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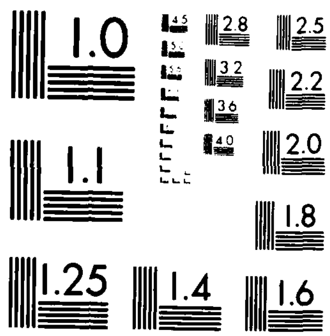

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On Filtered Binary Processes

R. F. Pawula and S. O. Rice

Abstract. The problem of calculating the probability density function of the output of an RC filter driven by a binary random process with intervals generated by an equilibrium renewal process is studied. New integral equations, closely related to McFadden's original integral equations, are derived, and solved by a matrix approximation method and by iteration. Transformations of the integral equations into differential equations are investigated. Some numerical results which compare the matrix and iteration solutions with both exact solutions and approximate solutions based upon the Fokker-Planck equation are presented.

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I. INTRODUCTION

We consider the problem of determining the probability density function of the output of an RC filter when the input is a binary random process. Such problems have been around for forty or so years, and arose during that time largely out of intrinsic interest in the development of the theory of random processes. Their solutions and the methods used to solve them have provided insight, and have led to many useful applications [1, 2]. More recently, filtered binary processes have received attention as theoretical models of a single channel in a nerve membrane distorted by the low-pass filtering action of recording equipment [5, 15]. Most previous work has focused attention on the special case in which the binary input process is the random telegraph signal (with exponentially distributed intervals), and for this case results are available for the output density function as well as for related quantities such as the interval statistics and level crossings of the output [5, 9-11, 13, 14, 17]. Few results have been obtained in the more difficult case in which the intervals of the input are not exponentially distributed.

For independent and identically distributed intervals with arbitrary statistics, McFadden [6] derived integral equations from which the output density can be obtained. However, because of the complexity of these integral equations, he was able to find a solution in only one special, non-trivial case. Another case that has received considerable attention, but with limited success, is that in which the binary process is the result of hard-limiting of a stationary Gaussian noise with exponential correlation [4, 7, 9]. In this case, the intervals of the binary process are no longer statistically independent. Although exact results are available for only one special ratio of time constants [4], approximations have been obtained by use of the Fokker-Planck equation [9].

In this paper, attention will be confined to binary processes of the type considered by McFadden; i.e., each binary process will be assumed to be constructed from an equilibrium renewal process [3, 16] so that its intervals will be independent of one another and of the state of the binary process at the transition times. We will extend McFadden's work in several directions: (i) by generalizing the binary input process to have different probability density functions for the up and down intervals, (ii) by finding approximations based upon the Fokker-Planck equation, (iii) by deriving new integral equations for the relevant density functions, (iv) by developing matrix and iterative methods for solutions of the integral equations, and (v) by investigating transformations of the integral equations into differential equations. Some numerical results which compare the various approaches will be given.

The paper begins in the next section with a brief review of the system model, the integral equations of McFadden, and a summary of known results. Sec. III discusses the moments of the filtered process, and Sec. IV gives approximations based upon the Fokker-Planck equation. In Sec. V, new integral equations are derived, and methods for their solution considered. Transformation of the integral equations into differential equations is examined in Sec. VI, numerical results are presented in Sec. VII, and the final section summarizes and discusses the results.

II. SYSTEM MODEL & KNOWN RESULTS

This section gives a brief summary of the system model and known results, and also serves to define the various quantities that enter into the analyses.

The system is governed by the differential equation

$$\frac{dy(t)}{dt} + \beta y(t) = \beta x(t) , \qquad \beta > 0 \tag{1}$$

where β^{-1} is the RC filter time constant, $x(t) = \pm 1$ is the binary input process and $y(t)$, $|y(t)| < 1$, is the filter output. We shall frequently use $T = \beta^{-1}$.

A. The Binary Input Process. The binary process $x(t)$ is characterized by the time intervals between its transitions, which are taken to be independent random variables. Further, the time intervals corresponding to the $x(t) = -1$ state will be taken to be identically distributed with one probability density function, while those corresponding to the $x(t) = +1$ state to be identically distributed with a different density. These densities and some quantities later needed are

$$f_0(t) = \text{p.d.f. of } x(t) = -1 \text{ interval length} \tag{2a}$$

$$f_1(t) = \text{p.d.f. of } x(t) = +1 \text{ interval length} \tag{2b}$$

$$\mu_i = \int_0^\infty t\, f_i(t)\, dt \tag{2c}$$

$$F_i(s) = \mathcal{L}\{f_i(t)\} = \int_0^\infty e^{-st} f_i(t)\, dt \tag{2d}$$

$$\mathcal{F}_i(t) = \int_t^\infty f_i(t')\, dt' \tag{2e}$$

All of the f's are to be considered as being defined on $[0, \infty]$, and to be zero for negative arguments, μ_i is the mean value of the up- or down interval, $F_i(s)$ the Laplace transform of the interval density, and $\mathfrak{F}_i(t)$ the complement of the interval probability distribution function. The quantities in (2a)-(2e) will sometimes be written without subscripts in the symmetric case in which $f_0(t)=f_1(t)=f(t)$, $\mu_0=\mu_1=\mu$, $F_0(s)=F_1(s)=F(s)$, and $\mathfrak{F}_0(t)=\mathfrak{F}_1(t)=\mathfrak{F}(t)$.

B. The Output Process. The output process $y(t)$ consists of segments of rising and decaying exponentials, and consequently will have local minima and maxima at the transition points of $x(t)$. The probability density functions of $y(t)$ at these transition points enter into the analyses. At this point, it is convenient to define six different probability density functions associated with $y(t)$, and these are

$p_0(y)$ = p.d.f. of $y(t)$ at a minimum point (3a)

$p_1(y)$ = p.d.f. of $y(t)$ at a maximum point (3b)

$p_-(y)$ = p.d.f. of $y(t)$ at a time picked at random during an $x(t) = -1$ interval (3c)

$p_+(y)$ = p.d.f. of $y(t)$ at a time picked at random during an $x(t) = +1$ interval (3d)

$p(y)$ = unencumbered p.d.f. of $y(t)$ at a time picked at random (3e)

$p_{FP}(y)$ = Fokker-Planck approximation to $p(y)$ (3f)

All of the p's are zero outside the interval $[-1, 1]$ - this will be tacitly assumed throughout and will not be stated each time an expression for one of the p's is given. In the symmetric case, we have $p_0(y) = p_1(-y)$ and $p_-(y) = p_+(-y)$.

