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APPROXIMATION IN OPTIMAL CONTROL AND IDENTIFICATION OF  
LARGE SPACE STRUCTURES(U) CALIFORNIA UNIV LOS ANGELES  
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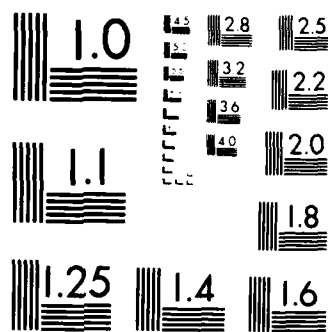
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APPROXIMATION IN OPTIMAL CONTROL AND IDENTIFICATION OF  
LARGE SPACE STRUCTURES

FINAL SCIENTIFIC REPORT

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J.S. Gibson  
Mechanical, Aerospace and Nuclear Engineering  
University of California  
Los Angeles CA 90024

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## REPORT DOCUMENTATION PAGE

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| 19. ABSTRACT (Continue on reverse if necessary and identify by block number)<br>This project dealt with the application of distributed system theory to control and identification of large flexible space structures. The main analytical tools were control theory for infinite dimensional systems and approximation theory for distributed systems. Both theoretical results and practical numerical approximation schemes were developed. The research dealt with both continuous-time and discrete-time control and identification. In each case, an ideal infinite dimensional compensator was used to guide the design of implementable finite dimensional compensators. Most of the research dealt with optimal linear-quadratic control theory, but significant preliminary results were obtained on infinite dimensional autoregressive-moving-average models of distributed systems. These models will be used in adaptive control and identification of flexible space structures. |       |                                                                  |                                                                                                                    |                                                                                                                         |                          |
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AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)

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This technical report has been reviewed and is approved for public release IAW AFR 190-12.

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MATTHEW J. KERPER

Chief, Technical Information Division

## 1. Research Objectives

1. Establish mathematical properties of optimal infinite dimensional compensators based on linear-quadratic-gaussian(LQG) theory for distributed systems, and characterize the resulting closed-loop systems. Such properties include the form of control and estimator gains and the stability of the closed-loop systems. Extend previous results on infinite dimensional Riccati operator equations, which determine the gains for the infinite dimensional compensators. This work includes both continuous-time and discrete-time control of flexible structures.

2. Develop approximation theory for numerical solution of the infinite dimensional Riccati equations and construction of the corresponding compensators. This research concerns convergence analysis to determine necessary and sufficient conditions for convergence of approximate solutions to infinite dimensional Riccati equations and the study of the types of convergence produced by different approximation schemes.

3. Develop widely applicable formulas for computing approximations to functional control and estimator gains from finite dimensional approximations to the solutions to

infinite dimensional Riccati equations.

4. Investigate infinite dimensional autoregressive-moving-average (ARMA) models of distributed systems with applications to large flexible structures. This discrete-time representation of the distributed model of a flexible structure will serve as the basis for adaptive parameter identification and control of large space structures.

Several difficult existence and uniqueness questions arise in the infinite dimensional case that do not arise in finite dimensions.

5. Develop approximation theory for computing infinite dimensional ARMA representations for flexible structures using finite element approximations of the structures.

## II. Status of Research

*Objective 1:* For the continuous-time case with bounded input operator, the results are practically complete. Their application was discussed in [5] and will be illustrated in [3]. Some preliminary discrete-time results are given in [1,2]. A comprehensive treatment will be presented in [4].

*Objectives 2 and 3:* Preliminary results were given in [5,6], and complete results will be contained in [3,4].

*Objective 4:* Results are reported in [1,2]. A journal paper extending these results is planned for SIAM J. Contr. Opt.

*Objective 5:* While initial results based on modal approximation were used in [1,2], significant work remains to develop theory and methods for arbitrary finite element approximations.

Research in discrete-time control and parameter estimation of flexible space structures is continuing under AFOSR Grant 840309, "Optimal Control and Identification of Large Space Structures: Distributed System Theory and Numerical Approximation." Also, problems with unbounded input operators are being investigated, especially in the discrete-time case.



## References

- 1.\* J. S. Gibson and F. Jabbari, "Discrete-time optimal control of flexible structures," 22nd IEEE Conference on Decision and Control, San Antonio, TX, December 1983.
- 2.\* J. S. Gibson and F. Jabbari, "An ARMA model for a class of distributed systems," 22nd IEEE Conference on Decision and Control, Las Vegas, NE, December 1984.
3. J. S. Gibson and M. Navid, "Approximation theory for linear-quadratic optimal control of hyperbolic systems," to be submitted to SIAM J. Contr. Opt.
4. I. G. Rosen and J. S. Gibson, "Approximation theory for the discrete-time infinite dimensional regulator problem," to be submitted to SIAM J. Contr. Opt.
5. J. S. Gibson, "Control of flexible space structures," Second International Conference on Control Theory for Distributed Systems, Vorau, Austria, July 1984.
6. J. S. Gibson, D. L. Mingori, A. Adamian and F. Jabbari, "Approximation of infinite dimensional compensators for flexible structures," Workshop on Identification and Control of Space Structures, San Diego, CA, June 1984.

\*Included in Appendix.

## APPENDIX

J. S. Gibson and F. Jabbari, "Discrete-time optimal control of flexible structures," 22nd IEEE Conference on Decision and Control, San Antonio, TX, December 1983.

J. S. Gibson and F. Jabbari, "An ARMA model for a class of distributed systems," 22nd IEEE Conference on Decision and Control, Las Vegas, NE, December 1984.

# DISCRETE-TIME OPTIMAL CONTROL OF FLEXIBLE STRUCTURES

J. S. Gibson and F. Jabbari

Mechanics and Structures Department  
University of California, Los Angeles  
Los Angeles, California 90024

## Summary

Approximation of an ideal infinite dimensional compensator for an infinite dimensional system is discussed. Two approaches are considered; one is to design a compensator for a finite dimensional model of the system, and the other approach, which is stressed here, is to truncate an infinite dimensional ARMA representation of the ideal compensator. Performance of the closed-loop system is discussed, and an example from control of flexible structures is presented.

