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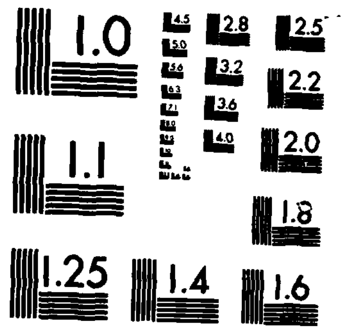
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NONLINEAR 0-1 PROGRAMMING:

I. LINEARIZATION TECHNIQUES

by

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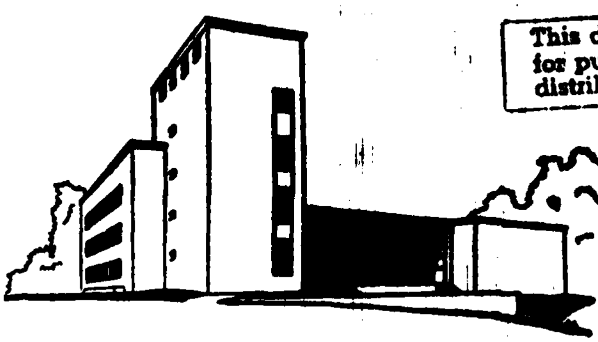

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Abstract

Any real-valued nonlinear function in 0-1 variables can be rewritten as a multilinear function. We discuss classes of lower and upper bounding linear expressions for multilinear functions in 0-1 variables. For any multilinear inequality in 0-1 variables, we define an equivalent family of linear inequalities. This family contains the well known system of generalized covering inequalities, as well as other linear equivalents of the multilinear inequality that are more compact, i.e., of smaller cardinality. In a companion paper ~~[7]~~, we discuss dominance relations between various linear equivalents of a multilinear inequality, and describe a class of algorithms for multilinear 0-1 programming based on these results.

NONLINEAR 0-1 PROGRAMMING:

I. LINEARIZATION TECHNIQUES

by

Egon Balas and Joseph B. Mazzola

1. Introduction

It is well known [15] that a real-valued function f(x) in 0-1 variables can be rewritten as a multilinear function in the same variables, i.e.,

$$(1) \qquad f(x) = \sum_{j \in N} a_j \left(\prod_{i \in Q_j} x_i \right), \quad x_i = 0 \text{ or } 1, \quad i \in \bigcup_{j \in N} Q_j,$$

where a_j, $j \in N$, are real numbers, and π means product. Thus, without loss of generality, when discussing nonlinear 0-1 programs it is sufficient to consider the general multilinear program

$$\text{(MLP)} \qquad \max\{f_0(x) \mid f_k(x) \le b_k,\ k \in K,\ x \text{ binary}\},$$

where f_0 and f_k, $k \in K$, are multilinear functions of the form (1).

The subject of nonlinear 0-1 programming has received considerable attention in the literature (see for example [8-9, 20, 1-3, 17, 14, 19, 10-11, 5, 12-13, 16]. For a survey of the area, see Hansen [18].

Applications involving nonlinear 0-1 programs arise in a variety of areas. For a partial list of these, along with references, see our companion paper [7].

The most frequently used approach to solving (MLP) consists of linearizing the problem and then solving the resulting linear 0-1 program. An early linearization, due to Fortet [8, 9], involves the replacement of every distinct product of 0-1 variables by a new 0-1 variable and a pair of new inequalities. This approach, while useful when there are only a few distinct products of 0-1 variables, becomes less and less practical as the number of such products increases. Glover and Woolsey [10, 11] pointed out that under certain conditions the number of new constraints can be reduced, and the new 0-1 variables can be replaced by continuous variables.

A linearization of (MLP) that does not require new variables was given by Granot and Hammer [14], who have shown (MLP) to be equivalent to a generalized covering problem, i.e., a set covering problem in the original variables and their complements. While the number of covering inequalities in this formulation tends to be exponential in the number of variables, Granot, Granot and Kallberg [13] (see also Granot and Granot [12]) have recently used this linearization in an algorithm that generates the constraints sequentially, as needed, and for relatively sparse problems manages to avoid producing an excessively large number of them.

In this paper we develop a new linearization for nonlinear functions and inequalities in 0-1 variables, which uses only the original variables. Our results were first presented in [5] and then circulated under [6].