Some elementary relations between $p_-(y)$, $p_+(y)$ and $p(y)$ are

$$p(y) = \frac{\omega_0}{\omega_0 + \omega_1}\, p_-(y) + \frac{\omega_1}{\omega_0 + \omega_1}\, p_+(y) \tag{4a}$$

$$p_-(y) = \frac{\omega_0 + \omega_1}{2\omega_0}\,(1 - y)\, p(y) \tag{4b}$$

$$p_-(y) = \frac{\omega_0 + \omega_1}{2\omega_1}\,(1 - y)\, p(y) \tag{4c}$$

Equations (4b) and (4c) follow from a theorem on conditional expectation of Mazo and Salz [8] which says* that $E[\dot{y} \mid y] = 0$. Applying this to (1) gives $E[x(t) \mid y(t)] = y(t)$ which, in turn, implies (4b) and (4c) (see the derivation in [11] for a similar situation). From (4b) and (4c) it follows that

$$\omega_0(1+y)p_-(y) = \omega_1(1-y)p_+(y) \tag{4d}$$

which also can be obtained by equating the average number per second of the upward and downward crossings of the level y.

C. <u>The Basic Integral Equations</u>. The basic integral equations relating the interval densities and densities of the output process are, with $T = 1/\beta$,

$$p_0(y) = \frac{T}{1+y} \int_y^1 d\eta\; p_1(\eta)\, f_0\!\left(T \ln \frac{1+\eta}{1+y}\right) \tag{5a}$$

$$p_1(y) = \frac{T}{1-y} \int_{-1}^{y} d\eta\; p_0(\eta)\, f_1\!\left(T \ln \frac{1-\eta}{1-y}\right) \tag{5b}$$

$$p_-(y) = \frac{T}{\omega_0(1-y)} \int_y^1 d\eta\; p_1(\eta)\, \mathfrak{F}_0\!\left(T \ln \frac{1+\eta}{1+y}\right) \tag{5c}$$

$$p_+(y) = \frac{T}{\omega_1(1-y)} \int_{-1}^{y} d\eta\; p_0(\eta)\, \mathfrak{F}_1\!\left(T \ln \frac{1-\eta}{1-y}\right) \tag{5d}$$

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*The precise mathematical conditions required are satisfied for the processes considered here.

Equations (5b) and (5d) were derived by McFadden (cf. (5) and (7) of [6]), who wrote them in a slightly different form, and are the result of considering the output $y(t)$ over a time interval during which $x(t) = +1$. Exactly following the same steps for an $x(t) = -1$ interval leads to (5a) and (5c). In the symmetric case, McFadden changed the independent variable from y to t by means of

$$y = 1 - 2e^{-\beta t} \tag{6}$$

and, by Laplace transforming with respect to the new variable t, transformed (5b) and (5d) into algebraic expressions which enabled him to obtain a solution in one special case. To do the same type of transformation in connection with (5a) and (5c) would require the slightly different change of variable

$$y = -1 + 2e^{-\beta t} \tag{7}$$

D. <u>Known Results</u>. The only nontrivial cases in which exact results are known are those of exponentially distributed intervals and McFadden's special case. A brief summary of these follows.

(i) Symmetric Case: $f(t) = a e^{-at}$, [10, 17]

$$p(y) = \frac{(1 - y^2)^{\alpha - 1}}{2^{2\alpha - 1} B(\alpha, \alpha)}; \quad \alpha = aT \tag{8a}$$

$$p_0(y) = (1 - y)\, p(y) \tag{8b}$$

where $B(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u + v)$ is the beta function.

(ii) Asymmetric Case: $f_i(t) = a_i e^{-a_i t}$ [5]

$$p(y) = \frac{(1+y)^{\alpha_0 - 1}(1-y)^{\alpha_1 - 1}}{2^{\alpha_0 + \alpha_1 - 1} B(\alpha_0, \alpha_1)} ; \quad \alpha_i = a_i T \tag{9a}$$

$$p_0(y) = p_-(y) ; \quad p_1(y) = p_+(y) \tag{9b}$$

(iii) Symmetric Case: McFadden Interval PDF* [6]

$$f(t) = \frac{e^{-a\beta t}(1 - e^{-\beta t})^{b-a-1}}{B(a, b-a)T} ; \quad b > a , \tag{10a}$$

$$p(y) = \frac{1 - I_{(1+y)/2}(b, a) - I_{(1-y)/2}(b, a)}{\beta\mu(1 - y^2)} ; \quad \beta\mu = \psi(b) - \psi(a) \tag{10b}$$

$$p_0(y) = \frac{(1+y)^{a-1}(1-y)^{b-1}}{2^{a+b-1} B(a, b)} \tag{10c}$$

Because of the $(1 - e^{-\beta t})$ term in $f(t)$, McFadden's results hold only for the special case that the input interval density is related to the filter time constant through this term.

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* $I_x(p, q) = \int_0^x u^{p-1}(1-u)^{q-1} du / B(p, q)$ is the incomplete beta function [12] and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. Our "a" and "b" are McFadden's "a+1" and "b+1."

III. MOMENTS

The moments $E[y^n(t)]$, $n = 1, 2, \ldots$ can often be evaluated in situations of the present kind without first getting the density $p(y)$ [7]. In the symmetric case, all of the odd moments are zero. In general, the moments can be expressed in terms of the Laplace transforms of the interval densities, $F_0(s)$ and $F_1(s)$, and the first two nonzero moments in the symmetric and asymmetric cases are:

(i) Symmetric Case:

$$\overline{y^2} = 1 - \frac{2}{\beta\mu}\,\frac{1 - F(\beta)}{1 + F(\beta)} \tag{11a}$$

$$\overline{y^4} = 1 - \frac{8}{3\beta\mu}\,\frac{[1 - 2F(\beta) + 2F(2\beta) - F(\beta)F(2\beta)][1 - F(3\beta)]}{[1 + F(\beta)][1 - F(2\beta)][1 + F(3\beta)]} \tag{11b}$$

(ii) Asymmetric Case:

$$\overline{y} = \frac{\mu_1 - \mu_0}{\mu_0 + \mu_1} \tag{11c}$$

$$\overline{y^2} = 1 - \frac{4}{\beta(\mu_0 + \mu_1)}\,\frac{[1 - F_0(\beta)][1 - F_1(\beta)]}{1 - F_0(\beta)F_1(\beta)} \tag{11d}$$

Equation (11a) is derived in the Appendix, and its generalization to (11d) is also outlined there. A recursive method for obtaining all of the moments of $y(t)$ as well as all of the conditional moments $E[y^n(t) \mid x(t) = \pm 1]$ has recently been developed by A. Munford [19], and (11b) has been deduced from his work. The n-th moment is a function of $F_0(k\beta)$ and $F_1(k\beta)$ for $k = 1, 2, \ldots n-1$; and becomes increasingly complicated as the order of the moment increases. The moments are significantly more complicated in the asymmetric case than in the symmetric case.

IV. FOKKER-PLANCK APPROXIMATIONS

Approximations to $p(y)$ based upon a certain linearity assumption can be obtained by use of the Fokker-Planck equation. The Fokker-Planck approximations are, in fact, exact in the case that $x(t)$ has exponentially distributed intervals [5, 10], and are close approximations when $x(t)$ is hard-limited Gaussian RC noise [7, 9]. It will later be shown that the Fokker-Planck approximation is also exact in the case of the McFadden interval density with $b = a + 1$ and $b = a + 2$. In some other cases, the approximations are accurate to within a few percent. Also, the approximations provide starting points for iterative solutions to the integral equations, to be considered later. Our use of the Fokker-Planck equation closely parallels that in [9].

