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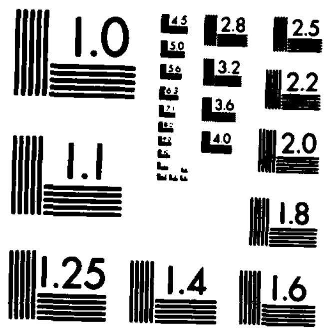

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RECURSIVE MOMENT FORMULAS FOR REGENERATIVE SIMULATION

by

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RECURSIVE MOMENT FORMULAS FOR REGENERATIVE SIMULATION

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1. Introduction

Let f be a real-valued function defined on the state space of a regenerative process $\mathbf{X} = \{X(t) : t \geq 0\}$ with regeneration times $0 = T_0 < T_1 < \cdots$, and suppose that

$$r_t = \frac{1}{t}\int_0^t f(X(s))ds \to r \quad \text{a.s.} \tag{1.1}$$

as $t \to \infty$. The problem of estimating r via a simulation of $\mathbf{X}$ is called the steady state simulation problem.

Relation (1.1) implies that r_t is a strongly consistent point estimator for r. To obtain confidence intervals for r, set (for $k \geq 1$)

$$Y_k(f) = \int_{T_{k-1}}^{T_k} f(X(s))ds$$

$$\tau_k = T_k - T_{k-1}$$

$$Z_k = Y_k(f) - r\tau_k .$$

The regenerative structure of X guarantees that $r = E\{Y_1(f)\}/E\{\tau_1\}$ and that $\{(Y_k(f), \tau_k) : k \geq 1\}$ is a sequence of i.i.d. random vectors. Standard arguments (see CRANE and IGLEHART (1975)) show that if $E\{Y_1^2(f) + \tau_1^2\} < \infty$, then

$$\sqrt{t}(r_t - r) \Rightarrow \sigma N(0,1) \tag{1.2}$$

as $t \to \infty$, where $\sigma^2 = \sigma^2\{Z_1\}/E\{\tau_1\}$. To use (1.2) for confidence intervals, the regenerative cycle structure of X is exploited to obtain a strongly consistent estimator v_t for σ^2.

These confidence intervals, while asymptotically correct, often have poor small-sample behavior. For example, such confidence intervals often tend to significantly undercover the parameter r. Several recent studies have examined this problem. GLYNN (1982), in considering regenerative confidence intervals on the time scale of regenerative cycles, obtained asymptotic expansions for the coverage error which indicated that skewness/kurtosis effects play a significant role in determining quality of the confidence interval. To be more precise, the error, to a first approximation, is determined by the magnitude of quantities of the form $E\{Z_1^m \tau_1^n\}$ for $m + n \leq 4$. GLYNN and IGLEHART (1984) obtain expressions for the asymptotic covariance between r_t and v_t, and the variance of v_t; these expressions also involve mixed moments of the form $E\{Z_1^m \tau_1^n\}$. Consequently, in studying small-sample behavior of regenerative confidence intervals, it is of some interest to be able to calculate the exact values of the mixed moments for some "test case" stochastic models. HORDIJK, IGLEHART, and SCHASSBERGER (1976) showed how to do this for $m + n \leq 2$, when X is a discrete or continuous time Markov chain with countably many states. In this note, we show how to calculate such quantities when X is a semi-Markov process with countably many states; discrete and continuous time Markov chain results follow as special cases.

2. Statement of the Recursive Moment Formulas

Let $X = \{X(t) : t \geq 0\}$ be an irreducible non-explosive regenerative semi-Markov process on countable state space E. Thus, $X(t)$ may be represented as

$$X(t) = \sum_{k=0}^{\infty} R_k \, I(S_k \leq t < S_{k+1}) ,$$

where:

(i) $R = \{R_n : n \geq 0\}$ is a discrete-time Markov chain on E with transition matrix $P = (p_{xy} : x,y \in E)$

(ii) $S = \{S_n : n \geq 0\}$ is an increasing sequence of jump times with $S_0 = 0$ and differences $\alpha_n = S_{n+1} - S_n$ which are conditionally independent r.v.'s given R.

