))\, A^* = \nabla P_n(x), \quad A \in \mathcal{A}, \quad j \in \mathbb{Z}^2.$$

Let I denote the unit matrix. Identity (7) implies in particular that

$$\nabla P_n(x) \in \ker(I - A) \quad \text{if} \quad (I - A^*)x = 2\pi j. \tag{8}$$

For $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{A}$ it follows from (8) that ∇P_n vanishes at the points $\pi\, \mathbb{Z}^2$ and $\pm(\frac{2\pi}{3}, \frac{2\pi}{3}) + 2\pi\, \mathbb{Z}^2$ respectively. For $A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in \mathcal{A}$, the matrices $(I - A)$ have rank one and it follows from (8) that for $k \in \mathbb{Z}$,

$$\begin{aligned}
(0,1)\nabla P_n(x) &= 0 \quad \text{if} \quad (1,2)x = 2\pi k, \\
(1,0)\nabla P_n(x) &= 0 \quad \text{if} \quad (2,1)x = 2\pi k, \\
(1,-1)\nabla P_n(x) &= 0 \quad \text{if} \quad (1,-1)x = 2\pi k, \\
(1,1)\nabla P_n(x) &= 0 \quad \text{if} \quad (1,1)x = 2\pi k, \\
(2,-1)\nabla P_n(x) &= 0 \quad \text{if} \quad (1,0)x = 2\pi k, \\
(1,-2)\nabla P_n(x) &= 0 \quad \text{if} \quad (0,1)x = 2\pi k.
\end{aligned} \tag{9}$$

The remaining 4 matrices in $\mathcal{A}$ give no further information.

Let Ω denote the (closed) triangle with vertices $(0,0)$, $(\pi,0)$, $(\frac{2\pi}{3},\frac{2\pi}{3})$. The set

$$\Omega^* = \bigcup_{A \in \mathcal{A}} A\Omega,$$

which is the convex hull of the six points $\pm(\frac{2\pi}{3},\frac{2\pi}{3})$, $\pm(\frac{4\pi}{3},\frac{-2\pi}{3})$, $\pm(\frac{2\pi}{3},\frac{-4\pi}{3})$, is a fundamental domain, i.e. its translates form an essentially disjoint partition of $\mathbb{R}^2$. Therefore, to complete the proof of Theorem 2, it is sufficient to show that

$$(10) \qquad \nabla P_n(x) \neq 0 \quad \text{for} \quad x \in \Omega \setminus \{(0,0), (\pi,0), (\tfrac{2\pi}{3},\tfrac{2\pi}{3})\}$$

and that

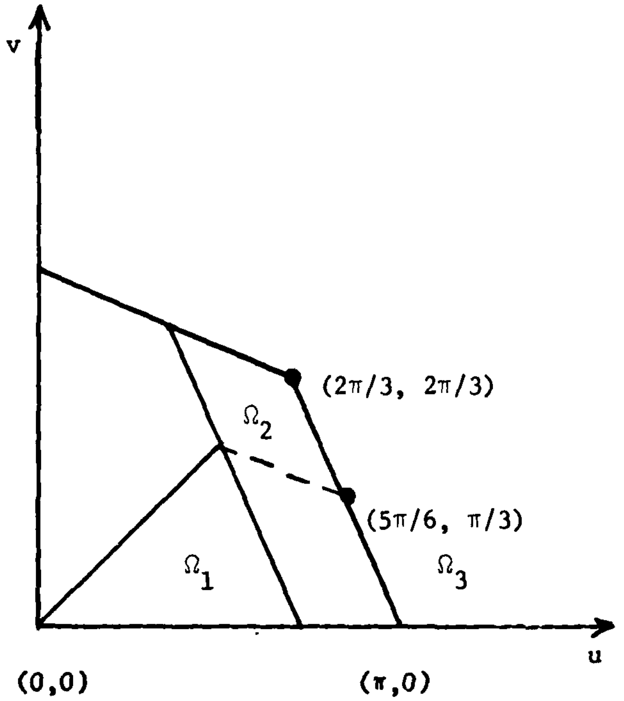
$$(11) \qquad P_n(\tfrac{2\pi}{3},\tfrac{2\pi}{3}) < P_n(\pi,0) < P_n(0,0).$$

