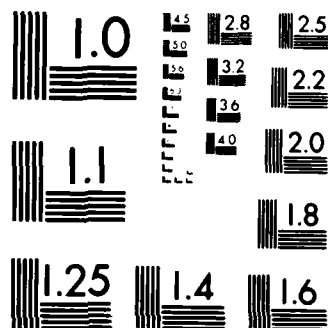END

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HE AMBIGUITY FUNCTION AND
THE WIGNER DISTRIBUTION

by

David C. Kirk

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Thesis submitted to the Faculty of the Graduate School
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This paper compares the radar ambiguity function and the Wigner distribution. Both functions are time-frequency representations of a given signal. The properties of each function are presented and their relationship is demonstrated. Emphasis is placed on how each function relates to the Heisenberg uncertainty principle.

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THE AMBIGUITY FUNCTION AND
THE WIGNER DISTRIBUTION

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A thesis submitted to the University of Maryland
College Park, Maryland in partial fulfullment
of the requirements for the degree of Master of Arts

TABLE OF CONTENTS

I. INTRODUCTION AND NOTATION

It has been known for many years that displaying the frequency components of a signal as a function of time is a valuable signal analysis technique. This is particularly true for nonstationary signals, such as speech, whose frequency structure is different from moment to moment. In the case of speech, the spectrograph has been widely used to plot a time-frequency representation of a spoken signal.

In this paper two time-frequency functions will be studied; the ambiguity function and the Wigner distribution. These functions are closely related and yet are dissimilar enough so that each provides a valuable perspective to signal analysis problems. Of particular interest is how these functions characterize the Heisenberg uncertainty principle.

§1. Notation

The following notation, definitions and theorems will be used frequently.

a. Given $f \in L^2(\mathbb{F})$, the Fourier transform F of f, sometimes noted as $F = \hat{f}$, is defined as

$$(1) \qquad F(v) = \int f(t)e^{-ivt} \, dt.$$

Here, as throughout this paper, unless othewise noted the limits of integration are assumed to be from $-\infty$ to $+\infty$.

The inverse Fourier transform is defined as

$$(2) \qquad f(t) \;=\; \frac{1}{2\pi} \int F(v)\, e^{ivt}\, dv.$$

b. The L^2-norm of f is

$$(3) \qquad \|f\| \;=\; \left(\int |f(t)|^2\, dt \right)^{1/2}.$$

The L^2-norm of $F = \hat{f}$ is

$$(4) \qquad \|F\| \;=\; \left(\frac{1}{2\pi} \int |F(v)|^2\, dv \right)^{1/2}.$$

c. Convolution Theorem: If $\hat{f} = F$ and $\hat{g} = G$ are Fourier transform pairs and $h = fg$ then the Fourier transform of h is

$$(5) \qquad H(v) \;=\; \frac{1}{2\pi} \int F(r)G(v-r)\, dr \;=\; (F * G)(v).$$

d. Plancherel's Theorem: Given $F \in L^2(\mathbb{R})$ then there is a function $f \in L^2(\mathbb{R})$ such that $F = \hat{f}$ and $\|f\| = \|F\|$.

e. The Schwarz inequality for L^2-functions is

$$(6) \qquad \left| \int fg \right|^2 \;\leq\; \int |f|^2 \int |g|^2.$$

f. $\delta(t)$ is defined to be the Dirac measure having the property that

$$(7) \qquad \int f(t)\delta(t)\, dt \;=\; f(0).$$

II. AMBIGUITY FUNCTION

In this chapter the ambiguity function (AF) will be studied. Some of its more useful properties will be stated and proved. The AF will be shown to arise naturally from the output of a matched filter and its specific application to the radar problem will be discussed. In the course of calculating several examples, an interesting theorem concerning the linear transformation of the coordinate axes of the AF will be stated and proved. Finally, the AF will be related to signal duration and shown to characterize the Heisenberg uncertainty principle.

The AF was originally introduced in 1948 by Ville in the context of a general signal analysis tool. Application of the AF to the radar problem was given by Woodward in 1950 [21]. Subsequently, many of the properties and theorems associated with the AF have been presented by Siebert [16] and Wilcox [20].

§2. Definitions

The AF for a given signal f has several common definitions. The definition most similar to the Wigner distribution was chosen for this paper. Given $f \in L^2(\mathbb{R})$ and continuous, the AF χ of f is

$$(1) \qquad \chi(u,\tau) = \int f(t + \tfrac{\tau}{2})\overline{f(t - \tfrac{\tau}{2})}e^{-iut} \, dt.$$

Although the definition easily generalizes to a function of two continuous, L^2-functions, such a definition does not

have a simple physical interpretation and will not be
studied. If f is not continuous it will be treated as
a generalized function.

Other definitions are similar. For example, in the
following section, the AF will be shown to be the output
modulation of a matched filter. In that section the
ambiguity function will be defined as

$$(2) \qquad \theta(u,\tau) = \int f(t)\overline{f(t-\tau)}e^{-iut} \, dt.$$

X is related to θ by the formula,

$$(3) \qquad X(u,\tau) = e^{-i\frac{u\tau}{2}} \theta(u,\tau).$$

In radar waveform design, it is the magnitude of the AF
which provides the measure of resolution and ambiguity in
f. Thus, in this context definitions (2.1) and (2.2) are
equivalent. In Papoulis' text the AF is normalized [12].
This definition is not selected so that generalized functions
may be studied.

Consider now the expression for X in terms of the
Fourier transform of f. First, let the kernel function
γ be defined. Given f,

$$(4) \qquad \gamma(t,\tau) = f(t+\tfrac{\tau}{2})\overline{f(t-\tfrac{\tau}{2})}.$$

The AF of f is the Fourier transform of γ for fixed
τ. That is, $\forall \tau \in \mathbb{F}$

$$(5) \qquad \chi(u,\tau) = \int \gamma(t,\tau)e^{-iut} \, dt.$$

This expression is valid, if for each $\tau \in \mathbb{F}$, $\gamma(\cdot,\tau) \in L^1(\mathbb{F})$. To see that this is the case, fix $\tau \in \mathbb{F}$ and use Cauchy-Schwarz in the following;

$$(6) \quad \left(\int |\gamma(t,\tau)| \, dt\right)^2 = \left(\int |f(t+\tfrac{\tau}{2})| \, |\overline{f(t-\tfrac{\tau}{2})}| \, dt\right)^2$$

$$\leq \int |f|^2 \int |f|^2 = \left(\int |f|^2\right)^2.$$

Since $f \in L^2(\mathbb{F})$, then $\gamma(\cdot,\tau) \in L^1(\mathbb{F})$ for each $\tau \in \mathbb{F}$ and (2.5) is valid. Now define the two-dimensional Fourier transform Γ of the function γ;

$$(7) \qquad \Gamma(u,v) = \iint \gamma(t,\tau)e^{-i(ut+v\tau)} \, dt \, d\tau.$$

Calculate this integral,

$$\Gamma(u,v) = \iint f(t+\tfrac{\tau}{2})\overline{f(t-\tfrac{\tau}{2})}e^{-i(ut+v\tau)} \, dt \, d\tau$$

$$= \int \left(\int f(r)\overline{f(r-\tau)}e^{-iur} \, dr\right)e^{-i\tau(v-u/2)} \, d\tau.$$

If $\hat{f} = F$ then $\overline{f(r-\tau)} \leftrightarrow e^{-iu\tau}\,\overline{F(-u)}$. By the convolution theorem,

$$\Gamma(u,v) = \int \left(\tfrac{1}{2\pi} \int \overline{F(-s)} \, e^{-is\tau} \, F(u-s) \, ds\right)e^{-i\tau(v-u/2)} \, d\tau.$$

Since $e^{-iat} \leftrightarrow 2\pi\delta(v+a)$, we may formally write,

$$\Gamma(u,v) = \int \overline{F(-s)}F(u-s)\delta(s+v-u/2) \, ds.$$

Hence,

(8) $\Gamma(u,v) = F(v+u/2)\overline{F(v-u/2)}$.

The two-dimensional Fourier transform is an iterated integral, consequently (2.5) and (2.7) may be combined to give,

(9) $\gamma(t,\tau) \underset{t}{\leftrightarrow} X(u,\tau) \underset{\tau}{\leftrightarrow} \Gamma(u,v)$.

By Cauchy-Schwarz we see that for fixed $u \in \mathbb{R}$,

(10) $(\int |\Gamma(u,v)| dv)^2 \le (\int |F(v)|^2 dv)^2$.

Since $f \in L^2(\mathbb{R})$, by Plancherel's theorem, $F \in L^2(\mathbb{R})$ and we conclude that for every $u \in \mathbb{R}$ $\Gamma(u,\cdot) \in L^1(\mathbb{R})$. We may then write from (2.9),

(11) $X_f(u,\tau) = \frac{1}{2\pi} \int F(v + \frac{u}{2})\overline{F(v - \frac{u}{2})} e^{iv\tau} dv$.

If definition (2.1) is used to define the AF of the Fourier transform F, we have

(12) $X_F(u,\tau) = \int F(v + \frac{\tau}{2})\overline{F(v - \frac{\tau}{2})} e^{-iuv} dv$.

X_f is related to X_F by the formula,

(13) $X_f(u,\tau) = \frac{1}{2\pi} X_F(-\tau,u)$.

Thus the AF of the function f is a scaled and rotated version of the AF for its Fourier transform F.

§3.   AF in Radar

In this section the AF will be shown to be the
complex modulation function out of a matched filter
receiver. This will help interpret the time-frequency
plane on which the AF is defined and provide a foundation
for an interpretation of the function's properties.

A radar (an acronym for RAdio Detection And Ranging)
transmits electromagnetic energy which propagates in the
atmosphere. Depending on the reflective properties of the
objects that this energy contacts, some of the signal is
reflected (or reradiated) back to the receiving system.
Based upon the difference between the transmitted waveform
and the reflected waveform, the radar system extracts
information about the target. Of interest is the target's
location in range and its relative motion (radial velocity).

The radar will be modeled as a matched filter.
Although there are may ways to design a receiving system,
the matched filter is the most common because it has been
shown to be the optimum filter for a very large class of
problems. In particular, the matched filter has been
shown to maximize the signal-to-noise ratio in the presence
of additive, Gaussian white noise [10]. Also, it has been
shown to maximize the probability of detection using the
maximum likelihood criterion [4]. Finally, in a deter-
ministic sense, given a known signal input, the matched
filter has been shown to be the impulse response which
maximizes the output at a given time with respect to all

other linear, time-invariant systems [12].

Two assumptions are made in this development. First, the signals are "narrowband" signals. This means that given the radar signal is

(1) $s(t) = a(t)\cos(\omega t+\theta(t))$,

then $a(t)$ and $\theta(t)$ vary slowly relative to the high carrier frequency ω. (A more precise definition of narrowband may be found in reference 4). This permits the cumbersome expression (3.1) for the radar signal to be written

(2) $f(t) = c(t)e^{i\omega t}$,

where ω is fixed, the complex modulation is

(3) $c(t) = a(t)e^{i\theta(t)}$,

and

(4) $\text{Re } f(t) = s(t)$.

(3.2) is the complex signal representation of the signal $s(t)$ and is discussed further in Appendix A.

The narrowband assumption is also necessary to represent the doppler effect on the reflected signal as merely a shift in frequency. The doppler approximation is developed in Appendix B. Fortunately, the narrowband assumption is valid for almost all radar applications. An application where such an assumption is not valid is in sonar.

Secondly, for reasons of clarity, the transmitter and the target from which the electromagnetic energy is reflected are modeled as point sources. Modeling the transmitter as a point source is accurate since the antenna characteristics do not effect the time delay or doppler shift of the returned signal. Modeling the target as a point source ignores relatively small effects on the returned signal which do not serve to clarify the meaning of the ambiguity function.

To begin, define the transmitted waveform

$$(5) \qquad f_t(t) = c(t)e^{i\omega t}.$$

The subscript t denotes transmitted. By definition of a matched filter, the impulse response of a linear, time-invariant filter matched to the transmitted waveform f_t is

$$(6) \qquad h(t) = \overline{f_t(-t)}.$$

If $\hat{f}_t = F_t$, then the transfer function of the matched filter is

$$H(u) = \overline{F_t(u)}.$$

The radar sends out f_t. What returns to the radar (assuming a target is present) is a time delayed, frequency shifted version of f_t. The time delay τ is related to the range of the target R, by

$$(7) \qquad \tau = \frac{2R}{v}.$$

ν is the velocity of propagation of the signal and τ is the time it takes the signal to make the trip out to the target and back. The doppler shift u is related to the radial velocity v of the target by,

$$(8) \qquad u = \frac{\pi \omega V}{\nu}.$$

ω is the fixed carrier frequency in radians. The resulting reflected signal, when it enters the matched filter, is

$$(9) \qquad f_r(t) = c(t-\tau_r)e^{i(\omega-u_r)(t-\tau_r)}.$$

Subscript r denotes returned.

The radar designer wishes to maximize the probability of detection, hence, would like to match the receiver's filter to the return signal f_r. But the parameters τ_r and u_r are dependent on the target and therefore not known a priori. Consequently, the designer must anticipate a time delay τ_m and a frequency shift u_m. Then the signal to which the filter is matched is

$$(10) \qquad f_m(t) = c(t-\tau_m)e^{i(\omega-u_m)(t-\tau_m)}.$$

By (3.6), the impulse response of the filter matched to f_m is

$$(11) \qquad h_m(t) = \overline{c(-t-\tau_m)}e^{-i(\omega-u_m)(-t-\tau_m)}.$$

The output y of the filter h_m with the input f_r is

$$y(t) = (h_m * f_r)(t)$$

$$= \int f_r(x) h_m(t-x)\,dx$$

$$= \int c(x-\tau_r) c^{i(\omega-u_r)(x-\tau_r)} \overline{c(-t-\tau_m+x)} e^{-i(\omega-u_m)(-t-\tau_m+x)}\,dx.$$

Let $\eta = x - \tau_r$,

$$y(t) = \int c(\eta) e^{i(\omega-u_r)\eta} \overline{c(\eta+\tau_r-t-\tau_m)} e^{-i(\omega-u_m)(\eta+\tau_r-t-\tau_m)}\,d\eta.$$

Let $\tau = t + \tau_m - \tau_r$ and $u = u_r - u_m$, then

$$(12) \quad y(t) = e^{+i\tau(\omega-u_m)} \int c(\eta)\overline{c(\eta-\tau)} e^{-iu\eta}\,d\eta.$$

Compare the integral on the r.h.-side of (3.12) to (2.2). We see that

$$(13) \qquad y(t) = e^{+i\tau(\omega-u_m)} \Theta(u,\tau).$$

For a given target, u_r and τ_r are constants, ω, u_m and τ_m are fixed design parameters and therefore the AF, $\Theta(u,\tau)$, is a function of time $(\tau = t+\tau_m-\tau_r)$. Compare (3.13) to (3.2) and we see that $y(t)$ may then be interpreted as having two components; the carrier and the complex modulation function $\Theta(u,\tau)$. If $y(t)$ is viewed as an information‑bearing signal we might expect to extract that information from the modulation function $\Theta(u,\tau)$. How well that target information is recovered will depend on the character $\Theta(u,\tau)$.

