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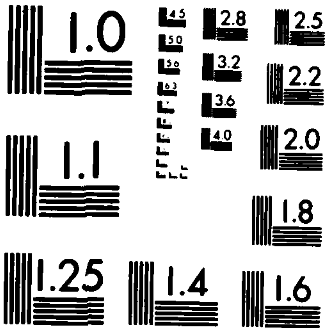

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ON LIMIT OF THE LARGEST EIGENVALUE OF THE LARGE DIMENSIONAL SAMPLE COVARIANCE MATRIX

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ABSTRACT

In this paper, the authors showed that the largest eigenvalue of the sample covariance matrix tends to a limit under certain conditions when both the number of variables and the sample size tend to infinity. The above result is proved under the mild restriction that the fourth moment of the elements of the sample sums of squares and cross products (SP) matrix exist.

Keywords: Largest eigenvalue, sample covariance matrix, large dimensional random matrices, limit.

1. INTRODUCTION

The distribution of the largest eigenvalue of the sample covariance matrix is useful in certain problems of inference in the area of multivariate analysis. For example, it is useful in testing the hypothesis that the eigenvalues of the covariance matrix are equal to a specified value. Geman (1980) showed that the largest eigenvalue of $A = WW'/n$ tends to $(1 + \sqrt{y})^2$ almost surely when $\lim (p/n) = y$, $W = (w_{ij})$ and w_{ij}'s are distributed independently with mean zero and variance one. In proving the above result, Geman assumed that $E|w_{11}|^n \leq n^{\alpha n}$ for $n = 1, 2, \ldots$ and a positive constant α. Jonsson (1983) announced the above result under the weaker condition that $E(|w_{11}|^7) < \infty$ by using a "truncation" method. Recently, Silverstein (1984) proved the same result under the condition that $E|w_{11}|^{6+\varepsilon} < \infty$ where $\varepsilon > 0$ is arbitrary. In the present paper, we prove the above result under a much weaker condition that $E|w_{11}|^4 < \infty$.

2. PRELIMINARIES

The following results are needed in the sequel:

Lemma 2.1. For any $\delta > 0$, we have

$$\binom{k}{r}\binom{k}{\ell-r}\binom{2k-\ell}{\ell}\delta^{k-\ell} \le (1+\sqrt{\delta})^{2k}\binom{k}{\ell}\binom{2\ell}{2r}$$

for all $1 \le r \le \ell \le k$, where $\binom{n}{m} = n!/m!(n-m)!$.

Proof. Let

$$I(r) = \binom{k}{r}\binom{k}{\ell-r}\Big/\binom{2\ell}{2r} \quad \text{for } r = 0, 1, \ldots, \ell.$$

Then, we have for $r = 0, 1, 2, \ldots, \ell - 1$,

$$\frac{I(r+1)}{I(r)} = \frac{(2r+1)(k-r)}{[2(\ell-r)-1][k-(\ell-r)+1]}.$$

But

$$(2r+1)(k-r) - [2(\ell-r)-1][k-(\ell-r)+1]$$
$$= (2r-\ell+1)(2k-2\ell+1) \ge 0 \quad \text{iff } 2r \ge \ell - 1.$$

So,

$$\frac{I(r+1)}{I(r)} \ge 1 \quad \text{iff} \quad 2r \ge \ell - 1.$$

Hence $I(r)$ has its maximum at $I(0) = I(\ell)$, and so

$$\left(\binom{k}{r}\binom{k}{\ell-r}\binom{2k-\ell}{\ell}\Big/\binom{k}{\ell}\binom{2\ell}{2r}\right)\delta^{k-\ell} \le \binom{2k-\ell}{\ell}\delta^{k-\ell} \le \sum_{\ell=0}^{k}\binom{2k-\ell}{\ell}\delta^{k-\ell}$$

$$= \sum_{\ell=0}^{k}\binom{k+\ell}{\ell}\delta^{\ell} \le \sum_{\ell=0}^{2k}\binom{2k}{\ell}\delta^{\ell/2} = (1+\sqrt{\delta})^{2k}.$$

In studying strong limit properties of random matrix, the techniques of truncation and centralization play an important role.