## Introduction

This paper treats compensator design for a discrete-time infinite-dimensional linear control system. Usually, the compensator which solves the optimal LQG problem for such a system is infinite dimensional; in other words, it has an irrational transfer function. We will refer to an infinite dimensional compensator as an ideal compensator because, in applications, only finite dimensional compensators can be built, preferably of low order. The design philosophy of this paper is to construct finite dimensional compensators which approximate the ideal infinite dimensional compensators.

For the discrete-time problem discussed here, we present two approaches to approximating an infinite dimensional compensator. The first approach is to approximate the infinite dimensional control system by a finite dimensional model and then design a corresponding finite dimensional compensator. We call the resulting implementable compensator an indirect approximation of the ideal compensator. This design procedure is common practice among engineers who use finite dimensional control methods for flexible structures and often works quite well - although the finite dimensional compensator is rarely thought of as approximating some ideal compensator. Our only suggestion is to try to ensure that the finite dimensional compensator will perform well by selecting the approximation scheme so that, as the dimension of the model increases, the finite dimensional compensator converges, in an appropriate sense, to the ideal infinite dimensional compensator.

The second approach to designing a finite

dimensional compensator produces what we call a direct approximation of the ideal compensator. In this approach, an infinite dimensional ARMA representation of the ideal compensator is constructed and then truncated to obtain the ARMA representation of the finite dimensional compensator. For computational purposes, the ARMA representation of the infinite dimensional compensator is obtained as the limit of the ARMA representations of the finite dimensional compensators obtained by the indirect approach. But these compensators are discarded once the ARMA coefficients for the infinite dimensional compensator are obtained, and the ideal compensator itself is truncated to obtain the direct approximation.

In this paper, we are most interested in the direct approximation of the ideal compensator. We will discuss the performance of the closed-loop system with the resulting finite dimensional compensator, showing that stability and near-optimality can be preserved. Finally, we give an example in which a flexible beam is controlled.

## The Infinite Dimensional Compensator

Let  $E$  be an infinite dimensional Hilbert space,  $T$  a bounded linear operator on  $E$ ,  $B$  a linear operator from  $R^m$  to  $E$ ,  $C$  a bounded linear operator from  $E$  to  $R^p$ , and  $D$  a linear operator from  $R^m$  to  $R^p$ . We consider the discrete-time control system

$$z(t+1) = T z(t) + B u(t), \quad t=0,1,2,\dots, \quad (1)$$

$$y(t) = C z(t) + D u(t), \quad (2)$$

where the state vector  $z(t) \in E$ , the control vector  $u(t) \in R^m$  and the measurement vector  $y(t) \in R^p$ .

Our ideal compensator for this system is based on the infinite dimensional discrete-time state estimator

$$\hat{z}(t+1) = T \hat{z}(t) + B u(t) + G(y(t) - C \hat{z}(t) - D u(t)), \quad (3)$$

where the estimator gain  $G$  is a linear operator from  $R^p$  to  $E$ . The feedback control is

$$u(t) = -K \hat{z}(t) \quad (4)$$

where the control gain  $K$  is a bounded linear operator from  $E$  to  $R^m$ . Our first representation of the ideal compensator is given by (3)

and (4). We will call this compensator optimal if G and K are chosen according to an infinite-dimensional discrete-time LQG problem for (1) and (2) (with added noise). This is not necessary for the discussion in this paper, although we have chosen G and K this way in the Example. We call the compensator in (3) - (4) ideal because it involves infinite-dimensional dynamics.

We will assume

**Hypothesis 1.** The spectral radius of T-GC is less than 1.  $\square$

This means that there exist constants M and r,  $r < 1$ , such that

$$\| (T-GC)^n \| \leq M r^n, \quad n = 1, 2, \dots \quad (5)$$

In applications to flexible structures, Hypothesis 1 implies that all but a finite number of modes are uniformly exponentially stable due to inherent system damping, and that any unstable modes are observable. Also, from (5) it follows that  $\hat{z}(t)$  approaches  $z(t)$  exponentially; i.e.,

$$\| z(t) - \hat{z}(t) \| \leq M r^t \| z(0) - \hat{z}(0) \|, \quad t = 1, 2, \dots \quad (6)$$

To derive the infinite dimensional ARMA representation of (3) - (4), we define

$$S = T - GC \quad \text{and} \quad \hat{B} = B - GD, \quad (7)$$

and write (3) as

$$\hat{z}(t+1) = S \hat{z}(t) + \hat{B} u(t) + G y(t). \quad (8)$$

Repeated application of (8) yields

$$\hat{z}(t) = \sum_{j=1}^n S^{j-1} (\hat{B} u(t-j) + G y(t-j)) + S^n z(t-n), \quad (9)$$

$$n = 1, 2, \dots$$

If we set  $\hat{z}(t) = 0$ ,  $u(t) = 0$ , and  $y(t) = 0$  for  $t \leq 0$ , then from (9) we have

$$\hat{z}(t) = \sum_{j=1}^{\infty} S^{j-1} (\hat{B} u(t-j) + G y(t-j)), \quad 0 < t < \infty. \quad (10)$$

Furthermore, if the sequences  $r^j \|\hat{z}(-j)\|$ ,  $r^j \|u(-j)\|$ , and  $r^j \|y(-j)\|$ , ( $j = 0, 1, 2, \dots$ ) are summable, then we still can define  $\hat{z}(t)$  by (10), and (6) will hold.

With (10), the control law in (4) becomes

$$u(t) = -\sum_{j=1}^{\infty} \alpha_j u(t-j) - \sum_{j=1}^{\infty} \beta_j y(t-j), \quad t = 0, 1, 2, \dots \quad (11)$$

where each  $\alpha_j$  is an  $m \times m$  matrix and each  $\beta_j$  is an  $m \times p$  matrix. The control law in (11) is the ARMA representation of the ideal compensator in (3) - (4). The matrices  $\alpha_j$  and  $\beta_j$  are given by

$$\alpha_j = K S^{j-1} \hat{B} \quad \text{and} \quad \beta_j = K S^{j-1} G, \quad (12)$$

and in view of (5) and (7), we have

$$\|\alpha_j\| \leq \|K\| \cdot \|B\| \cdot M r^{j-1} \quad \text{and} \quad \|\beta_j\| \leq \|K\| \cdot \|G\| \cdot M r^{j-1}. \quad (13)$$