Resolution and ambiguity are two measures of how well information can be extracted from the function Θ. Consider first resolution. Given design parameters u_m and τ_m, the AF at a specific time t is a function of the target parameters u_r and τ_r. Without loss of generality, we can study the $\Theta(u,\tau)$ at $t = 0$. Then the origin of the (u,τ)-plane represents the point where the radar designer has precisely anticipated the reflected signal's parameters. Points in $\mathbb{R}^2$ away from the origin represent mismatches. The shape of the AF near the origin will dictate how well one can distinguish the actual return from $(0,0)$. This is a measure of the resolution of f.

Next consider the ambiguity of f. It will be shown that all AFs have a maximum at the origin. Should the AF have peaks away from the origin, then for a given threshold a, the set $E = \{(u,\tau) : |\Theta(u,\tau)| \ge a\}$ may be disjoint components of the plane. Ambiguity then arises in determining in which component the actual returned parameters lie. Both concepts will be discussed in greater detail in Section 6.

§4. Properties

1. χ has a global maximum at the origin;

$$(1) \qquad |\chi(u,\tau)| \le \chi(0,0), \qquad \forall\, (u,\tau) \in \mathbb{R}^2.$$

Proof. Let $g_1(t) = f(t + \frac{\tau}{2})e^{-iut}$ and $g_2(t) = \overline{f(t - \frac{\tau}{2})}$. Then by (1.6)

$$|\chi(u,\tau)|^2 = \left|\int g_1(t)g_2(t)dt\right|^2 \le \int |g_1(t)|^2 dt \int |g_2(t)|^2 dt.$$

Substituting for g_1 and g_2 we obtain,

$$(2) \quad |\chi(u,\tau)|^2 \le \left(\int |f(t)|^2 dt\right)^2 = (\chi(0,0))^2. \qquad \#$$

The result may be stated as $|\chi(u,\tau)| \le \|f\|^2$, since

$$(3) \qquad \chi(0,0) = \int_\infty^\infty |f(t)|^2\, dt = \|f\|^2.$$

2. Symmetry.

$$(4) \quad a.) \qquad \chi(u,\tau) = \overline{\chi(-u,-\tau)}.$$

<u>Proof</u>. Note that $\overline{\gamma(t,-\tau)} = \overline{f(t-\frac{\tau}{2})f(t+\frac{\tau}{2})} = \gamma(t,\tau)$. Then

$$\overline{\chi(-u,-\tau)} = \int \overline{\gamma(t,-\tau)}e^{-iut}\, dt = \chi(u,\tau). \qquad \#$$

b.) In general χ is a complex-valued function. If f is real valued and either $f(t) = f(-t)$ or $f(-t) = -f(t)$ then χ is real-valued. That is,

$$(5) \qquad \chi(u,\tau) = \overline{\chi(u,\tau)}.$$

<u>Proof</u>. Only the case of odd symmetric f will be verified. The case of even symmetric f easily follows. Note that f real and odd implies

$$\overline{\gamma(-t,\tau)} = \overline{f(-t+\frac{\tau}{2})f(-t-\frac{\tau}{2})} = f(t+\frac{\tau}{2})\overline{f(t-\frac{\tau}{2})} = \gamma(t,\tau).$$

Then

$$\overline{\chi(u,\tau)} = \int \overline{\gamma(t,\tau)}e^{iut}\, dt = \int \overline{\gamma(-t,\tau)}e^{-iut}\, dt = \chi(u,\tau). \qquad \#$$

3. Translations of $f(t)$ and $F(\omega)$.

    a.) Let T_a be the translation operator, that is

$$T_a f(t) \;=\; f(t-a).$$

Then

(6) $$\chi_{T_a f}(u,\tau) \;=\; e^{iau}\,\chi_f(u,\tau).$$

Proof. The result follows from the fact that

$$\gamma_{T_a f}(t,\tau) = \gamma_f(t-a,\tau). \qquad\qquad\qquad \#$$

    b.) If $T_b F(v) = F(v-b)$ and $g(t) \leftrightarrow F(v-b)$ then

(7) $$\chi_g(u,\tau) \;=\; e^{ib\tau}\,\chi_f(u,\tau).$$

Proof. It is known that $e^{ibt} f(t) \leftrightarrow F(v-b)$. Hence

$$\gamma_g(t,\tau) \;=\; e^{ib(t+\frac{\tau}{2})} f(t+\tfrac{\tau}{2}) e^{-ib(t-\frac{\tau}{2})} \overline{f(t-\tfrac{\tau}{2})}$$

$$=\; e^{ib\tau}\,\gamma_f(t,\tau).$$

    The result follows by the definition (2.1).   #

4. Modulations of f and F by simple sinusoids.

    Let M_a be the modulation operator defined as

$$M_a f(t) \;=\; e^{iat} f(t),$$

and $$M_b F(v) \;=\; e^{ibv} F(v).$$

(8) Then $$\chi_{M_a f}(u,\tau) \;=\; e^{ia\tau}\,\chi_f(u,\tau).$$

Furthermore, if $g \leftrightarrow M_b F$ then

(9) $\qquad \chi_g(u,\tau) = e^{ibu} \chi_f(u,\tau).$

Proof. The proof of (4.8) is the same as that for (4.7), and the proof of (4.9) is the same as that for (4.6). #

Concise statements may also be made for the AFs of Fourier transforms. For example, property 3b may be written as

(10) $\qquad \chi_{T_a F}(u,\tau) = e^{-iau} \chi_F(u,\tau).$

Proof. Let $g \leftrightarrow T_a F$, then from (2.13),

$$\chi_{T_a F}(u,\tau) = 2\pi \chi_g(\tau,-u).$$

From (4.7) and again (2.13) we conclude,

$$\chi_{T_a F}(u,\tau) = 2\pi e^{-iau} \chi_f(\tau,-u)$$

$$= e^{-iau} \chi_F(u,\tau). \qquad\qquad \#$$

We conclude from equations (4.6) and (4.7) that the modulus of the AF is invariant to translations of f or F. This is a significant feature of AFs.

5. Multiplication of functions.

The previous results can be generalized to modulation of f by any function g.

Let $h = fg$, then

(11) $\chi_h(u,\tau) = \frac{1}{2\pi} \int \chi_f(r,\tau)\chi_g(u-r,\tau)dr.$

<u>Proof</u>. Notice that $\gamma_h(t,\tau) = \gamma_f(t,\tau)\gamma_g(t,\tau)$ then

$$\chi_h(u,\tau) = \int \gamma_f(t,\tau)\gamma_g(t,\tau)e^{-iut} dt.$$

Hold τ fixed and use the convolution theorem and the
result follows.   #

6. Linear filtering.

Let

$$h(t) = (f*g)(t),$$

where g is the impulse response of the linear, time-
invariant filter. Then

(12) $\chi_h(u,\tau) = \int \chi_f(u,r)\chi_g(u,\tau-r)dr.$

<u>Proof</u>. It is known that with $h(t)$ so defined,

$$H(\omega) = F(\omega)G(\omega),$$

where $\hat{f} = F$ and $\hat{g} = G$. Use (2.11) to write,

$$\chi_h(u,\tau) = \frac{1}{2\pi} \int H(v+\tfrac{u}{2}) \overline{H(v-\tfrac{u}{2})} e^{iv\tau} dv$$

$$= \frac{1}{2\pi} \int F(v+\tfrac{u}{2}) \overline{F(v-\tfrac{u}{2})}G(v+\tfrac{u}{2}) \overline{G(v-\tfrac{u}{2})} e^{iv\tau} dv.$$

The result follows from the convolution theorem and (2.8)
and (2.9) which state that for fixed $u \in \mathbb{R}$

$$\chi_f(u,\tau) \underset{\tau}{\leftrightarrow} F(v + \tfrac{u}{2}) \overline{F(v - \tfrac{u}{2})}$$

and

$$\chi_h(u,\tau) \underset{\tau}{\leftrightarrow} G(v + \tfrac{u}{2}) \overline{G(v - \tfrac{u}{2})}.$$

7. Dilation.

Let $g(t) = f(at)$, for some $a \in \mathbb{R}$,

(13) then $\chi_g(u,\tau) = |\tfrac{1}{a}| \chi_f(\tfrac{u}{a}, a\tau)$.

Proof. By definition,

$$\chi_g(u,\tau) = \int f(at + \tfrac{a\tau}{2}) \overline{f(at - \tfrac{a\tau}{2})} e^{-iut} dt.$$

For $a > 0$ we conclude that

$$\chi_g(u,\tau) = \int f(r + \tfrac{a\tau}{2}) \overline{f(r - \tfrac{a\tau}{2})} e^{-iu\frac{r}{a}} \tfrac{1}{a} dr$$

$$= \tfrac{1}{a} \chi_f(\tfrac{u}{a}, a\tau).$$

For $a < 0$ we conclude that,

$$\chi_g(u,\tau) = - \tfrac{1}{a} \chi_f(\tfrac{u}{a}, a\tau).$$

So $\forall a \in \mathbb{R}$,

$$\chi_g(u,\tau) = \tfrac{1}{|a|} \chi_f(\tfrac{u}{a}, a\tau).$$

8. Invertibility and Uniqueness.

Given an ambiguity function χ, the generating function f may be uniquely recovered to within a multiplicative constant $c \in \mathbb{C}$ such that $|c| = 1$.

Proof. Assume two functions f and g generate the same AF so that

$$\chi_f = \chi_g.$$

Then by the uniqueness of the Fourier transform we can conclude that $\gamma_f = \gamma_g$. That is

$$(14) \qquad f(t + \tfrac{\tau}{2}) \, \overline{f(t - \tfrac{\tau}{2})} = g(t + \tfrac{\tau}{2}) \, \overline{g(t - \tfrac{\tau}{2})}.$$

Let $t_1 = t + \tfrac{\tau}{2}$ and $t_2 = t - \tfrac{\tau}{2}$. Then (4.14) is,

$$\frac{f(t_1)}{\overline{g(t_1)}} = \overline{g(t_2)}/\overline{f(t_2)}, \qquad \forall t_1, t_2 \in \mathbb{R}.$$

Hence, $\forall t \in \mathbb{R}$, $g = cf$ and

$$\frac{f}{cf} = \frac{\overline{cf}}{\overline{f}}.$$

$\frac{1}{c} = \overline{c}$ so $1 = |c|^2$. $\qquad\qquad\qquad\qquad$ #

This raises the question as to how one identifies an ambiguity function. That is, what are the necessary and sufficient conditions for a function $\chi(u,\tau)$ to be an AF. This is of particular interest in radar waveform design where one would like to establish the AF so that χ displays the desired resolution and ambiguity character-istics. After inverting χ, the designer would have f, the suitable radar signal. Much work has been devoted to this effort but the best that can be said about the

sufficient conditions for χ to be an ambiguity function is that when the inverse Fourier transform is performed on $\chi(u,\tau)$, with respect to the u variable, the result is the factored form of the generating function $\gamma(t,\tau)$. To be more precise:

9. $\chi(u,\tau)$ is an ambiguity function if and only if

$$(15) \quad f(t_1) \overline{f(t_2)} = \frac{1}{2\pi} \int \chi(u, t_2-t_1) e^{i \frac{u}{2}(t_1+t_2)} du.$$

Proof. The necessary and sufficient conditions are just a restatement of the definition. Let $t_1 = t - \frac{\tau}{2}$ and $t_2 = t + \frac{\tau}{2}$, and rely on the uniqueness of Fourier transforms.

10. The squared-magnitude of χ has the unusual property of being, after a coordinate transform, self-reciprocal in the two dimensional Fourier transform. That is,

$$(16) \quad \iint |\chi(u,\tau)|^2 e^{-i(uv+t\tau)} du\, d\tau = 2\pi |\chi(t,-v)|^2.$$

Proof. Use definition (2.11) for χ in terms of $F = \hat{f}$, and property (4.4) to obtain,

$$(17) \quad \overline{\chi(u,\tau)} = \chi(-u,-\tau) = \frac{1}{2\pi} \int F(v - \frac{u}{2}) \overline{F(v + \frac{u}{2})} e^{-iv\tau} dv.$$

Use this expression in the expansion of (4.16) which follows.

$$\iint |\chi(u,\tau)|^2 \, e^{-i(uv+\tau t)} du d\tau \;\; =$$

$$\iint \left(\int f(x + \tfrac{\tau}{2}) \overline{f(x - \tfrac{\tau}{2})} e^{-iux} dx \right) \left(\tfrac{1}{2\pi} \int F(y - \tfrac{u}{2}) \overline{F(y + \tfrac{u}{2})} e^{-iy\tau} \, dy \right)$$
$$\times \; e^{-i(uv+\tau t)} du d\tau.$$

Use the change of variables $r = x + \tfrac{\tau}{2}$ and $s = y - \tfrac{u}{2}$, then

(18)

$$\iint |\chi(u,\tau)|^2 \, e^{-i(uv+\tau\tau)} du d\tau \;\; =$$

$$\iint f(r)F(s) \left(\int \overline{f(r-\tau)} e^{-i\tau(s+t)} d\tau \right) \left(\tfrac{1}{2\pi} \int \overline{F(s+u)} e^{-iu(r+v)} du \right) dr ds.$$

The first inner integral is,

$$\int \overline{f(r-\tau)} \, e^{-i\tau(s+t)} d\tau \;\; = \;\; e^{-ir(s+t)} \, \overline{F(s+t)}.$$

The second inner integral is

$$\tfrac{1}{2\pi} \int \overline{F(s+u)} e^{-iu(r+v)} du \;\; = \;\; e^{is(r+v)} \overline{f(r+v)}.$$

Substituting these back into (4.18) and rearranging terms, we see that

$$\iint |\chi(u,\tau)|^2 \, e^{-i(uv+\tau\tau)} du d\tau \;\; = \;\; \chi_f(t,-v) \, \overline{\chi_F(v,t)}.$$

The result follows from the formula (2.13). $\qquad\qquad$ #

11. The radar uncertainty principle.

Integrating $|\chi|^2$ over all of $\mathbb{R}^2$ we have the interesting result,

(19) $\qquad \tfrac{1}{2\pi} \iint |\chi(u,\tau)|^2 \, du d\tau \;\; = \;\; |\chi(0,0)|^2 = \|f\|^4.$

Proof. This is a special case of (4.16) with $v = t = 0$. #

This property can be interpreted as the "conservation of ambiguity property." It says that the best a radar designer can do, given a specific energy constraint, is to shift the ambiguities inherent in f to the unused parts of the (u,τ) -plane. Another interpretation is that the amount of ambiguity in f is invariant over the class of functions whose L^2 -norm is the same. This will be further discussed in Section 6.