A. The Symmetric Case. The (extended) Fokker-Planck equation for $p(y)$ is readily shown to be [9, 11]

$$\frac{d}{dy}\left[\frac{1}{2}\beta^{2}(1 - y^{2})\, p(y)\right] - A(y)\, p(y) = 0 \tag{12}$$

in which $A(y)$ is defined as the limit of a conditional expectation as

$$A(y) = \lim_{\Delta \to 0} \frac{1}{\Delta^{2}} E\left[y(t+\Delta) - y(t) \mid y(t)\right] \tag{13}$$

In general, it is not known how to evaluate $A(y)$. An assumption that has led to an excellent approximation in one case [9] is that $A(y)$ is approximately a linear function of y, and, for that reason, we make the same assumption here. Setting

$$A(y) = -\beta^2 u y\ , \quad u = \text{constant} \tag{14}$$

in (11) and solving the ensuing equation then gives

$$p_{FP}(y) = \frac{(1 - y^2)^{u-1}}{2^{2u-1}\ B(u, u)} \tag{15}$$

The subscripts "FP" on the density function are intended to emphasize the fact that the density function is an approximation based upon the Fokker-Planck equation, and not necessarily an exact result. The unknown constant u can be found by evaluating $\overline{y^2}$ by two different ways (see (11a) and the Appendix), and comes out to be

$$u = \frac{1 - F(\beta)}{\beta\mu[1 + F(\beta)] - 2[1 - F(\beta)]} \tag{16}$$

It will later be shown in Sec. VI-A (see (43) et seq.) that an exact expression for $A(y)$ in terms of $p(y)$ and $p_0(y)$ is

$$\frac{A(y)}{\beta^2} = \frac{p_0(y) - p_0(-y)}{2\beta\mu\ p(y)} \tag{17}$$

which provides a way of comparing the assumption (14) with the exact $A(y)$ when $p(y)$ and $p_0(y)$ are known.

B. <u>The Asymmetric Case</u>. The Fokker-Planck approximation in the asymmetric case follows in much the same way as in the symmetric case, and so the details will not be given. The results are

$$p_{FP}(y) = \frac{(1+y)^{u-1}(1-y)^{v-1}}{2^{u+v-1}B(u,v)} \qquad (18)$$

where

$$v = \frac{\mu_1}{\mu_0}u \qquad (19)$$

$$u = \frac{(\mu_0+\mu_1)(1-\overline{y^2})}{\overline{y^2}(\mu_0+\mu_1)^2-(\mu_0-\mu_1)^2} \qquad (20)$$

$$\overline{y^2} = \frac{u+v+(u-v)^2}{u+v+(u+v)^2} \qquad (21)$$

and, the analog of (17) is

$$\frac{A(y)}{\beta^2} = \frac{p_0(y)-p_1(y)}{\beta(\mu_0+\mu_1)\,p(y)} \qquad (22)$$

Equation (11d) is an alternate expression for $\overline{y^2}$ which can be used in (20) to give the value of the parameter u in terms of the Laplace transforms of the interval densities.

V. INTEGRAL EQUATION APPROACH

A. Some New Integral Equations. In this section, we explore ways of further using the basic integral equations (5a) - (5d). Equations (5a) and (5b) can be used in the integrands of one another to give integral equations in either $p_0(y)$ or $p_1(y)$ only; viz.,

$$p_0(y) = \frac{T^2}{1+y} \int_y^1 \frac{d\xi}{1-\xi}\, f_0\left(T \ln \frac{1+\xi}{1+y}\right) \int_{-1}^{\xi} d\eta\, p_0(\eta)\, f_1\left(T \ln \frac{1-\eta}{1-\xi}\right) \tag{23a}$$

$$p_1(y) = \frac{T^2}{1-y} \int_{-1}^{y} \frac{d\xi}{1+\xi}\, f_1\left(T \ln \frac{1-\xi}{1-y}\right) \int_{\xi}^{1} d\eta\, p_1(\eta)\, f_0\left(T \ln \frac{1+\eta}{1+\xi}\right) \tag{23b}$$

By interchanging orders of integration, these can also be written as

$$p_0(y) = \int_{-1}^{1} K_0(y,\eta)\, p_0(\eta)\, d\eta \tag{24a}$$

$$p_1(y) = \int_{-1}^{1} K_1(y,\eta)\, p_1(\eta)\, d\eta \tag{24b}$$

where the kernels are*

$$K_0(y,\eta) = \frac{T^2}{1-y} \int_{\max(y,\eta)}^{1} f_0\left(T \ln \frac{1+\xi}{1+y}\right) f_1\left(T \ln \frac{1-\eta}{1-\xi}\right) \frac{d\xi}{1-\xi} \tag{25a}$$

$$K_1(y,\eta) = \frac{T^2}{1-y} \int_{-1}^{\min(y,\eta)} f_0\left(T \ln \frac{1+\eta}{1+\xi}\right) f_1\left(T \ln \frac{1-\xi}{1-y}\right) \frac{d\xi}{1+\xi} \tag{25b}$$

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*Eqs. (24a) and (24b) were derived independently by R. FitzHugh in a private correspondence.

The forms of (24a) and (24b) lend themselves more directly to numerical techniques than do (23a) and (23b). A matrix technique for solving them will be discussed later in this section.

A second way of using the set of basic integral equations is that of inverting (5c) and (5d) to express $p_0(y)$ and $p_1(y)$ as integrals of $p_-(y)$ and $p_+(y)$:

$$p_0(y) = \frac{\omega_0 T}{1+y} \int_y^1 d\tau \; p_-(\tau) \; h_0\left(T \ln \frac{1+\tau}{1+y}\right) \tag{26a}$$

$$p_1(y) = \frac{\omega_1 T}{1-y} \int_{-1}^y d\tau \; p_+(\tau) \; h_1\left(T \ln \frac{1-\tau}{1-y}\right) \tag{26b}$$

in which

$$h_i(t) = \mathcal{L}^{-1}\left\{\frac{s\,F_i(s)}{1-F_i(s)}\right\} \;; \quad i = 0, 1 \tag{27}$$

Equation (26a) can be obtained by using the change of variable (7) in (5a) and (5c), and then Laplace transforming both with respect to the new variable. The resulting two equations can be solved for the transform of $p_0(y)$ in terms of the transform of $p_-(y)$. Inversion then leads to (26a). Equation (26b) can be obtained in the same way by starting with (5b) and (5d) and using the change of variable (6).