## The Closed-loop System with the ARMA Compensator

Henceforth, we will assume that the sequences  $u(-j)$  and  $y(-j)$ ,  $j = 0, 1, 2, \dots$ , are in  $\ell_2(\mathbb{R}^m)$  and  $\ell_2(\mathbb{R}^p)$ , respectively. In other words, the control and measurement histories are square-summable. Thus, if  $\hat{z}(t)$  is given by (10), (6) holds. For  $t \geq 0$ , we define

$$U(t) = \{u(t), u(t-1), u(t-2), \dots\} \in \ell_2(\mathbb{R}^m) \quad (14)$$

and

$$Y(t) = \{y(t), y(t-1), y(t-2), \dots\} \in \ell_2(\mathbb{R}^p) \quad (15)$$

It is now an easy exercise to show that, when the compensator in (11) is used, there exists a constant  $M_1$  such that

$$\| -K z(t) - u(t) \| \leq M_1 r^t (\|z(0)\| + \|U(0)\| + \|Y(0)\|), \quad t = 1, 2, \dots \quad (16)$$

and

$$\| (T-BK)z(0) - z(0) \| \leq M_1 r^t (\|z(0)\| + \|U(0)\| + \|Y(0)\|), \quad t = 1, 2, \dots \quad (17)$$

For direct approximation of the ideal compensator, it will be useful to define the space  $Z = E \times \ell_2(\mathbb{R}^m) \times \ell_2(\mathbb{R}^p)$  and to define the unique bounded linear operator  $T$  on  $Z$  such that, for  $t = 0, 1, 2, \dots$ ,

$$(z(t+1), U(t+1), Y(t+1)) = T(z(t), U(t), Y(t)). \quad (18)$$

It would be easy to write out  $T$  explicitly, but this should not be necessary. Let us note only that if, in addition to Hypothesis 1,  $T-BK$  is uniformly exponentially stable, then so is  $T$ ; i.e., the spectral radius of  $T$  is less than 1. The purpose of (18) is to allow us to view the closed-loop system as an autonomous discrete-time process evolving on the state space  $Z$ . When the ARMA compensator is used, the closed-loop system at time  $t$  is the vector  $(z(t), U(t), Y(t)) \in Z$ ; i.e., the vector consisting of the current state of the control system in (1) and the current histories of the control and the measurement.

## Indirect Approximation of the Ideal Compensator

The idea here is to approximate the infinite dimensional control system in (1) and (2) by a sequence of control systems, each of finite dimension  $N$ . Thus we will have sequences of approximating operators  $T_N$ ,  $B_N$ ,  $C_N$ , and  $D_N$ . For each  $N$ , an  $N$ -dimensional compensator of the form (3) - (4) is designed with estimator and control gains  $G_N$  and  $K_N$ , respectively. This  $N^{\text{th}}$  compensator is then

$$\hat{z}_N(t+1) = T_N \hat{z}_N(t) + B_N u(t) + G_N (y(t) - C_N \hat{z}_N(t) - D_N u(t)), \quad (19)$$

$$u(t) = -K_N \hat{z}_N(t). \quad (20)$$

If the approximation is done correctly (see [1,2] for related continuous-time problems

and [3] for the discrete-time case), then the compensator in (19) - (20) will approximate the ideal compensator in (3) - (4) as  $N$  becomes large. Although stronger convergence often can be obtained, here we will need only

Hypothesis 2. Let

$$S_N = T_N - G_N C_N \quad \text{and} \quad \hat{B}_N = B_N - G_N D_N. \quad (21)$$

As  $N \rightarrow \infty$ ,

$$S_N \rightarrow S \text{ strongly}, \quad (22)$$

$$\hat{B}_N \rightarrow \hat{B} \text{ strongly}, \quad (23)$$

$$G_N \rightarrow G \text{ strongly}, \quad (24)$$

and

$$K_N \rightarrow K \text{ strongly}. \quad \square \quad (25)$$

The statement that the finite dimensional compensator in (19) - (20) converges to the infinite dimensional compensator in (3) - (4) means that  $S_N$ ,  $\hat{B}_N$ ,  $G_N$ , and  $K_N$  converge in some sense to  $S$ ,  $\hat{B}$ ,  $G$ , and  $K$ , respectively. Also, under additional but often realistic hypotheses, it can be shown that the closed-loop system consisting of (1), (2), (19), and (20) converges to the ideal closed-loop system consisting of (1), (2), (3), and (4) in such a way as to preserve stability and near-optimality for  $N$  sufficiently large. However, in this paper we will pursue such convergence only for the subsequent direct compensator approximation.

#### Direct Approximation of the Ideal Compensator

Truncation. The idea now is to compute, as discussed later, the matrices  $\alpha_j$  and  $\beta_j$  for the ARMA compensator in (11), and then to approximate this compensator directly. The obvious way to approximate (11) is simply to truncate all but a finite number of terms in each of the series in (11). That is, replace (11) by the control law

$$u(t) = -\sum_{j=1}^{N_1} \alpha_j u(t-j) - \sum_{j=1}^{N_2} \beta_j y(t-j), \quad t = 0, 1, 2, \dots \quad (26)$$

For any positive integers  $N_1$  and  $N_2$ , (26) is the finite dimensional ARMA approximation of the ideal compensator.

#### Convergence and Performance of the Closed-loop System.

When the ARMA compensator in (26) is used to control the system in (1), we again will call the triple  $(z(t), U(t), Y(t))_{N_1, N_2}$  the closed-loop system at time  $t$ . Note that  $U(t)$  and  $Y(t)$  are still the infinite histories in (14) and (15), so that  $(z(t), U(t), Y(t))$  is a vector in  $Z$  whether (11) or (26) is used.

