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Methods of Progressing Waves in Turbulent Media

with Applications to Acoustics* *No DTIC*

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Abstract. A method of constructing progressing wave solutions is introduced for treating random linear hyperbolic systems which govern the wave propagation in turbulent media. The method is based on the sample-path asymptotic solution, the diffusion approximation for the characteristic equations, and the Wiener-functional integrals. It is then applied to some problems concerning acoustic waves in turbulent media.

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1. Introduction

In the area of wave propagation in random media, most workers assumed that the fluctuations are independent of time. For a short-range propagation at high frequency, the omission of temporal fluctuation may not cause serious errors. However, in the long-range transmission or in the propagation of pulses, the temporal fluctuations of media become important and should be taken into account. For physical reason a time-dependent random medium will be called a turbulent medium. For earlier papers of basic importance, one is referred to the classics by Keller [1, 2].

In this paper we shall develop a general effective method of computing the statistics of a pulsed wave travelling in a turbulent medium. To any experienced worker in this field, it seems clear that the goal is unattainable in the strong fluctuation case. Therefore our study will be confined to the case of weak fluctuation. Then it is possible to devise a method which will account for the accumulative effect of weak fluctuations in the long run. If the governing equations were ordinary, instead of partial differential equations, the powerful method of diffusion approximation developed by Khasiminskii [3], and Papanicolaou [4] would have been an answer. Here, though not directly applicable, it will also play an important role as to be seen. Our method consists of three basic steps: the deterministic method of characteristics for PDE's (partial differential equations), the diffusion approximation for stochastic ODE's (ordinary differential equations), and the method of functional integrals [5], in that order. As the first step, we seek a progressing wave solution which enables us to reduce a stochastic PDE to a system of stochastic ODE's, to which the diffusion approximation is applicable. By invoking the diffusion limit theorems

mentioned above, the amplitude and phase fluctuations may be described asymptotically in terms of some known diffusion processes. Thereby the statistics of the original random wave function can be computed approximately by evaluating some average functionals of the known diffusion processes. This last step may be viewed as the evaluation of certain functional integrals over the probability distributions of the diffusion processes. In probability theory, the last two steps correspond to the celebrated invariance principle of Donsker [6], in a generalized situation. We wish to point out that, even after an asymptotic reduction, the evaluation of a functional integral is by no means easy. But the reduced problem is much simpler than the original one which is totally intractable. The proposed method is well suited for treating the pulse propagation problem governed by a random hyperbolic system. It yields a statistical description of the physically meaningful quantities, such as the ray, amplitude and phase, in terms of which the statistical law for the wave function may be determined. This allows us, in principle, to compute more complicated statistics, other than the moments of the solution as commonly done.

The text of the paper consists of three sections. In Section 2 we develop the basic ideas via a simple model equation. The first-order random PDE is to be solved by the method of characteristics. By applying the appropriate limit theorems, the fluctuations of amplitude and phase functions may be determined along the characteristic curves in an asymptotic sense. As examples, some statistical functionals are evaluated. The general method of progressing waves for a random hyperbolic system is presented in Section 3. Due to a marked difference between one and higher dimensional cases, they will be discussed separately. There the three basic steps stated above will be carried out in detail. To illustrate the

computational procedures, the method is applied to problems in the acoustic wave propagation through a turbulent medium. This is done in Section 4. To avoid obscuring the basic ideas involved, some technical details concerning two main theorems that we rely on heavily are omitted in the text, and they are presented in the appendix.

2. Simple Model Equation

To illustrate the basic ideas, let us consider a first-order partial differential equation with a random coefficient.

$$(2.1) \quad \begin{cases} [\partial_t + \eta^\epsilon(t,x,\omega)\partial_x]u = b(t,x)u, & t > 0, \\ u(0,x,\omega) = a(x), & x \in \mathbb{R}. \end{cases}$$

Here η^ϵ , for each $\epsilon > 0$, is a known random function (field) of $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, defined over a probability space (Ω, \mathcal{F}, P) with the sample space Ω containing the sample point ω . The given functions $a: \mathbb{R} \rightarrow \mathbb{R}$ and $b: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ or \mathbb{C} are deterministic and smooth. The solution $u(t,x,\omega)$ is thus a random field. The statistical law and averages of u are of our primary concern. By the standard notation, a statistical average of $\{\cdot\}$ will be denoted by $E\{\cdot\}$. Also, in this paper, ∂_t, ∂_x denote the partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$, etc.

As is well-known, the system (2.1) can be integrated by introducing the characteristic equation:

$$(2.2) \quad \begin{cases} \frac{dy}{ds} = \eta^\epsilon(s,y,\omega), & 0 \leq s < t, \\ y(t) = x, \end{cases}$$

which is a stochastic system.

Let

$$(2.3) \quad y = \xi_s^\epsilon(t,x,\omega), \quad 0 \leq s \leq t$$

be a random solution of (2.2), which defines a characteristic curve $\Gamma_{t,x}^\epsilon(\omega)$ passing through (t,x) . Along such a curve, the system (2.1) can be integrated to give the "wave function":

$$(2.4) \quad u(t,x,\omega) = a[\xi_0^\epsilon(t,x,\omega)] \exp\{\theta^\epsilon(t,x,\omega)\},$$

where

$$(2.5) \quad \theta^\varepsilon(t, x, \omega) = \int_0^t b[s, \xi_s^\varepsilon(t, x, \omega)] ds,$$

will be called a "phase function", (which may not be physically meaningful as shown in the next section). We note that the representation of solution (2.4) is not constructive, since, given η^ε , the probability distribution of ξ_s^ε can not be determined easily in case of strong fluctuations. Suppose that

$$(2.6) \quad \eta^\varepsilon(t, x, \omega) = \hat{\eta} + \varepsilon \eta(t, x, \omega),$$

where $\hat{\eta}$ is a constant and $0 < \varepsilon \ll 1$ is the small scale of fluctuations. Now we let

$$(2.7) \quad \begin{aligned} y(s) &= \hat{\xi}_s(t, x) + z(s), \\ \hat{\xi}_s(t, x) &= x - \hat{\eta}(t-s). \end{aligned}$$

Then, in view of (2.2), $z_s = z(t-s)$ satisfies

$$(2.8) \quad \begin{cases} \frac{dz_s}{ds} = Z_t(s, z_s, \omega), \\ z_0 = x, \end{cases}$$

where

$$(2.9) \quad Z_t(s, z_s, \omega) = \eta(t-s, \hat{\xi}_{t-s} + z_s, \omega).$$

To the system (2.8), we can apply Theorem A.1 to get a diffusion approximation. To this end we assume that the random field η is homogeneous and stationary such that

$$(2.10) \quad \begin{aligned} E\eta &= 0, \\ E\eta(t, x)\eta(s, y) &= \gamma(t-s, x-y), \end{aligned}$$

and the following limits exist

$$(2.11) \quad \begin{aligned} \beta &= v^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_0^T \gamma[t-s, \hat{\eta}(t-s)] ds dt, \\ \kappa &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_0^T \gamma_x[t-s, \hat{\eta}(t-s)] ds dt, \end{aligned}$$

$$\text{with } \gamma_x(t, x) = \frac{\partial}{\partial x} \gamma(t, x).$$

Then, under a strong mixing condition among others, one can apply the Khasiminskii type of limit theorem to the system (2.8), as shown in the appendix. According to Theorem A.1 there, we can assert that, as $\varepsilon \downarrow 0$, $t > s \uparrow \infty$ with $\tau = \varepsilon^2 t$ and $\sigma = \varepsilon^2 t$ held fixed, we have

$$(2.12) \quad z_s = \zeta_s^\varepsilon(t, x) \Rightarrow \zeta_\sigma = \zeta_\sigma(\tau), \quad 0 \leq \sigma \leq \tau$$

where $\{\zeta_\sigma, \sigma \leq \tau\}$ is a Wiener process with the mean and variance parameters κ and ν , respectively, and $\zeta_\tau = 0$. Let $\{w(t), t \geq 0\}$ be the standard Wiener process or the Brownian motion with $w(0) = 0$. Then ζ_σ will satisfy the Itô equation

$$(2.13) \quad \begin{aligned} d\zeta_\sigma &= \kappa d\sigma + \nu dw(\tau - \sigma), \quad \sigma < \tau, \\ \zeta_\tau &= 0, \quad \text{with } \nu^2 = \beta. \end{aligned}$$

Therefore, in the diffusion limit, the solution of (2.1), noting (2.3), can be approximated asymptotically as follows:

$$(2.14) \quad y(s) = \xi_s^\varepsilon(t, x) \sim \xi_s(t, x),$$

where, in view of (2.7) and (2.12),

$$(2.15) \quad \xi_s(t, x) = x - (c + \varepsilon^2 \kappa)(t - s) + \varepsilon \nu w(t - s).$$