Lemma 2.2. (Truncation Lemma). Let r be a number belonging to the interval $[\frac{1}{2},2]$ and let $\{w_{ij}, i, j = 1, 2, \ldots\}$ be a collection of iid random variables with $Ew_{11} = 0$ and $E|w_{11}|^{2/r} < \infty$. For each n, define

W_n to be $p \times n$ matrix whose (i,j)th entry is w_{ij}, where $p = p(n)$ is such that $p/n \to y \in (0,\infty)$ as $n \to \infty$. Then there exists a sequence of positive numbers $\delta = \delta_n$ such that

(a) $\delta \to 0$ as $n \to \infty$,

(b) The convergence rate of δ can be slower than any preassigned rate,

(c) $P(W_n \neq \hat{W}_n, \text{ i.o.}) = 0$

where $\hat{W}$ is the $p \times n$ matrix whose (i,j)-th entry is $\hat{w}_{ij} = w_{ij} I_{[|w_{ij}|<\delta n^r]}$ and I_A denotes the indicator of the set A.

<u>Proof</u>. Since $E|w_{11}|^{2/r} < \infty$, we have for any $\varepsilon > 0$

$$\sum_{m=1}^{\infty} 2^{2m} P(|w_{11}| \geq \varepsilon 2^{mr}) < \infty .$$

Because of the arbitrariness of ε in the above inequality, there exists a sequence of positive numbers $\varepsilon = \varepsilon_m$ such that

(1) $\varepsilon_m \to 0$, when $m \to \infty$,

(2) The rate of convergence is slower than any preassigned rate,

(3) $\sum_{m=1}^{\infty} 2^{2m} P(|w_{11}| \geq \varepsilon_m 2^{mr}) < \infty$.

Define $\delta = \delta_n = 2\varepsilon_m$ for $2^{m-1} \leq n < 2^m$. It is obvious that such a sequence of δ satisfies the requirements (a) and (b). Define $\hat{W}_n$ with this δ. Because $\frac{p}{n} \to y$, $0 < y < \infty$, we have $p \leq 2yn$ when n is large enough. Thus

$$\begin{aligned}
&P(W_n \neq \hat{W}_n, \text{ i.o.}) \\
&\leq \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{2^{m-1}\leq n<2^m} \bigcup_{i=1}^{p} \bigcup_{j=1}^{n} (|w_{ij}| \geq \delta n^r)\Big) \\
&\leq \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{2^{m-1}\leq n<2^m} \bigcup_{i=1}^{2y2^m} \bigcup_{j=1}^{2^m} (|w_{ij}| \geq \varepsilon_m 2^{mr})\Big) \\
&= \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{i=1}^{2y2^m} \bigcup_{j=1}^{2^m} (|w_{ij}| \geq \varepsilon_m 2^{mr})\Big) \\
&\leq \lim_{k\to\infty} 2y \sum_{m=k}^{\infty} 2^{2m} P(|w_{11}| \geq \varepsilon_m 2^{mr}) = 0
\end{aligned}$$

which completes the proof.

Lemma 2.3. (Centralization Lemma) Under the assumptions of Lemma 2.2, we have

$$|\tilde{\lambda}_{max}(n) - \hat{\lambda}_{max}(n)| \longrightarrow 0. \quad \text{a.s.} \tag{2.1}$$

where $\tilde{\lambda}_{max}(n)$ and $\hat{\lambda}_{max}(n)$ are the largest eigenvalues of $\frac{1}{n^{2r}}\tilde{W}_n\tilde{W}_n'$ and and $\frac{1}{n^{2r}}\hat{W}_n\hat{W}_n'$, respectively, and $\tilde{W}_n$ is the $n \times n$ matrix whose (i,j)th entry is $\hat{w}_{ijn} - E\hat{w}_{ijn}$.

Proof. Denote by M_n the $n \times n$ matrix whose entries are all $E\hat{w}_{11n}$. Since

$$|E\hat{w}_{11n}| \le E|w_{11}I_{[|w_{11}| \ge \delta\sqrt{n}]}| \le E|w_{11}|^{2/r}(\delta\sqrt{n})^{-\frac{2}{r}+1},$$