To this end we prove the following estimates:

(i) $D_u P_n(u,v) \,/\, (2u + v) < 0$ for $(u,v) \in \Omega_1 := \{(u,v):$
$0 \leq v \leq u, 2u + v \leq \frac{3\pi}{2}, \ u > 0\}$,

(ii) $D_u P_n(u,v) \,/\, (2\pi - 2u - v) < 0$ for $(u,v) \in \Omega_2 := \{(u,v):$
$\frac{3\pi}{2} \leq 2u + v < 2\pi, \ 0 \leq v, \ u + 2v \leq 2\pi\}$,

(iii) $D_v P_n(u,v) \,/\, v < 0$ for $(u,v) \in \Omega_3 := \{(u,v): \ 0 < v = 2\pi - 2u \leq \frac{\pi}{3}\}$.



< Figure 2 >

Note that, since $P_n(u,v) = P_n(v,u)$, it follows from (ii) that

$$D_v P_n\,(\pi - v/2,\, v)\ /\ (\pi - 3v/2) < 0, \quad \pi/3 \le v < 2\pi/3.$$

For small n the inequalities (i) - (iii) can be verified numerically and we shall assume in the sequel that n is sufficiently large ($n \geq 5$). Using the Poisson summation formula and (2), we write P_n in the form

$$(12) \qquad P_n(u,v) = \sum_{(k,\ell)\in\Lambda} S(u+k)^n\, S(v+\ell)^n\, S(u+v+k+\ell)^n$$

where $\Lambda := 2\pi\, Z^2$. For $(u,v) \in \Omega$ and large n, the terms with $|k| + |\ell|$ small dominate in the expression for P_n. This fact is crucial for the subsequent estimates.

Proof of (i). We write

$$D_u P_n(u,v) = n \sum_{(k,\ell)\in\Lambda} a_{k,\ell} b_{k,\ell} \tag{13}$$

with

$$\begin{aligned} a_{k,\ell} &:= S(u+k)^{n-1}\, S(v+\ell)^n\, S(u+v+k+\ell)^{n-1} \\ b_{k,\ell} &:= S'(u+k)\, S(u+v+k+\ell) + S(u+k)\, S'(u+v+k+\ell). \end{aligned} \tag{14}$$

Using the inequalities

$$\begin{gathered} |S(w)|, |S'(w)| \le \min\,(1, 2/|w|), \\ -\frac{1}{12} w \le S'(w) \le -\frac{1}{16} w, \quad 0 \le w \le \pi, \end{gathered} \tag{15}$$

for $(u,v) \in \Omega_1$, we obtain the estimates

$$b_{0,0} \ge -\frac{1}{12}(2u+v),$$

$$b_{0,0} \le -\frac{1}{16}\, u\, S(\pi) - \frac{1}{16}\,(u+v)\, S(\tfrac{3\pi}{4}) \le -\frac{1}{8\pi}\,(2u+v),$$

(16)

$$\left|\frac{b_{k,\ell}}{b_{0,0}}\right| \le \frac{4 \cdot 8\pi}{|u+k|\,|u+v+k+\ell|}\left(\frac{\sin(u/2)}{2u+v} + \frac{\sin((u+v)/2)}{2u+v}\right)$$

$$\le \frac{16\pi}{|u+k|\,|u+v+k+\ell|}\,.$$

For $(u,v) \in \Omega_1$ and $(k,\ell) \neq (0,0)$, we have

$$\frac{1}{|u+k|\,|u+v+k+\ell|}\left|\frac{v}{v+\ell}\right| \le \pi^{-2}.$$

Combining this inequality with (16), we see from the definition of $a_{k,\ell}$ and S that