This shows clearly that the original characteristic

$$(2.16) \quad \Gamma_{t,x}^\varepsilon : y = \xi_s^\varepsilon(t, x), \quad 0 \leq s \leq t,$$

is close to the asymptotic characteristic

$$(2.17) \quad \Gamma_{t,x} : y = \xi_s(t, x), \quad 0 \leq s \leq t,$$

as given explicitly by (2.15). It contains a known fluctuation, a Brownian motion with the parameter ν about an effective (in general, not the unperturbed) characteristic

$$(2.18) \quad \tilde{\Gamma}_{t,x} : y = \tilde{\xi}_s(t, x) = x - (\hat{\eta} + \varepsilon^2 \kappa)(t - s).$$

Suppose that the functions a and b in the system (2.1) are so smooth as required by Theorem A.2. Then this theorem implies that the random solution (2.4) yields

$$(2.19) \quad \begin{aligned} u(t,x) &\sim a[\xi_0(t,x)] \exp\{\theta(t,x)\} , \\ \theta(t,x) &= \int_0^t b[s, \xi_s(t,x)] ds. \end{aligned}$$

Here, for brevity, the sample point ω in the argument of all the relevant functions is suppressed. This will often be done hereafter so long as there is no confusion. In view of (2.15), the result (2.19) shows that the wave function is in fact a functional of Wiener process or Wiener functional in short. Hence the computation of asymptotic statistics of the wave function u amounts to the evaluation of certain functional integrals with respect to the Wiener measure. Furthermore the statistical laws of the amplitude and phase become completely known.

As an example, let b be imaginary,

$$(2.20) \quad b(t,x) = i \phi(t,x), \quad \phi \text{ real.}$$

Let $\langle u \rangle = E u$ denote the mean or coherent wave function. Then

$$(2.21) \quad \begin{aligned} \langle u \rangle \sim \langle v \rangle &= E_w \{ a[\tilde{\xi}(t,x) - \epsilon v w(t)] \\ &\times \exp i \int_0^t \phi[s, \tilde{\xi}(t-s,x) - \epsilon v w(t-s)] ds \} , \end{aligned}$$

where E_w denotes the expectation with respect to the Wiener process $w(t)$.

We note that the average Wiener functional in (2.21) is just a modified Feynman-Kac formula along the effective characteristic $\tilde{\Gamma}_{t,x}$. It can be

shown, similar to the conventional case [7], that $\langle v \rangle$ satisfies the differential equation:

$$(2.22) \quad \begin{aligned} [\partial_t + (c + \epsilon^2 \kappa) \partial_x] \langle v \rangle &= \frac{1}{2} \beta \epsilon^2 \partial_x^2 \langle v \rangle + i \phi(t,x) \langle v \rangle , \\ \langle v \rangle |_{t=0} &= a(x). \end{aligned}$$

We remark that the equation (2.3) can be derived by the so-called method of smooth perturbation introduced by Keller [1.2]. In fact, higher moments of v can be treated in a similar manner.

However, in contrast with the traditional perturbation methods for moments, the current method yields other important statistics than just the first few moments of u . For instance, the statistics of the wave amplitude

$$(2.23) \quad |u(t,x)|^2 \sim I(t,x) = a^2[\tilde{\xi}(t,x) - \epsilon v w(t)]$$

can be computed easily. Suppose $a^2(x) = \rho(|x|)$ and ρ is strictly increasing. Then the probability

$$\begin{aligned} P\{I(t,x) \geq r\} &= P\{|\tilde{\xi}(t,x) - \epsilon v w(t)| \leq \rho^{-1}(r)\} \\ &= \frac{1}{(2\pi v \epsilon^2 t)^{1/2}} \int_{\xi^-}^{\xi^+} \exp\left\{-\frac{s^2}{v \epsilon^2 t}\right\} ds \end{aligned}$$

where $\xi^\pm = \tilde{\xi}(t,x) \pm \rho^{-1}(r)$ and ρ^{-1} is the inverse of ρ .

Thus the distribution function for $I(t,x)$ is given by

$$F(r;t,x) = 1 - P\{I(t,x) \geq r\}$$

from which all statistics of I at any point (t,x) is computable. Similarly

the joint distributions at two points $F(r_1, r_2; t_1, x_1, t_2, x_2)$
 $= P\{I(t_1, x_1) < r_1, I(t_2, x_2) < r_2\}$, and multiple points may be obtained.

The computation becomes tedious but elementary.

Since the phase θ given in (2.19) is a Wiener functional, its statistics are more difficult to compute. However, there exist an extensive list of literatures on this subject, (e.g., see the reference in [5,8]).

3. Method of Progressing Waves

Let $A_j = A_j^\epsilon(t, x, \omega)$, $j = 1, 2, \dots, n$, be some given $m \times n$ matrix-valued random functions of $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$, depending on a small parameter $\epsilon > 0$, and let $B(t, x)$ be a $m \times n$ matrix-valued, deterministic function on $\mathbb{R}^+ \times \mathbb{R}^n$. We consider the random linear hyperbolic system in \mathbb{R}^n for the wave function u in \mathbb{R}^m :

$$(3.1) \quad Lu = (\partial_t - \sum_{j=1}^n A_j \partial_j - B)u = 0,$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$. The random solution $u(t, x)$, in which ω is suppressed, is subject to appropriate initial-boundary conditions. For clarity we shall present the method of progressing wave solution to (3.1) in three steps as indicated in §1. As a matter of convenience, a vector will sometimes be regarded as a column matrix, such as in (3.1), and the indicial notations will also be used as one sees fit.

3.1. Sample Progressing Wave Solution

To study the propagation of weak singularities, such as pulses, the progressing wave solution is an effective technique in the deterministic case [9]. For a fixed $\omega \in \Omega$, we may apply the method of solution to the system as described in [10] and [11]. To make the paper self-contained, we shall present a derivation based on Lax's approach [11].

For the system (3.1), we seek an approximate solution \tilde{u} of the form:

$$(3.2) \quad \tilde{u}(t, x, \omega) = g[\phi(t, x, \omega)]v(t, x, \omega),$$

where g is a scalar, possibly generalized function of a real or complex variable, ϕ a scalar random function, while v a m -vector valued, random function on $\mathbb{R}^+ \times \mathbb{R}^n$.

Apply L to \tilde{u} as defined in (3.1) and (3.2) to get:

$$(3.3) \quad L\tilde{u} = g'Mv + gLv,$$

where $g'(t) = \frac{dg(t)}{dt}$, and

$$(3.4) \quad Mv = (\phi_t I - \sum_{j=1}^n \phi_j A_j)v,$$

with $\phi_t = \partial_t \phi$, $\phi_j = \partial_j \phi$ and I a $n \times n$ identity matrix. Since g may be a generalized function, for the function Lu to be less singular, we set

$$(3.5) \quad Mv = 0,$$

so that (3.3) becomes

$$(3.6) \quad L\tilde{u} = gLv.$$

For (3.5) to have a nontrivial solution, we must have

$$(3.7) \quad \partial_t \phi = \lambda^\epsilon(t, x, \partial_x \phi, \omega),$$

and

$$(3.8) \quad v = \alpha(t, x, \omega)r(t, x, \omega),$$

where λ is an eigenvalue, with the corresponding right eigenvector r ,

of the random matrix $\sum_{j=1}^n \phi_j A_j$, and α is a scalar random function. To determine α , we seek the next order of approximation by writing

$$(3.9) \quad u = \tilde{u} + q.$$

In view of (3.1) and (3.6), the remainder q satisfies

$$(3.10) \quad Lq = -gLv.$$

Assume the second-order approximation \tilde{q} to q has a form similar to \tilde{u} :

$$(3.11) \quad \tilde{q} = h(\phi)v_1$$

where h and v_1 are scalar and vector-valued functions to be determined

so that the first-order residual term on the right-hand side of (3.10) is removed. This leads to the choices:

$$(3.12) \quad h' = g,$$

$$(3.13) \quad Mv_1 = -Lv.$$

Let $\ell(t,x,\omega)$ be a left eigenvector associated with r so that their inner product

$$(3.14) \quad \ell \cdot r = 1 .$$

Then, since r is a solution of the homogeneous equation of (3.13), the solvability condition implies that

$$(3.15) \quad \ell \cdot Lv = 0$$

or, noting (3.8)

$$(3.16) \quad \left[\partial_t - \sum_{j=1}^n (\ell \cdot A_j r) \partial_j \right] \alpha + (\ell \cdot Lr) \alpha = 0 ,$$

which determines the amplitude function in the first-order approximation.

To find the phase function ϕ , we can integrate the Hamilton-Jacobi equation (3.7) by the method of characteristics [10]. The characteristic (or bi-characteristic) equations read:

$$(3.17) \quad \left\{ \begin{array}{l} \frac{dy}{ds} = -\lambda_q^\epsilon(s,y,q,\omega) , \\ \frac{dq}{ds} = \lambda_y^\epsilon(s,y,q,\omega) , \quad 0 \leq s < t, \\ y(t) = x, \\ q(t) = p . \end{array} \right.$$

In the physical space, the characteristic curve

$$(3.18) \quad \Gamma_{t,x} : y = \xi_s(t,x) , \quad 0 \leq s \leq t$$

is called a ray through the point (t,x) . As in the theory of geometric optics [12], the equation (3.7) will be called the eikonal equation for the phase θ , the equation (3.16) the transport equation for the amplitude α . It is easy to check that, along a ray $\Gamma_{t,x}$, θ is conserved and that the transport equation (3.16) can be integrated similarly to the problem in §2, where the characteristic equation now becomes a bicharacteristic equation (3.17).