we obtain

$$\begin{aligned}
&|\tilde{\lambda}_{max}(n) - \hat{\lambda}_{max}(n)| \\
&\le n^{-2r}\left\{2 \sup_{\|a\|=1} |a'\hat{W}_n M_n a| + \sup_{\|a\|=1} |a' M_n^2 a|\right\} \\
&= n^{-2r}\left\{2 \sup_{\sum_{i=1}^p a_i^2=1} |\sum_{i=1}^p a_i| \, |\sum_{i=1}^p \sum_{j=1}^n a_i \hat{w}_{ijn} E\hat{w}_{11n}| + n|E\hat{w}_{11n}|^2 \right.\\
&\qquad \left. \times \sup_{\sum_{i=1}^p a_i^2=1} \left(\sum_{i=1}^p a_i\right)^2\right\} \\
&\le cn^{-2r}\left\{n^{-\frac{1}{r}+1}\delta^{-\frac{2}{r}+1}\left(\sum_{i=1}^p \sum_{j=1}^n w_{ij}^2\right)^{1/2} + n^{-\frac{2}{r}+3}\delta^{-\frac{4}{r}+2}\right\} \\
&\le c\left\{n^{-\frac{1}{r}(r-1)^2-1}\delta^{-\frac{2}{r}+1}\left(n^{-2r}\sum_{i=1}^p \sum_{j=1}^n w_{ij}^2\right)^{1/2} \right.\\
&\qquad \left. + n^{-\frac{2}{r}(r-1)^2-1}\delta^{-\frac{4}{r}+2}\right\} \longrightarrow 0, \text{ a.s. if } r > 1.
\end{aligned}$$

$$\leq c \left\{ n^{-\frac{1}{r}(r-1)^2 - r} \delta^{-\frac{2}{r}+1} \left(n^{-2} \sum_{i=1}^{p} \sum_{j=1}^{n} w_{ij}^2 \right)^{1/2} + n^{-\frac{2}{r}(r-1)^2 - 1} \delta^{-\frac{4}{r}+2} \right\} \longrightarrow 0, \quad \text{a.s. if } r \leq 1. \tag{2.2}$$

In proving the above result, we have used the facts that

$$n^{-2r} \sum_{i=1}^{p} \sum_{j=1}^{n} w_{ij}^2 \longrightarrow 0, \quad \text{a.s. if } r > 1,$$

and

$$n^{-2} \sum_{i=1}^{p} \sum_{j=1}^{n} w_{ij}^2 \to Ew_n^2 \quad \text{a.s. if } r \leq 1,$$

which can be seen from Marcinkiewicz strong law of large numbers.

Remark 1. Throughout this paper, we will use Lemmas 2.2 and 2.3 with $r = \frac{1}{2}$, and the requirement (b) is specified as

$$\text{(b')} \ \delta \log n \to \infty, \text{ when } n \to \infty. \tag{2.3}$$

Remark 2. From Lemmas 2.2 and 2.3 we can easily see that

$$\lambda_{max}(n) - \tilde{\lambda}_{max}(n) \longrightarrow 0, \quad \text{a.s. as } n \longrightarrow \infty,$$

if the conditions of Theorem 1 hold, where $\lambda_{max}(n)$ is the largest eigenvalue of $\frac{1}{n}W_n W_n'$

3. SOME RESULTS ON GRAPH THEORY

Given a sequence $(i_1,j_1,i_2,j_2,\ldots,i_k,j_k)$, where $i_1,\ldots,i_k$ are integers in the set $\{1,\ldots,p\}$, and $j_1,\ldots,j_k$ are integers in the set $\{1,\ldots,n\}$, we define a directed multigraph as follows. We draw two parallel real lines, I-line and J-line. We plot $i_1,\ldots,i_k$ on the I-line and plot $j_1,\ldots,j_k$ on the J-line. These are vertices and they are split into two disjoint classes on the two parallel lines. So even if the two integers i_a and j_b are equal, they will not be the same vertex because i_a belongs to I-line and j_b belongs to J-line. But if $i_a = i_b$ (or $j_a = j_b$), we regard these two vertices identical. Edges of the directed bigraph will be the directed segments $\overrightarrow{i_1j_1}$, $\overrightarrow{j_1i_2}$, $\overrightarrow{i_2j_2}$, $\overrightarrow{j_2i_3},\ldots,\overrightarrow{i_kj_k}$, $\overrightarrow{j_ki_1}$. They are 2k in number and they should be regarded different from each other, even in the case when two edges have the same initials and ends.

Sometimes it is conveniant to denote i_a by v_{2a-1} and j_a by v_{2a}. So the vertices of the graph are $v_1,v_2,\ldots,v_{2k}$, and the edges are v_1v_2, $v_2v_3,\ldots,v_{2k-1}v_{2k}$, $v_{2k}v_1$. Notice that when we write an edge as v_av_{a+1}, we always mean that v_a is the initial vertex and v_{a+1} is the end vertex, the direction of the edge is from v_a to v_{a+1}.