As with $f_i(t)$, whenever $h_i(t)$ is used, it will be tacitly assumed to be zero for negative t. When (26a) and (26b) are substituted into the integrands of (5c) and (5d), the results are two equations containing only $p_-(y)$ and $p_+(y)$. Employing (4b) and (4c), the first of these gives an equation involving only the unencumbered density $p(y)$; i.e.,

$$p(y) = \frac{T^2}{1 - y^2} \int_{-1}^{y} \frac{d\xi}{1+\xi} \; \mathcal{F}_1\left(T \ln \frac{1-\xi}{1-y}\right) \int_{\xi}^{1} d\eta\,(1-\eta)\,p(\eta)\; h_0\left(T \ln \frac{1+\eta}{1+\xi}\right) \tag{28}$$

which can also be written as

$$p(y) = \int_{-1}^{1} K(y, \eta)\, p(\eta)\, d\eta \tag{29}$$

where

$$K(y, \eta) = \frac{(1-\eta)\,T^2}{1 - y^2} \int_{-1}^{\min(y,\eta)} h_0\left(T \ln \frac{1+\eta}{1-\xi}\right) \; \mathcal{F}_1\left(T \ln \frac{1-\xi}{1-y}\right) \frac{d\xi}{1-\xi} \tag{30}$$

Any of (23a), 23b) or (29) can be solved by the matrix method of the next sub-section.

The $h(t)$ functions will usually be more complicated than the interval densities, however, there are some cases in which the $h(t)$ are particularly simple. The Laplace transform of the McFadden interval density (10a) is

$$F(s) = \frac{\Gamma(b)\,\Gamma(sT + a)}{\Gamma(a)\,\Gamma(sT + b)} \tag{31}$$

and the Laplace transform of the gamma interval density

$$f(t) = \frac{a^{n+1}\, t^n\, e^{-at}}{n!}\;; \qquad n = 0, 1, \ldots \tag{32}$$

is

$$F(s) = \left(\frac{a}{s - a}\right)^{n+1}$$

Cases in which these lead to simple forms for $h(t)$ are:

(i) McFadden, $b = a + 1$

$$h(t) = a\beta\delta(t) \tag{34a}$$

(ii) McFadden, $b = a + 2$

$$h(t) = a\beta(a+1)\, e^{-(2a+1)\beta t} \tag{34b}$$

(iii) McFadden, $b = a + 3$

$$h(t) = \frac{2a\beta(a+1)(a+2)}{\sqrt{3a^2 + 6a - 1}}\, e^{-3(a+1)\beta t/2} \sin(\sqrt{3a^2 + 6a - 1}\,\beta t/2) \tag{34c}$$

(iv) Gamma, $n = 0$

$$h(t) = a\,\delta(t) \tag{34d}$$

(v) Gamma, $n = 1$

$$h(t) = a^2\, e^{-2at} \tag{34e}$$

(vi) Gamma, $n = 2$

$$h(t) = (2a^2/\sqrt{3})\, e^{-3at/2} \sin(\sqrt{3}\,at/2) \tag{34f}$$

For either the McFadden density with $b - a$ an integer, or the gamma density, $\mathcal{L}\{h(t)\}$ is, in general, the ratio of two polynomials in s. An explicit form for $h(t)$ as the sum of complex exponentials can be written in the case of the gamma density for arbitrary n.

B. Solution by a Matrix Method. Any of (26a), (26b) or (29) can be solved by a numerical matrix method [18]. Consider (29) which, by approximating the integral by a sum, can be written in the form

$$p(y_i) = \frac{2}{N}\sum_{j=1}^{N} K(y_i, y_j)\, p(y_j)\;; \qquad i = 1, \ldots, N \tag{35}$$

where $y_k = -1 + (2k-1)/N$. This is a homogeneous set of N simultaneous linear equations in the $p(y_i)$, $i = 1, \ldots, N$ and can be solved by solving the associated matrix eigenvalue problem

$$(\underset{\sim}{K} - \lambda \underset{\sim}{I})\underset{\sim}{P} = 0 \tag{36}$$

Here $\underset{\sim}{P}$ is an $N \times 1$ column vector with components $p(y_i)$, $\underset{\sim}{I}$ is the identity matrix, $\underset{\sim}{K}$ is the $N \times N$ matrix with element $2K(y_i, y_j)/N$ and λ denotes an eigenvalue. The desired solution is the one for which λ is nearest to one under the normalization

$$\frac{2}{N}\sum_{i=1}^{N} p(y_i) = 1 \tag{37}$$

The eigenvalue problem can be solved by standard techniques from the theory of linear algebra [18].

C. Solution by Iteration. The integral equations lend themselves to solution by iteration in several ways depending upon which of the equations one chooses to work with. Such a choice will be influenced, in part, by the kernels of the equations, which are simplest in the basic equations (5a)-(5d) and somewhat more complicated in (24a), (24b), (26a), (26b) and (29). We will describe in detail only one of these possible iteration schemes - that which can be applied to the basic equations.

With the idea of iteration in mind, (5a) and (5b) will be written as [18]

$$p_{0,n+1}(y) = \frac{T}{1+y} \int_y^1 d\eta\, p_{1n}(\eta)\, f_0\left(T \ln \frac{1-\eta}{1+y}\right) \tag{38a}$$

$$p_{1,n+1}(y) = \frac{T}{1-y} \int_{-1}^y d\eta\, p_{0n}(\eta)\, f_1\left(T \ln \frac{1-\eta}{1-y}\right) \tag{38b}$$

Here, $p_{00}(y)$ and $p_{10}(y)$ represent initial starting estimates of $p_0(y)$ and $p_1(y)$ respectively. One way of getting these initial estimates is by using the Fokker-Planck approximation $p_{FP}(y)$ and (4b) and (4c) in the integrands of (26a) and (26b); viz.,

$$p_{00}(y) = \frac{(\mu_0 + \mu_1)T}{2(1+y)} \int_y^1 d\eta\,(1-\eta)\, p_{FP}(\eta)\; h_0\left(T \ln \frac{1+\eta}{1+y}\right) \tag{39a}$$

$$p_{10}(y) = \frac{(\mu_0 + \mu_1)T}{2(1-y)} \int_{-1}^y d\eta\,(1-\eta)\, p_{FP}(\eta)\; h_1\left(T \ln \frac{1-\eta}{1-y}\right) \tag{39b}$$

These then lead to $p_{01}(y)$ and $p_{11}(y)$ by (38a) and (38b), and the results can again be used back in the integrands of (38a) and (38b) to give $p_{02}(y)$ and $p_{12}(y)$, etc. The general procedure is illustrated in Fig. 1, which also shows the final step of getting an estimate $\hat{p}(y)$ of $p(y)$ at the culmination of the iteration. It is reasonable to suppose that (see [18])

$$p_0(y) = \lim_{n \to \infty} p_{0n}(y) \tag{40a}$$

$$p_1(y) = \lim_{n \to \infty} p_{1n}(y) \tag{40b}$$

and that the convergence will be rapid if the initial guess $p_{FP}(y)$ is good. The results of some calculations using the iteration procedure will be given in Sec. VI. As will later be seen, a single iteration is sufficient in some cases.