Also, we can define a bounded linear operator  $T_{N_1, N_2}$  on  $Z$  such that the closed-loop system resulting from (26) can be written

$$(z(t+1), U(t+1), Y(t+1))_{N_1, N_2} = T_{N_1, N_2} (z(t), U(t), Y(t))_{N_1, N_2} \quad (27)$$

Now let us write (11) and (26) as

$$u(t+1) = K_1 U(t) + K_2 Y(t) \quad (28)$$

and

$$u(t+1) = K_{1N_1} U(t) + K_{2N_2} Y(t), \quad (29)$$

respectively, where  $K_1, K_{1N_1} \in L(L_2(R^m), R^m)$  and  $K_2, K_{2N_2} \in L(L_2(R^p), R^m)$ . From (13), it follows that

$$\|K_1 - K_{1N_1}\| \rightarrow 0 \quad \text{as} \quad N_1 \rightarrow \infty \quad (30)$$

and

$$\|K_2 - K_{2N_2}\| \rightarrow 0 \quad \text{as} \quad N_2 \rightarrow \infty. \quad (31)$$

Finally, it is straightforward to show

$$\|T - T_{N_1, N_2}\| \leq \|K_1 - K_{1N_1}\| + \|K_2 - K_{2N_2}\|, \quad (32)$$

so that

$$\|T - T_{N_1, N_2}\| \rightarrow 0 \quad \text{as} \quad N_1, N_2 \rightarrow \infty. \quad (33)$$

From (33), we see that the response of the closed-loop system obtained by using the finite dimensional compensator in (26) can be made arbitrarily close to the response of the ideal closed-loop system obtained by using the ideal compensator in (11). If the ideal closed-loop system is uniformly exponentially stable, then so is the closed-loop system with the finite dimensional ARMA compensator for  $N_1$  and  $N_2$  sufficiently large. Also, if the ideal closed-loop system is optimal in some sense, then the closed-loop system obtained with (26) can be made arbitrarily close to optimal. Of course, we hope that in applications, stability and near optimality can be obtained with reasonably small  $N_1$  and  $N_2$ .

#### Computation of $\alpha_j$ and $\beta_j$

For the finite dimensional state estimator and control law in (19) and (20), we can retrace the development in (7) - (11) to obtain an approximation of (11) with  $\alpha_j$  and  $\beta_j$  replaced respectively by

$$\alpha_{jN} = K_N S_N^{j-1} \hat{S}_N \quad \text{and} \quad \beta_{jN} = K_N S_N^{j-1} G_N, \quad (34)$$

where  $S_N$  and  $\hat{S}_N$  are given by (21). In other words, after computing the gains  $G_N$  and  $K_N$  for (19) and (20), we replace the operators  $K$ ,  $G$ ,  $S$ , and  $\hat{S}$  in (12) by  $N$ th order approximations. Hypothesis 2 then guarantees

$$\alpha_{jN} \rightarrow \alpha_j \text{ and } \beta_{jN} \rightarrow \beta_j \text{ as } N \rightarrow \infty. \quad (35)$$

#### An Infinite Dimensional MA Compensator

In general, stability of both T-GC and T-BK does not guarantee stability of the operator

$$\hat{S} = T - GC - BK - GDK \quad (36)$$

However,  $\hat{S}$  often is uniformly exponentially stable. When this is the case, it is convenient to write (3) and (4) as

$$\hat{z}(t+1) = \hat{S} \hat{z}(t) + Gy(t), \quad (37)$$

$$u(t) = -K \hat{z}(t). \quad (38)$$

Then, retracing the steps to (11) leads to

$$u(t) = \sum_{j=1}^{\infty} \gamma_j y(t-j), \quad t = 0, 1, \dots, \quad (39)$$

where each  $\gamma_j$  is an  $m \times p$  matrix given by

$$\gamma_j = K \hat{S}^{j-1} G. \quad (40)$$

The control law in (39) is the MA representation of the ideal compensator.

Of course, we approximate (39) by

$$u(t) = -\sum_{j=1}^{N_3} \gamma_j y(t-j). \quad (41)$$

The analysis of the convergence and performance of the closed-loop system produced by (26) is easily modified to obtain the same results when the finite dimensional MA compensator in (41) is used. Also, the matrices  $\gamma_j$  are computed as the limits of

$$\gamma_{jN} = K_N \hat{S}_N^{j-1} G_N. \quad (42)$$

#### Example

We will control the transverse vibrations of a simply supported Euler-Bernoulli beam represented by the partial differential equation

$$\ddot{w}(t, \eta) + 2\zeta A_0^{1/2} \dot{w}(t, \eta) + A_0 w(t, \eta) = 0 \quad (43)$$

where  $\eta \in (0, 1)$  is measured along the beam. The basic space for  $w(t, \cdot)$  is  $L_2(0, 1)$ , and the operator  $A_0$  is defined by

$$D(A_0) = \{\phi \in H^4(0, 1) : \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0\}, \quad (44)$$

$$A_0 \phi = \phi^{(4)}. \quad (45)$$

We take the damping ratio  $\zeta = .01$ .

Our control is a moment on the left end

of the beam. Hence the boundary condition

$$\frac{\partial^2 w(t, \eta)}{\partial \eta^2} \Big|_{\eta=0} = u(t). \quad (46)$$

This control is a piecewise constant function of time  $t$ , and the length of the time step is .03.

To put the system in first order form, we write  $z(t) = (w, \dot{w})$  and take the space  $E$  to be  $H_0^2(0, 1) \times L_2(0, 1)$ , where  $H_0^2(0, 1) = \{\phi \in H^2 : \phi(0) = \phi(1) = 0\}$ . Then (43) becomes

$$\dot{z} = Az \quad (47)$$

where  $A$  (see [1]) generates a  $C_0$ -semigroup  $T(\cdot)$  on  $E$ . The operator  $T$  in  $\mathcal{O}(1)$  is  $T(.03)$ , and the operator  $B$  can be obtained from the solution to (43) for constant  $u(t)$  in (46). Note that  $B$  is bounded for the discrete-time problem, although the input operator is unbounded for the continuous-time problem.

Our measurement is the slope at the left end of the beam:

$$y(t) = \frac{\partial w(t, \eta)}{\partial \eta} \Big|_{\eta=0} = C z(t). \quad (48)$$

This operator  $C$  is a bounded linear functional on  $E = H^2 \times L_2$ .

For the estimator gain  $G$  we solved for the optimal filter for (1) and (2) when the process noise for (1) has covariance operator  $I$  and the measurement noise for (2) has covariance  $1$ . Because the resulting estimator had unsatisfactory eigenvalues (the eigenvalues of  $S$  in (7)), we added an  $\alpha$ -shift to obtain a  $G$  which made the eigenvalues of  $S$  all have magnitude less than .9.