3.2. Diffusion Approximation

To study the effect of weak fluctuations, we assume that the random matrices in (3.17) take the special form:

$$(3.19) \quad A_j = A_j^\varepsilon(t, x, \omega) = \hat{A}_j + \varepsilon R_j(t, x, \omega), \quad j=1, 2, \dots, n$$

where \hat{A}_j is a constant matrix, and R_j a spatially homogeneous random, matrix-valued function satisfying

$$E R_i = 0 ,$$

$$E\{\rho_{i, k\ell}(t, x) \rho_{j, pq}(s, y)\} = \gamma_{ij, k\ell, pq}(t, s; x, y).$$

where $\{\rho_{i, k\ell}\}$ are the entries of R_i , $i, j, \ell, q=1, 2, \dots, n$, $k, p=1, 2, \dots, m$.

To stress the dependence on ε , the associated quantities will be indexed by ε , such as ℓ^ε , r^ε , M^ε , etc. In view of (3.4) and (3.14), we can write

$$(3.21) \quad M^\varepsilon r^\varepsilon = (\lambda^\varepsilon I - \sum_{j=1}^n p_j A_j^\varepsilon) r^\varepsilon = 0,$$

$$(3.22) \quad \ell^\varepsilon \cdot r^\varepsilon = 1.$$

from which we get

$$\ell^\varepsilon \cdot M^\varepsilon r^\varepsilon = \lambda^\varepsilon - \ell^\varepsilon \cdot (p \cdot A^\varepsilon) r^\varepsilon = 0 ,$$

or

$$(3.23) \quad \lambda^\varepsilon = \ell^\varepsilon \cdot (p \cdot A^\varepsilon) r^\varepsilon ,$$

$$\text{where } p \cdot A^\varepsilon = \sum_{j=1}^n p_j A_j^\varepsilon .$$

Expand λ^ε in the small parameter and write

$$(3.24) \quad \lambda^\varepsilon = \lambda + \varepsilon \lambda_1 + 0(\varepsilon^2).$$

For simplicity, we put $\lambda = \lambda^0$, $\ell = \ell^0$, and so on. For $\varepsilon = 0$ and by

(3.19), (3.22) and (3.23) yield

$$(3.25) \quad \ell \cdot r = 1$$

and

$$(3.26) \quad \lambda = \ell \cdot (p \cdot \hat{A}) r = \ell \cdot \left(\sum_{j=1}^n p_j \hat{A}_j \right) r.$$

By differentiating (3.22) and (2.23) with respect to ϵ at $\epsilon=0$, we get

$$(3.27) \quad \lambda_1 = \ell \cdot (p \cdot R) r = \ell \cdot \left(\sum_{j=1}^n p_j R_j \right) r.$$

Note that since, by assumption \hat{A} is constant, the eigenvectors ℓ and r may depend only on p , not on x . Similarly, by differentiating (3.22) and (3.23) with respect to p and then to x , and invoking (3.21) and (3.24)-(3.27), we can express the characteristic equations (3.17), keeping only terms up to $0(\epsilon)$, in the explicit form:

$$(3.28) \quad \begin{cases} \frac{dy}{ds} = \hat{Y}(q) + \epsilon Y(s, y, q, \omega) \\ \frac{dq}{ds} = \epsilon Q(s, y, q, \omega), \\ y(t) = x, \\ q(t) = p, \end{cases}$$

where

$$(3.29) \quad \begin{cases} \hat{Y}_i(p) = -\ell \cdot \hat{A}_i r = - \sum_{j,k=1}^n \hat{A}_{i,jk} \ell_j(p) r_k(p), \\ Y_i(t, x, p, \omega) = -\ell \cdot R_i r = - \sum_{j,k=1}^n \rho_{i,jk}(t, x, \omega) \ell_j(p) r_k(p), \\ Q_i(t, x, p, \omega) = \ell \cdot (p \cdot \partial_i R) r = \sum_{j,k,h=1}^n \partial_i \rho_{j,hk}(t, x, \omega) p_j \ell_k(p) \ell_h(p), \end{cases}$$

$i=1, 2, \dots, n.$

From the above expressions, we see that, in general, the characteristic equations are coupled. In case that they do not, the asymptotic analysis becomes much simpler. This special case will be treated first.

(a) Decoupled Bi-characteristics

It is possible that λ_p^ϵ , hence, Y and \hat{Y} in (3.28) may be independent of p , (see an example in §4). Then we can decouple the first equation

from the second in (3.28) to get the physical characteristic curve:

$$(3.31) \quad \Gamma_{t,x}^\varepsilon : \begin{cases} \frac{dy}{ds} = \hat{Y} + \varepsilon Y(s,y,\omega), & 0 \leq s < t, \\ y(t) = x, \end{cases}$$

where \hat{Y} is a constant vector. This is a special form of the stochastic equation (1) treated in the appendix. Thus if the random field Y (or R) satisfies the conditions of Theorem A.1, then, as $\varepsilon \downarrow 0$, $t \geq s \uparrow \infty$ with $\tau = \varepsilon^2 t$ and $\sigma = \varepsilon^2 s$ held fixed, the solution of (3.31) approaches the asymptotic form

$$(3.22) \quad y(s) = \xi_s^\varepsilon(t,x) \sim \xi_s(t,x) = \hat{\xi}_s(t,x) + \zeta_\sigma(\tau),$$

where

$$(3.33) \quad \hat{\xi}_s(t,x) = x - \hat{A}(t-s),$$

and the process $\{\zeta_\sigma(\tau), 0 \leq \sigma \leq \tau\}$ is a diffusion process satisfying the Itô equation (6). Here the drift vector κ and the diffusion matrix β , referring to (A.6), are:

$$(3.34) \quad \kappa_i = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{s_1} E \sum_{j=1}^n \partial_j Y_i(T-s_1, x - \hat{A}s_1) Y_j(T-s_2, x - \hat{A}s_2) ds_1 ds_2.$$

$$(3.35) \quad \beta_{ij} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} E Y_i(T-s_1, x - \hat{A}s_1) Y_j(T-s_2, x - \hat{A}s_2) ds_1 ds_2, \\ i, j = 1, 2, \dots, n,$$

where Y_i is defined in (3.29), and, by the assumption of homogeneity, κ and β are constant.

(b) Coupled Bi-characteristics

In the general situation, Theorem A.1 does not apply directly to the bi-characteristic equations (3.28). However, since $(q-p) = o(\varepsilon)$, if we linearize \hat{Y} about p ,

$$(3.36) \quad \hat{Y}(p+q') = \hat{Y}(p) + \hat{Y}_1 q' + o(\varepsilon^2), \text{ with } q' = q-p,$$

then the system (3.28) becomes

$$(3.37) \quad \begin{cases} \frac{dy}{ds} = \hat{Y}(p) + \hat{Y}_1 q' + \varepsilon Y(s, y, p+q', \omega) \\ \frac{dq'}{ds} = \varepsilon Q(s, y, p+q', \omega) \\ y(t) = x, \\ q'(t) = 0, \end{cases}$$

where

$$\hat{Y}_1 = \partial_q \hat{Y}(q) |_{q=p}$$

is a $n \times n$ matrix.