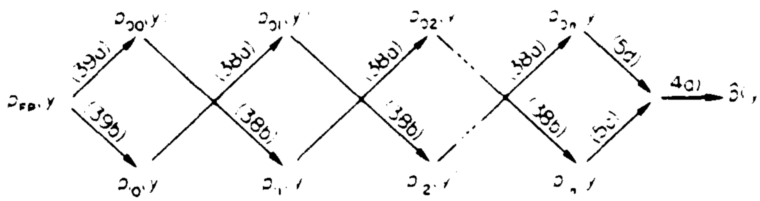

Fig. 1. Illustration of iteration procedure. The numbers above the arrows are the equations to be used in going from one step to the next, and $\hat{p}(y)$ is the final estimate of $p(y)$.

It is also possible to iterate one of (24a), (24b) or (29). An example will later be given in which $p(y)$ will be estimated by the single iteration of (29); i.e.,

$$\hat{p}(y) = \int_{-1}^{1} K(y, \eta)\, p_{FP}(\eta)\, d\eta \tag{41a}$$

which is the same as

$$\hat{p}(y) = \frac{2T}{(\mu_0 + \mu_1)(1 - y^2)} \int_{-1}^{y} d\xi\, p_{00}(\xi)\, \mathcal{F}_1\left(T \ln \frac{1 - \xi}{1 - y} \right) \tag{41b}$$

and it will be seen that the iteration is good to three significant figures. Such accuracy is possible because the Fokker-Planck approximation is excellent in many cases.

VI. DIFFERENTIAL EQUATION APPROACH

In certain cases, the integral equations (5a)-(5d) can be turned into differential equations, and this approach is the main concern of this section. First, however, we consider some general relations, which can be deduced by differentiation, for arbitrary interval densities.

A. Some Differential Formulas. Multiplying through by $(1+y)$, and differentiating with respect to y, (5c) leads to

$$\beta\mu_0 \frac{d}{dy}\left[(1+y)\, p_-(y)\right] = p_0(y) - p_1(y) \tag{42a}$$

In a similar way, (5d) yields

$$\beta\mu_1 \frac{d}{dy}\left[(1-y)\, p_+(y)\right] = p_0(y) - p_1(y) \tag{42b}$$

Adding and making use of (4b) and (4c) then gives

$$\frac{\beta(\mu_0+\mu_1)}{2} \frac{d}{dy}\left[(1-y^2)\, p(y)\right] = p_0(y) - p_1(y) \tag{43}$$

Equation (43), when combined with (12) leads directly to (17) and (22). This derivation of (43) illustrates the general method of turning the integral formulas into differential formulas, and will be used in essentially the same way in the remainder of this section.

B. Intervals with Gamma Densities. Aside from (43), it does not appear possible to use the differential equation approach without specifying the interval statistics of the input process, so we here take $f_0(t)$ and $f_1(t)$ to be Gamma distributed with densities given by (32). Then it can be shown by repeated differentiation that (5a) and (5b) lead to the differential equations

$$\mathcal{L}_+ \, p_0(y) = (-\alpha_0)^{n+1} \, p_1(y) \tag{44a}$$

$$\mathcal{L}_- \, p_1(y) = \alpha_1^{n+1} \, p_0(y) \tag{44b}$$

where $\mathcal{L}_-$ and $\mathcal{L}_+$ are linear differential operators defined by

$$\mathcal{L}_-(\cdot) = (1-y)^{\alpha_1} \underbrace{\frac{d}{dy}(1-y) \cdots \frac{d}{dy}(1-y)}_{\text{n times}} \frac{d}{dy} (1-y)^{1-\alpha_1}(\cdot) \tag{45a}$$

$$\mathcal{L}_+(\cdot) = (1+y)^{\alpha_0} \underbrace{\frac{d}{dy}(1+y) \cdots \frac{d}{dy}(1+y)}_{\text{n times}} \frac{d}{dy} (1+y)^{1-\alpha_0}(\cdot) \tag{45b}$$

Operating with $\mathcal{L}_-$ on (44a) and $\mathcal{L}_+$ on (44b), it follows that

$$\mathcal{L}_- \mathcal{L}_- \, p_0(y) - (-\alpha_0 \alpha_1)^{n+1} \, p_0(y) = 0 \tag{46a}$$

$$\mathcal{L}_+ \mathcal{L}_- \, p_1(y) - (-\alpha_0 \alpha_1)^{n+1} \, p_1(y) = 0 \tag{46b}$$

These are linear differential equations of order $2n+2$. When $n = 0$, we have the case of exponentially distributed intervals, and it is straightforward to show that solving (46a) and (46b) along with (43) leads to (9a) and (9b). The next simplest case is that of $n = 1$, to which we now turn.

C. <u>The Case $f(t) = a^2 t e^{-at}$</u>. For the sake of simplicity, we consider only the symmetric case. Equation (46a) for $p_0(y)$ then becomes

$$(1-y^2)^2 p_0^{(4)} + 2(2-5y+2\alpha y)(1-y^2) p_0'' + [(6\alpha^2 - 24\alpha + 25)y^2 + 8(\alpha-2)y - (2\alpha^2 - 8\alpha - 5)] p_0'$$

$$+ (3-2\alpha)[(2\alpha^2 - 6\alpha + 5)y - 3 + 2\alpha] p_0' + [(1-\alpha)^4 - \alpha^4] p_0 = 0 \qquad (47)$$

This equation is somewhat formidable, and we have not been successful in finding general solutions. However, the form of the asymptotic behavior of $p(y)$ around the endpoints $y = \pm 1$ can be determined from it. Setting $x = 1 - y$, $1 + y = 2 - x$, $d/dy = -d/dx$ in (47), and assuming a solution of the form x^{ν} leads to the indicial equation

$$\nu(\nu + 1)[\nu - (\alpha + 1)]^2 = 0 \qquad (48)$$

Consequently, a power series solution to (47) will have the leading terms

$$p_0(y) \sim c_1 + c_2(1-y) + c_3(1-y)^{\alpha+1} + c_4(1-y)^{\alpha-1} \ln(1-y) \text{ around } y = 1 \qquad (49a)$$

Similarly

$$p_0(y) \sim c_{-1} + c_{-2}(1+y) + c_{-3}(1+y)^{\alpha-1} + c_{-4}(1+y)^{\alpha-1} \ln(1+y) \text{ around } y = -1 \qquad (49b)$$

The c's are constants (which could depend upon α).