For the control gain  $K$ , we solved the optimal regulator problem for (1) and the performance index

$$J = \sum_{k=0}^{\infty} [\|z(.03k)\|_E^2 + u^2(.03k)]. \quad (49)$$

To compute  $G$  and  $K$  we approximated the solutions to infinite dimensional discrete-time Riccati equations, as discussed in (3). Basically, we projected the infinite dimensional LQG problem on subspaces spanned by  $N$  natural modes of the beam,  $N = 1, 2, \dots$ . The ARMA gains  $\alpha_j$ ,  $\beta_j$ , and  $\gamma_j$  were computed as the limits of (34) and (42) for increasing  $N$ . We found that  $\hat{S}$  (see (36)) is exponentially stable for this example.

The following tables give the values of the ARMA and MA compensator gains.

$$a_{jN}$$

| j  | $a_{j10}$ | $a_{j12}$ | $a_{j14}$ |
|----|-----------|-----------|-----------|
| 1  | -.019     | -.020     | -.020     |
| 2  | .114      | .117      | .119      |
| 3  | .045      | .049      | .051      |
| 4  | .027      | .022      | .021      |
| 5  | .198      | .193      | .191      |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| 15 | .029      | .028      | .028      |
| 16 | .034      | .034      | .034      |
| 17 | -.024     | -.023     | -.023     |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| 35 | -.001     | -.001     | -.001     |
| 36 | .003      | .003      | .003      |
| 37 | .001      | .001      | .001      |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| .  | .         | .         | .         |

$$-y_{jN}$$

| j  | $y_{j10}$ | $y_{j12}$ | $y_{j14}$ |
|----|-----------|-----------|-----------|
| 1  | -1.387    | -1.384    | -1.383    |
| 2  | -.643     | -.638     | -.638     |
| 3  | 1.444     | 1.442     | 1.444     |
| 4  | .615      | .620      | .623      |
| 5  | -2.147    | -2.151    | -2.154    |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| 15 | .080      | -.082     | -.079     |
| 16 | -.214     | -.221     | -.223     |
| 17 | .050      | .049      | .048      |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| 35 | .007      | .008      | .008      |
| 36 | -.010     | -.008     | -.008     |
| 37 | .001      | .0001     | -.0001    |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| .  | .         | .         | .         |

$$B_{jN}$$

| j  | $B_{j10}$ | $B_{j12}$ | $B_{j14}$ |
|----|-----------|-----------|-----------|
| 1  | -1.387    | -1.384    | -1.383    |
| 2  | -.616     | -.611     | -.611     |
| 3  | 1.298     | 1.292     | 1.291     |
| 4  | .450      | .448      | .449      |
| 5  | -2.061    | -2.056    | -2.055    |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| 15 | -.047     | -.047     | -.047     |
| 16 | -.148     | -.148     | -.148     |
| 17 | -.039     | -.039     | -.038     |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| 35 | .009      | .009      | .009      |
| 36 | .002      | .002      | .002      |
| 37 | -.006     | -.006     | -.006     |
| .  | .         | .         | .         |
| .  | .         | .         | .         |
| .  | .         | .         | .         |

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# AN ARMA MODEL FOR A CLASS OF DISTRIBUTED SYSTEMS

BY

J. S. Gibson and F. Jabbari

Mechanical, Aerospace and Nuclear Engineering  
University of California, Los Angeles 90024

## Abstract

This paper presents an infinite dimensional ARMA model for infinite dimensional time-invariant linear systems in which the state transition operator is trace class. The ARMA model appears to be the natural extension of the finite dimensional ARMA model, and it is computed as the limit of a sequence of finite dimensional ARMA models. In a flexible structure example, numerical results are given for the ARMA models of both the open-loop plant and an optimal compensator.

## 1. Introduction

In [4], we introduced an infinite dimensional ARMA model for a class of infinite dimensional discrete-time control systems. The method there, based on an infinite dimensional observer for the state space representation of the system, can produce an infinite number of different ARMA models of the same system, but none of those ARMA models is a natural extension of the minimal order ARMA model for a finite dimensional system represented in state space form. For a large class of infinite dimensional systems, we present in this paper the natural extension of the minimal order finite dimensional ARMA model. This paper deals exclusively with single-input/single-output systems, but the main ideas and results can be extended to the multi-input/multi-output case.

That the infinite dimensional ARMA model of this paper is the natural extension of the finite dimensional case should be clear from the way the present ARMA model is defined in Section 2 and from certain of its properties. One of the most important properties is that the ARMA coefficients corresponding to the infinite dimensional state space representation of the system are, under appropriate convergence criteria, the limits of the ARMA coefficients corresponding to increasingly large finite dimensional approximation of the infinite dimensional system. See Section 4.

After defining the ARMA model and giving some of its important characteristics in Sections 2 and 3, we sketch the relevant approximation theory in Section 4. Because of

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of limited space, we will not prove all of the results stated here, but we will indicate the main ideas of the missing proofs. In Section 5, we apply the results to obtain the ARMA model of a flexible structure and the ARMA representation of an optimal (infinite dimensional) LQG compensator for the structure.

As the example illustrates, we are interested in both the ARMA model for the open-loop version of a distributed parameter control system and the ARMA model for an infinite dimensional compensator for such a system. The ARMA model of the open-loop system should be useful for infinite dimensional adaptive control and parameter identification theory, and as a limit that reveals the limiting properties of the ARMA representations of finite dimensional approximations to the distributed system. The ARMA representation of an infinite dimensional compensator for a distributed system gives a concrete representation of an otherwise abstract entity, and can serve as a basis for deriving implementable (finite dimensional) approximations to such a compensator.

We consider control systems whose state space representations have the form

$$z(t+1) = Tz(t) + Bu(t), \quad t = 0, 1, 2, \dots \quad (1.1)$$

$$y(t) = Cz(t), \quad (1.2)$$

where the state vector  $z(t)$  is in a Hilbert space  $E$ ,  $T$  is a bounded linear operator on  $E$ ,  $B$  is a linear operator from the real line into  $E$  (i.e.,  $B \in E$ ) and  $C$  is a bounded linear functional on  $E$ . The scalar  $u(t)$  is the control, and the scalar  $y(t)$  is the measurement. To define the ARMA model of this paper, we must assume that the operator  $T$  is trace class (see [2,5] and Section 2). Also, we assume that the system (1.1)-(1.2) is observable in the sense that there is no nonzero  $z(0)$  such that  $y(t) = 0$  for each  $t \geq 0$ .