Let

$$(3.38) \quad \begin{cases} y^* = (y, q'), \\ \hat{Y}^*(y^*) = A^* y^* + C^*, \\ A^* y^* = (\hat{Y}_1 q', 0), \quad C^* = (\hat{Y}(p), 0), \\ \text{and} \\ Y^*(s, y^*, \omega) = (Y(s, y, p+q', \omega), Q(s, y, p+q', \omega)). \end{cases}$$

Then the system (3.36) can be written as

$$(3.39) \quad \begin{cases} \frac{dy^*}{ds} = \hat{Y}^*(y^*) + \varepsilon Y^*(s, y^*, \omega), \\ y^*(t) = x^* = (x, 0) \end{cases}$$

which is of the same form as the standard equation (1) in the appendix with y replaced by y^* . Hence we can apply Theorem A.1 to (3.39), provided that the corresponding conditions (A.1)-(A.6) are met. To this end let K_t^* be a fundamental matrix for the system (3.40) at $\varepsilon=0$ with $K_0^* = I^*$. It is simple to check that

$$(3.40) \quad K_t^* = \begin{bmatrix} I & t \hat{Y}_1 \\ 0 & I \end{bmatrix} \text{ with } (K_t^*)^{-1} = K_{-t}^*.$$

Similar to Z_t as in (6), define

$$(3.41) \quad Z_t^*(s, z^*, \omega) = (K_{t-s}^*)^{-1} Y^*[t-s, \hat{\xi}_{t-s}^*(t, x) + K_{t-s}^* z_1^*, \omega],$$

where

$$(3.42) \quad \xi_s^*(t, x) = (x + \hat{Y}(p)(t-s), 0).$$

Then, according to Theorem A.1, the solution of (3.40) has the following asymptotic form

$$(3.43) \quad y^* = \xi_s^*(t, x, \epsilon) \sim \xi_s^*(t, x) + K_{t-s}^* \zeta_\sigma^*(t),$$

where $\{\zeta_\sigma^*(t), 0 \leq \sigma \leq \tau\}$ is a diffusion process in \mathbb{R}^{2n} with the drift vector and the diffusion matrix given, respectively, by

$$(3.44) \quad \begin{cases} \kappa_i = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{s_1} E \sum_{j=1}^{2n} \left[\frac{\partial}{\partial \xi_j} (Z_T^*)_i(s_1, \xi) \right] (Z_T^*)_j(s_2, \xi) ds_1 ds_2 \\ \beta_{ij} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} E (Z_T^*)_i(s_1, \xi) (Z_T^*)_j(s_2, \xi) ds_1 ds_2. \end{cases}$$

The equation (3.43) defines an asymptotic bi-characteristic curve in the phase space. Its projection onto the physical space yields the characteristic curve $\Gamma_{t,x}^\epsilon$:

$$(3.45) \quad y = \xi_s^\epsilon(t, x) \sim \hat{\xi}_s(t, x) + \zeta_\sigma(\tau) - (t-s) \hat{Y}_1 \zeta_\sigma'(\tau) = \xi_s(t, x),$$

where

$$(3.46) \quad \begin{cases} \xi_s(t, x) = \hat{\xi}_s(t, x) = x + \hat{Y}(p)(t-s), \\ \zeta_\sigma^* = (\zeta_\sigma, \zeta_\sigma'). \end{cases}$$

From (3.45) we see that, in contrast with the one-dimensional case, the random fluctuation about the unperturbed characteristic is a diffusion in the phase, instead of physical space. However, as to be shown later in 4§ in a special case, the physical characteristics, or ζ_σ , is asymptotically decoupled from ζ_σ' under appropriate conditions.

3.3. Asymptotic Statistics

In the first-order approximation, noting (3.2) and (3.8), the progressing wave solution of (3.1) takes the form

$$(3.47) \quad u(t,x,\omega) \sim \alpha^\epsilon(t,x,\omega)g[\phi^\epsilon(t,x,\omega)]r(x)$$

where, within $O(\epsilon)$, the unperturbed initial eigenvector $r(x)$ is used in place of $r^\epsilon(t,x,\omega)$. That is, $r(x)$ satisfies the equation (3.21) with $\epsilon=0$ and $p_j = \partial_j \theta(x)$. Suppose that the initial state of u may be represented as:

$$(3.48) \quad u(0,x,\omega) \sim a(x)g[\theta(x)]r(x),$$

with

$$(3.49) \quad \begin{cases} \alpha^\epsilon(0,t,x,\omega) = a(x), \\ \phi^\epsilon(0,t,x,\omega) = \theta(x), \\ r^\epsilon(0,t,x,\omega) = r(x). \end{cases}$$

By integrating the eikonal equation (3.7) and the transport equation (3.16) along the physical characteristic curve $\Gamma_{t,x}^\epsilon$, (3.47) and (3.48) yield, within an error of $O(\epsilon)$,

$$(3.50) \quad \begin{aligned} u(t,x) &\sim a[\xi_0^\epsilon(t,x)]g\{\theta[\xi_0^\epsilon(t,x)]\}r(x) \\ &\times \exp\left\{\int_0^t b[s,\xi_s^\epsilon(t,x)]ds\right\} \end{aligned}$$

where

$$(3.51) \quad b(t,x) = \ell(x) \cdot B(t,x)r(x) + \sum_{j=1}^n \ell(x) \cdot \hat{A}_j \partial_j r(x).$$

Therefore, if the functions a,b,g,θ are such that the conditions for Theorem A.2 are satisfied, then, in the diffusion limit, we have

$$(3.52) \quad \begin{aligned} u(t,x) &\sim a[\xi_0(t,x)]g\{\theta[\xi_0(t,x)]\}r(x) \\ &\times \exp\left\{\int_0^t b[s,\xi_s(t,x)]ds\right\}, \end{aligned}$$

where, depending on the bi-characteristics being uncoupled (case a) or otherwise (case b), the asymptotic expression ξ_s is given by (3.32) or (3.45), respectively.

Suppose that $\{r_1, r_2, \dots, r_k\}$ and $\{\ell_1, \ell_2, \dots, \ell_k\}$, the right and left eigenvectors correspond to the distinct, nonzero eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$, respectively, are such that

$$(3.53) \quad \ell_i \cdot r_j = \delta_{ij}, \quad \text{for } i, j = 1, 2, \dots, k \leq n.$$

If the initial state, instead of (3.48), can be represented as

$$(3.54) \quad u(0, x, \omega) \sim \sum_{j=1}^m a_j(x) g[\theta_j(x)] r_j(x),$$

then, by superposition, the corresponding progressing wave solution (3.51) becomes

$$(3.55) \quad u(t, x, \omega) \sim \sum_{j=1}^k a_j[\xi_{j,0}(t, x, \omega)] g\{\theta_j[\xi_{j,0}(t, x, \omega)]\} r_j(x) \\ \times \exp\left\{\int_0^t b[s, \xi_{j,s}(t, x, \omega)] ds\right\}$$

where $y = \xi_{j,s}$ is the asymptotic solution of the characteristic equation when $\lambda^E = \lambda_j^E$.

For each component (mode) of u in (3.55), the asymptotic statistics of the progressing wave may be computed in a similar manner as illustrated for the model problem in §2. However, in order to compute the statistics involving more than one component in (3.55), one has to obtain the joint probability distributions of $\xi_{1,s}, \dots, \xi_{k,s}$. This can be done by augmenting the single system (3.17) to k -simultaneous systems of bi-characteristic equations, corresponding to the k distinct eigenvalues. Then apply the usual asymptotic analysis to the simultaneous systems to derive the joint

diffusion processes $\{\xi_{1,s}, \dots, \xi_{k,s}\}$. This involves no new concept, but does increase the computational complexity.

We remark that, as in the deterministic case, it is possible to carry out the higher order approximation. But the statistical problem would become too complicated to be of practical value. Also we note that the asymptotic symbol, such as in (3.55), has a double meaning. That is, the asymptotic sequence is ordered by the strength of singularity of the phase function g and the small parameter ε . In what follows we shall apply the method to some problems in acoustic wave propagation.

4. Acoustic Waves in Turbulent Media

Consider the acoustic wave propagation through a weakly turbulent fluid governed by the following system

$$(4.1) \quad \begin{cases} \partial_t v + \tilde{\rho}^{-1} \partial_x p = 0, \\ \partial_t p + \tilde{\rho} \tilde{c}^2 \partial_x \cdot v = 0, \quad t > 0, x \in \mathbb{R}^n, n \leq 3, \end{cases}$$

where v is the acoustic velocity, p the acoustic pressure, $\tilde{\rho}$ the density and \tilde{c} is the local speed of sound. In this section, the symbol p will be reserved exclusively for the pressure. The symbols ∂_x and $\partial_x \cdot$ denote the gradient and divergent operators, respectively. Due to the weak turbulence, the density $\tilde{\rho}$ and the sound speed \tilde{c} fluctuate randomly. So we assume that the random functions

$$(4.2) \quad \begin{cases} \tilde{\rho} = \rho^\varepsilon(t, x, \omega) = \hat{\rho} + \varepsilon \rho(t, x, \omega) \\ \tilde{c} = c^\varepsilon(t, x, \omega) = \hat{c} + \varepsilon c(t, x, \omega) \end{cases}$$

depend on a small parameter $\varepsilon > 0$ as given above. For convenience, we define

$$(4.3) \quad \begin{cases} \mu_1^\varepsilon = (\rho^\varepsilon)^{-1} \\ \mu_2^\varepsilon = \rho^\varepsilon (c^\varepsilon)^2, \end{cases}$$

so that

$$(4.4) \quad \mu_1^\varepsilon \mu_2^\varepsilon = (c^\varepsilon)^2 .$$

To put the system (4.1) in the standard form (3.1), let us define u and A_j^ε as follows:

$$(4.5) \quad u = (v_1, v_2, \dots, v_n, p),$$

$$(4.6) \quad A_j = A_j(\mu^\varepsilon) = (-1) \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & \mu_1^\varepsilon \delta_{1j} \\ 0 & 0 & & & & & \mu_1^\varepsilon \delta_{2j} \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ 0 & & & \cdot & \cdot & \cdot & 0 & \mu_1^\varepsilon \delta_{nj} \\ \mu_2^\varepsilon \delta_{1j} & \mu_2^\varepsilon \delta_{2j} & \cdot & \cdot & \cdot & \mu_2^\varepsilon \delta_{nj} & 0 \end{bmatrix}, j=1,2,\dots,n,$$

where $\delta_{ij} = 0$ or 1 according to $i=j$ or $i \neq j$. Let us rewrite (4.3) as

$$(4.7) \quad \mu_i^\varepsilon = \hat{\mu}_i + \varepsilon \mu_i + o(\varepsilon^2), \quad i=1,2,$$

where, in view of (4.2),

$$(4.8) \quad \hat{\mu}_1 = \hat{\rho}^{-1}, \quad \hat{\mu}_2 = \hat{\rho}(\hat{c})^2,$$

and

$$(4.9) \quad \begin{aligned} \mu_1 &= -\rho(t,x,\omega)/\hat{\rho}, \\ \mu_2 &= \hat{c}[\hat{c}\rho(t,x,\omega) + 2\hat{\rho}c(t,x,\omega)]. \end{aligned}$$