$$\left|\frac{D_u P_n(u,v)}{n a_{0,0} b_{0,0}} - 1\right| \le \sum_{\Lambda\setminus(0,0)} \left|\frac{a_{k,\ell}}{a_{0,0}}\right|\left|\frac{b_{k,\ell}}{b_{0,0}}\right|$$

$$\le \sum \left|\frac{u}{u+k}\right|^{n-1}\left|\frac{v}{v+\ell}\right|^{n}\left|\frac{u+v}{u+v+k+\ell}\right|^{n-1}\frac{16\pi}{|u+k|\,|u+v+k+\ell|}$$

$$\le \frac{16}{\pi}\sum \left|\frac{3\pi/4}{3\pi/4+k}\right|^{n-1}\left|\frac{\pi/2}{\pi/2+\ell}\right|^{n-1}\left|\frac{\pi}{\pi+k+\ell}\right|^{n-1}.$$

The last right-hand-side is less than 1 for $n \ge 5$. Therefore, inequality (i) follows from the second inequality in (16) and the fact that $a_{0,0}$ is positive on Ω_1.

Proof of (ii). In expression (13) for D_uP_n we split the index set Λ into the three parts

$$\Lambda_0 := \{(k,\ell):\ 2k + \ell + 2\pi = 0\},$$

$$\Lambda_\pm := \{(k,\ell):\ \pm(2k + \ell + 2\pi) > 0\}.$$

The sets Λ_+ and Λ_- are related by the bijective mapping

$$(k,\ell) \in \Lambda_+ \longleftrightarrow (k',\ell') = (-k - \ell - 2\pi, \ell) \in \Lambda_-.$$

Therefore, we can write D_uP_n in the form

(17) $$D_uP_n(u,v) = n \sum_{\Lambda_0} a_{k,\ell}b_{k,\ell} + n \sum_{\Lambda_+} a_{k,\ell}\tilde{b}_{k,\ell}$$

where (c.f. (14))

$$\tilde{b}_{k,\ell} := b_{k,\ell} + \frac{a_{-k-\ell-2\pi,\ell}}{a_{k,\ell}} b_{-k-\ell-2\pi,\ell}$$

$$= [S'(u+k)S(u+v+k+\ell) + S(u+k)S'(u+v+k+\ell)]$$

$$+ \zeta^{n-1}[S'(u-k-\ell-2\pi)S(u+v-k-2\pi) + S(u-k-\ell-2\pi)S'(u+v-k-2\pi)]$$

with

$$\zeta := \frac{u+k}{u-k-\ell-2\pi} \; \frac{u+v+k+\ell}{u+v-k-2\pi} .$$

Observe that for $(u,v) \in \Omega_2$ and $(k,\ell) \in \Lambda_+$

$$0 \le \zeta \quad \text{and} \quad 1 - \zeta = \frac{2\pi-v-2u}{u-k-\ell-2\pi} \; \frac{2k+\ell+2\pi}{u+v-k-2\pi} . \tag{18}$$

Since the numerator in $1 - \zeta$ is positive, letting $\Lambda_*:= \{(k,\ell) \in \Lambda_+: (k+\ell+\pi)(k+\pi) > 0\}$, we have

$$(19) \qquad \begin{aligned} &0 \le \zeta \le 1, && (k,\ell) \in \Lambda_*, \\ &0 \le 1/\zeta \le 1, && (k,\ell) \in \Lambda_+ \setminus \Lambda_*. \end{aligned}$$