In section 2, we introduce some families of lower and upper bounding affine functions for the multilinear function f(x) of (1). We start by assuming that $a_j > 0$, $j \in N$, and define a family $\mathcal{L}$ of affine functions $g_M(x)$, one for every $M \subseteq N$, such that $g_M(x) \leq f(x)$ for all binary x, and $\mathcal{L}$ is complete in the sense that for every binary x some function in $\mathcal{L}$ is equal to f(x). We then identify a proper subfamily of $\mathcal{L}$ that is also complete. Next we define a family $\mathcal{U}$ of linear functions $h_\varphi(x)$, one for every mapping

φ that associates to any set Q_j one of its elements, such that $h_\varphi(x) \geq f(x)$ for all binary x, and $\mathcal{U}$ is complete. We then identify a proper subfamily of $\mathcal{U}$ that is also complete. We establish several properties of these families of lower and upper bounding functions, and then combine the above results to define a (complete) family $\mathcal{B}_0$ of lower bounding affine functions for a multilinear function f(x) with coefficients a_j of arbitrary sign.

In section 3, we turn to inequalities of the form

(2) $\quad f(x) \leq b$

where f(x) is the multilinear function given by (1) and x is binary. For a multilinear inequality (2) in 0-1 variables, we introduce a family $\mathcal{F}$ of linear inequalities, equivalent to (2) in the sense of having the same solution set. The members of $\mathcal{F}$ correspond to subsets of N that are covers for the inequality (2). The family $\mathcal{F}$ strictly contains the family of generalized covering inequalities defined in [14]. More specifically, the generalized covering inequalities are those members of $\mathcal{F}$ associated with minimal covers for (2). Furthermore, $\mathcal{F}$ typically contains more compact linearizations of (2) than the family of generalized covering inequalities, i.e., linear equivalents of (2) that are of smaller cardinality than the latter family. These more compact linearizations are associated with covers for (2) that are not minimal.

In a companion paper [7], we characterize certain dominance relations between members of the family $\mathcal{F}$, and give a procedure for strengthening inequalities of $\mathcal{F}$ that satisfy certain conditions. Based on this, we have developed and implemented a class of algorithms for multilinear 0-1 progrms that we describe in [7], where we also discuss our computational experience.

2. Lower and Upper Bounding Affine Functions

Consider the multilinear function (1) of section 1, and for any $M \subseteq N$, let $Q_M = \bigcup_{j \in M} Q_j$, and let $Q = Q_N$, $q = |Q|$. Also, for any $x \in \{0,1\}^q$, let $Q(x) = \{i \in Q | x_i = 1\}$ be the support of x. A function g(x) is said to be a lower (upper) bounding function for f(x) if $f(x) \geq g(x)$ $(f(x) \leq g(x))$ for all $x \in \{0,1\}^q$.

In what follows, summation over the empty set is always taken to yield zero.

Lemma 1. For $j \in N$, define

$$(3) \qquad p_j(x) = \prod_{i \in Q_j} x_i \quad , \quad s_j(x) = \sum_{i \in Q_j} x_i - |Q_j| + 1.$$

Then for every $j \in N$ and $x \in \{0,1\}^q$,

$$(4) \qquad p_j(x) \geq 0 \quad , \quad p_j(x) \geq s_j(x),$$

and

$$(5) \qquad \begin{aligned} p_j(x) &= 0 \qquad \text{if and only if} \quad |Q_j \backslash Q(x)| \geq 1, \\ p_j(x) &= s_j(x) \quad \text{if and only if} \quad |Q_j \backslash Q(x)| \leq 1. \end{aligned}$$

Proof. (4) follows from (3), while (5) follows from (3), (4) and the definition of $Q(x)$.||

Next we introduce a family of lower bounding affine functions for a given f(x) with positive coefficients. The family has a member for every subset of N.

Theorem 1. Let f(x) be as in (1), with $a_j > 0$, $j \in N$, and for every $M \subseteq N$, define

$$(6) \qquad g_M(x) = \sum_{i \in Q_M} \Big(\sum_{j \in M | i \in Q_j} a_j\Big) x_i - \sum_{j \in M} (|Q_j| - 1) a_j.$$

Then every $x \in \{0,1\}^q$ satisfies the inequality

$(7)_M$ $\qquad f(x) \geq g_M(x)$

for every $M \subseteq N$, and $(4)_M$ holds with equality if and only if

(8) $\qquad \{j \in N \,\big|\, |Q_j \setminus Q(x)| = 0\} \subseteq M \subseteq \{j \in N \,\big|\, |Q_j \setminus Q(x)| \leq 1\}.$

<u>Proof</u>. Since

$$f(x) = \sum_{j \in N} a_j p_j(x)$$

and for any $M \subseteq N$

$$g_M(x) = \sum_{j \in M} a_j s_j(x),$$

$(7)_M$ follows from (4). Further, from the definitions $f(x) = g_M(x)$ if and only if $p_j(x) = 0$ for $j \in N \setminus M$ and $p_j(x) = s_j(x)$ for $j \in M$, and therefore (5) implies (8).||

<u>Remark 1</u>. For every $x \in \{0,1\}^q$, there exists some $M \subseteq N$ such that $f(x) = g_M(x)$.