For $\alpha = 1$, (47) can be once integrated and leads to the somewhat simpler third order equation

$$(1-y^2)^2 p_0''' + 2(2-y)(1-y^2) p_0' + (3+y^2) p_0' - (1-y) p_0 = \text{const.}, \quad \alpha = 1 \qquad (50)$$

It is possible to get a differential equation for the unencumbered density

$p(y)$ in the following way. Using (34e) along with (4b) in (26a) leads to the integral formula

$$p_0(y) = 2\alpha(1+y)^{2\alpha-1}\int_y^1 d\eta\,(1-\eta)(1+\eta)^{-2\alpha}\,p(\eta) \tag{51}$$

When this is substituted back into the right-hand side of (43) for $p_0(y)$ and $p_1(y) = p_0(-y)$, the resulting expression contains only the unknown $p(y)$. The integrals can be eliminated by differentiation, which then gives the third order differential equation for $p(y)$

$$0 = (1-y^2)^2 p''' - 2(5-2\alpha)y(1-y^2)p'' + 2[(3\alpha^2 - 11\alpha + 12)y^2 - \alpha^2 + 3\alpha - 4]p'$$
$$+ 2(3-2\alpha)(\alpha^2 - 2\alpha + 2)yp \tag{52}$$

As was done above for $p_0(y)$, the form of the asymptotic behavior of $p(y)$ in the vicinity of $y = \pm 1$ can be determined from this differential equation, and is

$$p(y) \sim k_1 + k_2(1-y^2)^{\alpha-1} + k_3(1-y^2)^{\alpha-1}\ln(1-y^2) \quad \text{around } y = \pm 1 \tag{53}$$

From this, it is apparent that $p(y) \to \infty$ as $|y| \to \pm 1$ if $\alpha < 1$, and that $p(y)$ remains finite for all y if $\alpha > 1$.

In the special case $\alpha = 1/2$, (45) can be once integrated, and doing so results is

$$\frac{d}{dy}[(1-y^2)^2 p'] + \frac{1}{2}(5y^2 - 3)p = \text{constant} \quad (\alpha = 1/2) \tag{54a}$$

The constant can be evaluated in terms of $\overline{y^2}$ by integrating the differential equation over $[-1, 1]$. The result of this is the differential equation

$$\frac{d}{dy}\left[(1-y^2)^2 p'\right] - \frac{1}{2}(5y^2-3)p = \frac{1}{4}(5\overline{y^2} - 3) \quad (\gamma = 1/2) \tag{54b}$$

and appropriate boundary conditions for the solution are that $p(y)$ be symmetric and integrate to one.

Just how appropriate boundary conditions can be found from the integral equations, in general, is not obvious. For the gamma density with $n = 1$, it is straightforward to show from the basic integral equations that $p_0(1) = p_0'(1) = 0$. But, the differential equation for $p_0(y)$ is of fourth order. For the gamma density with arbitrary n, it can be shown that $p_0(y)$ and its first n derivatives must vanish at $n = 1$; however, since the corresponding differential equation is of order $2n-2$, an additional $n-1$ conditions must be specified (one of these is that the density integrate to one).

VII. EXAMPLES

The results of some calculations will be given in this section in the two cases of intervals with McFadden's density and intervals with a gamma density with $n = 1$. Only symmetric cases will be considered. Since exact results are known for the McFadden interval density, it will be possible to use them to see just how accurate the approximate techniques of Sec. IV are. First, though, we consider cases in which the Fokker-Planck "approximation" is, in fact, exact.

A. <u>McFadden Interval PDF, $b = a - 1$ and $b = a - 2$</u>. The McFadden interval density (10a) degenerates to the exponential density when $b = a - 1$, and it is known that the Fokker-Planck density is then exact. What is somewhat surprising is that the Fokker-Planck "approximation" $p_{FP}(y)$ is also exact when $b = a - 2$. From (10b), when $b = a - 2$,

$$3\mu = \frac{1}{a} + \frac{1}{a-1} \tag{55}$$

and then using this and (31) in (10) leads to $u = a$. Consequently,

$$p(y) = p_{FP}(y) = \frac{(1-y^2)^{a-1}}{2^{2a-1}\,B(a,a)} \tag{56}$$

The Fokker-Planck result comes from (15), and it can be verified that (10b) for $p(y)$ is the same as (56) by first changing variables of integration in one of the incomplete beta functions (by letting $u \to 1-u$) in (10b), and then integrating by parts twice.

B. <u>McFadden Interval PDF, $b = a+3$</u>. In this case,

$$3\mu = \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} \tag{57}$$

$$u = \frac{3a(a+1)(a+2)}{3a^2 + 10a + 6} \tag{58}$$

and, again changing variables of integration in one of the incomplete beta functions in (10b), and integration by parts three times gives*

$$p(y) = \frac{(2a+1)(1-y^2)^{a-1}(3a+4+ay^2)}{(3a^2+6a-2)\,2^{2a}\,B(a,a)} \tag{59}$$

This is not of the same form as $p_{FP}(y)$.

For the special case $\beta = 1$ and $a = 1$, we have $\mu = 11/6$, $u = 18/19$, and

---

*The probability density $p(y)$ can be expressed as a polynomial in y whenever $b = a+m$ and m is an integer - then m integrations by part are required to get the polynomial form from (10b).

$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t} \tag{60a}$$

$$\mathfrak{F}(t) = 3e^{-t} - 3e^{-2t} + e^{-3t} \tag{60b}$$

$$h(t) = 3\sqrt{2}\, e^{-3t} \sin(\sqrt{2}t) \tag{60c}$$

$$p_{FP}(y) = \frac{(1-y^2)^{-1/19}}{2^{17/19}\, B(18/19, 18/19)} \tag{60d}$$

and the exact $p(y)$ is*

$$p(y) = \frac{3}{44}(7 + y^2) \tag{60e}$$

Table I compares values of $p(y)$, $p_{FP}(y)$ and $\hat{p}(y)$, the result of a single iteration by means of (41b). As can be seen from the table, even though $p_{FP}(y) \to \infty$ as $y \to 1$, $\hat{p}(y)$ approaches the correct value. Also, $\hat{p}(y)$ is accurate to three significant figures.

C. Gamma Interval PDF, $n = 1$. For the gamma interval density with $n = 1$, (32), (33) and (16) give

$$\mu = \frac{2}{a}; \quad u = \frac{\alpha}{2}\frac{1 + 2\alpha}{1 + \alpha}; \quad \alpha = aT \tag{61}$$

The Fokker-Planck approximation is especially simple when $u = 1$; i.e., when $\alpha = (1 + \sqrt{17})/4$. Then $p_{FP}(y) = 1/2$, and (39a) gives

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*In this case, (60e) and (11b) each result in $\overline{y^4} = 81/385$.