The condition that  $T$  be trace class holds for the discrete-time state space representation of the solutions to many generalized wave equations, heat equations and time-delay equations. As discussed in Section 5, a common modal damping model ensures that  $T$  is trace class for the linear distributed model of a flexible structure.



Under the above hypotheses, the ARMA model of the system (1.1)-(1.2) has the form

$$y(t) = \sum_{i=1}^{\infty} [a_i y(t-1) + b_i u(t-1)], \quad (1.3)$$

where the ARMA coefficients  $a_i$  and  $b_i$  are scalars. The purpose of this paper is to define these coefficients so that the measurement in (1.2) does satisfy (1.3), to establish the decay rate for  $a_i$  and  $b_i$  with increasing  $i$ , to indicate how the coefficients can be computed, and to illustrate the results with a numerical example.

## II. Definitions of the ARMA Coefficients

### 2.1. Trace Class Operators

We will need some standard results on trace class operators on Hilbert spaces. (See [2, p.1088-1105] and [5, p.521-524].) First, a linear operator is trace class if it is compact and its singular values are summable. In this case, the trace norm of  $T$ , denoted by  $\|T\|_1$ , is equal to the sum of the singular values repeated according to multiplicity. From here on,  $T$  will be trace class.

Let  $\lambda_1, \lambda_2, \dots$  be the nonzero eigenvalues of  $T$ . Then

$$\text{tr } T = \sum_{i=1}^{\infty} \lambda_i \quad (2.1)$$

and 
$$\sum_{i=1}^{\infty} |\lambda_i| \leq \|T\|_1 < \infty. \quad (2.2)$$

The set of trace class operators on a Hilbert space  $E$  is a linear space, and the composition of a trace class operator with a bounded linear operator is trace class.

### 2.2. Recursive Definition of the ARMA Coefficients

For  $i = 1, 2, 3, \dots$ , we define

$$a_i = \text{tr } S_i / i \quad (2.3)$$

where

$$S_1 = T \text{ and } S_{i+1} = T(S_i - a_i I). \quad (2.4)$$

Also,

$$b_1 = CB \text{ and } b_i = C(S_{i-1} - a_{i-1} I)B, \quad i \geq 2. \quad (2.5)$$

These  $a_i$ 's and  $b_i$ 's are the coefficients that we use in (1.3). If the state space  $E$  has finite dimension  $n$ , then  $a_i = b_i = 0$  for  $i > n$ . In this case, the  $a_i$ 's are the negatives of the coefficients in the characteristic polynomial of  $T$ , and (2.3)-(2.5) constitute a well known algorithm for computing the ARMA coefficients. The algorithm here then would appear to be the natural way to define the ARMA coefficients for the infinite dimensional case. However, that the output

$y(t)$  in (1.2) indeed satisfies (1.3) requires a nontrivial proof along with precise definitions of the infinite output and input histories assumed in (1.3). Because of limited space, we only will outline this proof in Section 4.

### III. Decay Rates for the ARMA Coefficients

**Theorem 1.** For each  $r > 0$ ,  $\lim_{i \rightarrow \infty} r^i a_i = 0$ .

In other words, the  $a_i$ 's decay faster than any exponential.

The proof of Theorem 1 requires the following lemma, which is the generalization of the finite dimensional situation and can be proved using (2.2).

**Lemma 1.** For  $i = 1, 2, \dots$ , the coefficient  $a_i$  is  $(-1)^{i-1}$  times the absolutely convergent countable sum of all products of  $i$  distinct eigenvalues of  $T$ .

**Proof of Theorem 1.** For  $i = 1, 2, \dots$ , let the products of  $i$  distinct eigenvalues of  $T$  be ordered and denoted by  $\Pi_k^{(i)}$ ,  $k = 1, 2, \dots$ , so that

$$a_i = \sum_{k=1}^{\infty} \Pi_k^{(i)}, \quad i = 1, 2, \dots. \quad (3.1)$$

Next, define

$$\delta_i = \sum_{k=1}^{\infty} |\Pi_k^{(i)}|, \quad i = 1, 2, \dots. \quad (3.2)$$

that each  $\delta_i$  is finite can be seen by defining the infinite dimensional diagonal matrix whose eigenvalues are the magnitudes of the eigenvalues of  $T$ . This matrix represents a trace class operator  $\tilde{T}$  on  $\mathcal{L}_2$ , and the  $\delta_i$ 's in (3.2) are the coefficients generated by (2.3) and (2.4) for  $\tilde{T}$ . Of course,

$$|a_i| \leq \delta_i, \quad i = 1, 2, \dots. \quad (3.3)$$

For each  $\Pi_k^{(i)}$ , let  $q_k^{(i)}$  be the sum of the absolute values of all eigenvalues of  $T$  except those in  $\Pi_k^{(i)}$ . Hence

$$q_k^{(i)} \leq \delta_1, \quad i, k = 1, 2, \dots. \quad (3.4)$$

The key identity here is

$$\sum_{k=1}^{\infty} |\Pi_k^{(i)}| q_k^{(i)} = (i+1) \sum_{k=1}^{\infty} |\Pi_k^{(i+1)}|, \quad i = 1, 2, \dots. \quad (3.5)$$

From (3.2), (3.4) and (3.5), we have

$$\delta_{i+1} \leq \frac{\delta_1}{i+1} \delta_i, \quad i = 1, 2, \dots. \quad (3.6)$$

which yields

$$\lim_{i \rightarrow \infty} r^i \delta_i = 0 \quad (3.7)$$

for any  $r > 0$ . Then the theorem follows from (3.3) and (3.7).

Our numerical experience with examples suggests that the coefficients  $b_i$  also decay faster than any exponential, under our hypotheses on  $T$ ,  $B$  and  $C$ . So far, however, we can prove this only when the eigenvectors of  $T$  form an orthonormal basis for  $E$ . In this case, an argument similar to the proof of Theorem 1 shows that  $\|S_i\|_1$  decays faster than any exponential. We hope to extend this result for any  $T$  of trace class.

When the spectral radius of  $T$  is less than 1, we can show at least that  $b_i$  approaches zero. This applies to the compensator of the example in Section 5.