The conditional distribution of α_n is given by $F(R_n, R_{n+1}, dt) = P\{\alpha_n \in dt | R\}$, where $F(x,y,0) = 0$ for all $x,y \in E$. Note that $S_n \to \infty$ a.s., since X is non-explosive by assumption. Fix $z \in E$ as the regenerative state; let $T(z) = \inf\{t > 0 : X(t-) \neq z, X(t) = z\}$ and set

$$Y(u) = \int_0^{T(z)} u(X(t))dt$$

where $u : E \to \mathbb{R}$ is an arbitrary function. We wish to study mixed moments of the form

$$a_{ij}(x) = E_x\{Y(g)^i \; Y(h)^j\}$$

for $x \in E$, $0 \leq i \leq m$, and $0 \leq j \leq n$, when g and h are fixed functions, and m and n are non-negative integers. Throughout the paper we shall use $P_x\{\cdot\}$ and $E_x\{\cdot\}$ to denote conditional probabilities and expectations, given $X(0) = R_0 = x$. Note that by choosing $g(\cdot) = f(\cdot) - r$ and $h(\cdot) = 1$, $a_{ij}(z)$ yields $E_z\{Z_1^i \; \tau_1^j\}$.

To state our result, let $a \vee b$ denote $\max(a,b)$ and set $b_{mn}(x) = E_x\{(Y(|g|) \vee 1)^m \; (Y(|h|) \vee 1)^n\}$. Let G_n and β_n be the matrix and function, respectively, defined by

$$G_n(x,y) = \begin{cases} p_{xy} \; \mu_n(x,y) \; ; & y \neq z \\ 0 \; ; & y = z \end{cases}$$

$$\beta_n(x) = \sum_{y \in E} \mu_n(x,y) \; p_{xy}$$

where $\mu_n(x,y) = \int_0^\infty t^n \, F(x,y,dt)$. Also, we shall identify real-valued functions $u(\cdot)$ on E with column vectors $\mathbf{u}$, and shall use the notation $\mathbf{u} \circ \mathbf{v}$ to denote the vector with x^{th} component $(\mathbf{u} \circ \mathbf{v})(x) = u(x)v(x)$. Set $\mathbf{u}^0(\cdot) = 1$ and $\mathbf{u}^{n+1} = (\mathbf{u} \circ \mathbf{u}^n)$ for $n \geq 0$.

Let $\mathcal{C}$ denote the class of all $m \times n$ matrix-valued functions, $\mathbf{C}$, on E. Then set

$$\mathcal{C}_{mn} = \{\mathbf{C} \in \mathcal{C} \;:\; \sum_{k=0}^{i} \sum_{\ell=0}^{j} |g(x)|^{i-k} \cdot |h(x)|^{j-\ell} \cdot \sum_{y \in E} G_{i+j-k-\ell}(x,y)|c_{k\ell}(y)| < \infty$$

$$\text{and } (G_0^k c_{ij})(x) \to 0 \text{ as } k \to \infty,$$

$$\text{for all } x \in E,\ 0 \le i \le m, \text{ and } 0 \le j \le n\} .$$

(2.1) Theorem. If $b_{mn}(z) < \infty$, then $\mathbf{A} = \{a_{ij} : 0 \le i \le m,\ 0 \le j \le n\}$ is the unique solution in $\mathcal{C}_{mn}$ to the system:

$$(2.2)\quad c_{ij} = g^i \circ h^j \circ \beta_{i+j} + \sum_{(k,\ell)\in B_{ij}} \binom{i}{k}\binom{j}{\ell}(g^{i-k} \circ h^{j-\ell} \circ G^{i+j-k-\ell} c_{k\ell}),$$

where $0 \le i \le m$, $0 \le j \le n$, and $B_{ij} = \{(k,\ell) : 0 \le k \le i,\ 0 \le \ell \le j,\ k+\ell > 0\}$.