§5. Examples of AFs.

In this section examples of AFs will be calculated. Each illustrates some of the properties of Section 4. The significance of some examples will be discussed more fully in Sections 6 and 7.

Example 1.

Let $f(t) = p_T(t) = \begin{cases} 1 & \text{for } |t| < T \\ 0 & \text{elsewhere.} \end{cases}$

Then by (2.4), $\gamma(t,\tau) = p_T(t + \frac{\tau}{2})p_T(t - \frac{\tau}{2})$. $\gamma(t,\tau)$ is unity inside the rhombus below and zero elsewhere.
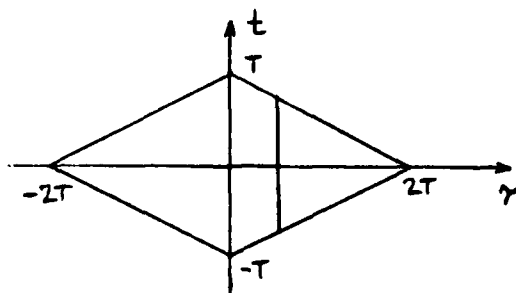

Figure 1

By definition (2.5),

$$\chi(u,\tau) = \int \gamma(t,\tau)e^{-iut}\, dt.$$

For fixed $\tau \in [-2T,2T]$ the limits of integration are $t = -T + |\tau|/2$ to $t = T - |\tau|/2$. Thus

$$\chi(u,\tau) = \int_{-T+|\tau|/2}^{T-|\tau|/2} e^{iut}\, dt,$$

and we conclude

$$(1)\quad \chi(u,\tau) = \begin{cases} \dfrac{2}{u}\sin[u(T - \dfrac{|\tau|}{2})] & \text{for} \quad |\tau| \leq 2T \\[2ex] 0 & \text{for} \quad |\tau| \geq 2T. \end{cases}$$

Notice that for $\tau = 0$ and $u = 0$ we have

$$\chi(u,0) = \frac{2}{u}\sin uT \quad \text{for all} \quad u \in \mathbb{R}.$$

and

$$\chi(0,\tau) = \begin{cases} 2T - |\tau| & \text{for} \quad |\tau| \leq 2T \\[1ex] 0 & \text{elsewhere.} \end{cases}$$
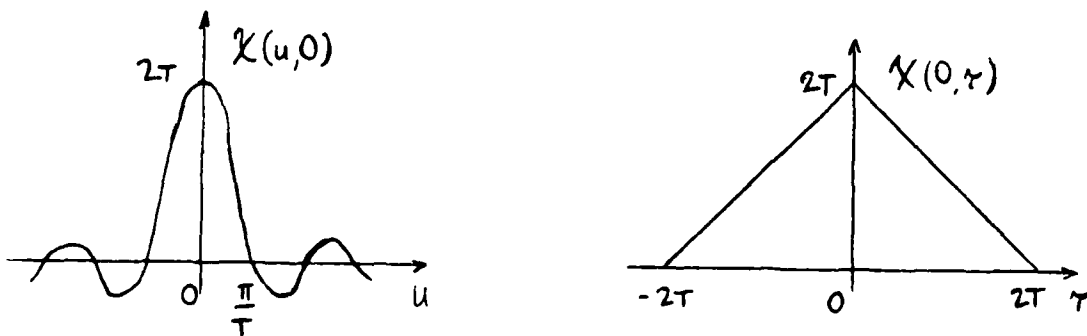
These are sketched in Figure 2.

Figure 2

Example 2. Let $f(t) = e^{iat} p_T(t)$.

The using property 4 and the previous example,

$$(2) \quad \chi(u,\tau) = \begin{cases} e^{ia\tau} \frac{2}{u} \sin[u(T - \frac{|\tau|}{2})] & \text{for} \quad |\tau| \leq 2T \\ 0 & \text{for} \quad |\tau| \leq 2T. \end{cases}$$

Example 3. Let $f(t) = e^{iat}$ for all $t \in \mathbb{R}$.

Then, formally,

$$(3) \quad \chi(u,\tau) = \int e^{ia(t+\tau/2)} e^{-ia(t-\tau/2)} e^{-iut} dt.$$

Recalling $1 \leftrightarrow 2\pi\delta(v)$, (5.3) is

$$\chi(u,t) = 2\pi e^{ia\tau} \delta(u), \quad \forall \tau \in \mathbb{R}.$$

This means that the AF of f concentrates all of its mass on the $u = 0$ axis. Hence, f has perfect resolution of the u variable, but no resolution of the variable τ.

Example 4. Let $f(t) = \sum_{n=0}^{N} s(t-nT),$

where $s(t) = \begin{cases} 1 & \text{for} \quad t \in [0, \frac{T}{2}] \\ 0 & \text{elsewhere.} \end{cases}$

f is a coherent train of $N + 1$ pulses of width $\frac{T}{2}$, and separated by a gap $\frac{T}{2}$. (Figure 3).
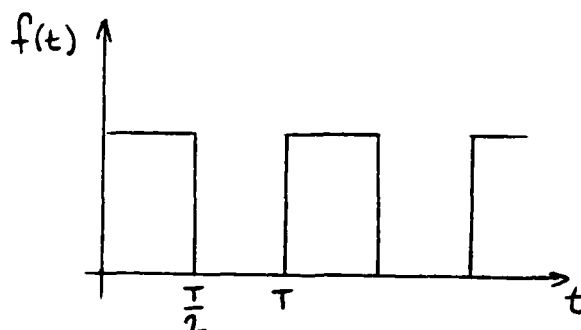
Figure 3

There are several references for calculating X. The most general is in Bird [2]. The following is a special case of Vakman's calculation [19].

First calculate the F.T. of f.

$$F(v) \;=\; \int \sum_{n=0}^{N} s(t-nT)e^{-iut}\, dt \;=\; \sum_{n=0}^{N} S(v)e^{ivnT}.$$

Then from (2.8) we see that

$$\Gamma(u,v) = F(v + \tfrac{u}{2})\overline{F(v - \tfrac{u}{2})}$$

$$= \sum_{n=0}^{N} \sum_{m=0}^{N} S(v + \tfrac{u}{2})\overline{S(v - \tfrac{u}{2})}\; e^{ivT(m-n)}\; e^{-i\frac{u}{2}T(m+n)}.$$

By definition (2.11), the AF of f is

$$X(u,\tau) = \sum_{n=0}^{N} \sum_{m=0}^{N} e^{-i\frac{uT}{2}(m+n)} \frac{1}{2\pi}\int S(v + \tfrac{u}{2})\overline{S(v - \tfrac{u}{2})}e^{ivT(m-n)}e^{iv\tau}\, dv.$$

The inner integral is the AF of s, where $\hat{s} = S$,

evaluated at $\tau - T(n-m)$ i.e.,

$$\chi(u,\tau) = \sum_{n=0}^{N} \sum_{m=0}^{N} e^{-i\frac{uT}{2}(m+n)} X_s(u,\tau-T(n-m)).$$

Change the variable n for the variable k by letting $k = n - m$: then k takes on integer values $-N$ to N, and

$$\chi(u,\tau) = \sum_{k=-N}^{N} X_s(u,\tau-kT) e^{-i\frac{ukT}{2}} \sum_{m=0}^{N} e^{-imuT}.$$

Define

$$\phi_k(u) = e^{-i\frac{uT}{2}k} \sum_{m=0}^{N} e^{-imuT}.$$

The sum is easily computed so that,

$$\phi_k(u) = e^{-i\frac{uT}{2}(k+N)} \frac{\sin[u(N+1)T/2]}{\sin\frac{uT}{2}},$$

and

$$\chi(u,\tau) = \sum_{k=-N}^{N} X_s(u,\tau-kT)\phi_k(u).$$

Compare the mangitudes of $|\phi_k(u)|$ and $|X_s(u,0)|$. For large N, $|\phi_k(u)|$ is a periodic function having spikes of magnitude $N + 1$. (Figure 4).
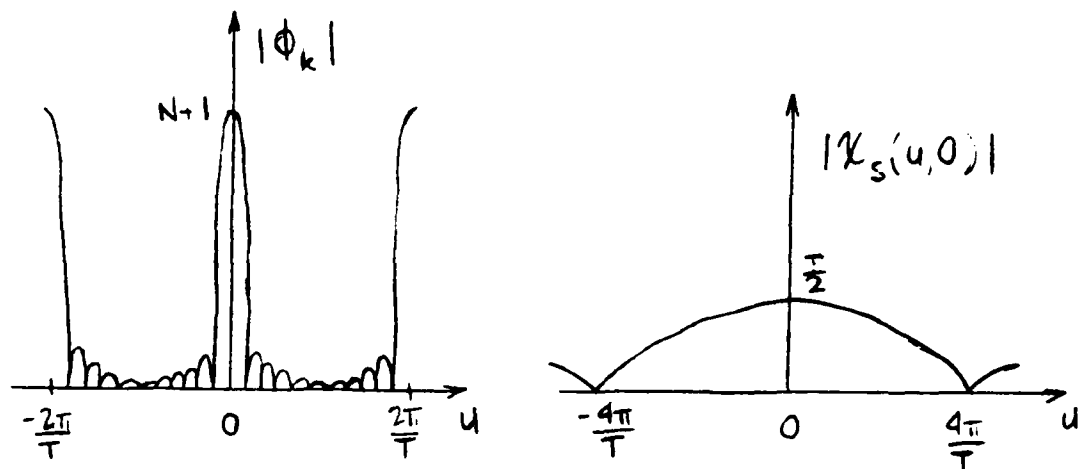
Figure 4

For large N, $|X_s(u,0)|$ is dominated by $|\phi_k|$ at the points $\frac{2n\pi}{T}$, $n = 0$ and n odd. The zeros of $|X_s(u,0)|$ at $\frac{2n\pi}{T}$, n even cause the AF to vanish there. Figure 5 is the level curves of X.
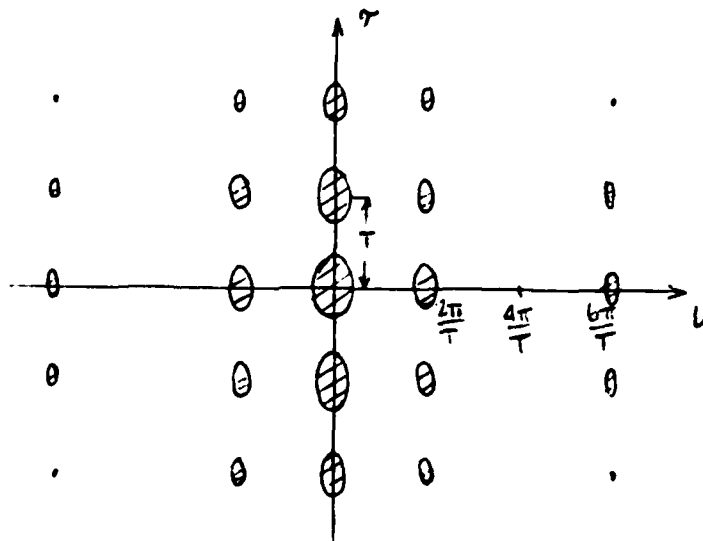

Figure 5

§6. Resolution and Ambiguity

The concepts of resolution and ambiguity were intro-
duced in Section 3. These two ideas will be studied in
detail in this section. The level curves of the AF near
the origin will be shown to always be ellipses and several
examples will be calculated.

§6.1. Resolution

A more precise definition of resolution than that
introduced in Section 3 will be used here. The resolution
of f is the width of the AF at the origin along the
coordinate axes. The smaller the width, the better the
resolution.

Since we are interested in the shape of $\chi(u,\tau)$ near
the origin, let us approximate $\chi(u,\tau)$ by a truncated
2-dimensional Taylor expansion about $(0,0)$. Let χ be
an arbitrary AF and f be the corresponding generating
function. Let subscripts u, and τ denote partial
differentiation. Then by the definition of the Taylor
series,

(1) $\quad \chi(u,\tau) = \chi(0,0) + \chi_u(0,0)u + \chi_\tau(0,0)\tau + \frac{1}{2}\chi_{uu}(0,0)u^2$

$$+ \chi_{u\tau}(0,0)u\tau + \frac{1}{2}\chi_{\tau\tau}(0,0)\tau^2 + \dots .$$

It is assumed, and will be verified later, that for the
given function f, $\chi_{u\tau} = \chi_{\tau u}$. Equation (6.1) may be
normalized by dividing through by $\chi(0,0)$. This was shown
in (4.3) to be $\|f\|^2$. Henceforth, assume that we have

normalized the function f by $\|f\|^2$ and therefore $\chi(0,0) = 1$.