Statistically, we assume that the random functions ρ and c are centered, homogeneous and satisfy the strong mixing and other smooth properties so that the diffusion approximation holds. For any homogeneous random functions ξ, η , let us introduce the following notations:

$$(4.10) \quad \begin{aligned} \gamma_{\xi\eta}(t,s;x-y) &= E\xi(t,x)\eta(s,y), \\ \gamma_{\xi\eta}^*(t,s;x-y) &= \gamma_{\xi\eta}(s,t;y-x) = \gamma_{\eta\xi}(t,s;x-y), \text{ by definition.} \end{aligned}$$

Then, by assumption, we have

$$(4.11) \quad E\rho = Ec = 0,$$

and

$$\gamma_{\rho\rho}(t,s;x-y) = E\{\rho(t,x)\rho(s,y)\}.$$

$$(4.12) \quad \gamma_{cc}(t,s;x-y) = E\{c(t,x)c(s,y)\},$$

$$\gamma_{\rho c}(t,s;x-y) = E\{\rho(t,x)c(s,y)\}.$$

Now, noting (4.7), the matrices A_j^ϵ in (4.6) takes the following form:

$$(4.13) \quad A_j^\epsilon = \hat{A}_j + \epsilon R_j + O(\epsilon^2),$$

where

$$(4.14) \quad \begin{aligned} \hat{A}_j &= A_j(\hat{\mu}), \\ R_j &= A_j(\mu), \quad j=1,2,\dots,n. \end{aligned}$$

After taking the above preliminary steps, we are ready to apply the method of §3 to construct a progressing wave solution in one and three dimensions.

(a) One-Dimensional Problem

In one space-dimension (3.5) and (3.6) reduce to

$$(4.15) \quad u = (v,p)$$

$$(4.16) \quad A^\epsilon = A(\mu^\epsilon) = (-1) \begin{bmatrix} 0 & \mu_1^\epsilon \\ \mu_2^\epsilon & 0 \end{bmatrix}$$

and (4.13) becomes

$$(4.17) \quad A^\epsilon = \hat{A} + \epsilon R + O(\epsilon^2)$$

where, noting (4.14)

$$(4.18) \quad \begin{aligned} \hat{A} = A(\hat{\mu}) &= \begin{bmatrix} 0 & \hat{\mu}_1 \\ \hat{\mu}_2 & 0 \end{bmatrix}, \\ R = A(\mu) &= \begin{bmatrix} 0 & \hat{\mu}_1 \\ \mu_2 & 0 \end{bmatrix}. \end{aligned}$$

For the eigenvalue problem

$$(4.19) \quad M^\epsilon(q)r^\epsilon = (\gamma^\epsilon I - qA^\epsilon)r^\epsilon = 0,$$

the eigenvalues and the associated normalized right and left eigenvectors are

$$(4.20) \quad \lambda^\epsilon = \lambda_{1,2}^\epsilon = \pm c^\epsilon q,$$

$$(4.21) \quad r^\epsilon = r_{1,2}^\epsilon = \pm \begin{bmatrix} -(\rho^\epsilon c^\epsilon) \hat{q} \\ 1 \end{bmatrix}, \quad \ell_\epsilon = \ell_{1,2}^\epsilon = \pm \frac{1}{2}[-\rho^\epsilon c^\epsilon \hat{q} \quad 1].$$

From (4.20), $\lambda_q^\epsilon = \pm c^\epsilon$, so that we have the decoupled characteristic equation:

$$(4.22) \quad \begin{cases} \frac{dy}{ds} = \pm c^\epsilon(s, y, \omega), & 0 \leq s < t, \\ y(t) = x, \end{cases}$$

where, as is well known, the different signs correspond to two waves travelling in the opposite directions.

For simplicity, let us first consider only one mode, say $\lambda^\epsilon = \lambda_1^\epsilon = c^\epsilon q$. To apply the results in (case a) of §3, we note that, by (4.2), \hat{Y} and Y in (3.31) are given simply by

$$(4.23) \quad \begin{aligned} \hat{Y} &= \hat{c}, \\ Y &= c(t, x, \omega). \end{aligned}$$

Thus the asymptotic ray (3.32) becomes

$$(4.24) \quad y = \xi_s(t, x) = x - \hat{c}(t-s) + \zeta_\sigma(\tau),$$

where ζ_σ satisfies the Itô equation

$$(4.25) \quad \begin{cases} d\zeta_\sigma = \kappa d\sigma - \nu dw(\tau-\sigma), & \text{with } \beta = \nu^2, \\ \zeta_\tau = 0. \end{cases}$$

The parameters κ and β are defined as in (3.34) and (3.35)

$$(4.26) \quad \begin{aligned} \kappa &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{s_1} \gamma'_{cc} [T-s_1, T-s_2; \hat{c}(s_1-s_2)] ds_1 ds_2 \\ \beta &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \gamma_{cc} [T-s_1, T-s_2; \hat{c}(s_1-s_2)] ds_1 ds_2 \end{aligned}$$

in which $\gamma'_{cc}(t,s;x) = \partial_x \gamma_{cc}(t,s;x)$.

Let the initial condition be given by

$$(4.27) \quad u(0,x,\omega) \sim a(x)g[\theta(x)]r(x).$$

Here $r(x)$ is the initial right eigenvector of $(\theta_x \hat{A})$. As seen from (4.21), for $\epsilon=0$, the resulting eigenvectors

$$(4.28) \quad r = \begin{bmatrix} -(\hat{\rho}\hat{c})^{-1} \\ 1 \end{bmatrix}, \quad \ell = \frac{1}{2} [-\hat{\rho}\hat{c} \quad 1]$$

are constant. Consequently, the function b defined by (3.51) vanishes, or

$$(4.29) \quad b(t,x) \equiv 0.$$

Hence the asymptotic solution (3.52) reduces to

$$(4.30) \quad u(t,x,\omega) \sim a[\xi_0(t,x)]g\{\theta[\xi_0(t,x)]\} \begin{bmatrix} -(\hat{\rho}\hat{c})^{-1} \\ 1 \end{bmatrix}$$

from which the second component gives the acoustic pressure

$$(4.31) \quad p(t,x,\omega) \sim a[\xi(t,x)]g\{\theta[\xi(t,x)]\}$$

where $\xi(t,x) = \xi_0(t,x)$ is the diffusion process given in terms of the Wiener process $\{w(t), t \geq 0\}$:

$$(4.32) \quad \xi(t,x) = x - (\hat{c} + \epsilon^2 \kappa)t + \epsilon v w(t).$$

Now it is clear that, if, instead of (4.27), the initial state consists of two modes:

$$(4.33) \quad u(0,x,\omega) \sim \sum_{j=1}^2 a_j(x)g_j[\theta_j(x)]r_j$$

the pressure field is simply

$$(4.34) \quad p(t,x,\omega) \sim \sum_{j=1}^2 a_j[\xi_j(t,x)]g_j\{\theta_j[\xi_j(t,x)]\},$$

where

$$(4.35) \quad \xi_j(t,x) = x + (-1)^j [(\hat{c} + \epsilon^2 \kappa)t + \epsilon v w(t)].$$

Therefore the statistics of p are certain Wiener functionals which can be evaluated in exactly the same manner as that of the model problem in §2. Also it is interesting to observe that the asymptotic pressure wave p given by (4.30) depends only on the fluctuation of the local sound speed c^ε , but not on the density fluctuation ρ , as assumed in (4.2). This also turns out to be true in higher dimensions as to be shown below.