<u>Proof</u>. Set $M = \{j \in N | Q_j \subseteq Q(x)\}$. Then x and M satisfy (8), hence $f(x) = g_M(x)$.||

Denoting by $f_j(x)$ the j^{th} term of $f(x)$ in (1) and applying Theorem 1 to $f_j(x)$, we obtain $f_j(x) \geq g_{\{j\}}(x)$. Furthermore, we have

<u>Remark 2</u>. $g_M(x) = \sum_{j \in M} g_{\{j\}}(x)$.

<u>Proof</u>. By applying the definition of $g_M(x)$ to $\{j\}$ for each $j \in M$.||

Remark 3. For all $M \subseteq N$ and $\lambda \in R^{|M|}$ such that $0 \leq \lambda_j \leq 1$, $j \in M$,

$$(7)^{\lambda}_{M} \qquad f(x) \geq \sum_{j \in M} \lambda_j \, g_{\{j\}}(x)$$

for all $x \in \{0,1\}^q$.

Proof. Follows from $\lambda_j \geq 0$, $j \in M$, $f_j(x) \geq g_{\{j\}}(x)$, $j \in M$, and $f(x) \geq \sum_{j \in M} f_j(x)$.||

When $f(x)$ is a quadratic function, the family of lower bounding functions given by $(7)^{\lambda}_{M}$ specializes to the one defined by Hammer, Hansen and Simeone [16].

A set $\mathcal{H}$ of lower (upper) bounding functions $h(x)$ for $f(x)$ will be called complete if for every $x \in \{0,1\}^q$ there exists $h \in \mathcal{H}$ such that $h(x) = f(x)$. From Remark 1, the set $\mathcal{L} = \{g_M(x) | M \subseteq N\}$ is complete. Since $\mathcal{L}$ is fairly large ($|\mathcal{L}| = 2^{|N|}$), it is of interest to find proper subsets of $\mathcal{L}$ that are complete. Next we identify one such subset.

For any $M \subseteq N$, define

$$E(M) = \{j \in N \big| |Q_j \setminus Q_M| \leq 1\}.$$

Clearly, $M \subseteq E(M)$ for all $M \subseteq N$. Also, note that for arbitrary subsets $M_1, M_2 \subseteq N$, $M_1 \neq M_2$ does not imply $E(M_1) \neq E(M_2)$.

Consider now the family

$$\mathcal{L}_0 = \{g_{E(M)}(x) | M \subseteq N\}$$

of lower bounding functions for $f(x)$, whose cardinality is typically much smaller than that of $\mathcal{L}$.

Theorem 2. Let $f(x)$ be as in (1), with $a_j > 0$, $j \epsilon N$. Then $\mathcal{L}_0$ is a complete set of lower bounding functions for $f(x)$.

Proof. For a given $x \epsilon \{0,1\}^q$, define $L = \{j \epsilon N | Q_j \subseteq Q(x)\}$. Then $g_{E(L)} \epsilon \mathcal{L}_0$, and condition (8) is satisfied for $M = E(L)$, which implies $f(x) = g_{E(L)}(x)$. Since this is true for every $x \epsilon \{0,1\}^q$, $\mathcal{L}_0$ is complete.||

Remark 4. For every $M \subseteq N$, there exists some $x \epsilon \{0,1\}^q$ such that $f(x) = g_{E(M)}(x)$.

Proof. For given $M \subseteq N$, let $\hat{x}$ be defined by $Q(\hat{x}) = \bigcup_{j \epsilon M} Q_j$. Then

$$\{j \epsilon N \,\big|\, |Q_j \backslash Q(\hat{x})| = 0\} \subseteq E(M) = \{j \epsilon N \,\big|\, |Q_j \backslash Q(\hat{x})| \leq 1\},$$

and hence, from Theorem 1, $f(\hat{x}) = g_{E(M)}(\hat{x})$.||

Note that, while every lower bounding function in $\mathcal{L}_0$ is "attained" by $f(x)$ for some $x \epsilon \{0,1\}^q$, the same is not true in general with respect to the larger family $\mathcal{L}$. For example, let

$$f(x) = x_1x_2x_3 + x_4x_5 + x_1x_4 + x_1x_5 + x_2x_5 + x_3x_4,$$

and choose $M = \{1,2\}$, where $Q_1 = \{1,2,3\}$, $Q_2 = \{4,5\}$. Then the lower bounding function

$$g_{\{1,2\}}(x) = x_1 + x_2 + x_3 + x_4 + x_5 - 3$$

is not equal to $f(x)$ for any $x \epsilon \{0,1\}^5$.