Calculation of the first partial derivatives is straightforward and,

(2) $\qquad \chi_u(0,0) = -i \int t|f(t)|^2 \, dt.$

Define

(3) $\qquad a = \int t|f(t)|^2 \, dt$

then

(4) $\qquad a = i\chi_u(0,0).$

Ultimately we only will be interested in the magnitude of χ. From (4.6) we know

(5) $\qquad |\chi_f(u,\tau)| = |\chi_{T_a f}(u,\tau)|.$

Consequently, without loss of generality, assume the function we are considering in (6.1) is $f(t-a)$. In that case, from (4.6) we calculate the partial derivative of the corresponding AF as,

$$\frac{\partial}{\partial u} \chi_{T_a f}(0,0) = \frac{\partial}{\partial u}[e^{iau}\chi_f(u,\tau)]\Big|_{u=\tau=0} = ia + \chi_u(0,0) = 0.$$

Similarly, using (2.11),

(6) $\qquad \chi_\tau(0,0) = \frac{i}{2\pi} \int v|F(v)|^2 \, dv.$

Define

(7) $\qquad b = \frac{1}{2\pi} \int v|F(v)|^2 \, dv = -i\chi_\tau(0,0).$

Assume that the F.T. of the function f which we are considering in (6.1) is suitably translated by b, then it can be shown that,

$$\chi_\tau(0,0) = 0.$$

Henceforth assume f and F have been so translated and the first partial derivatives vanish. (6.1) may now be simplified to

(8) $\quad \chi(u,\tau) = 1 + \frac{1}{2}\chi_{uu}(0,0)u^2 + \chi_{u\tau}(0,0)u\tau + \frac{1}{2}\chi_{\tau\tau}(0,0)\tau^2 + \dots \;.$

The second partial derivatives are

(9) $\qquad \chi_{uu}(0,0) = -\int t^2|f(t)|^2 \, dt$

and

(10) $\qquad \chi_{\tau\tau}(0,0) = -\frac{1}{2\pi}\int v^2|F(v)|^2 \, dv.$

The mixed partial derivatives are

(11) $\quad \chi_{u\tau}(0,0) = \frac{i}{4\pi}\int v[F(v)\overline{F'(v)} - F'(v)\overline{F(v)}]dv$

and

(12) $\quad \chi_{\tau u}(0,0) = \frac{i}{2}\int t[f'(t)\overline{f(t)} - f(t)\overline{f'(t)}]dt.$

Notice using the transform pairs, $-itf(t) \leftrightarrow F'(v)$ and $f'(t) \leftrightarrow ivF(v)$ and Plancherel's theorem that

(13) $\quad \frac{i}{4\pi}\int v[F(v)\overline{F'(v)} - F'(v)\overline{F(v)}]dv = \frac{i}{2}\int t[f'(t)\overline{f(t)} - f(t)\overline{f'(t)}]dt.$

Hence $\chi_{u\tau}(0,0) = \chi_{\tau u}(0,0)$.

Let the constants in (6.9), (6.10) and (6.11) be defined $-d^2$, $-D^2$ and $-\mu$ respectively. Then the Taylor expansion up to the quadratic terms in (6.8) may be written.

(14) $\qquad \chi(u,\tau) = 1 \quad \frac{1}{2}[d^2u^2 + 2\mu u\tau + D^2\tau^2].$

For χ equal to a constant, this is the equation of an ellipse in the (u,τ)-plane. That is, given $c \in [0,1]$, the level curves of χ near the origin are

(15) $\qquad d^2u^2 + 2\mu u\tau + D^2\tau^2 = c^2.$

It is known that when $\mu = 0$ the major and minor axes of the ellipse in (6.15) lie along the coordinate axes of the (u,τ)-plane. To see what conditions need to be met to have $\mu = 0$, let

$$f(t) = a(t)e^{i\theta(t)}$$

and $\qquad F(v) = A(v)e^{i\phi(v)}.$

By (6.12)

$$\mu = -\int ta^2(t)\theta'(t)dt.$$

By (6.11),

$$\mu = -\frac{1}{2\pi}\int vA^2(v)\phi'(v)dv.$$

Hence $\mu = 0$ if f is real or F is real.

Let us apply these results to some specific AFs.
Recall Example 1. Since f is real-valued, $\mu = 0$ and
the ellipse around the origin has its axes on the coordinate
axes. In this case it is simpler to use the zeros of the
AF along the coordinate axes than to calculate d^2 and
D^2. Refer to Figure 2 and we see that the width of the
main lobe is

$$\Delta\tau = 4T$$

and
$$\Delta u = 2\pi/T.$$

Therefore, for a large pulse width T and a given c,
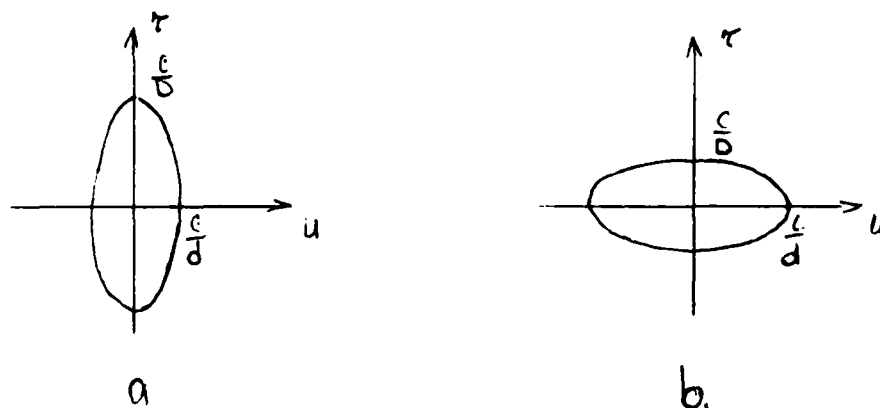the level curve near the origin is depicted in Figure 6a.

Figure 6

Hence, the larger the pulse width the better the resolution
of u (target velocity). Conversely, the shorter the
pulse width the better the resolution of the τ variable
(target range) (Figure 6b).

We showed, in Example 3, a signal that exists for all time results in perfect resolution of the variable in (doppler shift). This is equivalent to letting the pulse width of Example 1 go to infinity. The consequences are intuitively satisfying for if the transmitted signal is a single frequency for all time, then any shift in frequency of the returned signal would be readily apparent. Conversely, an attempt to resolve the time delay variable from such a signal would be futile.

The last example concerning resolution will demonstrate how to simultaneously achieve good resolution along both axes. The idea of using a linear FM signal to achieve such resolution characteristics was a major development in radar technology. Linear FM signals solved the problem of extending the range of a radar system without concurrently degrading range resolution.

Example 5. Let

$$g(t) = e^{i\frac{a}{2}t^2} f(t),$$

where

$$f(t) = P_T(t) \quad \text{as in Example 1.}$$

Use definition (2.4) to calculate the kernel function $\gamma_g(t,\tau)$ as

(16)
$$\gamma_g(t,\tau) = e^{ia\tau t} \gamma_f(t,\tau).$$

Then

$$\chi_g(u,\tau) = \int \gamma_f(t,\tau)e^{-it(u-a\tau)} \, dt = \chi_f(u-a\tau,\tau).$$

By (5.1)

$$\chi_g(u,\tau) = \begin{cases} \dfrac{2}{(u-a\tau)} \sin[(u-a\tau)(T - \frac{\lfloor\tau\rfloor}{2})] & \text{for } |\tau| \leq 2T \\ 0 & \text{elsewhere.} \end{cases}$$

$g(t)$ is not real-valued and it can be shown $G(v)$ is not real-valued, hence, $\mu \neq 0$. The level curve of the AF is a rotated ellipse. Again it is simpler to calculate the first zeros along each axis than to calculate d^2 and D^2. Let $\tau = 0$, then

$$|\chi_g(u,0)| = \frac{2}{u}|\sin(uT)|.$$

So the width of the main lobe along the u axes is

$$\Delta u = 2\pi/T$$

This is unchanged from Example 1.

Let $u = 0$, then

$$|\chi_g(0,\tau)| = \frac{2}{a\tau} \sin[a\tau(T - \frac{\lfloor\tau\rfloor}{2})] \ .$$

For small τ

$$T - \frac{\lfloor\tau\rfloor}{2} \cong T,$$

hence the first zero along the τ-axis is approximately $T/\pi a$ and

$$\Delta\tau = 2T/\pi a.$$

Consequently, for $a > 1$ the resolution of τ has been improved. The level curve for given c and large T is depicted in Figure 7.
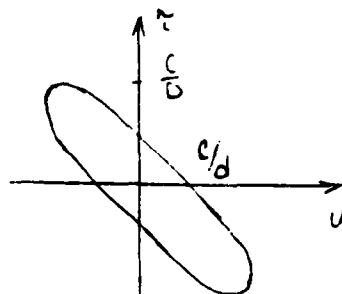

Figure 7

§6.2. Ambiguities

The AF is also useful in characterizing the inherent ambiguities in a given radar signal. Should a given function have peaks indistinguishable from the peak at the origin, then identifying the point which corresponds to the actual target parameters becomes unclear.

Consider Example 4. Here $f(t)$ is a pulse train. The pulses are of width $T/2$ and are periodic with period T. Intuitively, one might reason that if the first pulse leaves the receiver at $t = 0$ and does not return until $t = 2T$, then it will be unclear whether the returned pulse is a reflection of the first pulse trans-mitted or the second pulse transmitted at $t = T$. The reflected energy must return to the receiver before time

$t = T$ for the receiver to unambiguously measure the time delay of the returned pulse. The AF (Figure 5) characterizes this uncertainty because it has peaks along the τ-axis centered at $\tau = kT$, $k = 0, \pm 1, \ldots$.

The AF also characterizes uncertainty in the u parameter. This fact is not so readily apparent when one considers $f(t)$ in isolation. Hence the AF gives the waveform designer a visual tool to characterize, simultaneously, the ambiguities in both parameters.

Example 8. Let

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT).$$
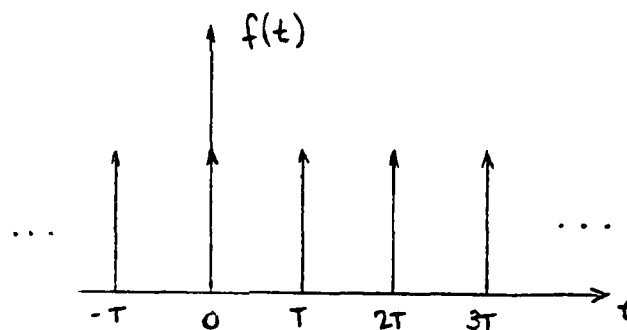

Figure 8

To derive the AF for f, calculate the kernel as

$$\gamma(t,\tau) = \sum_{n=-\infty}^{\infty} \delta(t + \frac{\tau}{2} - nT) \sum_{m=-\infty}^{\infty} \delta(t + \frac{\tau}{2} - mT).$$

$\gamma(t,\tau)$ is zero except when $\tau = kT$, $k = 0, \pm 1, \ldots$. Therefore,

$$\gamma(t,\tau) = \sum_{n=-\infty}^{\infty} \delta(t-nT), \quad \tau = kT, \; k = 0,\pm 1, \ldots .$$

The AF is the F.T. of $\gamma(t,\tau)$ and it is known that

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(u - \frac{2n\pi}{T}).$$

Consequently for each $\tau = kT$, $k = 0,\pm 1, \ldots$ $\chi(u,\tau)$ is an impulse train and the AF is a lattice of points. (Figure 9).

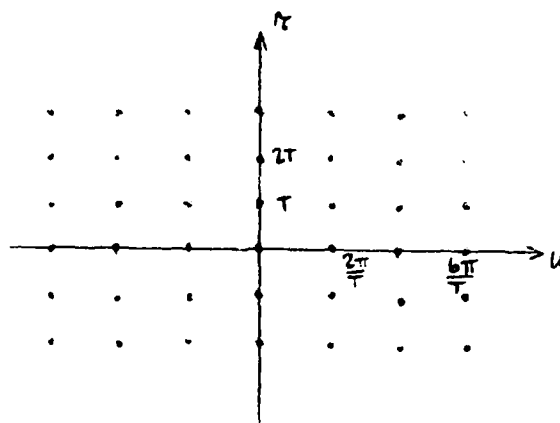$$\chi(u,\tau) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(u - \frac{2n\pi}{T}, \tau - nT).$$

Figure 9

§7. The AF as a Signal Analysis Tool

The AF will be studied as a general analytic tool in this section. The quadratic phase character of the AF will be generalized and studied. It will be shown how the AF characterizes the Heisenberg uncertainty principle and several examples will be calculated.

§7.1. Decomposition of a Linearly Transformed AF [12].

Let χ be a known AF of a given function f. Let L be a real-valued matrix,

$$(1) \qquad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Furthermore let L map the coordinates (u,τ) to (U,T), i.e.

$$(2) \qquad \begin{pmatrix} U \\ T \end{pmatrix} = L \begin{pmatrix} u \\ \tau \end{pmatrix}.$$

Then define

$$(3) \qquad \chi_L(u,\tau) = \chi(U,T),$$

so $\qquad \chi_L(u,\tau) = \chi(au+b\tau, cu+d\tau).$

Theorem. Let χ be the AF of f and define χ_L as in (7.2). χ_L is an AF if and only if $\det L = 1$.

Proof. First the necessity of the condition will be verified.

Since $L(0) = 0$, then $|\chi_L(0,0)| = |\chi(0,0)|$. Therefore

from (4.19) we conclude

(4) $\iint |X(u,\tau)|^2 \, dud\tau = \iint |X_L(u,\tau)|^2 \, dud\tau.$

By definition (7.3) we may write

(5) $\iint |X(u,\tau)|^2 \, dud\tau = \iint |X(U,T)|^2 \, dUdT.$

Assume the Jacobian does not vanish in $\mathbb{R}^2$ then we may write

(6) $\iint |X(u,\tau)|^2 \, dud\tau = \iint |X(u,\tau)|^2 \left[\frac{\partial(U,T)}{\partial(u,\tau)} \right] dud\tau,$

where the Jacobian is

$$\left[\frac{\partial(U,T)}{\partial(u,\tau)} \right] = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Thus (7.6) is only true when $ad - bc = 1$.

Next, given that the $\det L = 1$, it will be shown that X_L, defined in (7.3) must be an AF. This will be demonstrated by decomposing L into "elementary" transforms which are equivalent to operations on f which do not alter the integrability of f. Therefore the resulting function, f_L, may be used to generate an AF.