When two edges v_av_{a+1}, v_bv_{b+1} have the same vertex sets, i.e. $\{v_a,v_{a+1}\} = \{v_b,v_{b+1}\}$, we cannot conclude that $v_av_{a+1} = v_bv_{b+1}$, since $v_av_{a+1} = v_bv_{b+1} \iff a = b$. When $\{v_a,v_{a+1}\} = \{v_b,v_{b+1}\}$, we say that the two edges coincide.

The graph we just constructed will be called a W-graph.

A W-graph will be called canonical, if $v_a \le \max\{v_{a-2},v_{a-4},\ldots\} + 1$, for each $a > 2$, and $v_1 = 1$, $v_2 = 1$.

In the following, we will get a bound for the number of canonical W-graphs.

In a canonical W-graph, an edge $v_{a-1}v_a$ ($a \geq 2$) will be called an innovation, if v_a does not occur in $v_1, v_2, \ldots, v_{a-1}$. Suppose $v_{a-1}v_a$ is an innovation. If a is odd, $v_{a-1}v_a$ is called a row innovation, and if a is even, a column innovation. Note that v_1v_2 is a column innovation according to the above definition.

In a W-graph, an edge $v_{a-1}v_a$ ($a \geq 2$) is said to be single up to v_b ($b \geq a$), if there is no edge $v_{c-1}v_c$ with $1 < c \leq b$ such that $v_{c-1}v_c$ coincides with $v_{a-1}v_a$. An edge $v_{b-1}v_b$ ($b \geq 3$) will be called a T_3-edge if there is an innovation $v_{a-1}v_a$, single up to v_{b-1}, and $v_{b-1}v_b$, $v_{a-1}v_a$ coincide.

An edge will be called a T_4-edge if it is not an innovation and not a T_3-edge.

A consecutive segment $v_a v_{a+1} \ldots v_{b-1} v_b$ of the whole W-graph will be called a chain.

<u>Lemma 3.1</u>. Let $v_a v_{a+1} \ldots v_c$ be a chain such that

(1) $v_a v_{a+1}$ is single up to v_c,

(2) v_c has been visited by $v_1 v_2 \ldots v_a$.

Then the chain contains at least one T_4-edge.

<u>Proof</u>. When $c - a = 1$, evidently $v_a v_{a+1} = v_a v_c$ is an edge of T_4.

We know that $v_{c-1}v_c$ must be a T_4 or a T_3 since it cannot be an innovation. If it is a T_4, the proof is completed. If $v_{c-1}v_c$ is a T_3, then there is a single innovation $v_{b-1}v_b$ coincident with $v_{c-1}v_c$ such that $b < c$.

<u>Case 1</u>. $b \geq a+1$. Since $v_c = v_{b-1}$ or $v_c = v_b$, then either $v_a v_{a+1} \ldots v_{b-1}$ or $v_a v_{a+1} \ldots v_b$ has the properties (1),(2) and is shorter than $v_a v_{a+1} \ldots v_c$. By induction hypothesis $v_a v_{a+1} \ldots v_b$ contains a T_4-edge, but it is a part of the chain $v_a v_{a+1} \ldots v_c$. So the original path contains a T_4-edge.

Case 2. $b < a+1$, i.e. $v_{b-1}v_b$ is in the path $v_1v_2\ldots v_a$, and then $v_{c-1} = (v_b$ or $v_{b-1})$ is visited by $v_1v_2\ldots v_a$. Thus $v_av_{a+1}\ldots v_{c-1}$ has the properties (1),(2). By induction, the lemma is proved.

Note: In a W-graph, the chain $v_1v_2\ldots v_{a+1}$ determines completely whether the edge v_av_{a+1} is an innovation, a T_3 or a T_4.

Lemma 3.2. If in the chain $v_1v_2\ldots v_a$, there are s edges, each of which is single up to v_a and has a vertex equal to v_a and if t is the number of noncoincident T_4 edges in $v_1v_2\ldots v_a$, then $s \leq t + 1$.