Set $\tau = \inf\{n \ge 1 : R_n = z\}$ and observe that

$$G_0^k(x,y) = P_x\{R_k = y,\ \tau > k\} .$$

Hence, $G_0^k \to 0$ as $k \to \infty$. It follows that if E has a finite number of elements, $\mathcal{C}_{mn} \equiv \mathcal{C}$, so that the a_{ij}'s are unique in the class of all possible solutions to (2.2). Also, in the presence of a finite state space, it is well known that $(I - G_0)^{-1}$ exists, so that (2.2) may be re-written as

$$(2.3) \qquad e_{ij} = (I - G_0)^{-1}\{g^i \circ h^j \circ \beta_{i+j} + \sum_{(k,\ell)\in A_{ij}} \binom{i}{k}\binom{j}{\ell}(g^{i-k} \circ h^{j-\ell} \circ (G_{i+j-k-\ell}e_{k\ell}))\} ,$$

where $0 \leq i \leq m$, $0 \leq j \leq n$, $i + j > 0$, and $A_{ij} = \{(k,\ell) : 0 \leq k \leq i, 0 \leq \ell \leq j, 0 < k+\ell < i+j\}$. Also observe that $e_{00} \equiv 1$. Note that the system of equations (2.3) is recursive in $i+j$, in the sense that the e_{ij}'s may be solved in terms of the $e_{k\ell}$'s, where $k+\ell < i+j$. By successively solving for the $e_{k\ell}$'s with fixed $k+\ell$ on each iteration, one eventually obtains e_{mn}.

Formula (2.3) can be further simplified when X has special structure. Note that if X is a continuous time Markov chain, then

$$F(x,y,dt) = \lambda(x) \exp(-\lambda(x)t)dt$$

for $t > 0$, so that

$$\mu_n(x,y) = n!/\lambda(x)^n \equiv \eta_n(x) .$$

We find that (2.3) can be re-written as

$$e_{ij} = (I - G_0)^{-1}\{g^i \circ h^j \circ \eta_{i+j} + \sum_{(k,\ell)\in A_{ij}} \binom{i}{k}\binom{j}{\ell}(g^{i-k} \circ h^{j-\ell} \circ \eta_{i+j-k-\ell} \circ G_0 e_{k\ell}))\} ,$$

where $0 \leq i \leq m$, $0 \leq j \leq n$, and $i+j > 0$.

For discrete time Markov chains, $\beta_n \equiv 1$ and $G_n = G_0$, so (2.3) takes the form

$$c_{ij} = (I - G_0)^{-1}\{g^i \circ h^j$$

$$+ \sum_{(k,\ell)\in A_{ij}} \binom{i}{k}\binom{j}{\ell}(g^{i-k} \circ h^{k-\ell} \circ (G_0\, c_{k\ell}))\}\ ,$$

where $0 \leq i \leq m$, $0 \leq j \leq m$, and $i+j > 0$.

Relation (2.4) expresses c_{ij} in terms of $G_0\, c_{k\ell}$, where $(k,\ell) \in A_{ij}$. Equation (2.4) can be re-written, when $g = h$, so that the c_{ij}'s are written directly in terms of the $c_{k\ell}$'s. If $g = h$, we write c_{ij} as c_{i+j}, and observe that (2.4) takes the form

$$(2.5) \qquad c_i = (I - G_0)^{-1}\{g^i + \sum_{k=1}^{i-1} \binom{i}{k}(g^{i-k} \circ G_0\, c_k)\}, \qquad 1 \leq i \leq n\ .$$

Recall also that from (2.2), $c_0 \equiv 1$. We claim that the system (2.5) can be re-written as

$$(2.6) \qquad c_i = (I - G_0)^{-1}\{\sum_{k=1}^{i} (-1)^{k+1}\binom{i}{k}\, g^k \circ c_{i-k}\}, \qquad 1 \leq i \leq n\ .$$

The proof is by induction. For $n = 1$, the result is obvious, so suppose (2.5) and (2.6) are equivalent systems for $n = m$. To check the $(m+1)$'st equation in the $(m+1)$'st system, observe that a solution of (2.5) satisfies

$$(2.7) \qquad c_{m+1} = (I - G_0)^{-1}\{g^{m+1} + \sum_{i=1}^{m} \binom{m+1}{i}(g^{m+1-i} \circ G_0\, c_i)\}\ .$$