#### IV. Approximation

We assume that we have sequences of finite rank operators  $T_n$ ,  $B_n$ ,  $C_n$  such that

$$\|T_n - T\|_1 \rightarrow 0, \quad (4.1)$$

$$\|B_n - B\| \rightarrow 0, \quad (4.2)$$

$$\|C_n - C\| \rightarrow 0, \quad (4.3)$$

as  $n \rightarrow \infty$ . For each  $n$ , we compute ARMA coefficients  $a_{in}$  and  $b_{in}$  according to (2.3)-(2.5). Then (4.1)-(4.3) guarantee that, for each  $i$ ,

$$\lim_{n \rightarrow \infty} a_{in} = a_i \quad (4.4)$$

$$\text{and } \lim_{n \rightarrow \infty} b_{in} = b_i. \quad (4.5)$$

In applications, (4.2) and (4.3) almost always hold because  $B$  and  $C$  have finite rank. However, (4.1) is stronger than the convergence obtained with many approximation schemes, and we hope to be able to weaken this condition. Our approximation of the open-loop system in the example of the next section illustrates the important class of applications where (4.1) holds because the eigenvectors of  $T$  form an orthonormal basis for  $E$  and the approximation consists of projecting onto the eigenspaces of  $T$  (modal approximation).

Also, if the transition operator is  $T = T_0 + F$ , where the eigenvectors of  $T_0$  form an orthonormal basis and  $F$  has finite rank, then approximation by projecting onto the eigenspaces of  $T_0$  will yield (4.1). This is the case in our approximation of the compensator in the example.

For each  $n$ , we have the finite dimensional system

$$z_n(t+1) = T_n z_n(t) + B_n u(t), \quad (4.6)$$

$$y_n(t) = C_n z_n(t). \quad (4.7)$$

The ARMA model for this system is

$$y_n(t) = \sum_{i=1}^n [a_{in} y_n(t-i) + b_{in} u(t-i)], \quad (4.8)$$

where

$$a_{in} = b_{in} = 0, \quad i = n+1, n+2, \dots \quad (4.9)$$

Recall that, before now, it has not been shown that the  $y(t)$  of (1.2) satisfies (1.3). If, for some  $t_0$ , we choose  $z_n(t_0)$  such that

$$\lim_{n \rightarrow \infty} z_n(t_0) = a(t_0), \quad (4.10)$$

then (4.1)-(4.3) along with (4.6)-(4.7) ensure that

$$\lim_{n \rightarrow \infty} y_n(t) = y(t), \quad t \geq t_0. \quad (4.11)$$

If

$$y(t) = y_n(t) = u(t) = 0, \quad t < t_0, \quad (4.12)$$

then (4.1), (4.2) and (4.11) imply that each term in (4.8) converges to each term in (1.3). Without (4.10) and (4.12), we justify (1.3) in a similar way, but only as  $t \rightarrow \infty$ . The decay rates for  $a_i$  and  $b_i$  are then useful.

#### V. Application to Flexible Structures

##### 5.1. Abstract Structure Model

An important class of applications for infinite dimensional ARMA models is the control of flexible structures whose linear distributed model has the form of the differential equation

$$M_0 \ddot{x} + D_0 \dot{x} + A_0 x = B_0 u, \quad (5.1)$$

where the stiffness operator  $A_0$  usually contains partial differential operators for the elastic components of the structure. In applications  $A_0$  is selfadjoint with compact resolvent and  $M_0$  selfadjoint and coercive. For details of the first order form of the equation, see [3]. We only note that, with reasonable conditions on the damping operator  $D_0$ , we can take the state vector

$$z(t) = (x(t), \dot{x}(t))$$

and write the solution to (5.1) as (1.1) for piecewise constant control  $u(t)$ . Of course, we scale the time variable so that the input/sampling interval is 1. For the observability condition that we assumed in Section 1, it is sufficient that no natural frequency of the undamped structure be a multiple of the sampling frequency.

A common damping model for flexible structures is modal damping that provides the same damping ratio for each mode. This means (see [1]) that

$$D_0 = c_0 A_0^{1/2} \quad (5.2)$$

in (5.1), with  $c_0$  a positive real number. It can be shown that - at least for structures whose flexible components are beams, plates, strings or membranes - the damping operator in (5.2) causes the operator  $T$  in (1.1) to be trace class.

We should note, however, that linear

viscoelastic damping represented by (see [3])

$$D_0 = c_0 A_0 \quad (5.3)$$

produces a non-trace class  $T$  because a subsequence of the eigenvalues of  $T$  converges to  $-1/c_0$ .

For (5.1) with either (5.2) or (5.3), the eigenvectors, or modes, of the open-loop system form an orthonormal basis for the state space, so that modal approximation yields (4.1)-(4.3). We used modal approximation in the following example, but we are particularly interested in being able to weaken (4.1) to justify more general finite element approximations for approximating ARMA models of flexible structures.

## 5.2. Example

The structure in this example consists of an Euler-Bernoulli beam cantilevered to a rigid disc which is free to rotate about its fixed center. In-plane motion is modeled, including linear transverse vibrations of the beam. The rotation of the disc gives a rigid-body mode. An actuator applies a control torque to the disc, and a sensor measures the rotation of the disc.

For this problem, we used two levels of approximation. First, we used a finite element approximation to compute the natural mode shapes and frequencies of the composite system. Then we used the system modes as the basis vectors in the approximations of the ARMA model and the optimal control problem. For computing both ARMA coefficients and control and estimator gains, we found that the first 25 system modes were easily sufficient for convergence. We made sure that our original finite element approximation had converged for the first 25 modes.

The lowest natural frequency of the structure was 4.9 rad/sec. We used the modal damping model in (5.2) with both  $c_0 = .02$  (1% critical damping) and  $c_0 = .1$  (5% critical damping).