Proof. Let $v_{a_1-1}v_{a_1}, v_{a_2}v_{a_2+1}, \ldots, v_{a_s}v_{a_s+1}$ be all the single edges such that $a_1 < a_2 < \ldots < a_s < a$ and $v_{a_2}v_{a_2+1}, \ldots, v_{a_s}v_{a_s+1}$ are single up to v_a, and

$v_{a_1} = v_{a_2} = \ldots = v_{a_s} = v_a$.

Consider chains $L_2 = v_{a_2}v_{a_2+1} \ldots v_{a_3}$, $L_3 = v_{a_3}v_{a_3+1} \ldots v_{a_4}, \ldots,$ $L_s = v_{a_s}v_{a_s+1} \ldots v_a$. By Lemma 3.1, L_2 has a T_4-edge E_2. Let v_{b_3} be the first vertex in L_3 which belongs to L_2. Then by Lemma 3.1 $v_{a_3}v_{a_3+1} \ldots v_{b_3}$ contains an edge of T_4 and we denote it by E_3. Evidently, E_3 and E_2 are not coincident. Let v_{b_4} be the first vertex of L_4 which also belongs to $L_2 \cup L_3$. Then $v_{a_4}v_{a_4+1} \ldots v_{b_4}$ has an edge of T_4, by Lemma 3.1. Let it be denoted by E_4. Evidently no two of E_2, E_3, E_4 are coincident. Continue this procedure. Finally, we get $s-1$ edges of T_4, which are not coincident with each other. So $s-1 \leq t$, $s \leq 1+t$.

A T_3 edge v_av_{a+1} is called regular, if there is more than one innovation with a vertex equal to v_a and single up to v_a.

Lemma 3.3. In any W-graph there is a mapping Φ from regular T_3 edges to T_4 edges such that for any T_4 edge E, there are at most two regular T_3 edges whose Φ image is E.

Proof. Define Φ as follows.

Let $v_{a_1}v_{a_1+1}, v_{a_2}v_{a_2+1}, \ldots, v_{a_s}v_{a_s+1}$ be the set of all innovations single up to v_a such that

(1) $v_{a_2} = v_{a_3} = \ldots = v_{a_s} = v_a$

(2) $v_{a_1} = v_a$ or $v_{a_1+1} = v_a$

(3) $a_1 < a_2 < \ldots < a_s < a_{s+1} = a$.

We note that there is at most one innovation inward to v_a and if there is such an innovation, it must be the foremost one among innovations with a vertex v_a.

In this W-graph, $v_a v_{a+1}$ must coincide with one of $v_{a_1}v_{a_1+1}, \ldots, v_{a_s}v_{a_s+1}$. Let that one be $v_{a*}v_{a*+1}$. Also, let

$$\nu = \nu(a) = \begin{cases} i + 1, \text{ if } a* = a_i, \text{ for } i = 1,\ldots,s-1, \\ s, \quad \text{if } a* = a_s. \end{cases}$$

By Lemma 3.1, in the path $C_a = v_{a_\nu}v_{a_\nu+1} \cdots v_{a_\nu+1}$, there is at least one edge of T_4. Let the first one be E_a. Define $\Phi(v_a v_{a+1}) = E_a$.

At first we prove that if $v_a v_{a+1}$, $v_b v_{b+1}$ are two regular T_3 edges and $v_a \neq v_b$ then $E_a \neq E_b$ (E_a, E_b may coincide). Suppose $C_a = v_{a'}v_{a'+1}\cdots v_{a''}$, $C_b = v_{b'}v_{b'+1}\cdots v_{b''}$. Then, we have the following possibilities to consider:

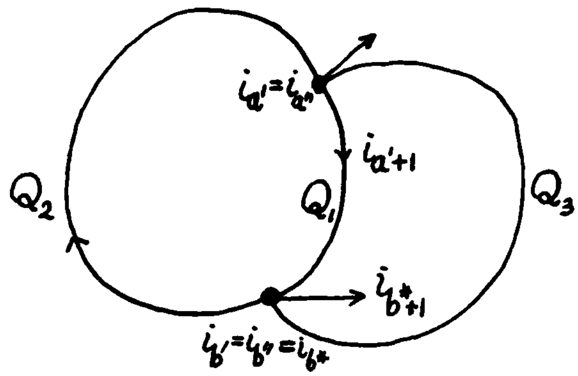
(1) $a' < a'' < b' < b''$,

(2) $a' < b' < a'' < b''$,

(3) $a' < b' < b'' < a''$.