By the inductive hypothesis, (2.6) shows that

$$G_0\, c_i = -\sum_{k=0}^{i} (-1)^{k+1}\binom{i}{k} g^k \circ c_{i-k} \tag{2.8}$$

for $i \leq m$. Substituting (2.8) into (2.7), we get that $(I - G_0)c_{m+1}$ equals

$$g^{m+1} + \sum_{i=1}^{m}\sum_{k=0}^{i} \binom{m+1}{i}\binom{i}{k}(-1)^{i-k}\, g^{m+1-k} \circ c_k$$

$$= \sum_{k=0}^{m} \binom{m+1}{k}(g^{m+1-k} \circ c_k) \sum_{j=1}^{m+1-k} \binom{m+1-k}{j}(-1)^{m+1-k-j}$$

$$= \sum_{k=0}^{m} \binom{m+1}{k}(g^{m+1-k} \circ c_k)(-1)^{m+2-k}$$

$$= \sum_{\ell=1}^{m+1} \binom{m+1}{\ell}(g^{\ell} \circ c_{m+1-\ell})(-1)^{\ell+1} ,$$

which is equivalent to the $(m+1)$'st relation of (2.6) (the binomial identity was used for the third equality). The steps being reversible, this proves the claimed result. We remark that (2.6) yields the equations of [4] for $i = 1, 2$.

3. Proof of the Theorem

We proceed via a series of lemmas.

(3.1) Lemma. If $b_{mn}(z) < \infty$, then the r.v.'s $Y(g)^i\, Y(h)^j$ are integrable under the probability distribution P_x for $0 \leq i \leq m$, $0 \leq j \leq n$, $x \in E$.

Proof. Note that

$$(Y(|g|) \vee 1)^{i} \, (Y(|h| \vee 1)^{j} \leq (Y(|g| \vee 1)^{m} \, (Y(|h| \vee 1)^{n}$$

so that $b_{ij}(z) < \infty$ for $0 \leq i \leq m$, $0 \leq j \leq n$. Since X is irreducible, it follows that $P_z\{T(y) < T(z)\} > 0$ for all $y \in E$, for otherwise the regenerative property guarantees that $P_z\{T(y) = \infty\} = 1$, which violates our irreducibility assumption.

Then, by the strong Markov property applied at time $T(y)$,

$$E_z\{(\int_{T(y)}^{T(z)} |g(X(s)|ds \vee 1)^{i} \, (\int_{T(y)}^{T(z)} |h(X(s))|ds \vee 1)^{j}; \, T(y) < T(z)\}$$

$$= b_{ij}(y) \, P\{T(y) < T(z) | X(0) = z\} \leq b_{ij}(z)$$

so that $b_{ij}(y) < \infty$ for all $y \in E$, which proves the result. □

(3.2) Proposition. If $b_{mn}(z) < \infty$, then $a_{ij}(x)$ exists and is finite for $0 \leq i \leq m$, $0 \leq j \leq n$, $x \in E$. Furthermore, A solves (2.2).

Proof. The first part follows immediately from Lemma 3.1. For the second part, the integrability of $Y(g)^{i} \, Y(h)^{j}$ ensures that the following manipulations of conditional expectations are valid:

$$
\begin{aligned}
a_{ij}(x) &= E_x\{Y(g)^i\ Y(h)^j\} \\
&= E_x\{Y(g)^i\ Y(h)^j;\ \tau = 1\} + E_x\{Y(g)^i\ Y(h)^j;\ \tau > 1\} \\
&= g^i(x)\ h^j(x)\ \mu_{i+j}(x,z)\ p_{xz} \\
&\quad + E_x\Big\{(g(R_0)\ \alpha_0 + \int_{S_1}^{T(z)} g(X(t))dt)^i\ (h(R_0)\ \alpha_0 \\
&\quad + \int_{S_1}^{T(z)} h(X(t))dt)^j;\ \tau > 1\Big\} \\
&= g^i(x)\ h^j(x)\ \mu_{i+j}(x,z)\ p_{xz} \\
&\quad + \sum_{k=0}^{i} \sum_{\ell=0}^{j} \binom{i}{k}\binom{j}{\ell}\ E_x\Big\{g(R_0)^{i-k}\ h(R_0)^{j-\ell}\ \alpha_0^{i+j-k-\ell}(\int_{S_1}^{T(z)} g(X(t))dt)^k \\
&\qquad\quad \cdot (\int_{S_1}^{T} h(X(t))dt)^{\ell};\ \tau > 1\Big\} \\
&= g^i(x)\ h^j(x)\ \mu_{i+j}(x,z)\ p_{xz} \\
&\quad + \sum_{k=0}^{i} \sum_{\ell=0}^{j} \binom{i}{k}\binom{j}{\ell}\ g^{i-k}(x)\ h(x)^{j-\ell} \sum_{y \neq z} p_{xy}\ \mu_{i+j-k-\ell}(x,y)\ a_{k\ell}(y) \\
&= (g^i \circ h^j \circ \beta_{i+j})(x) + \sum_{(k,\ell)\in B_{ij}} \binom{i}{k}\binom{j}{\ell}(g^{i-k} \circ h^{j-\ell} \\
&\qquad\quad \circ (G_{i+j-k-\ell}\ a_{k\ell})(x))\ ;
\end{aligned}
$$

the strong Markov property at time S_1 was used to obtain the second last equality. □