Let L be a matrix of form (7.1) such that $\det L = 1$. Let P be defined for some $\alpha \in \mathbb{R}$,

(7) $P(\alpha) = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$

Let Q be defined for some $\beta \in \mathbb{R}$,

$$(8) \qquad Q(\beta) = \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix}.$$

Then if $c \neq 0$, by substitution it can be shown,

$$(9) \qquad L = P(\alpha)Q(\beta)P(\gamma),$$

where $\alpha = \frac{1}{c}(1-a)$, $\beta = -c$ and $\gamma = \frac{1}{c}(1-d)$,

If $b \neq 0$, then it can be shown that

$$(10) \qquad L = Q(\alpha)P(\beta)Q(\gamma),$$

where $\alpha = \frac{1}{b}(1-d)$, $\beta = -b$ and $\gamma = \frac{1}{b}(1-a)$.

If both $b = 0$ and $c = 0$, let $a = \frac{1}{k}$, then

$$(11) \qquad L(k) = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & k \end{pmatrix}.$$

We will show that for some $\lambda \in \mathbb{R}$,

$$(12) \qquad L(k) = Q(\alpha)P(\beta)Q(\gamma)P(-\lambda),$$

where $\alpha = \frac{1}{k\lambda}(k-1)$, $\beta = k\lambda$ and $\gamma = \frac{1}{k\lambda}(\frac{1}{k}-1)$. This may be shown by fixing $\lambda \in \mathbb{R}$ and noting

$$(13) \qquad L(k)P(\lambda) = \begin{pmatrix} \frac{1}{k} & -k\lambda \\ 0 & k \end{pmatrix}.$$

This is in the form for the decomposition (7.10). Equation (7.12) follows by noting that $P^{-1}(\lambda) = P(-\lambda)$.

Thus L can be decomposed into elementary transforms of the form P and Q. Example 5 demonstrated that transform P was equivalent to multiplying f by a linear FM signal. That is, given a function f, its AF, χ, and the transformation $P(\alpha)$ as defined in (7.7), then

$$(14) \qquad \chi_p(u,\tau) = \chi(u-\alpha\tau,\tau)$$

and

$$(15) \qquad f_p(t) = f(t)e^{i\frac{\alpha}{2}t^2}.$$

Similarly, given f, χ and the transform $Q(\beta)$ defined in (7.8), we will show that

$$(16) \qquad \chi_Q(u,\tau) = \chi(u,\tau-\beta u),$$

and

$$(17) \qquad f_Q(t) = (2\pi\beta)^{-\frac{1}{2}} f(t) * e^{i\frac{1}{2\beta}t^2}.$$

Use the F.T. pair $e^{i\frac{1}{2\beta}t^2} \leftrightarrow \sqrt{i2\pi\beta}\, e^{-i\frac{\beta}{2}v^2}$ and the convolution theorem to calculate the F.T. of f_Q as

$$F_Q(v) = (i)^{\frac{1}{2}} F(v)e^{-i\frac{\beta}{2}v^2}.$$

Then by definition (2.11), the AF of f_q is

$$\chi_Q(u,\tau) = \frac{1}{2\pi}\int F(v+\frac{u}{2})\overline{F(v-\frac{u}{2})}e^{i\frac{\beta}{2}(v-\frac{u}{2})^2}e^{-i\frac{\beta}{2}(v+\frac{u}{2})^2}e^{iv\tau}\,dv,$$

$$= \frac{1}{2\pi}\int F(v+\frac{u}{2})\overline{F(v-\frac{u}{2})}\,e^{iv(\tau-u\beta)}dv,$$

$$= \chi(u,\tau-u\beta).$$

f_P and f_Q are still L^2-functions therefore χ_L is well defined and the theorem is proved.

An interesting application of the theorem is in property 7. There we saw that if

$$(18) \qquad\qquad f_L(t) = f(at)$$

then

$$(19) \qquad\qquad \chi_L(u,\tau) = \frac{1}{|a|} \chi_f(\frac{u}{a}, au).$$

This is transform (7.11). Therefore, according to (7.12),

$$(20) \quad f(at) = A(((f(t)e^{-i\frac{\lambda}{2}t^2}) \ast e^{i\frac{1}{2\gamma}t^2})e^{i\frac{\beta}{2}t^2}) \ast e^{i\frac{1}{2\alpha}t^2},$$

where $\alpha = \frac{1}{a\lambda}(a-1)$, $\beta = a\lambda$, $\gamma = \frac{1}{a\lambda}(\frac{1}{a}-1)$, λ is arbitrary, and A is the constant of proportionality dependent on a and λ.

Example 7. From (2.13) we know $\chi_F(u,\tau) = 2\pi \chi_f(\tau,-u)$. Thus the AF of F results in the linear transform of the AF of f represented by,

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From (7.9) we may decompose L into,

$$L = P(-1)Q(1)P(-1).$$

Therefore a real-time spectrum analyzer may be constructed as follows;

$$F(v) = 2\pi[(f(t)e^{-i\frac{t^2}{2}}) * e^{i\frac{t^2}{2}}]e^{-i\frac{t^2}{2}}.$$
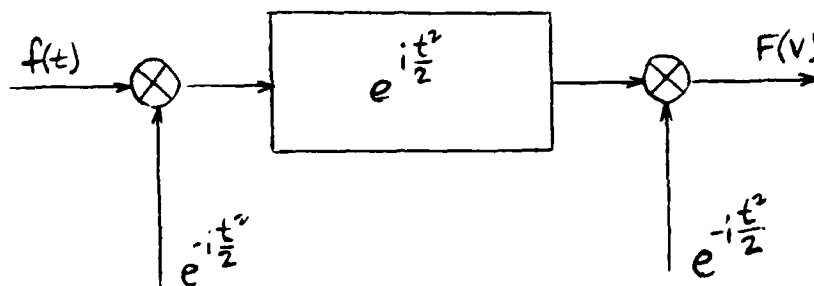
This is depicted in Figure 10.

Figure 10

7.2. Central Moments and the Uncertainty Principle

Signal duration and bandwidth are frequently used quantities in signal analysis. Perhaps the most common way to characterize these features is in the moments of inertia. In the following, the central moments will be shown to be the derivatives of the AF at the origin. The shape of the AF will be related to the Heisenberg uncertainty principle and several interesting examples will be presented.

In most signal analysis applications, $f(t)$ is the complex representation of the real-valued signal of interest. Therefore the central moments will be defined in terms of $|f|^2$. Define the energy of the function as

$$(21) \qquad m_0 = \int |f(t)|^2 \, dt.$$

By definition (1.3) and equation (4.3) we also note

$$(22) \qquad m_0 = \|f\|^2 = \chi(0,0).$$

The center of gravity is defined

$$(23) \qquad m_1 = \int t |f(t)|^2 \, \frac{dt}{m_0} \, .$$

We will henceforth assume $m_1 = 0$. This is a valid assumption because we may always translate f to make $m_1 = 0$. Finally, define the moment of inertia or effective signal duration as,

$$(24) \qquad m_2 = \int t^2 |f(t)|^2 \frac{dt}{m_0}.$$

This was defined as d^2 in Section 5. By definition (1.3) we may also write

$$(25) \qquad m_2 = \frac{1}{m_0} \|tf(t)\|^2 = d^2.$$

Similarly, the moments of $F = \hat{f}$ are defined;

$$(26) \quad M_0 = \frac{1}{2\pi} \int |F(v)|^2 \, dv = \|F\|^2 = \chi(0,0),$$

$$(27) \quad M_1 = \frac{1}{2\pi} \int v |F(v)|^2 \frac{dv}{M_0} = 0 \quad \text{(by assumption)},$$

and

$$(28) \quad M_2 = \frac{1}{2\pi} \int v^2 |F(v)|^2 \frac{dv}{M_0} = \frac{1}{M_0} \|vF(v)\|^2 = D^2.$$

Recalling equations (6.9) and (6.10), we see that

$$(29) \qquad m_2 = -\chi_{uu}(0,0)$$

and

$$(30) \qquad M_2 = -\chi_{\tau\tau}(0,0)$$

Thus the width of the AF function along the τ-axis depends on the spectrum of the generating function and the width along the u-axis depends on the duration of the signal. The Heisenberg uncertainty principle (UP) is

$$(31) \qquad \|tf(t)\|^2 \|vF(v)\|^2 \geq \frac{1}{2} \|f\|^2,$$

or

(32) $$m_2 M_2 \geq \frac{1}{2}.$$

Thus the curvature of χ at the origin is constrained by the UP;

(33) $$\chi_{uu}(0,0)\chi_{\tau\tau}(0,0) \geq \frac{1}{2}.$$

Unlike the Wigner distribution of the next chapter, this does not restrict χ from becoming concentrated about the origin. Klauder demonstrated a compressible AF in 1960 [7].

We next consider the signal duration and bandwidth of the function f_L. f_L is the generating function of the linearly transformed AF, χ_L, defined in (7.3). To simplify notation, given f, let the moments of inertia of f and F be

(34) $\quad w = m_2 = -\chi_{uu}(0,0) \qquad$ (from (7.29)),

and

(35) $\quad W = M_2 = -\chi_{\tau\tau}(0,0) \qquad$ (from (7.30)),

respectively.

Recall in Section 6 we defined

$$\mu = \chi_{u\tau}(0,0) = \chi_{\tau u}(0,0).$$

This is also known as the mixed moment of f or F. Note the corresponding moments of f_L and F_L as w_L, W_L and μ_L. Then using the chain rule we can differen-

tiate (7.3) to show

$$(36) \qquad w_L = a^2 w + 2ac\,\mu + c^2 W,$$

$$(37) \qquad W_L = b^2 w + 2bd\,\mu + d^2 W,$$

and

$$(38) \qquad \mu_L = abw + (ab+bc)\mu + cdW.$$

These relationships will be applied in the following examples.

Example 8. Let $r \in L^2(\mathbb{R})$ and real-valued. Define

$$(39) \qquad f(t) = r(t)e^{i\frac{\alpha}{2}t^2}.$$

The F.T. of f is

$$(40) \quad F(v) = \int r(t)e^{i\frac{\alpha}{2}t^2}e^{-ivt}dt = e^{-\frac{iv^2}{2\alpha}}\int r(t)e^{i\frac{\alpha}{2}(t-\frac{v}{2})^2}dt.$$

This is not, in general, computable in closed form. Usually α is assumed large enough so that stationary phase arguments may be applied. Then

$$(41) \qquad F(v) \cong \sqrt{\frac{i2\pi}{\alpha}}\, e^{-i\frac{1}{2\alpha}v^2}\, r(\frac{v}{\alpha}).$$

This may be used in (7.28) to approximate the effective bandwidth of F as

$$(42) \quad W_F = \frac{1}{2\pi}\int v^2 |F(v)|^2 dv = \frac{1}{\alpha}\int v^2 |r(\frac{v}{\alpha})|^2\, dv = \alpha^2\, w_r,$$

where w_r was defined in (7.24).

We may use the results of the previous analysis to get an exact expression for effective bandwidth. From equation (7.15) and (7.7) we see that the transform matrix is $P(\alpha)$, and $a = 1$, $b = -\alpha$, $c = 0$ and $d = 1$. Hence from (7.37) we see that

$$(43) \qquad W_L = \alpha^2 w_r - 2\alpha\mu_r + W_r.$$

r is real-valued, hence $\mu_r = 0$ and

$$(44) \qquad W_L = \alpha^2 w_r + W_r.$$

For large α, the equation (7.44) agrees with the approximation (7.42).

Consider equation (7.43) as a quadratic in α. For fixed r (not necessarily real-valued) w_r, μ_r and W_r are fixed and (7.43) is a parabola opening upward.

For

$$\mu_r = 0; \quad \min W_F(\alpha) = W_r \quad \text{at} \quad \alpha = 0, \qquad \text{and for}$$

$$\mu_r \neq 0; \quad \min W_F(\alpha) = W_r - \mu_r^2/w_r \quad \text{at} \quad \alpha = \mu_r/w_r.$$

Hence we see the effective bandwidth decreases for a $\mu_r > 0$ (Figure 11).
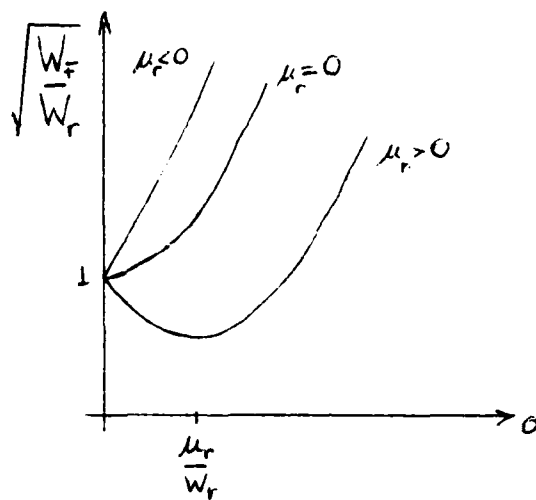
Figure 11

Example 9. Let $r \in L^2(\mathbb{R})$ and real-valued. Define

$$(45) \qquad f_L(t) = [r(t)e^{-i\frac{\alpha}{2}t^2}] * e^{i\frac{1}{2\beta}t^2}.$$

Using (7.7) and (7.8) we see that

$$(46) \qquad L = P(-\alpha)Q(\beta) = \begin{pmatrix} 1-\alpha\beta & \alpha \\ -\beta & 1 \end{pmatrix}$$

and

$$(47) \qquad \chi_L(u,\tau) = \chi_r(u-\alpha\beta u+\alpha\tau, -\beta u+\tau).$$

r is real, so $\mu = 0$. Fix $\alpha \in \mathbb{R}$ then by (7.36),

$$(48) \qquad w_L = w_L(\beta) = (1-\alpha\beta)^2 w_r + \beta^2 W_r.$$

Expanding we obtain,

$$(49) \qquad w_L(\beta) = (\alpha^2 w_r + W_r)\beta^2 - 2\alpha w_r \beta + w_r.$$

This is a quadratic opening upward with a minimum at

$$(50) \qquad \beta_{min} = \frac{\alpha w_r}{\alpha^2 w_r + W_r}.$$

For $\alpha^2 w_r \gg W_r$, this may be approximated as

$$(51) \qquad \beta_{min} = \frac{1/\alpha}{1 + W_r/\alpha^2 w_r} \cong \frac{1}{\alpha}.$$

Hence, we see that although a signal disperses when passed through a quadratic phase filter, in this case we get some signal compression (Figure 12).
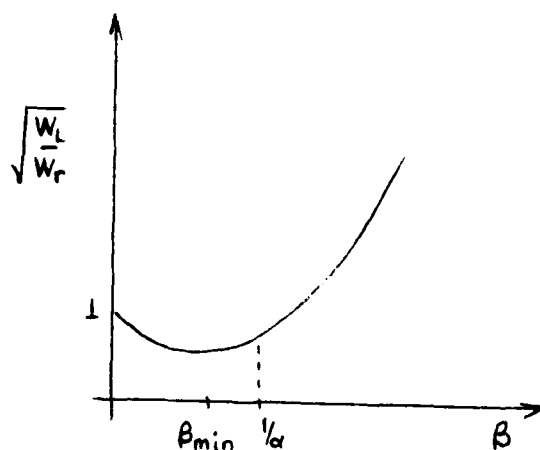

Figure 12

From (7.51) and (7.48) we find the minimum signal duration to be,

$$W_r(\beta_{min}) \cong \frac{W_r}{\alpha^2}.$$

III WIGNER DISTRIBUTION

In this chapter the Wigner distribution will be presented. It will be defined, its salient properties will be stated and proved, and examples, which will serve to illuminate those properties, will be calculated. Finally some interesting results concerning the moments of the Wigner distribution (WD) will be presented.