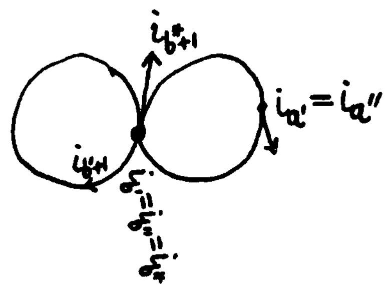
For case (1), $E_a \neq E_b$ is evident.

Consider case (2). C_a, C_b are divided into three parts as $Q_1 = v_{a'}v_{a'+1}\cdots v_{b'}$, $Q_2 = v_{b'}v_{b'+1}\cdots v_{a''}$, $Q_3 = v_{a''}v_{a''+1}\cdots v_{b''}$ as given in the following diagram.



It is enough to show that E_a is in Q_1, E_b is in $Q_2 \cup Q_3$. By definition, there is an innovation $v_{b^*}v_{b^*+1}$ with $b^* < b'$, $v_{b^*}=v_{b'}=v_b$, single up to v_b. If $b^* \leq a'$, by Lemma 3.1, Q_1 contains a T_4-edge. If $a' < b^* < b'$, we can consider the chain $v_{b^*}v_{b^*+1}\cdots v_{b'}$; it is part of Q_1 and it contains a T_4-edge by Lemma 3.1. So $E_a \in Q_1$. It is obvious that E_b is in $Q_2 \cup Q_3$. Thus $E_a \neq E_b$.

Consider (3). As before, we can show that E_a is in $v_{a'}v_{a'+1}\cdots v_{b'}$ and E_b is in $v_{b'}v_{b'+1}\cdots v_{b''}$.



Then we consider the case $v_a = v_b$ and $a < b$. Now v_av_{a+1}, v_bv_{b+1} both are regular T_3 edges and so they coincide with single innovations $v_{a^*}v_{a^*+1}$ and $v_{b^*}v_{b^*+1}$. $v_{a^*}v_{a^*+1}$ and $v_{b^*}v_{b^*+1}$ cannot coincide. So $E_a \neq E_b$ except $v_{a^*}v_{a^*+1}$, $v_{b^*}v_{b^*+1}$ are the last two single innovations. In the last case $\Phi^{-1}(E_a)$ has cardinal 2.

At last we get that the mapping Φ: $v_a v_{a+1} \longrightarrow E_a$ for regular T_3's has the property that $\Phi^{-1}(E_a)$ has at most two edges of T_3. The proof of Lemma 3.3 is completed.

4. LIMIT OF THE LARGEST EIGENVALUE

We now prove the main result of our paper.

Theorem 3.1. Let $\{w_{ij}: \ i,j = 1,2,\ldots\}$ be an infinite matrix of iid random variables, $Ew_{11} = 0$ and $Ew_{11}^4 < \infty$. If $\lambda_{max}(n)$ denotes the largest eigenvalue of the matrix $\frac{1}{n} W_n W_n'$, here W_n denotes the $p \times n$ random matrix $\{w_{ij};\ i = 1,\ldots,p;\ j = 1,\ldots,n\}$, *then*

$$\lim_{n\to\infty} \lambda_{max}(n) = (1 + \sqrt{y})^2 \, Ew_{11}^2 \quad \text{a.s.}$$

as $n \to \infty$, $p \to \infty$ and $p/n \to y$.

Proof. Without loss of generality, we assume that $Ew_{11}^2 = 1$. We only have to prove that $\overline{\lim}\ \lambda_{max}(n) \leq (1 + \sqrt{y})^2$ a.s. . But, according to Remark 2 of Section 2, it is sufficient to show that

$$\overline{\lim}\ \tilde{\lambda}_{max}(n) \leq (1 + \sqrt{y})^2 \quad \text{a.s. .}$$

In other words, we assume in the sequel that

(1) $|w_{ij}| < \delta\sqrt{n}$,

(2) $Ew_{ij} = 0$,

(3) $Ew_{ij}^2 \leq 1$,

(4) $E|w_{ij}|^{\ell} \leq (\delta\sqrt{n})^{\ell-1}$, for $\ell \geq 2$,

(5) $E|w_{ij}|^{\ell} \leq c(\delta\sqrt{n})^{\ell-3}$, for $\ell \geq 3$.