Table I. McFadden Interval PDF, $a = 1$, $b = 4$, $\partial = 1$. Comparison of Approximate and Exact Densities

y	$p(y)$	$\hat{p}(y)$	$p_{FP}(y)$
0	.4773	.4775	.4836
.2	.4800	.4801	.4847
.4	.4882	.4883	.4881
.6	.5018	.5018	.4951
.8	.5209	.5209	.5103
.9	.5325	.5324	.5278
.99	.5441	.5440	.5943
.999	.5453	.5452	.6707
1	.5455	.5454	∞

$$p_{00}(y) = \alpha \left\{ \frac{[(1+y)/2]^{2\alpha-1}}{(2\alpha-1)(\alpha-1)} - \frac{(1+y)/2}{\alpha-1} + \frac{2}{2\alpha-1} \right\} \quad ; \quad \alpha = (1+\sqrt{17})/4 \qquad (62)$$

This $p_{00}(y)$ was used to do one step of the iteration as outlined in Fig. 1, and the results are plotted in Fig. 2. Also shown on the figure are the results of an 8×8 matrix approximation to the integral equation for $p_0(y)$. As the curves show, both $p_{00}(y)$ and $p_{01}(y)$ agree well with the matrix approximation except in the vicinity of $y = -1$. In this region, only $p_{01}(y)$ has the correct asymptotic behavior as predicted by (48a). Also, $\hat{p}(y)$ has the correct asymptotic form as given by (52).

The conditional moment $A(y)$ was computed from (17) using $p_{01}(y)$ and $\hat{p}(y)$ for $p_0(y)$ and $p(y)$ respectively, and is shown in Fig. 3. As the figure shows, $A(y)$ is nearly linear except in the vicinity of the endpoints. By (43) and the asymptotic form (49a), it can be shown that $p_0(y)/p(y) \to 4$ as $y \to -1$, and consequently from (17) that

$$\frac{-A(y)}{\beta^2} \to \alpha \quad \text{as} \quad y \to 1 \tag{63}$$

which is consistent with the behavior depicted in Fig. 3. Similar computations were repeated for the case $\alpha = 1/2$, and are also shown in Fig. 3.

Fig. 4 contains curves of $p_{01}(y)$ for $\alpha = 1/2$, 1, 3 and 7 which were obtained by the same iteration procedure as just used for $\alpha = (1+\sqrt{17})/4$. These curves agreed to within a few percent with the results of an 8×8 matrix approximation except in the case $\alpha = 1/2$. Because $p_0(y)$ tends to bunch up around $y = -1$ for small α, a matrix approximation with more points, or with nonuniform spacing, becomes necessary. In contrast, iteration was found to work well even for the smaller values of α.

D. <u>McFadden Interval PDF, $a = 6$, $b = 41$</u>. In all of the previous cases, the Fokker-Planck approximations were themselves good approximations to $p(y)$. A case in which this is not so is that of the example used by McFadden in which $a = 6$, $b = 41$ and $\beta = 1$. In this case,

$$\upsilon = \sum_{n=0}^{34} \frac{1}{n+6} = 1.99520\ldots \tag{64}$$

$$u = \frac{35}{47\upsilon - 70} = 1.47214\ldots \tag{65}$$

Fig. 5 gives a comparison of $p(y)$ and $p_{FP}(y)$, and Fig. 6 shows the actual and Fokker-Planck approximation to $-A(y)/\beta^2$. The conditional moment is far from being approximately linear, and $p_{FP}(y)$ is not a good approximation to $p(y)$. Fig. 5 also shows some values of $\hat{p}(y)$ which were computed from (41b) and are listed in Table II. The agreement of this single iteration with the true $p(y)$ is excellent except in the region where $p(y)$ is rapidly changing.

In doing the calculations to get $\hat{p}(y)$, it was found easiest to evaluate the terms in the integrand of (41b) from

$$\mathfrak{F}(-\ln z) = 1 - \sum_{k=0}^{a-1} \binom{b-1}{k} z^k (1-z)^{b-1-k} \tag{66}$$

in which a and b are integers, and

$$p_{00}(-1 + 2e^{-\beta t}) = \frac{1}{2} u \, T e^{-\beta t} \, \mathcal{L}^{-1}\left\{ \frac{B(sT, 2u+1)}{B(sT, u)} \, \frac{sF(s)}{1-F(s)} \right\} \tag{67}$$

where $F(s)$ is given by (31). Eq. (66) is the result of repeated integration by parts when (10a) is used in the definition of $\mathfrak{F}(t)$, and (67) follows by recognizing that (39a) is a convolution when the change of variable (7) is employed for y and the change $\tau = -1 + 2e^{-\beta\tau}$ in the variable of integration. The inversion in (67) was done along the imaginary axis by means of the FFT.

Table II. McFadden Interval PDF, $a = 6$, $b = 41$, $\beta = 1$. Comparison of Approximate and Exact Densities

y	$p(y)$	$\hat{p}(y)$	$p_{FP}(y)$
0	.5012	.5012	.6297
.2	.5221	.5221	.6177
.4	.5955	.5956	.5800
.6	.7206	.7297	.5101
.8	.4321	.3933	.3888

VII. SUMMARY AND CONCLUSIONS

The problem of calculating the probability density function of the output of an RC filter driven by a class of binary random inputs has been studied in detail. Some new integral equations were derived, and methods for their solution were developed. Also, transformations of the integral equations into differential equations were investigated. Exact and approximate results were compared in several examples.

The matrix solution technique was seen to work well for the cases in which the densities being computed were smoothly varying, but requires a finer grid with more points to yield any erratic behavior. Iteration was seen to work well, with only a single iteration required when the initial estimate was good. In the last example of the McFadden interval PDF with $a = 6$, $b = 41$, even though the initial estimate was not particularly good, the single iteration was within 10% of the correct value. This example and the first example are felt to establish the veracity of the iteration process. The Fokker-Planck approximation was seen to not only be a good estimate for starting the iteration, but, in many cases, to be in itself an excellent approximation to the desired p(y).

All in all, the differential equation approach was disappointing since it had been hoped that it would lead to new closed form solutions. The differential equations are not only complicated in themselves, but correct boundary conditions for their solutions are not apparent in the general case. Nevertheless some useful asymptotic information was obtained.

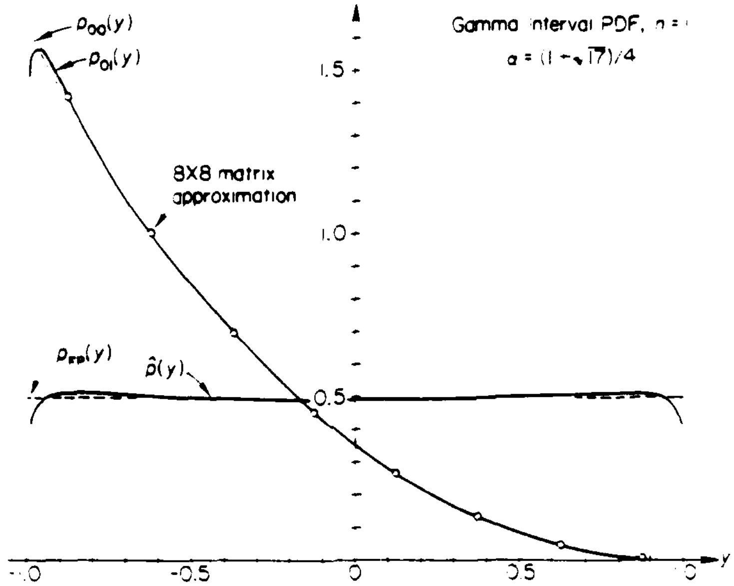

Fig. 2. Calculations for gamma interval PDF with $n = 1$ and $\alpha = (1 + \sqrt{17})/4$.