(b) Three-Dimensional Problem

In three dimensions, we set

$$(4.36) \quad u = (v_1, v_2, v_3, p),$$

$$(4.37) \quad A_j^\varepsilon = A_j(\mu) = (-1) \begin{bmatrix} 0 & 0 & 0 & \mu_1^\varepsilon \delta_{1j} \\ 0 & 0 & 0 & \mu_1^\varepsilon \delta_{2j} \\ 0 & 0 & 0 & \mu_1^\varepsilon \delta_{3j} \\ \mu_2^\varepsilon \delta_{1j} & \mu_2^\varepsilon \delta_{2j} & \mu_2^\varepsilon \delta_{3j} & 0 \end{bmatrix}$$

Analogous to (4.20) and (4.21), the non-zero eigenvalues and the eigenvectors for $(\sum_{j=1}^3 q_j A_j)$ are found to be

$$(4.38) \quad \lambda^\varepsilon = \lambda_{1,2}^\varepsilon = \pm c^\varepsilon |q|,$$

$$(4.39) \quad r = r_{1,2}^\varepsilon = \pm \begin{bmatrix} -\hat{q}/\rho^\varepsilon c^\varepsilon \\ 1 \end{bmatrix}, \quad \ell^\varepsilon = \ell_{1,2}^\varepsilon = \frac{1}{2}[-\rho^\varepsilon c^\varepsilon \hat{q} \quad 1],$$

where

$$(4.40) \quad \hat{q} = q/|q| = \frac{1}{|q|} (q_1, q_2, q_3).$$

From (4.38), we have

$$(4.41) \quad \partial_q \lambda^\varepsilon = \partial_q \lambda_{1,2}^\varepsilon = \pm c^\varepsilon \hat{q},$$

$$(4.42) \quad \partial_x \lambda^\varepsilon = \partial_x \lambda_{1,2}^\varepsilon = \pm |q| \partial_x c^\varepsilon.$$

Thus the bi-characteristic equations (3.28) takes the form:

$$(4.43) \quad \begin{cases} \frac{dy}{ds} = \mp [\hat{c}\hat{r} + \epsilon c(s,y,\omega)\hat{r}], \\ \frac{dr}{ds} = \pm \epsilon \partial_y c(s,y,\omega) |r|, \quad 0 \leq s < t, \\ y(t) = x, \\ r(t) = q, \end{cases}$$

which is of the case (b) in §3. To apply the results obtained there, let

$$(4.44) \quad r = q + q'.$$

Then by linearizing the term $\hat{c}\hat{r}$ in (4.43), we get the system (3.37), where

$$(4.45) \quad \begin{cases} \hat{Y}(q) = \mp \hat{c}\hat{q}, \\ \hat{Y}_1 q' = \mp \frac{1}{|q|} \{q' + (q' \cdot \hat{r})\hat{r}\}, \\ Y(s,y,q+q',\omega) = \mp c(s,y,\omega) \frac{(q+q')}{|q+q'|}, \\ Q(s,y,q+q',\omega) = \pm \partial_y c(s,y,\omega) |q+q'|. \end{cases}$$

Introduce

$$(4.46) \quad y^* = (y, q') = (y_1^*, \dots, y_6^*),$$

and define \hat{Y}^* , A^* and Y^* as in (3.38), and Z_t^* as in (3.41). That is

$$(4.47) \quad Y_j^* = \begin{cases} Y_j = \mp c(s,y,\omega) |q+q'|^{-1} (q+q')_j, & j=1,2,3, \\ Y_j^! = \pm \partial_{j-3} c(s,y,\omega) |q+q'|, & j=4,5,6, \end{cases}$$

$$(4.48) \quad (Z_t^*)_j = \begin{cases} Y_j(t-s, \cdot) - (t-s) Y_{j+3}^!(t-s, \cdot), & j=1,2,3, \\ Y_j^!(t-s, \cdot), & j=4,5,6. \end{cases}$$

where

$$(4.49) \quad Y_j \text{ (or } Y_j^!) (t-s, \cdot) = Y_j \text{ (or } Y_j^!) (t-s, \hat{\xi}_0(s,x) + y + (t-s)q', q', \omega),$$

$$(4.50) \quad \hat{\xi}_s(t,x) = x \pm \hat{c}\hat{q}(t-s).$$

To compute the drift vector and the diffusion matrix defined by (3.44), we make use of (4.45)-(4.50). In order to express the results concisely, let us introduce the function b_{ij} as follows:

(i) For $1 \leq i, j \leq 3$,

$$(4.51) \quad \begin{aligned} b_{ij}(s_1, s_2; q+q') &= \gamma_{cc}[s_1, s_2; \hat{c}(q+q')(s_2-s_1)](q+q')_i (q+q')_j \\ &+ (s_1-s_2) \partial_i \gamma_{cc}[\cdot] (q+q')_j \\ &- s_1 \cdot s_2 \partial_i \partial_j \gamma_{cc}[\cdot] |q+q'|^2 \end{aligned}$$

$$\begin{aligned} \text{where we set } \partial_i \gamma_{cc}[\cdot] &= \frac{\partial}{\partial y_i} \gamma_{cc}[s_1, s_2; y] \text{ at} \\ y &= \hat{c}(q+q')(s_2-s_1). \end{aligned}$$

(ii) For $1 \leq i \leq 3, 4 \leq j \leq 6$,

$$(4.52) \quad b_{ij}(s_1, s_2; q+q') = \partial_{j-3} \gamma_{cc}[\cdot] (q+q')_i + s_1 \partial_i \partial_j \gamma_{cc}[\cdot] |q+q'|^2.$$

(iii) For $4 \leq i, j \leq 6$,

$$(4.53) \quad b_{ij}(s_1, s_2; q+q') = -\partial_{i-3} \partial_{j-3} \gamma_{cc}[\cdot] |q+q'|^2.$$

In terms of b_{ij} , the equations (3.44) yield:

$$(4.54) \quad \begin{aligned} \beta_{ij}(q+q') &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} b_{ij}(s_1, s_2; q+q') ds_1 ds_2 \\ \kappa_i(q+q') &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{s_1} \left\{ \sum_{j=1}^3 \partial_j b_{ij}(s_1, s_2; q+q') \right. \\ &\quad \left. + \sum_{j=4}^6 \frac{\partial}{\partial q'_{j-3}} b_{ij}(s_1, s_2; q+q') \right\} ds_1 ds_2, \quad i, j=1, 2, \dots, 6. \end{aligned}$$

Assume all of the above limits exist. Then for large s_1 and s_2 , in view of (4.51) - (4.53), the components b_{ij} in (4.51) with $1 \leq i, j \leq 3$ are dominant. If the corresponding limits are finite, the rest must be zeros. That is,

$$(4.55) \quad \kappa_i = 0, \quad \beta_{ij} = 0 \quad \text{for } i \geq 4 \quad \text{or } j \geq 4.$$

This has an interesting consequence. Namely, in the asymptotic approximation, the process $\{y^*(s), 0 \leq s \leq t\}$ becomes degenerate and the time-averaging in (4.54) has the effect of projection

$$(4.56) \quad y^*(s) \mapsto (y(s), 0)$$

from the phase-space into the physical space, where

$$(4.57) \quad y(s) \sim \xi_s(t, s) = \hat{\xi}_s(t, x) + \zeta_\sigma(\tau).$$

Here ζ_σ is a diffusion in \mathbb{R}^3 defined by the Itô equation

$$(4.58) \quad \begin{cases} d\zeta_{\sigma, i} = \kappa_i d\sigma + \sum_{j=1}^3 v_{ij} dw_j(\tau - \sigma), \\ \zeta_{\tau, i} = 0, \quad i=1, 2, 3 \end{cases}$$

in which $\kappa_i = \kappa_i(q)$ and $\sum_{k=1}^3 v_{ik} v_{kj} = \beta_{kj}(q)$ with κ_i, β_{ij} defined by (4.54)

for $i, j=1, 2, 3$.

Suppose the initial state has only a single mode corresponding to the first eigenvalue $\lambda_1^E = c^E |q|$,

$$(4.59) \quad u(0, x, \omega) \sim a(x) g[\theta(x)] r(x),$$

where, noting (4.21)

$$(4.60) \quad r(x) = r_1(x) = \begin{bmatrix} -(\hat{\rho}\hat{c})^{-1} \hat{q} \\ 1 \end{bmatrix},$$

is the associated unperturbed right eigenvector with the left counterpart given by

$$(4.61) \quad \ell(x) = \frac{1}{2} [-\hat{\rho}\hat{c}\hat{q} \quad 1], \quad \hat{q} = \frac{\nabla\theta(x)}{|\nabla\theta|}.$$

With the aid of (4.60) and (4.61), we can easily compute b defined by (3.51) to get

$$(4.62) \quad b(x) = \frac{1}{2} \hat{c} \sum_{j=1}^3 \partial_j \hat{\theta}_j(x), \quad \text{with } \hat{\theta}_j = \partial_j \theta / |\nabla\theta|,$$

which, in contrast, is zero in one-dimension. This term reflects the curvature effect of the initial wave front. In view of (4.57), (4.58) and (4.62), the asymptotic progressing wave satisfying the condition (4.59) has exactly the same form as that of (3.52)

$$(4.63) \quad u(t,x,\omega) \sim a[\xi_0(t,x)]g\{\theta[\xi_0(t,x)]\} \begin{bmatrix} -(\hat{\rho}\hat{c})^{-1} \hat{\nabla}\theta(x) \\ 1 \end{bmatrix} \\ \times \exp \left\{ \int_0^t b[\xi_s(t,x)] ds \right\},$$

from which we get the acoustic pressure field:

$$(4.64) \quad p(t,x) \sim a[\xi_0(t,x)]g\{\theta[\xi_0(t,x)]\} \exp \left\{ \int_0^t b[\xi_s(t,x)] ds \right\}.$$

It is instructive to work out two concrete examples for which the phase factor g has a jump discontinuity:

$$(4.65) \quad g(t) = H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

where H is the Heaveside Function. Let us consider the following special cases of the phase function $\theta(x)$ corresponding to the plane and the spherical waves, respectively.