The Wigner distribution was originally introduced in the context of quantum mechanics in 1932. It was reintroduced by Ville in 1948 as a tool for signal analysis. The most recent applications of the Wigner distribution have been in the field of optics. In 1980 Claasen and Mechlenbräuker again studied the WD with regard to its potential as a signal analysis tool [3].

§8. Definition

The WD is defined for continuous functions f, mapping the real line to the complex plane, which are of finite energy. If $f \notin L^2(R)$ or if f is not continuous then f will be considered a generalized function and operations on f will be in the distribution sense.

The WD is defined as

(1) $$W(t,v) = \int f(t + \tfrac{\tau}{2})\overline{f(t - \tfrac{\tau}{2})}e^{-iv\tau}\,d\tau.$$

Note that the definition is stated as an "auto-Wigner distribution." This definition easily generalizes to a "cross-Wigner distribution."

As in the case of the AF we define

$$(2) \qquad \gamma(t,\tau) \;=\; f(t+\tfrac{\tau}{2})\overline{f(t-\tfrac{\tau}{2})}.$$

Then for fixed t, the WD is the Fourier transform with respect to the variable τ of the function $\gamma(t,\tau)$. That is $\forall t \in R$,

$$(3) \qquad W(t,v) \;=\; \int \gamma(t,\tau)e^{-iv\tau} \, d\tau.$$

Let $F = \hat{f}$. Then it can be shown that

$$(4) \qquad W_f(t,v) \;=\; \tfrac{1}{2\pi} \, W_F(v,-t).$$

If we define

$$(5) \qquad \Gamma(u,v) \;=\; F(v+\tfrac{u}{2})\overline{F(v-\tfrac{u}{2})}$$

and recall (2.6), which defined $\Gamma(u,v)$ as the two-dimensional Fourier transform of $\gamma(t,\tau)$, then

$$(6) \qquad \gamma(t,\tau) \underset{\tau}{\leftrightarrow} W(t,v) \underset{t}{\leftrightarrow} \Gamma(u,v).$$

This says that the WD is the intermediate result of the iterated two-dimensional F.T. of $\gamma(t,\tau)$ when the transformation with respect to τ is taken first. It was shown that for fixed $v \in \mathbb{R}$ $\Gamma(\cdot,v) \in L^1(\mathbb{R})$ hence,

$$(7) \qquad W(t,v) \;=\; \tfrac{1}{2\pi} \int F(v+\tfrac{u}{2})\overline{F(v-\tfrac{u}{2})}e^{iut} \, du$$

§9. Properties of WD

The properties will be listed in somewhat the same order as in the previous chapter. Where proof of the property is similar to the argument for the AF, the property will be stated without proof.

1. Global Maximum

The WD has global extrema at the origin only if f is real and symmetric. This property does not necessarily generalize to the "cross-Wigner distribution." If $f : \mathbb{R} \to \mathbb{R}$ and $f(t) = \pm f(-t)$ then

$$(1) \qquad |W(t,v)| \leq |W(0,0)|.$$

This property is stated in terms of the modulus of the WD. Although the WD is always real (property 2), its value at the origin may be negative, e.g., the case when f is odd.

2. Symmetry

a.) The WD, regardless of the nature of the function f, is always real valued. That is

$$(2) \qquad \overline{W(t,v)} = W(t,v).$$

Proof. Note that $\overline{\gamma(t,\tau)} = \gamma(t,-\tau)$. Then by definition (8.3),

$$\overline{W(t,v)} = \int \overline{\gamma(t,\tau)} e^{+iv\tau} \, d\tau = \int \gamma(t,-\tau) e^{iv\tau} \, d\tau = W(t,v). \qquad \#$$

This property generalizes to the "cross-Wigner distribution" since $\gamma_{fg} = \gamma_{gf}$.

b.) If f is real-valued and symmetric about the origin then,

$$(3) \qquad W(-t,-v) = W(t,v).$$

Proof. Consider only the case $f(t) = -f(-t)$. Then

$$\gamma(-t,\tau) = f(-t + \tfrac{\tau}{2})\overline{f(-t - \tfrac{\tau}{2})} = f(t - \tfrac{\tau}{2})\overline{f(t + \tfrac{\tau}{2})} = \overline{\gamma(t,\tau)}.$$

The property follows from definition (7.3).

3. Translation

a.) Translation of the function f results in a translation of the WD. Let T_a be the translation operator. Then

$$(4) \qquad W_{T_a f}(t,v) = W_f(t-a,v).$$

Proof. Notice that $\gamma_{T_a f}(t,\tau) = \gamma_f(t-a,\tau)$. Consequently,

$$W_{T_a f}(t,v) = \int \gamma_f(t-a,v)e^{-iv\tau}\,d\tau = W_f(t-a,v). \qquad \#$$

b.) Translation of the F.T. of f also results in a translation of the WD of f. If

$$T_a F(v) = F(v-a)$$

$$(5) \quad \text{and} \qquad h(t) \leftrightarrow F(v-a),$$

$$\text{then} \qquad W_h(t,v) = W_f(t,v-a).$$

Proof. It is known that $h(t) = e^{iat}f(t)$. Hence

$$W_h(t,v) = \int f(t + \tfrac{\tau}{2})\overline{f(t - \tfrac{\tau}{2})}e^{ia(t + \frac{\tau}{2})}e^{-ia(t - \frac{\tau}{2})}e^{-iv\tau}\,d\tau$$

$$= \int f(t + \tfrac{\tau}{2})\overline{f(t - \tfrac{\tau}{2})}e^{-i(v-a)\tau}\,d\tau = W_f(t,v-a). \qquad \#$$

4. Modulation by a simple sinusoid.

The previous property may also be stated in terms of a modulation operator. That is, let

$$M_a f(t) = e^{iat} f(t).$$

Then (9.5) may be written

$$(6) \qquad\qquad W_{M_a f}(t,v) = W_f(t,v-a).$$

Similarly if $M_{-a}F(v) = e^{-iav} F(v)$ and $h(t) \leftrightarrow e^{-iav} F(v)$ then (9.4) may be written

$$(7) \qquad\qquad W_h(t,v) = W_f(t-a,v).$$

An interesting computational rule is a result of (9.4) and (9.6). Combining these two properties we get

$$(8) \qquad\qquad W_{M_b T_a f}(t,v) = W_f(t-a,v-b).$$

Let $t = v = 0$ and change the sign of the parameters a and b. We obtain

$$(9) \qquad\qquad W_{M_{-b} T_{-a} f}(0,0) = W_f(a,b).$$

Hence, the WD of a function f may be evaluated at any point (a,b) in the plane by first translating and modulating f, then evaluating the WD of the resulting function at the point (0,0).

5. Multiplication of two functions.

If h(t) = g(t)f(t) then

$$(10) \qquad W_h(t,v) = \frac{1}{2\pi} \int W_g(t,r)W_f(t,v-r)dr.$$

6. Filtered Functions.

Let h(t) = (g*f)(t), where g is the impulse response to a linear, time-invariant filter. Then

$$(11) \qquad W_h(t,v) = \int W_g(r,v)W_f(t-r,v)dr.$$

7. Invertibility and Uniqueness.

From (8.3) it is apparent that given a WD W, one can uniquely invert the F.T. to recover the kernel function γ. Hence, given W(t,v) is a WD, then

$$(12) \qquad f(t+\frac{\tau}{2})\overline{f(t-\frac{\tau}{2})} = \frac{1}{2\pi} \int W(t,v)e^{iv\tau} dv.$$

Let $t_1 = t + \frac{\tau}{2}$, $t_2 = t - \frac{\tau}{2}$ and we see that

$$(13) \qquad f(t_1)\overline{f(t_2)} = \frac{1}{2\pi} \int W(\frac{t_1+t_2}{2},v)e^{iv(t_1-t_2)}dv.$$

If $t = t_1$, $t_2 = 0$ then

$$(14) \qquad f(t)\overline{f(0)} = \frac{1}{2\pi} \int W(\frac{t}{2},v)e^{ivt} dv.$$

Hence the generating function f(t) can be recovered to within a constant $\overline{f(0)}$. As in the case of the AF, it can be shown that this constant must have modulus 1. Furthermore, by letting $t_1 = t_2 = t$ in (9.13) we have the unusual result that $\forall t \in \mathbb{F}$

$$(15) \qquad |f(t)|^2 = \frac{1}{2\pi} \int W(t,v)dv.$$

The novelty of (9.15) lies in the fact that the WD is not always positive. Yet (9.15) says that regardless of the nature of f, at any particular t, the integral of the WD over all frequencies results in a nonnegative number.

8. Volume Invariance.

$$(16) \qquad \frac{1}{2\pi} \iint W(t,v)dtdv = \|f\|^2,$$

where the L^2-norm of f was defined in equation (1.3). This is a direct result of (9.15).

9. Analytic Signals.

Let f be the complex representation of the real signal s(t) as in (A.3). Then

$$(17) \qquad W_f(t,v) = \begin{cases} \frac{4}{\pi} \int W_s(t-r,v)h(r,v)dr & \text{for} \quad v > 0 \\ 0 & \text{for} \quad v < 0, \end{cases}$$

where $h(r,v) = \frac{1}{r}(\sin 2vr)$. Equation (9.17) is the convolution of the WD of the real-valued signal s(t) with the Fourier kernel h(r,v).

Proof. Recall (A.6) to see that

$$(18) \quad F(v + \frac{u}{2}) = \begin{cases} 2S(v + \frac{u}{2}) & \text{for } u > -2v \\ S(0) & u = -2v \\ 0 & u < -2v \end{cases} \quad \text{and}$$

$$F(v - \frac{u}{2}) = \begin{cases} 2S(v - \frac{u}{2}) & \text{for } u < 2v \\ S(0) & u = 2v \\ 0 & u > 2v. \end{cases}$$

Temporarily assume $v > 0$. Hence, using definition (8.7) we see

$$(19) \quad W_f(t,v) = \frac{2}{\pi} \int_{-2v}^{2v} S(v + \frac{u}{2})\overline{S(v - \frac{u}{2})}e^{iut}\, du.$$

Let

$$P_{2v}(u) = \begin{cases} 1 & \text{for } |u| < 2v \\ 0 & \text{elsewhere.} \end{cases}$$

Then (9.19) is

$$(20) \quad W_f(t,v) = \frac{2}{\pi} \int P_{2v}(u)S(v + \frac{u}{2})\overline{S(v - \frac{u}{2})}e^{iut}\, du.$$

It is known $(\frac{1}{\pi t}) \sin 2vt \underset{t}{\leftrightarrow} P_{2v}(u)$. By the convolution theorem, since $W_s \underset{u}{\leftrightarrow} \Gamma$, we conclude

$$(21) \quad W_f(t,v) = \frac{4}{\pi} \int (\frac{1}{r})\sin(2vr)W_s(t-r,v)\, dr, \quad v > 0.$$

Should $v < 0$ then (9.18) states that

$$F(v + \frac{u}{2}) \;=\; 0 \quad \text{for} \quad u < -2v$$

$$F(v - \frac{u}{2}) \;=\; 0 \quad \text{for} \quad u > 2v.$$

Hence $F(v + \frac{u}{2})\overline{F(v - \frac{u}{2})} = 0 \quad \forall u \in \mathbb{R},$ and the property is verified.

#

§10. Examples

Example 10. Let $f(t) = p_T(t)$. Then refer to Figure 1 and note that the limits of integration of the WD for fixed t are $\tau = -2T + 2|t|$ to $\tau = 2T - 2|t|$. Then for $T > 0$,

$$(1) \quad W(t,v) = \int_{-2(T-|t|)}^{2(T-|t|)} e^{-i\tau v} d\tau \;=\; \begin{cases} (\frac{2}{v}) \sin 2v(T-|t|) & \text{for } |t| < T \\ \\ 0 & \text{for } |t| > T. \end{cases}$$
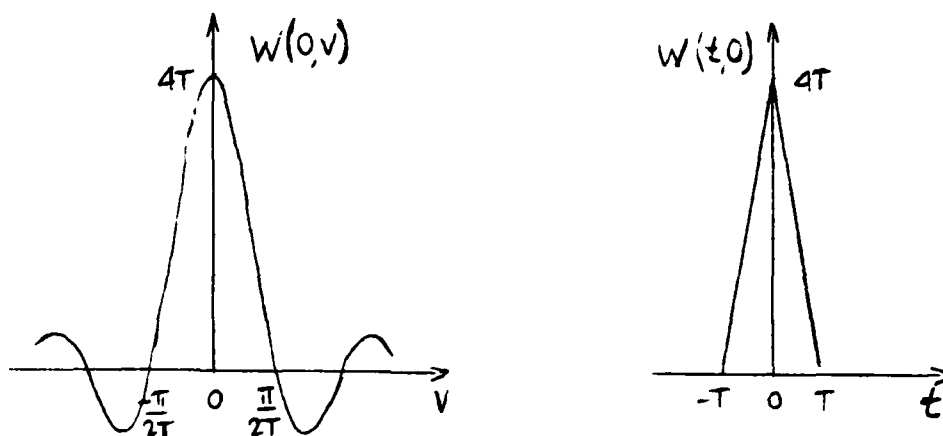

Figure 13

Note that since f is real and even, the WD has a global maximum at the origin. Except for a scale factor, the ·WD is the same as the AF.