Using the identity

$$S(p)S'(q) \pm S'(p)S(q) = \frac{2}{pq} \sin \frac{p \pm q}{2} - \frac{4(p \pm q)}{p^2 q^2} \sin \frac{p}{2} \sin \frac{q}{2}, \tag{20}$$

we can simplify the above expressions for $b_{k,\ell}$ and $\tilde{b}_{k,\ell}$ and obtain

$$(21) \quad b_{k,\ell} = \frac{2 \sin(u+v/2-\pi)}{(u+k)(u+v+k+\ell)} - \frac{4(2u+v-2\pi)}{(u+k)^2(u+v+k+\ell)^2} \sin \frac{u+k}{2} \sin \frac{u+v+k+\ell}{2} , \; (k,\ell) \in \Lambda_0,$$

$$(22)\qquad \tilde{b}_{k,\ell} = \frac{2(-1)^{\ell}\sin(u+v/2)}{(u+k)(u+v+k+\ell)}(1+\zeta^{n}) - \frac{4(-1)^{\ell}\sin(u/2)\sin((u+v)/2)}{(u+k)^{2}(u+v+k+\ell)^{2}}$$

$$\times\ [(2u+v+2k+\ell)+\zeta^{n+1}(2u+v-2k-\ell-4\pi)],\quad (k,\ell)\ \varepsilon\ \Lambda_{*}.$$

In the term in square brackets we add and subtract $(2u + v - 2k - \ell - 4\pi)$. Then a direct computation using (18) yields

$$(23)\qquad [\ldots] = (2u+v-2\pi)\left(2+\frac{(2k+\ell+2\pi)(2u+v-2k-\ell-4\pi)}{(u-k-\ell-2\pi)(u+v-k-2\pi)}\sum_{\nu=0}^{n}\zeta^{\nu}\right).$$

Analogous to case (i) we show that $a_{0,0}\tilde{b}_{0,0}$ is the dominant term for the right hand side of (17). Indeed,

$$(24)\qquad \tilde{b}_{0,0} \le -.6\,\frac{2\pi-2u-v}{u(u+v)}\qquad \text{for}\quad n \ge 5,$$

as one checks numerically for $n = 5$, and therefore has it for $n \ge 5$, since $\tilde{b}_{0,0}$ decreases as n increases as we see from (19), (22) and (23). For $(u,v)\ \varepsilon\ \Omega_2$ we have $\pi/3 \le u,\ u+v \le 4\pi/3$ and we obtain from (19)-(24) the estimates

$$(25)\qquad \left|\frac{b_{k,\ell}}{\tilde{b}_{0,0}}\right| \le \frac{2}{.6}\left|\frac{u}{u+k}\right|\left|\frac{u+v}{u+v+k+\ell}\right|,\quad (k,\ell)\ \varepsilon\ \Lambda_{0},$$

$$(26) \qquad \left|\frac{\tilde{b}_{k,\ell}}{\tilde{b}_{0,0}}\right| \leq \frac{3(n+1)(2k+\ell+2\pi)}{.6} \left|\frac{u}{u+k}\right| \left|\frac{u+v}{u+v+k+\ell}\right|, \ (k,\ell) \in \Lambda_*.$$

For $(k,\ell) \in \Lambda_+ \setminus \Lambda_*$ we estimate $\zeta^{-n}\tilde{b}_{k,\ell}$ in a similar way and obtain

$$(27) \qquad \left|\frac{\tilde{b}_{k,\ell}}{\tilde{b}_{0,0}}\right| \leq \zeta^n \frac{3(n+1)(2k+\ell+2\pi)}{.6} \left|\frac{u}{u+k}\right| \left|\frac{u+v}{u+v+k+\ell}\right|, \ (k,\ell) \in \Lambda_+ \setminus \Lambda_*.$$

For $(k,\ell) = (0,-2\pi)$ we obtain the sharper estimate

$$(28) \qquad \left|\frac{b_{0,-2\pi}}{\tilde{b}_{0,0}}\right| \leq .6 \, \frac{u+v}{2\pi-u-v}, \quad n \geq 5,$$

numerically for $n = 5$, hence valid for $n \geq 5$ since $|\tilde{b}_{0,0}|$ increases with n.