(case i) The Plane Wave

In this case, we assume

$$(4.66) \quad \theta(x) = \alpha \cdot x, \quad |\alpha| = 1.$$

The above implies that $q = \nabla\theta = \alpha$ and $b = 0$.

The pressure wave (4.64) reduces to

$$(4.67) \quad p(t,x) \sim a[\xi_0(t,x)]H[\alpha \cdot \xi_0(t,x)],$$

where

$$(4.68) \quad \xi_0(t,x) = x - \hat{c}_\epsilon \alpha t + \epsilon N w(t), \\ \hat{c}_\epsilon = \hat{c} + \epsilon^2 \kappa(\alpha), \quad N = [v_{ij}]^{3 \times 3} \text{ with } NN^* = \beta(\alpha),$$

and $w(t)$ is the standard Wiener process in R^3 .

The drift vector κ and the diffusion matrix β are determined by (4.54).

The statistics of p can be computed easily. For instance, let

$$(4.69) \quad I(t,x) = |p(t,x)|^2$$

be the intensity of p . Then, from (4.67), the mean and the covariance of I can be shown to be:

$$(4.70) \quad \begin{aligned} \hat{I}(t,x) &= E\{I(t,x)\} \\ &= \frac{1}{(2\pi t)^{3/2}} \int_{B(t,x)} |a(x - \hat{c}_\epsilon \alpha t + Nz)|^2 \exp\left\{-\frac{|z|^2}{2t}\right\} dz, \end{aligned}$$

and, for $0 \leq t \leq s$,

$$(4.71) \quad \text{Cov.}\{I(t,x), I(s,y)\} = M_4(t,s;x,y) - \hat{I}(t,x)\hat{I}(s,y),$$

where

$$(4.72) \quad \begin{aligned} M_4(t,s;x,y) &= E\{I(t,x)I(s,y)\} \\ &= \frac{1}{(2\pi)^3 [\tau(\sigma-\tau)]^{3/2}} \int_{B(t,x)} \int_{B(s,y)} |a(x - \hat{c}_\epsilon \alpha t + Nz)|^2 \\ &\quad \times |a(y - \hat{c}_\epsilon \alpha s + Nz')|^2 \exp\left\{-\frac{z^2}{2\tau} - \frac{|z'-z|^2}{2(\sigma-\tau)}\right\} dz dz'. \end{aligned}$$

Here $B(t,x)$ denotes the half-space:

$$(4.73) \quad B(t,x) = \{z \in \mathbb{R}^3 : \alpha \cdot (x - \hat{c}_\epsilon \alpha t + Nz) \geq 0\}.$$

In fact the multi-point statistical averages, such as

$$\Phi(t_1, t_2, \dots; x^{(1)}, x^{(2)}, \dots) = E\{F[p(t_1, x^{(1)}), p(t_2, x^{(2)}), \dots]\},$$

may be computed just as easily, since we know all the finite-dimensional joint probability distributions of the Wiener process.

(case ii) The Spherical Wave

Suppose

$$(4.74) \quad \theta(x) = r_0 - |x|.$$

Then

$$(4.75) \quad \nabla\theta = \frac{\mathbf{x}}{|\mathbf{x}|^3}, \quad \hat{\nabla}\theta = \frac{\mathbf{x}}{|\mathbf{x}|} = \hat{\mathbf{x}},$$

and

$$\nabla \cdot (\hat{\nabla}\theta) = \frac{2}{|\mathbf{x}|}.$$

Thus, by (4.62)

$$(4.76) \quad b(\mathbf{x}) = \frac{\hat{c}}{|\mathbf{x}|}.$$

The pressure field (4.64) now becomes

$$(4.77) \quad p(\mathbf{t}, \mathbf{x}) \sim a[\xi_0(\mathbf{t}, \mathbf{x})] H\{r_0 - |\xi_0(\mathbf{t}, \mathbf{x})|\} \\ \times \exp\{\hat{c} \int_0^{\mathbf{t}} |\xi_s(\mathbf{t}, \mathbf{x})|^{-1} ds\}$$

where

$$(4.78) \quad \xi_s(\mathbf{t}, \mathbf{x}) = \mathbf{x} - \hat{c}_\varepsilon \hat{\mathbf{x}}(\mathbf{t}-s) + N[w(\mathbf{t}) - w(s)],$$

$$(4.79) \quad \hat{c}_\varepsilon = \hat{c} + \varepsilon^2 \kappa(\mathbf{q}), \quad NN^* = \beta(\mathbf{q})$$

with

$$\mathbf{q} = \nabla\theta = \frac{\mathbf{x}}{|\mathbf{x}|^3}.$$

Therefore, unlike the previous case, the drift κ and the diffusion matrix β are now functions of \mathbf{x} . This is due to the variation of the normal vector field to the wave-front: $\theta(\mathbf{x}) = \text{const}$. Also, due to the wave-front curvature effect ($b \neq 0$), the average intensity I can no longer be calculated explicitly as done in (4.70). Instead, noting (4.77), we set

$$(4.80) \quad \hat{I}(\mathbf{t}, \mathbf{x}) = E\{a^2[\xi_0(\mathbf{t}, \mathbf{x})] H\{r_0 - |\xi_0(\mathbf{t}, \mathbf{x})|\} \\ \times \exp[2\hat{c} \int_0^{\mathbf{t}} |\xi_s(\mathbf{t}, \mathbf{x})|^{-1} ds]\}.$$

which, in view of (4.78), is an average Wiener functional or a Wiener integral. To evaluate (4.80) approximately, one may introduce an asymptotic expansion in ε , or adopt the method of differential equation, [13], [5].

In the former approach, the theory of large deviation [14] could be useful. Alternatively, we regard the average in (4.80) as taken with respect to the diffusion process $\xi_s(t,x)$ with the drift $\hat{c}_\varepsilon(x)$ and the diffusion matrix $\beta(x)$. By applying the Feynman-Kac formula to (4.80) along the effective characteristic, one can show that \hat{I} satisfies the following equation:

$$\partial_t \hat{I} + \sum_{j=1}^3 \partial_j [\hat{c}_\varepsilon(x) x_j \hat{I}] = \frac{\varepsilon^2}{2} \sum_{i,j=1}^3 \partial_i [\beta_{ij}(x) \partial_j \hat{I}] + \frac{2\hat{c}}{|x|} \hat{I} ,$$

$$\hat{I}(0,x) = |a(x)|^2 H(r_0 - |x|) .$$

to which one can apply some method of asymptotic solution for parabolic equations to yield an approximate evaluation of \hat{I} . It is also possible to derive differential equations for higher moments at one time, such as M_4 defined in (4.71) with $t=s$. However this will not be done here. A similar procedure was discussed in our papers [5,8].

Concluding Remarks

In closing we wish to make the following remarks:

- (i) The method of progressing waves introduced in this paper has a wide range of applications. Even though we have only discussed its application to acoustic problems, it is clear that the method applies to other problems in electromagnetic waves and other waves in continuous media.
- (ii) One of the shortcomings of the method is the linearization of the mean vector-field $\hat{Y}(q)$ in the bi-characteristic equations (3.28) in order to apply the existing diffusion limit theorems. But this excludes the strong interaction between the mean and fluctuating fields. The remedy to this problem requires a new type of limit theorem that generalizes the results of Kesten and Papanicolaou [15].

- (iii) As shown in the three-dimensional acoustic problem (b) in §4, the linearization of \hat{Y} leads an asymptotic decoupling of the bi-characteristic equations. This simplifies the subsequent statistical computation considerably. In fact this asymptotic reduction holds true in the general case as presented in §3.
- (iv) In view of the special cases (i) and (ii) in §4, the statistical problem for the plane wave is much simpler than the spherical one, or, for that matter, any non-plane waves. This suggests that the method of plane wave [16] may be used for simplifying the statistical computation.
- (v) In developing the progressing wave solution, we are mainly interested in propagation of weak singularity. For continuous waves, e.g., $g(\theta) = \exp\{k\theta\}$, it may be interpreted as an asymptotic expansion in the reciprocal powers of the large parameter k [17]. For time-independent media, the procedure yields the well-known geometric optics method. The asymptotic analysis of the stochastic ray equations near a caustic was done by White [18].