Example 11.

Let
$$f(t) = \begin{cases} e^{iat} & |t| < T \\ 0 & \text{elsewhere.} \end{cases}$$

From (9.10) and (10.1) we conclude

$$(2) \quad W(t,v) = \begin{cases} \frac{2}{(v-a)} \sin[2(v-a)(T-|t|)] & \text{for } |t| < T \\ 0 & \text{elsewhere.} \end{cases}$$

Example 12.

Let $f(t) = Ae^{iat}$, $\forall t \in \mathbb{R}$ and some $A \in \mathbb{C}$.

Note that $\gamma(t,\tau) = |A|^2 e^{ia\tau}$.

Then formally,

$$(3) \quad W(t,v) = |A|^2 \int e^{ia\tau} e^{-iv\tau} d\tau = 2\pi |A|^2 \delta(v-a),$$

where $\delta(v)$ has been previously defined in (1.7).

Example 13.

Let
$$f_1(t) = e^{i\frac{a}{2}t^2}, \quad |t| < T.$$

Define $g(t) = e^{i\frac{a}{2}t^2}$ and $f(t) = p_T(t)$. We will find the WD for $g(t)$ for all time, then use Example 10 and property 5 to find the WD of f_1. Formally,

$$W_g(t,v) = \int e^{i\frac{a}{2}(t+\frac{\tau}{2})^2} e^{-i\frac{a}{2}(t-\frac{\tau}{2})^2} e^{-iv\tau} d\tau$$

$$= \int e^{iat\tau} e^{-iv\tau} d\tau = 2\pi\delta(v-at).$$

Recalling (10.1) and (9.10) we conclude

$$W_{f_1}(t,v) = \frac{1}{2\pi} \int 2\pi\delta(r-at)\frac{2}{(v-r)} \sin[2(v-r)(t-|T|)]dr$$

$$= \begin{cases} (\frac{2}{v-at}) \sin[2(v-at)(t-|T|)] & |t| \leq T \\ 0 & \text{elsewhere.} \end{cases}$$

This may also be written,

$$(4) \qquad W_{f_1}(t,v) = W_f(t,v-at).$$

We see that multiplying a function by a linear FM signal results in a linear transform of the original WD.

Example 14. Let f be given, and $W(t,v)$ be the WD of f. Define

$$(5) \qquad f_L(t) = (2\pi b)^{-\frac{1}{2}} f(t) * e^{i\frac{1}{2b}t^2}.$$

We will calculate the WD of f_L.

Using the transform pair $e^{i\frac{1}{2b}t^2} \leftrightarrow \sqrt{2\pi ib}\, e^{-i\frac{b}{2}v^2}$, we can formally calculate the FT of f_L as

$$F_L(v) = (i)^{\frac{1}{2}} e^{-i\frac{b}{2}v^2} F(v).$$

Use the definition (8.7) and

$$\begin{aligned}
W_L(t,v) &= \frac{1}{2\pi} \int F_L(v+\frac{u}{2})\overline{F_L(v-\frac{u}{2})}e^{iut}\, du \\
&= \frac{1}{2\pi} \int F(v+\frac{u}{2})\overline{F(v-\frac{u}{2})}\, e^{-ibuv}\, e^{iut}\, du \\
&= W_f(t-bv,t).
\end{aligned}$$

§11. Central Moments

It is known that the central moments of a function provide some insight into the shape and character of a function without fully describing it. The following will generalize the notion of central moments in $\mathbb{R}$ to local and global moments in $\mathbb{R}^2$. We will see that these moments of the WD are some well known quantities in signal and network analysis.

Let the central moments of $|f|^2$ be m_0, m_1 and m_2 as defined in (7.21), (7.23) and (7.24). Let the moments of $|F|^2$ be M_0, M_1 and M_2 as defined in (7.26)-(7.28). We will first study the local moments of the WD.

For fixed t, the local average of the WD with respect to v has been calculated in (9.15) as

$$(1) \qquad n_0(t) = \int W(t,v)dv = |f(t)|^2 .$$

This is the power in f at time t. The local center of gravity with respect to v is

$$(2) \qquad n_1(t) = \int vW(t,v)\,\frac{dv}{n_0(t)} .$$

This may be formally calculated using Fubini's theorem and the F.T. pairs $v \leftrightarrow 2\pi\delta'(\tau)$ as

(3)
$$n_1(t) = \int [f(t + \tfrac{\tau}{2})\overline{f(t - \tfrac{\tau}{2})}]\delta'(t)\, \frac{d\tau}{n_0(t)},$$

$$= \tfrac{1}{2}\int [f'(t + \tfrac{\tau}{2})\overline{f(t - \tfrac{\tau}{2})} - f(t + \tfrac{\tau}{2})\overline{f'(t - \tfrac{\tau}{2})}]\delta(\tau)\, \frac{dt}{n_0(t)},$$

$$= \frac{1}{2n_0(t)}\, [f'(t)\overline{f(t)} - f(t)\overline{f'(t)}].$$

If f is real-valued then $n_1(t) \equiv 0$. If f is complex-valued then (10.3) can be put in a more meaningful form. Notice that since

$$[\overline{f}f' - f\overline{f}'] = \operatorname{Im} \overline{f}f',$$

then (10.3) is

(4) $$n_1(t) = \operatorname{Im} \frac{f'(t)}{f(t)} = \operatorname{Im}\{\frac{d}{dt} \ln f'(t)\}.$$

Therefore if $f(t) = a(t)e^{i\Theta(t)}$ we see that

(5) $$n_1(t) = \Theta'(t).$$

For complex representations of a real-valued signal (Appendix A) this is the instantaneous frequency of the function f. The WD permits the generalization of instantaneous frequency to arbitrary complex-valued functions.

The local moment of inertia for fixed t is

(6) $$n_2(t) = \frac{1}{2\pi} \int (v - n_1(t))^2\, W(t,v)\, \frac{dv}{n_0(t)}$$

$$= \frac{1}{2\pi} \int v^2\, W(t,v)\, \frac{dv}{n_0(t)} - n_1(t)^2.$$

The WD is not always positive so interpreting this as the spread of the WD is not entirely accurate. To derive some meaning from $n_2(t)$ we must put (10.6) in a different form. Use the F.T. $v^2 \leftrightarrow 2\pi\delta''(\tau)$ to formally calculate $n_2(t)$ in terms of f. We see that

$$(7) \quad n_2(t) = \frac{1}{4n_0(t)}[f''(t)\overline{f(t)} - 2|f'(t)|^2 + f(t)\overline{f''(t)}] - n_1(t)^2.$$

It can be shown that

$$(8) \qquad n_2(t) = -\frac{1}{2} \operatorname{Re}\{\frac{d}{dt} \frac{f'(t)}{f(t)}\}.$$

If $f(t) = a(t)e^{i\Theta(t)}$ then

$$(9) \qquad n_2(t) = -\frac{1}{2} \frac{d^2}{dt^2}\ln|a(t)|.$$

Hence the local second moment of the WD is independent of the phase of f. Furthermore, for any $\alpha \in \mathbb{R}$ and any $C > 0$,

$$(10) \quad n_2(t) \equiv 0 \quad \text{if and only if} \quad a(t) = Ce^{\alpha t}.$$

The sufficiency of the condition is shown by substitution. The necessity of the condition is shown by letting

$$(11) \qquad \frac{d^2}{dt^2} \ln|a(t)| = 0,$$

and integrating twice.

Consider now the local moments of the WD with respect

to t. Fix v, then

$$(12) \qquad N_0(v) = \int W(t,v)dt = |F(v)|^2.$$

Similarly, the local center of gravity is

$$(13) \qquad N_1(v) = \int tW(t,v) \frac{dt}{N_0(v)}.$$

This can be shown to be

$$(14) \qquad N_1(v) = \frac{1}{2N_0(t)} [F(v)\overline{F'(v)} - \overline{F(v)}F'(v)]$$

or

$$(15) \qquad N_1(v) = -Im\{\frac{F'(v)}{F(v)}\} = -Im\{\frac{d}{dv}\ln F(v)\}.$$

If $F(v) = A(v)e^{i\phi(v)}$ then

$$(16) \qquad N_1(v) = \phi'(v).$$

Should $F(v)$ be a systems transfer function, then (11.16) is the group delay. Again the WD permits a specific definition to be generalized.

By similar calculations, as in deriving (11.9) it can be shown that the local moment of inertia with respect to t is

$$(17) \qquad N_2(v) = -\frac{1}{2}\frac{d^2}{dv^2}\ln|A(v)|.$$

The global moments of the WD are taken over the entire (t,v)-plane, and will be shown to be moments of the generating function, f, and its F.T., F. The

global average of the WD was previously calculated in (9.16) as

$$(18) \qquad \bar{n}_0 = \frac{1}{2\pi} \iint W(t,v)dtdv = \|f\|^2.$$

Similarly

$$(19) \qquad \bar{N}_0 = \frac{1}{2\pi} \iint W(t,v)dtdv = \|F\|^2.$$

Hence $\bar{n}_0 = \bar{N}_0$.

The global mean or center of gravity with respect to the variable v is

$$(20) \quad \bar{n}_1 = \frac{1}{2\pi} \iint vW(t,v) \frac{dvdt}{\bar{n}_0} = \frac{1}{2\pi} \int v|F(v)|^2 \frac{dv}{\|F\|^2}.$$

The last equality results from interchanging the order of integration and (11.12). Hence the global center of gravity of the WD with respect to the frequency variable v is the same as the center of gravity of $|F(v)|^2$, where F is the F.T. of f.

In a similar manner, the global mean of the WD with respect to the variable t is defined as

$$(21) \quad \bar{N}_1 = \frac{1}{2\pi} \iint tW(t,v) \frac{dtdv}{\bar{N}_0} = \int t|f(t)|^2 \frac{dt}{\|f\|^2}.$$

The global moments of inertia are defined as

$$(22) \quad \bar{n}_2 = \frac{1}{2\pi} \iint (v-\bar{n}_1)^2 W(t,v)\frac{dvdt}{\bar{n}_0} = \frac{1}{2\pi} \int (v-\bar{n}_1)^2 |F(v)|^2 \frac{dv}{\|F\|^2}.$$

and

$$(23) \quad \bar{N}_2 = \frac{1}{2\pi} \iint (t-\bar{n}_1)^2 W(t,v) \frac{dt\,dv}{\bar{N}_0} = \int (t-\bar{n}_1)^2 |f(t)|^2 \frac{dt}{\|f\|^2}.$$

We see that the moments of inertia are non-negative and hence may be accurately interpreted as a measure of the spread of the WD.

§12 The Uncertainty Principle

The Heisenberg uncertainty principle (UP) constrains the moments of inertia of a function, f, and its F.T., F. Consequently the UP must also constrain the global moments of the WD. A form of the UP is

$$(1) \quad \|tf(t)\| \, \|vF(v)\| \geq \frac{1}{2}\|f\|^2.$$

In the case of non-centered moments (12.1) may be also written

$$(2) \quad \|(t-a)f(t)\| \, \|(v-b)F(v)\| \geq \frac{1}{2}\|f\|^2,$$

where a and b are the respective centers of gravity. Using similar notation, the global moments (11.22) and (11.23) may be written

$$(3) \quad \bar{n}_2 = \|(v-\bar{n}_1)F(v)\|^2 \frac{1}{\|F\|^2}; \quad \bar{N}_2 = \|(t-\bar{N}_1)f(t)\|^2 \frac{1}{\|f\|^2}.$$

Without loss of generality, assume f and F have henceforth been shifted so that $\bar{n}_1 = \bar{N}_1 = 0$. We will now show that the WD cannot be concentrated arbitrarily close

to the origin. Let x and y be fixed. Then it can be shown that,

(4)
$$\min_{p}(p^2 x + \frac{1}{p^2} y) = 2\sqrt{xy}.$$

If $x = \|tf(t)\|^2$ and $y = \|vF(v)\|^2$ then this implies that for any p,

(5)
$$p^2 \|tf(t)\|^2 + \frac{1}{p^2}\|vF(v)\|^2 \geq 2\|tf(t)\|\|vF(v)\|.$$

Then by (12.1) we conclude that

(6)
$$p^2 \|tf(t)\|^2 + \frac{1}{p^2}\|vF(v)\|^2 \geq \|f\|^2.$$

Using definitions (12.3) with $\bar{n}_1 = \bar{N}_1 = 0$ this may also be written

(7)
$$p^2 \bar{N}_2 + \frac{1}{p^2}\bar{n}_2 \geq 1.$$

In terms of the WD, this is

(8) $\frac{1}{2\pi} \iint (p^2 t^2 + \frac{1}{p^2} v^2) W(t,v) dt dv \geq \frac{1}{2\pi} \iint W(t,v) dt dv.$

(12.8) may have the following interpretation: For fixed $p \in \mathbb{R}$, let

(9)
$$g(t,v) = p^2 t^2 + \frac{1}{p^2} v^2.$$

Then $\forall r \in \mathbb{R}$, $g(t,v) = r^2$ is a weight function in the integral (12.8), which assigns the value r^2 to all values of the WD which lie on the ellipse,

$$(10) \qquad p^2 t^2 + \frac{1}{p^2} v^2 = r^2.$$

Hence $g(t,v)$ suppresses the values of the WD near the origin and amplifies the contribution of the WD away from the origin. The inequality in (12.8) means that the WD cannot be totally concentrated in an arbitrarily small region about the origin. If, for example, the WD were to vanish off an ellipse such that $r^2 < 1$, then inequality (12.8) would not hold.

The inequality (12.8) does not preclude a highly concentrated WD which has a small contribution far from the origin. If the WD is to characterize the Heisenberg uncertainty principle, then we must also prohibit just such a WD. To see that such WD's are in fact impossible we need the following theorem [5].