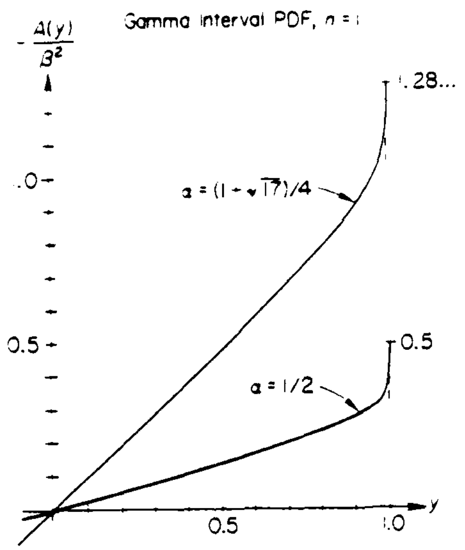

Fig. 3. Conditional moment of Fokker-Planck equation for gamma interval PDF with $n = 1$, $\alpha = (1 + \sqrt{17})/4$ and $\alpha = 1/2$.

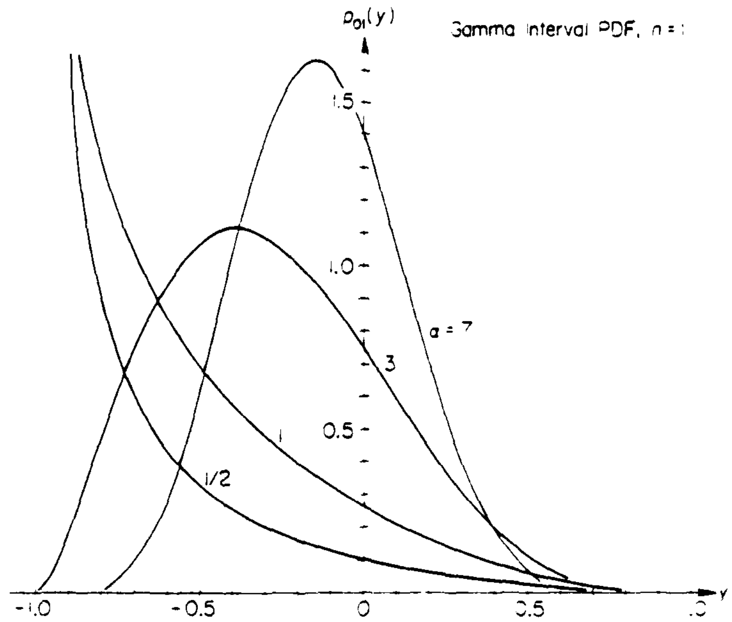

Fig. 4. The density $p_{01}(y)$ for a gamma interval PDF with $n = 1$ and α as a parameter.

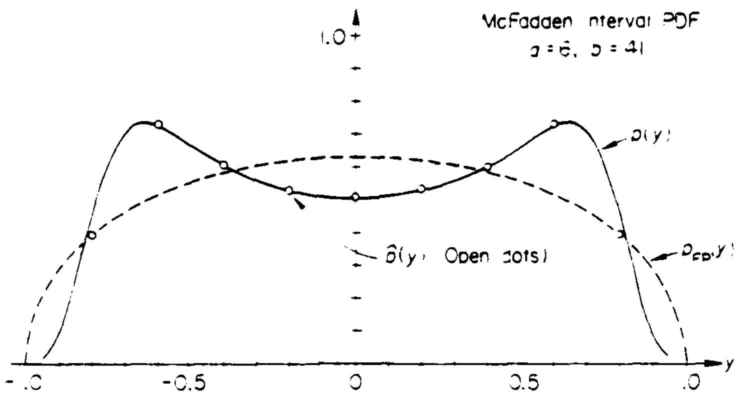

Fig. 5. Output probability density functions for McFadden interval PDF with a = 6, b = 41 and β = 1.

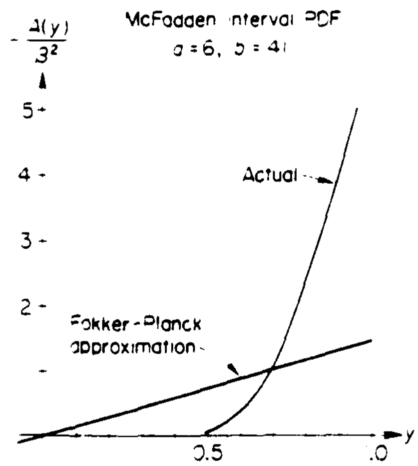

Fig. 6. Exact and approximate conditional moments for a McFadden interval PDF with a = 6, b = 41 and β = 1.

APPENDIX. CALCULATION OF $\overline{y^2}$

This appendix shows how $\overline{y^2}$ is determined from the system equation for arbitrary interval statistics of the binary input process. We will give the derivation only for the symmetric case, and rely heavily upon some results in Cox's book [3] on renewal theory.

The mean square of $y(t)$ can be obtained by integrating the power spectral density $S_y(f)$. Relating $S_y(f)$ to the spectral density $S_x(f)$ of the input binary process then results in

$$\overline{y^2} = \int_{-\infty}^{\infty} \frac{\beta^2 S_x(f)}{\beta^2 + \omega^2} \frac{d\omega}{2\pi} \tag{A1}$$

$S_x(f)$ can be further expressed in terms of the generating function of the number of renewals of an equilibrium renewal process as

$$S_x(t) = \int_0^{\infty} R_x(\tau)(e^{i\omega\tau} + e^{-i\omega\tau})\,d\tau$$

$$= G_e^*(-i\omega, -1) + G_e^*(i\omega, -1) \tag{A2}$$

where $R_x(\tau)$ is the autocorrelation function corresponding to $S_x(f)$, and $G_e^*(s, \lambda)$ is the generating function defined by Cox and evaluated by him as [3, eq. (3.2.6)]

$$G_e^*(s, -1) = \frac{1}{s} - \frac{2}{\mu s^2} \frac{1 - F(s)}{1 + F(s)} \tag{A3}$$

Using (A2) and (A3) in (A1) and doing the integrals by residue theory gives

$$\overline{y^2} = 1 - \frac{2}{3\mu} \frac{1 - F(3)}{1 + F(3)} \tag{A4}$$

Another expression for $\overline{y^2}$ follows from (14); viz.,

$$\overline{y^2} = \frac{1}{1 + 2u} \tag{A5}$$

Equating (A4) and (A5) yields (16).

The asymmetric case follows in much the same way using Cox's results on alternating renewal processes [3, Sec. 7.4].

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