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Appendix

Consider the random characteristic equation in \mathbb{R}^n :

$$(1) \quad \begin{cases} \frac{dy}{ds} = Y^\varepsilon(s, y, \omega), & 0 \leq s < t, \\ y(t) = x \in \mathbb{R}^n. \end{cases}$$

where $Y^\varepsilon = (Y_1^\varepsilon, \dots, Y_n^\varepsilon)$ is a given family of random fields for $\varepsilon \in [0, \varepsilon_0]$, defined over $\mathbb{R} \times \mathbb{R}^n \times \Omega$. Let \mathcal{F} be σ -field of measurable sets in the sample space Ω on which a probability measure P is prescribed.

Suppose that Y^ε satisfies the conditions

$$(A.1) \quad Y^\varepsilon(t, x, \omega) = \hat{Y}(t, x) + \varepsilon Y(t, x, \omega),$$

where

$$\hat{Y}(t, y) = A(t)y + c(t) .$$

and $A(t)$ is a $n \times n$ matrix, and $c(t) \in \mathbb{R}^n$ are bounded and continuous. Let $y = \hat{\xi}_s(t, x)$ be a solution of

$$(2) \quad \begin{cases} \frac{dy}{ds} = \hat{Y}(s, y) \\ y(t) = x. \end{cases}$$

Set

$$(3) \quad y = \hat{\xi}_s(t, x) + K_s(t)z,$$

where $K_s(t)$ is a fundamental matrix for the system (2) with $c(t) \equiv 0$ and $K_t(t) = I$. Then, in view of (1) and (2), it is easy to verify that $z(s)$ satisfies

$$(4) \quad \begin{cases} \frac{dz}{ds} = \varepsilon K_s^{-1} Y[s, \hat{\xi}_s + K_s z, \omega] , \\ z(t) = 0. \end{cases}$$

Let us set

$$(5) \quad z_s = z(t-s),$$

$$(6) \quad Z_t(s, z, \omega) = K_{t-s}^{-1}(0) Y[t-s, \hat{\xi}_{t-s}(t, x) + K_{t-s}(0)z, \omega].$$

Then (4) yields an initial-value problem:

$$(7) \quad \begin{cases} \frac{dz_s}{ds} = \varepsilon Z_t(s, z_s, \omega), & 0 < s \leq t, \\ z_0 = 0. \end{cases}$$

In this form, we can apply the limit theorem of Khasiminskii [3] or, more generally, that of Papanicolaou and Kohler [19]. To this end we assume that:

(A2) There exists a family F_s^t of σ -field for $\Omega \ni$

$$F_{s_1}^{t_1} \subset F_{s_2}^{t_2} \subset F, \quad \forall [s_1, t_1] \subset [s_2, t_2] \subset \mathbb{R}$$

(A3) F_s^t is strongly mixing in the sense that $\exists \rho(t) \ni$

$$\sup_{s \in \mathbb{R}} \sup_{\substack{A \in F_{s+t}^{\infty} \\ B \in F_{-\infty}^s}} |p(A|B) - p(A)| = \rho(t) \downarrow 0 \text{ as } t \uparrow \infty,$$

and

$$\int_0^\infty \rho^{1/2}(t) dt < \infty.$$

(A4) The random field Y in (A.1) is jointly measurable in its arguments and, for fixed (t, x) , $Y(t, x, \cdot)$ is F_t^t -measurable.

(A5) There is an absolute constant $C > 0 \ni |Y(t, x, \omega)| \leq C(1 + |x|)$,

and the partial derivatives up to order four are bounded, or

$$\left| \frac{\partial^\alpha Y(t, x, \omega)}{\partial x^\alpha} \right| \leq C, \text{ for } |\alpha| = 1, \dots, 4,$$

where $\alpha = (\alpha_1, \dots, \alpha_j)$ denotes a multi-index.

(A6) $EY(t,x) = 0,$

and the following limits exist

$$\beta_{ij}(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} EZ_T^i(s_1, z) Z_T^j(s_2, z) ds_1 ds_2$$

$$\kappa_i(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{s_1} E \sum_{j=1}^n \frac{\partial Z_T^i(s_1, z)}{\partial z_j} Z_T^j(s_2, z) ds_1 ds_2, \quad i, j=1, \dots, n$$

which are independent of t_0 , and (t,x) , (see (6)). Here Z_t^i denotes the i^{th} component of Z_t .

Under the above assumptions, we can easily check that the assumptions of the theorem of Papanicolaou and Kohler [19] are satisfied. Hence we have

Theorem A.1. Let the conditions (A.1) to (A6) be fulfilled, and let $\sigma = \epsilon^2 s$, $\tau = \epsilon^2 t$. Then, for $0 \leq \sigma \leq \tau < T$, the solution process $z_s = z_s^\epsilon(t,x)$ converges weakly as $\epsilon \downarrow 0$ to a diffusion process $\zeta_\sigma(\tau)$ with $\zeta_\tau(\tau) = 0$, for which the drift vector $\kappa(z)$ and the diffusion matrix $\beta(z)$ are given in (A6).

Remarks: (i) Let us set $\beta_{ij} = \sum_{k=1}^n v_{ik} v_{kj}$ or $\beta = NN^*$. Then the limiting process $\zeta_\sigma(\tau)$ will satisfy the Itô equation

$$(8) \quad \begin{cases} d\zeta_\sigma = \kappa(\zeta_\sigma) d\sigma + N(\zeta_\sigma) dw(\tau - \sigma), \\ \zeta_\tau = 0 \end{cases}$$

where $N = (v_{ij})$ and $w(t)$ is a standard Wiener process in \mathbb{R}^n .

(ii) As a corollary, in view of (3), (5), we have

$$y(s) \sim \hat{\xi}_s(t,x) + K_s \zeta_\sigma(\tau).$$

(iii) If $A(t) \equiv 0$, $c(t)$ is independent of t in (A.1),

$$\text{then } \hat{\xi}_s = x + c(t-s), \quad K_s = I \text{ (identity).}$$

(iv) If $\hat{Y}(t,y)$ in (A.1) is nonlinear, a diffusion limit may not exist.

For a special case, one is referred to the paper [19] by Kesten and

Papanicolaou concerning a stochastic acceleration problem.

To apply the above theorem to a stochastic wave problem, we must ensure the weak convergence of a certain function or functional of the solution process $y = \xi_s^\varepsilon(t, x)$ of the system (1), such as the amplitude and the phase:

$$(9) \quad \begin{aligned} a^\varepsilon(t, x) &= a[\xi_0^\varepsilon(t, x)], \\ \theta^\varepsilon(t, x) &= \int_0^t b[s, \xi_s^\varepsilon(t, x)] ds. \end{aligned}$$

Suppose the following conditions hold:

(A.7) There exist positive numbers $\alpha, \beta, M \ni$

$$E|\xi_0^\varepsilon(t_1, x) - \xi_0^\varepsilon(t_2, x)|^\alpha \leq M|t_1 - t_2|^{1+\beta}, \quad \forall t_1, t_2 \geq 0,$$

(A.8) $\lim_{h \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \sup_{|s_1 - s_2| \leq h} P\{|\xi_{s_1}^\varepsilon(t, x) - \xi_{s_2}^\varepsilon(t, x)| > \delta\} = 0$

$$\forall(t, x), \delta > 0.$$

(A.9) $a : \mathbb{R}^n \rightarrow \mathbb{R}$ and $b : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ are bounded continuous and

$$\exists g : \mathbb{R}^n \rightarrow \mathbb{R}^+ \ni$$

$$\lim_{N \rightarrow \infty} \sup_{\substack{t \geq 0 \\ |x| > N}} \frac{|b(t, x)|}{g(x)} = 0.$$

Then we have the following asymptotic result:

Theorem A.2. Let the conditions (a.7)-(A.9) be satisfied. Then, in the diffusion limit as in Theorem A.1, we have, referring to (9),

$$(a) \quad a[\xi_0^\varepsilon(t, x)] \sim a[\xi_0^\ell(t, x)],$$

$$(b) \quad \theta^\varepsilon(t, x) \sim \theta(t, x) = \int_0^t b[s, \xi_s(t, x)] ds,$$

where $\xi_s^\varepsilon(t, x) \sim \xi_s(t, x) = \hat{\xi}_s(t, x) + M_s \zeta_\sigma(t)$.

Remarks: (i) Here the asymptotic equality " \sim ", instead of the weak convergence " \Rightarrow ", means the weak convergence with respect to the centered

process $\xi_s^\varepsilon(t,x) - \hat{\xi}_s(t,x)$, since the latter part may not have a limit as $s \uparrow t$.

- (ii) The continuity assumption on the function a may be relaxed to the so-called a.s. B-continuity (for definition, see p.282, [20]). This will allow a to have simple jump discontinuities.

The proof of the theorem follows from two theorems in Gikhman and Skorokhod [21], (Theorem 2 on p. 450, and Theorem 2 on p. 486 with the obvious error $h \rightarrow \infty$ in (3) corrected to $h \downarrow 0$). The generalization from their version in one-dimension to one in several dimensions is immediate.

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