If $a > 0$, $b > 0$ and

$$(11) \quad W_{ab}(t,v) = \frac{1}{2\pi\sqrt{ab}} \iint e^{-\frac{\pi}{a}(t-r)^2 - \frac{\pi}{b}(v-s)^2} W(r,s)\,dr\,ds$$

is the Weierstrass transform of the WD $W(t,v)$, then

$$(12) \quad \forall t,v \in \mathbb{R}, \quad W_{ab}(t,v) \le \frac{2}{1+2\sqrt{ab}} \int |f(t)|^2 \, dt.$$

The proof of this statement depends upon expanding f in an orthogonal system in $L^2(\mathbb{R})$ which is related to Hermite polynomials and is beyond the scope of this paper. We may use this result for the specific values $t = 2r$ and $v = 2s$, then (12.11) is

$$(13) \quad W_{ab}(2r,2s) = \frac{1}{2\pi\sqrt{ab}} \iint e^{-\frac{\pi r^2}{a}} \, e^{-\frac{\pi s^2}{b}} \, W(r,s)drds.$$

(12.12) then becomes

$$(14) \quad \frac{1}{2\pi} \iint e^{-\frac{\pi r^2}{a} - \frac{\pi s^2}{b}} W(r,s)drds \leq \frac{2\sqrt{ab}}{1+2\sqrt{ab}} \int |f(t)|^2 \, dt.$$

Notice that

$$(15) \qquad \frac{2\sqrt{ab}}{1+2\sqrt{ab}} = \left(1 - \frac{1}{1+2\sqrt{ab}}\right),$$

hence (12.14) is

$$(16) \quad \iint \left(1 - e^{-\frac{\pi r^2}{a}} e^{-\frac{\pi s^2}{b}}\right) W(r,s)drds \geq \frac{1}{1+2\sqrt{ab}} \iint W(t,v)dtdv.$$

We now have an inequality using a weight function which, for fixed a and b, goes to 1 with increasing values of r and s. This precludes a WD as was previously described. Therefore from (12.8) and (12.16) we conclude that the WD cannot be arbitrarily concentrated about the origin (or in the case $\bar{n}_1 \neq 0$ or $\bar{N}_1 \neq 0$, about the center of gravity).

IV. EPILOGUE

In the previous chapters two time-frequency functions were studied. Their properties were reviewed and some applications shown. Attention was directed to their relationship to the Heisenberg uncertainty principle. In the next section we will state the relationship between these two transforms and consolidate their similarities and differences.

§13. WD vs. AF

Recall equations(2.9) and (8.6) and we see that the AF and the WD are related by a transform similar to a 2-dimensional F.T. (Figure 14).
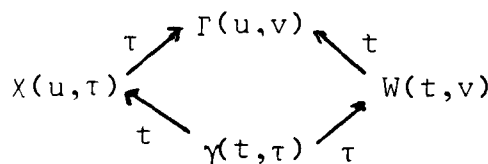

Figure 14

Thus

$$(1) \quad W(t,v) \;=\; \frac{1}{2\pi} \iint X(u,\tau)e^{i(ut-v\tau)}dud\tau,$$

and

$$(2) \quad X(u,\tau) \;=\; \frac{1}{2\pi} \iint W(t,v)e^{-i(ut-v\tau)}dtdv.$$

There are three basic differences between the WD and the AF. First, the WD is always real-valued, while the AF is, in general, a complex-valued function. The

modulus is frequently used in applications so it is
reasonable, in many cases, to compare the modulus of the
AF to the WD. Only when f is real-valued and
symmetric, is the AF the same as the WD (up to a
scale factor). This was demonstrated in Examples 1 and 10.

The second difference between the AF and the WD
is the manner in which these functions transform trans-
lated functions. The modulus of the AF is invariant to
translations of either f or its F.T. The WD shifts
as its generating function shifts. For this reason,
Claasen and Mecklenbraüker conclude the AF is not well
suited for general signal analysis [13].

Finally, the AF and WD each characterize signal
duration and bandwidth of the generating function f in
different ways. Effective duration and bandwidth of f
are equivalent to partial derivatives of the AF but
equivalent to the global moments of the WD. Consequently,
the WD was shown to characterize directly the Heisenberg
uncertainty principle. On the other hand, the AF has
an associated uncertainty principle (property 11) which
is an analogy to the Heisenberg uncertainty principle.

The WD and AF have three basic similarities.
First, each time-frequency transform of f is closely
related to the t-f transform of $F = \hat{f}$. The t-f
transform of F is a simple rotation of the t-f trans-
form of f. As such, Hermite polynomials are sometimes

associated with these functions. For instance, an
orthogonal system on L^2 based on Hermite polynomials
was used by Klauder to achieve highly concentrated AFs
about the origin [8]. This same orthogonal system was
used by De Bruijn in the proofs of his inequalities for
the WD [5].

Another similarity is that both functions transform
quadratic operations on the generating function to linear
operations on the AF and WD. Therefore, the decompo-
sitions of the AF, introduced in Section 7.2, may be
applied to the WD.

Finally, both t-f transforms yield similar constants
when integrated over the entire plane. In this case we
compare $|X|^2$ to the WD. For clarity we repeat (4.19)
and (9.16).

(4.19)
$$\frac{1}{2\pi} \iint |X(u,\tau)|^2 \, dud\tau = \|f\|^4,$$

and

(9.16)
$$\frac{1}{2\pi} \iint W(t,v)dtdv = \|f\|^2.$$

Let us define a norm on $\mathbb{R}^2$ as

(3)
$$\|X\|_2 = \left(\frac{1}{2\pi} \iint |X(u,\tau)|^2 \, dud\tau\right)^{1/2}.$$

Then $X \in L^2(\mathbb{R}^2)$ because $f \in L^2(\mathbb{R})$. Moyal's formula
is [3],

$$(4) \qquad \frac{1}{2\pi} \iint W^2(t,v)\,dt\,dv = \|f\|^4 .$$

This gives us a Parseval-like relationship in $\mathbb{R}^2$;

$$\|W\|_2 = \|X\|_2 .$$

Therefore, in the space of $L^2(\mathbb{R}^2)$, with the defined norm, the transform (13.1) is an isometry.

§14. Conclusions

Time-frequency functions like the AF and WD are important in pure mathematics and engineering. They are functions which transform a function of time into a function of time and frequency. They inherently embody much of the theory of Fourier transforms. Study of these functions provides a richer understanding of the uncertainty principle and is, therefore, valuable in the field of harmonic analysis. In the applied fields, a time-frequency transform helps to visualize the frequency content of non-stationary functions. These transforms help explain the intimate relationship between the time and frequency components of a signal.

The AF is a tool routinely used in radar waveform design. It is also used in developing the theory of Fresnel diffraction and Fourier optics [13]. It was shown to provide a better understanding of linear frequency modulation and pulse compression. The AF is an important function in both the applied and theoretic fields.

The WD has been for many years an asset to the
field of quantum physics. Recent work has shown it to
be a useful tool in signal analysis and suitable for
hardware implementation [3]. It was also shown to
generalize two common notions in signal analysis;
instantaneous frequency and group delay. The WD is
incompressible and reflects the Heisenberg uncertainty
principle. The WD is also an important function in
theoretic and applied fields.

Time-frequency transforms, like those presented in
this paper, have a broad application. It is the opinion
of this author that such functions may have even more to
contribute in such fields as spectrum estimation, and are
worthy of future study.

Appendix A. The Complex Signal

Frequent reference is made throughout this paper to the complex representation of a real signal. This is also known as the Gabor representation and the analytic signal. In this appendix we will define the analytic signal and explain why it is a useful tool.

In most applications of signal processing techniques, the signal of interest is a real-valued function of time, say s(t). As was stated in (3.1), for radar, s may be modeled as

$$(1) \qquad s(t) = a(t)\cos(\omega_0 t + \phi(t)).$$

This description is mathematically cumbersome and its Fourier transform has a lot of redundant information since

$$(2) \qquad S(-\omega) = \overline{S(\omega)}.$$

Consequently a complex-valued function f is formed so that Re{f} = s. The Gabor representation is such a complex-valued function and is defined

$$(3) \qquad f(t) = s(t) + i\check{s}(t)$$

where $\check{s}$ is the Hilbert transform of s. $\check{s}$ is defined

$$(4) \qquad \check{s}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s(r)}{r-t}\, dr = s(t) * \frac{1}{\pi t}.$$

The Hilbert transform has the useful property that

$$(5) \qquad \check{S}(\omega) \ = \ -i \ \text{sgn}(\omega) S(\omega),$$

where $\check{S}$ is the Fourier transform of $\check{s}$.

This may be verified by letting $x(t) = \dfrac{1}{\pi t}$ so that

$$\check{s}(t) \ = \ s(t) * x(t)$$

and

$$\check{S}(\omega) \ = \ S(\omega) X(\omega).$$

Then (A.5) follows from the Fourier transform pair
$\dfrac{1}{\pi t} \leftrightarrow -i \ \text{sgn} \ \omega$.

From (A.3) we see that

$$(6) \quad F(\omega) \ = \ S(\omega) + i\check{S}(\omega) \ = \ \begin{cases} 2S(\omega) & \omega > 0 \\ S(0) & \omega = 0 \\ 0 & \omega < 0. \end{cases}$$

The Gabor representation solves the problem of redundancy, but using (A.3) in calculations may be equally as cumbersome as using the original $s(t)$. Consequently, in many cases an approximation to (A.3) is used in calculations:

$$(7) \qquad f(t) \ = \ a(t) e^{i(\omega_0 t + \phi(t))}$$

This approximation becomes exact for narrowband signals. That is, define

$$F_+(\omega) \ = \ \frac{1}{2} \int f(t) e^{-i\omega t} \, dt$$

and

$$F_-(\omega) \ = \ \frac{1}{2} \int \overline{f(t)} \ e^{-i\omega t} \, dt.$$

Then f is narrowband if

$$F_+(\omega) \;=\; 0 \quad \text{for all} \quad \omega < 0,$$

and

$$F_-(\omega) \;=\; 0 \quad \text{for all} \quad \omega > 0.$$

In general, the error in the approximation (A.7) is

$$\varepsilon(t) \;=\; 2\, I_m \left\{ \frac{1}{2\pi} \int_{-\infty}^{0} F_+(\omega) e^{i\omega t}\, d\omega \right\}.$$

This is obviously zero for narrowband signals.

Appendix B. The Doppler Approximation

This appendix will explain the doppler approximation and why the effect of a moving target on a narrowband signal may be modeled as only a shift in the carrier frequency.

A radar transmits an electromagnetic signal, which when striking suitable surfaces is reflected and refracted similar to light. If the surface is moving, the reflected frequency of the signal will appear to be different than that which was transmitted. This is analogous to the commonly deserved phenomena of a constant-pitch train whistle appearing high as the train approaches and low as the train passes. This phenomena is known as the doppler effect.

In the case of radar, should the target be stationary (in the sense that the target velocity vector has no component in a radial direction to the radar), then the time delay of the returned signal will be

$$\tau = \frac{2}{c} R,$$

where R is the constant range and c is the velocity of propagation. The returned signal is then a time delayed, suitably attenuated version of the transmitted signal. If $s_t(t)$ is the transmitted signal then the returned signal is

$$s_r(t) = s_t(t-\tau).$$

If the reflecting surface is moving, then the range becomes a function of time and therefore the time delay becomes a function of time. Then

$$s_r(t) = s_t(t-T(t)).$$

The exact relation between the time delay $T(t)$ and the range to the target is [6]

$$(1) \qquad T(t) = \frac{2}{c} R[t - \frac{1}{2} T(t)].$$

Defining $T(\tau) = \tau$, (B.1) can be expanded in a Taylor series about this point;

$$T(t) = T(\tau) + \dot{T}(\tau)(t-\tau) + \frac{1}{2} \ddot{T}(\tau)(t-\tau)^2 + \dots .$$

The returned signal is then

$$s_r(t) = s_t(t-T(\tau)-\dot{T}(\tau)(t-\tau) - \frac{1}{2} \ddot{T}(\tau)(t-\tau)^2 - \dots).$$

Use $T(\tau) = \tau$ and this is

$$(2) \quad s_r(t) = s_t([1-\dot{T}(\tau)][t-\tau] - \frac{1}{2} \ddot{T}(\tau)[t-\tau]^2 - \dots).$$

Use standard notation for velocity and acceleration, i.e. $v(t) = \dot{R}(t)$ and $a(t) = \ddot{R}(t)$, and differentiate (E.1) to see that

$$(3) \qquad \dot{T}(\tau) = \frac{(\frac{2}{c}) v(\frac{\tau}{2})}{1 + \frac{v(\frac{\tau}{2})}{c}},$$

and

$$(4) \qquad \ddot{T}(\tau) = \frac{(\frac{2}{c})a(\frac{\tau}{2})}{[1+(\frac{1}{c})v(\frac{\tau}{2})]^3} .$$

In radar, the velocity of propagation. c, is nearly the speed of light, therefore c >> v(t). (B.3) and (B.4) are

$$(5) \qquad \dot{T}(\tau) = (\frac{2}{c})v(\frac{\tau}{2})$$

$$(6) \qquad \ddot{T}(\tau) = (\frac{2}{c})a(\frac{\tau}{2}) .$$

By definition, the doppler effect is the linear stretching of the time variable in (B.2). Therefore we only consider the first term in the series (B.2). This is equivalent to assuming the target velocity is constant near τ so that $v(\tau) = v(\frac{\tau}{2})$. The linear approximation of (B.2) is

$$(7) \qquad s_r(t) = s_t[w(t-\tau)],$$

where

$$w = 1 - \dot{T}(\tau) = 1 - \frac{2}{c} v(\tau).$$

Finally, assume s_t is a narrowband signal. Then we may write, for fixed ω,

$$(8) \qquad s_t(t) = u(t)e^{i\omega t}$$

where

$$(9) \qquad u(t) = a(t)e^{iO(t)} .$$

Then the doppler stretching w, of the time variable
of the returned signal may be assumed to apply only to
the carrier, $e^{i\omega t}$. This seems justified for the
perturbation of t would be less perceptible on the
slowly varying function $a(t)$ and $\Theta(t)$ relative to the
fast varying carrier. Hence

$$(10) \qquad s_r(t) \;=\; u(t-\tau)e^{i\omega(w)(t-\tau)}.$$

Define $\phi = \dfrac{2v\omega}{c}$ to be the doppler shift then

$$(11) \qquad s_r(t) \;=\; u(t-\tau)e^{i(\omega-\phi)(t-\tau)}.$$

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