(3.3) Lemma. If $b_{mn}(z) < \infty$, then $A \in \mathcal{C}_{mn}$.

Proof. *For the absolute summability observe that* $d_{ij}(x) = E_x\{Y(|g|)^i\ Y(|h|)^j\}$ satisfies

$$\mathbf{d}_{ij} = |\mathbf{g}|^i \circ |\mathbf{h}|^j \circ \beta_{i+j} + \sum_{(k,\ell)\in B_{ij}} \binom{i}{k}\binom{j}{\ell}(|\mathbf{g}|^{i-k} \circ |\mathbf{h}|^{j-\ell} \circ (\mathbf{G}_{i+j-k-\ell}\, \mathbf{d}_{k\ell}))\ .$$

But $d_{ij}(x) \le b_{ij}(x) < \infty$, so $|\mathbf{g}|^{i-k} \circ |\mathbf{h}|^{j-\ell} \circ (\mathbf{G}_{i+j-k-\ell}\, \mathbf{d}_{k\ell})$ is finite, proving the first part, since $|\mathbf{a}_{ij}| \le \mathbf{d}_{ij}$. For the second,

$$(G_0^k\, \mathbf{a}_{ij})(x) = E_x\{(\int_{S_k}^{T(z)} Y(g))^i\ (\int_{S_k}^{T(z)} Y(h))^j;\ \tau > k\}\ ,$$

which tends to zero by integrability of $Y(|g|)^i\ Y(|h|)^j$.

(3.4) Lemma. *If* $b_{mn}(z) < \infty$, *then* $\mathbf{A}$ *is the unique solution to* (2.2) *in* $\mathcal{C}_{mn}$.

Proof. Suppose that $\{\mathbf{a}_{rs} : 0 \le r \le k,\ 0 \le s \le \ell\}$ is unique in $\mathcal{C}_{k\ell}$ for $k+\ell < i+j$, where $0 \le i \le m$, $0 \le j \le n$. We shall prove uniqueness in $\mathcal{C}_{ij}$; "bootstrapping" this result yields the lemma. Since the a_{rs}'s are unique in $\mathcal{C}_{k\ell}$, any solution to the (i,j) equation must satisfy

$$\mathbf{c}_{ij} = \mathbf{g}^i \circ \mathbf{h}^j \circ \beta_{i+j} + \sum_{(k,\ell)\in A_{ij}} \binom{i}{k}\binom{j}{\ell}(\mathbf{g}^{i-k} \circ \mathbf{h}^{j-\ell} \circ \mathbf{G}_{i+j-k-\ell}\, \mathbf{a}_{k\ell}) + \mathbf{G}_0\, \mathbf{c}_{ij}\ . \tag{3.5}$$

By the absolute summability, any two solutions c_{ij}, c'_{ij} of (3.5) must satisfy $(c_{ij} - c'_{ij}) = G_0(c_{ij} - c'_{ij})$. Since $G_0^k(c_{ij} - c'_{ij}) \to 0$ as $k \to \infty$, it follows that $c_{ij} = c'_{ij}$, proving uniqueness in $\mathcal{C}_{ij}$. □

These above results prove all the assertions of the theorem.

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ABSTRACT: Let $X = \{X(t) : t \geq 0\}$ be a regenerative semi-Markov process with countable state space E and f a real-valued function on E. Denote by $Y_1(f)$ the area under the function $f(X(\cdot))$ in the first regenerative cycle. This paper gives a recursive method for computing moments of the form $E\{Y_1^m(f)Y_1^n(g)\}$ for arbitrary f and g, and $m, n \geq 1$. These moments are needed to improve the accuracy of confidence intervals for steady state parameters obtained when using the method of regenerative simulation.

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