We computed the open-loop ARMA coefficients as discussed in Section 4, with  $n$  being the number of system modes included in the approximation. Table 1 shows the converged (recall (4.4) and (4.5)) values of the ARMA coefficients, rounded to two significant digits.

$$c_0 = .02$$

|    | $a_1$     | $b_1$     |
|----|-----------|-----------|
| 1  |           |           |
| 2  | .43 E+1   | .29 E-2   |
| 3  | -.87 E+1  | -.11 E-1  |
| 4  | .11 E+1   | .21 E-1   |
| 5  | -.82 E+1  | .25 E-1   |
| 10 |           |           |
| 11 | .30 E+1   | -.16 E-1  |
| 12 | .39 E+1   | .17 E-1   |
| 20 |           |           |
| 21 | .32 E+0   | .19 E-2   |
| 22 | .84 E+0   | -.35 E-2  |
| 30 |           |           |
| 31 | -.48 E-1  | .14 E-3   |
| 32 | .32 E-1   | -.90 E-4  |
| 40 |           |           |
| 41 | -.39 E-5  | -.11 E-7  |
| 42 | .78 E-6   | -.22 E-8  |
| 50 |           |           |
| 51 | -.11 E-12 | .36 E-15  |
| 52 | .14 E-13  | -.36 E-16 |

Table 1

$$c_0 = .1$$

|    | $a_1$     | $b_1$     |
|----|-----------|-----------|
| 1  |           |           |
| 2  | .26 E+1   | .29 E-2   |
| 3  | -.22 E+1  | -.62 E-2  |
| 4  | -.51 E+1  | .38 E-2   |
| 5  | .16 E+1   | .34 E-3   |
| 10 |           |           |
| 11 | -.88 E-1  | -.21 E-2  |
| 12 | .24 E+0   | -.37 E-3  |
| 20 |           |           |
| 21 | -.15 E-3  | .44 E-6   |
| 22 | .21 E-4   | -.58 E-1  |
| 30 |           |           |
| 31 | -.51 E-14 | .16 E-16  |
| 32 | .47 E-16  | -.87 E-18 |

Table 2

## 5.3. Optimal LQG Control

For (1.1), we use the performance index

$$J = \int_0^\infty [\|z(t)\|^2 + u^2(t)] e^{-\alpha t} dt, \quad (5.4)$$

where  $\alpha > 0$ . The optimal control law then is

$$u(t) = -K z(t), \quad (5.5)$$

where

$$K = [1 + B_0^* P B_0]^{-1} B_0^* P T_0 \quad (5.6)$$

and the nonnegative selfadjoint operator  $P$  satisfies the infinite dimensional Riccati equation

$$P = I + T_0^* (P - P B_0 [1 + B_0^* P B_0]^{-1} B_0^* P) T_0. \quad (5.7)$$

where  $T_a = T/\alpha$  and  $B_a = B/\alpha$ . As usual, the  $\alpha$ -shift in this problem ensures that the spectral radius of the closed-loop operator  $T-BK$  will be no greater than  $\alpha$ . In this example, we used  $\alpha = .9$ .

The state estimator for (1.1) has the form

$$\hat{z}(t+1) = T \hat{z}(t) + Bu(t) + G(y(t) - C \hat{z}(t)). \quad (5.8)$$

To get the gain  $G$ , we solve the optimal estimator problem which is the dual of the preceding optimal control problem. This yields

$$G = T_a \hat{P} C_a^* [1 + C_a \hat{P} C_a^*]^{-1} \quad (5.9)$$

where  $\hat{P}$  satisfies

$$\hat{P} = I + T_a (\hat{P} - \hat{P} C_a^* [1 + C_a \hat{P} C_a^*]^{-1} C_a \hat{P}) T_a^* \quad (5.10)$$

and  $C_a = C/\alpha$ .

In state space form, the compensator is

$$\hat{z}(t+1) = (T - GC) \hat{z}(t) + Bu(t) + G y(t) \quad (5.11)$$

and

$$u(t) = -K \hat{z}(t). \quad (5.12)$$

Here, the  $\alpha$ -shift ensures that the spectral radius of closed-loop estimator operator  $T - GC$  is no greater than  $\alpha$ . Again, we used  $\alpha = .9$ .

Now, for (5.11)-(5.12), we think of  $y(t)$  as the input to the compensator and  $u(t)$  as the output. Applying the method of Section 2, we obtain the ARMA representation

$$u(t) = \sum_{i=1}^n [a_i^c u(t-1) + b_i^c y(t-1)]. \quad (5.13)$$

To compute the  $a_i^c$ 's and  $b_i^c$ 's for the example, we solved a sequence of finite dimensional LQG problems corresponding to the sequence of modal approximations to our flexible structure. (See [6] for details.) This yields a sequence of finite dimensional compensators of the form (5.11)-(5.12). For each such compensator, we computed the corresponding ARMA model of the form (5.13) with coefficients  $a_{in}^c$  and  $b_{in}^c$ , where  $n$  is the number of modes in the system approximation. As in the open-loop case,

$$\lim_{n \rightarrow \infty} a_{in}^c = a_i^c, \quad i = 1, 2, \dots, \quad (5.14)$$

and

$$\lim_{n \rightarrow \infty} b_{in}^c = b_i^c, \quad i = 1, 2, \dots, \quad (5.15)$$

For the case where the open-loop damping coefficient is  $c = .1$ , Table 3 gives the ARMA coefficients for (5.13).

|    |            |           |
|----|------------|-----------|
|    | $c_0 = .1$ |           |
|    | $a_1$      | $b_1^c$   |
| 1  |            |           |
| 2  | -.14 E+1   | .31 E+3   |
| 3  | -.44 E+0   | -.62 E+3  |
| 4  | .17 E+1    | .17 E+3   |
| 5  | -.98 E+0   | .19 E+3   |
| 6  |            |           |
| 7  |            |           |
| 8  |            |           |
| 9  |            |           |
| 10 | .78 E-1    | -.17 E+1  |
| 11 | -.81 E-1   | .24 E+2   |
| 12 |            |           |
| 13 |            |           |
| 14 |            |           |
| 15 |            |           |
| 16 |            |           |
| 17 |            |           |
| 18 |            |           |
| 19 |            |           |
| 20 | .15 E-3    | -.41 E-1  |
| 21 | -.19 E-4   | .64 E-2   |
| 22 |            |           |
| 23 |            |           |
| 24 |            |           |
| 25 |            |           |
| 26 |            |           |
| 27 |            |           |
| 28 |            |           |
| 29 |            |           |
| 30 | .55 E-14   | -.16 E-11 |
| 31 | -.22 E-16  | -.20 E-14 |

Table 3

### References

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