Now, choose $z > (1 + \sqrt{y})^2$ arbitrarily. We will now show that

(6) $$\sum_{n=1}^{\infty} E\left(\frac{\lambda_{max}(n)}{z}\right)^k < \infty$$

where $k = k_n$ satisfies

(7) $k_n / \log n \longrightarrow \infty$,

(8) $\delta^{1/6} k_n / \log n \longrightarrow 0$.

We have

$$E[\lambda_{max}(n)]^k \le E\, tr(\frac{1}{n} W\, W^T)^k = n^{-k}\, E\, tr(W\, W^T)^k$$

$$= n^{-k} \sum E\, w_{i_1 j_1}\, w_{i_2 j_1}\, w_{i_2 j_2} \cdots w_{i_k j_k}\, w_{i_1 j_k} .$$

Here the summation is taken in such a way that $i_1,\ldots,i_k$ run over all integers in $\{1,\ldots,p\}$, and $j_1,\ldots,j_k$ run over all integers in $\{1,\ldots,n\}$.

The above sum can be split in the following form.

$$E\, tr(\frac{1}{n} W\, W^T)^k = n^{-k} \sum{}' \sum{}'' \sum{}''' E\, w_{i_1 j_1}\, w_{i_2 j_1}\, w_{i_2 j_2} \cdots w_{i_k j_k}\, w_{i_1 j_k},$$

here

$\sum'$ - summation for different arrangements of four different types of elements at the 2k different positions.

$\sum''$ - summation for different canonical graphs Γ with given arrangement of the four types for 2k positions.

$\sum'''$ - summation of $E\, w_{i_1 j_1}\, w_{i_2 j_1} \cdots w_{i_1 j_k}$ for which the graph is isomorphic to the given canonical graph.

Let r denote the number of row innovations, ℓ denote the number of T_3 edges. Then there are $\ell - r$ column innovations and $(2k - 2\ell)T_4$ edges and so $\sum'$ is bounded by $\sum_{\ell=1}^{k} \sum_{r=1}^{\ell} \binom{k}{r}\binom{k}{\ell-r}\binom{2k-\ell}{\ell}$. Since every row innovation leads to a free i-index and every column innovation leads to a free j-index except the first column innovation $v_1 v_2$ which leads to an i-index and a j one. We know that $\sum'''$ is bounded by $p^{r+1}\, n^{\ell-r}$.

To bound $\sum''$, let t denote the number of noncoincident T_4 edges. By definition, each innovation in a canonical W-graph is uniquely determined by the chain before it and each nonregular T_3 edge is so done as innovations. If $t = 0$, i.e. $\ell = k$, then by Lemma 3.3, there are no T_4 edge and regular T_3 edge, so that $\sum''$ is only one summand. For $t \geq 1$, since each T_4 edge in a canonical W-graph must be one of the k^2 elements in the $k \times k$ matrix (w_{ij}) $i \leq k$, $j \leq k$, all the possibilities that $2k - 2\ell$ T_4 edges may take are less than $\binom{k^2}{t} t^{2k-2\ell}$. By Lemmas 3.2 and 3.3, all the possibilities that all the regular T_3 edges take is not more than $(t+1)^{4k-4\ell}$. Hence $\sum''$ is bounded by $\binom{k^2}{t} t^{2k-2\ell}(t+1)^{4k-4\ell}$ $\leq k^{2t}(t+1)^{6k-6\ell}$.

Finally, we bound the expectation $E\, w_{i_1j_1} w_{i_2j_1} \cdots w_{i_kj_k} w_{i_1j_k}$. If $t = 0$, each expectation is $(E\, w_{11}^2)^{\ell} \leq 1$. For $t \geq 1$, let μ denote the number of innovations which coincide with at least one T_4 edge and let n_i denote the number of T_4 edges which coincide with the i-th such innovation, $i = 1,2,\ldots,\mu$, respectively.