Similarly as for case (i), it follows from (17), (25)-(28), the definition of ζ, and the inequality

$$(v/(2\pi - v))^n((u + v)/(2\pi - u - v))^n \leq 1, \ (u,v) \in \Omega_2,$$

that

$$\left|\frac{D_u P_n(u,v)}{n a_{0,0} \bar{b}_{0,0}} - 1\right| \leq .6 +$$

$$\frac{2}{.6} \sum_{\Lambda_0 \setminus (0,-2\pi)} \left|\frac{\pi}{\pi+k}\right|^n \left|\frac{5\pi/6}{5\pi/6+\ell}\right|^n \left|\frac{4\pi/3}{4\pi/3+k+\ell}\right|^n +$$

$$\frac{3(n+1)}{.6} \sum_{\Lambda_* \setminus (0,0)} \left|\frac{\pi}{\pi+k}\right|^n \left|\frac{5\pi/6}{5\pi/6+\ell}\right|^n \left|\frac{4\pi/3}{4\pi/3+k+\ell}\right|^n (2k+\ell+2\pi) +$$

$$\frac{3(n+1)}{.6} \sum_{\Lambda_+ \setminus \Lambda_*} \left|\frac{\pi}{\pi+k+\ell}\right|^n \left|\frac{5\pi/6}{5\pi/6+\ell}\right|^n \left|\frac{4\pi/3}{2\pi/3+k}\right|^n (2k+\ell+2\pi).$$

The right-hand side is less than 1 for $n \geq 5$ and the inequality (ii) follows from (24) and the fact that $a_{0,0}$ is positive.

Proof of (iii). We have

$$D_v P_n(\pi - v/2, v) = n \sum_{\Lambda} a'_{k,\ell} b'_{k,\ell} \tag{29}$$

with

$$a'_{k,\ell} := S(\pi - v/2 + k)^{n-1} S(v + \ell)^{n-1} S(\pi + v/2 + k + \ell)^{n-1}$$

$$b'_{k,\ell} := S(\pi - v/2 + k) S'(v + \ell) S(\pi + v/2 + k + \ell) +$$

$$S(\pi - v/2 + k) S(v + \ell) S'(\pi + v/2 + k + \ell).$$

Note that $a_{0,0} = a_{-2\pi,0}$. It can be verified numerically that

$$(30) \qquad C := -\sup_{0<v\leq\pi/3} (b'_{0,0} + b'_{-2\pi,0})/v \geq .1.$$

To estimate the remaining terms in (29) we observe from the definition of S and (15) that

$$|b'_{k,\ell}| \leq 2v^2 \min \{1, \frac{2}{|\pi-v/2+k||v+\ell||\pi+v/2+k+\ell|}\}.$$

Therefore,

$$\left|\frac{a'_{k,\ell} b'_{k,\ell}}{a'_{0,0} Cv}\right| \leq \frac{1}{.2} \left|\frac{\pi}{\pi+k}\right|^n \left|\frac{\pi/3}{\pi/3+\ell}\right|^n \left|\frac{7\pi/6}{7\pi/6+k+\ell}\right|^n,$$

and we obtain

$$\left|\frac{D_v P_n(\pi-v/2,v)}{n a_{0,0}(b_{0,0}+b_{-2\pi,0})} - 1\right| \leq \frac{1}{.2} \sum_{\Lambda \{(0,0),(-2\pi,0)\}} \left|\frac{\pi}{\pi+k}\right|^n \left|\frac{\pi/3}{\pi/3+\ell}\right|^n \left|\frac{7\pi/6}{7\pi/6+k+\ell}\right|^n.$$

The right-hand side is less than 1 for $n \geq 2$ which, together with (30), implies the inequality (iii).

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Let M_n denote the bivariate box-spline corresponding to the directions (1.0), (0,1), (1,1), each occurring with multiplicity n. We determine all critical points of the polynomials

$$P_n(x) = \sum_{j\varepsilon Z^2} M_n(j)e^{ijx}, \quad n \varepsilon Z_+ .$$

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