Let m_j be the number of T_4 edges which coincide with each other but not with any innovation, $j = 1,2,\ldots,t-\mu$. Then we have

$$E\, w_{i_1j_1} w_{i_2j_1} \cdots w_{i_kj_k} w_{i_1j_k} = (E\, w_{11}^2)^{\ell-\mu} \prod_{i=1}^{\mu} (E\, w_{11}^{n_i+2}) \prod_{j=1}^{t-\mu} E\, w_{11}^{m_j}$$

where $2(\ell-\mu) + \sum_{i=1}^{\mu} (n_i + 2) + \sum_{j=1}^{t-\mu} m_i = 2k$ and $\mu \leq t$. By (3),(4) and (5) we have

$$|E\, w_{i_1j_1} w_{i_2j_1} \cdots w_{i_kj_k} w_{i_1j_k}| \leq c^{\mu}(\delta\sqrt{n})^{\sum_{i=1}^{\mu}(n_i-1)+\sum_{j=1}^{t-\mu}(m_j-1)}$$

$$= c^{\mu}(\delta\sqrt{n})^{2k-2\ell-t} \leq k^t(\delta\sqrt{n})^{2k-2\ell-t}$$

when k is large enough.

By the above argument, we obtain

$$E\,\mathrm{tr}(\tfrac{1}{n} W_n W_n^T)^k \le n^{-k} \sum_{\ell=1}^{k} \sum_{r=1}^{\ell} \binom{k}{r}\binom{k}{\ell-r}\binom{2k-\ell}{\ell} p^{r+1} n^{\ell-r} [\sum_{t=0}^{2k-2\ell} k^{2t}(t+1)^{6k-6\ell} k^t(\delta\sqrt{n})^{2k-2\ell-t}]$$

$$\le p \sum_{\ell=1}^{k} \sum_{r=1}^{\ell} \binom{k}{r}\binom{k}{\ell-r}\binom{2k-\ell}{\ell}\left(\frac{p}{n}\right)^r [\sum_{t=0}^{2k-2} k^{3t}(t+1)^{6k-6\ell}(\delta\sqrt{n})^{-t}]\ \delta^{2(k-\ell)}$$

Using the elementary inequality $a^{-t}(t+1)^b \le \left(\frac{b}{\log a}\right)^b$ for $a > 1$, $b > 0$, $t > 0$, we have

$$\sum_{t=0}^{2k-2\ell} k^{3t}(t+1)^{6k-6\ell}(\delta\sqrt{n})^{-t} \le 2k \left(\frac{6k-6\ell}{\log \frac{\delta\sqrt{n}}{k^3}}\right)^{6k-6\ell}$$

$$\le 2k \left(\frac{6k}{\frac{1}{2}\log n + \log\delta - 3\log k}\right)^{6(k-\ell)}$$

$$\le 2k \left(\frac{18k}{\log n}\right)^{6(k-\ell)} \quad \text{when } n \text{ is large enough.}$$

Using Lemma 2.1, we get for all large n

$$E\,\mathrm{tr}(\tfrac{1}{n} W_n W_n^T)^k$$

$$\le 2kp \sum_{\ell=1}^{k}\sum_{r=1}^{\ell} (1+\sqrt{\delta})^{2k} \binom{k}{\ell}\binom{2\ell}{2r}\left(\frac{p}{n}\right)^r [\frac{18\delta^{1/6}k}{\log n}]^{6(k-\ell)}$$

$$\le 2kp\,(1+\sqrt{\delta})^{2k}\,[(1+\sqrt{p/n})^2 + [\frac{18\delta^{1/6}k}{\log n}]^6]^k$$

$$= (2kp)^{1/k}(1+\sqrt{\delta})^2((1+\sqrt{p/n})^2 + [\frac{18\delta^{1/6}k}{\log n}]^6)^k$$

$$\le \eta^k,$$

where η is a constant satisfying $(1+\sqrt{y})^2 < \eta < z$. Here the last inequality follows from the following facts:

(a) $(2kp)^{1/k} \longrightarrow 1$, because $k/\log n \longrightarrow \infty$ and $p/n \longrightarrow y \in (0,\infty)$

(b) $(1+\sqrt{\delta})^2 \longrightarrow 1$, because $\delta \longrightarrow 0$

(c) $(1+\sqrt{p/n})^2 \longrightarrow (1+\sqrt{y})^2$, because $p/n \longrightarrow y \in (0,\infty)$

(d) $\left(\frac{18\delta^{1/6}k}{\log n}\right)^6 \longrightarrow 0$, because $\delta^{1/6}k/\log n \longrightarrow 0$.

This leads to (6) since $k/\log n \to \infty$ and the proof is thus complete.

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In this paper, the authors showed that the largest eigenvalue of the sample covariance matrix tends to a limit under certain conditions when both the number of variables and the sample limit size tend to infinity. The above result is proved under the mild restriction that the fourth moment of the elements of the sample sums of squares and cross products (SP) matrix exist.

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