Next we illustrate the families $\mathcal{L}$ and $\mathcal{L}_0$ on an example.

Example 1. Let

$$f(x) = 3x_1x_2x_3 + 2x_1x_4 + x_2x_3x_4.$$

Then $Q_1 = \{1,2,3\}$, $Q_2 = \{1,4\}$, $Q_3 = \{2,3,4\}$, and

$$\begin{aligned}
g_{\{1,2,3\}} &= 5x_1 + 4x_2 + 4x_3 + 3x_4 - 10\\
g_{\{1,2\}} &= 5x_1 + 3x_2 + 3x_3 + 2x_4 - 8\\
g_{\{2,3\}} &= 2x_1 + x_2 + x_3 + 3x_4 - 4\\
g_{\{1,3\}} &= 3x_1 + 4x_2 + 4x_3 + x_4 - 8\\
g_{\{1\}} &= 3x_1 + 3x_2 + 3x_3 - 6\\
g_{\{2\}} &= 2x_1 + 2x_4 - 2\\
g_{\{3\}} &= x_2 + x_3 + x_4 - 2\\
g_{\emptyset} &= 0
\end{aligned}$$

A complete system of lower bounding functions consists of

$$\mathcal{L}_0 = \{g_{\{1,2,3\}}, g_{\{2\}}, g_{\emptyset}\},$$

since for all $M \subset \{1,2,3\}$, $M \neq \{2\}, \emptyset$, we have $E(M) = \{1,2,3\}$.||

We now turn to upper bounding linear functions for $f(x)$. Let φ be a mapping that associates to every $j \in N$ some $i \in Q_j$, i.e., $\varphi(j) \in Q_j$, $j \in N$, and let Φ be the set of all such mappings.

<u>Theorem 3</u>. Let $f(x)$ be as in (1), with $a_j > 0$, $j \in N$. For $\varphi \in \Phi$, define

$$h_\varphi(x) = \sum_{j \in N} a_j x_{\varphi(j)} .$$

Then every $x \in \{0,1\}^q$ satisfies the inequality

$$(9)_\varphi \qquad f(x) \leq h_\varphi(x)$$

for every $\varphi \in \Phi$, and $(9)_\varphi$ holds as equality if and only if $\varphi(j) \in Q_j \setminus Q(x)$ for all $j \in N$ such that $Q_j \setminus Q(x) \neq \emptyset$.

Proof. For a given $x \in \{0,1\}^q$, define $M = \{j \in N | Q_j \subseteq Q(x)\}$. Then for every $\varphi \in \Phi$,

$$f(x) = \sum_{j \in M} a_j \qquad \text{(by the choice of M)}$$

$$\text{(10)} \qquad \leq \sum_{j \in N} a_j x_{\varphi(j)} = h_\varphi(x) \qquad \text{(since } a_j > 0,\ j \in N\text{)},$$

i.e., $(9)_\varphi$ is satisfied.

If $\varphi(j) \in Q_j \backslash Q(x)$ for all $j \in N$ such that $Q_j \backslash Q(x) \neq \emptyset$, then $x_{\varphi(j)} = 0$, $\forall\ j \in N \backslash M$, and the inequality in (10), hence in $(9)_\varphi$, holds as equality. Conversely, if $\varphi(j) \in Q_j \cap Q(x)$ for some $j \in N \backslash M$, then $x_{\varphi(j)} = 1$ and $(9)_\varphi$ holds as strict inequality since $a_j > 0$, $j \in N$.||

Remark 5. If $f(x)$ is as in (1) but with $a_j < 0$, $j \in N$, then for every $\varphi \in \Phi$, $h_\varphi(x)$ is a lower bounding linear function for $f(x)$.

Proof. Applying Theorem 3 to $-f(x)$ yields $-f(x) \leq -h_\varphi(x)$, $\forall\ \varphi \in \Phi$.||

Remark 6. For every $\varphi \in \Phi$ there exists some $x \in \{0,1\}^q$ for which $f(x) = h_\varphi(x)$.

Proof. Both $x = 0$ and $x = e$, where $e = (1,\ldots,1)$, produce equality in $(9)_\varphi$ for all $\varphi \in \Phi$.

Remark 7. For every $x \in \{0,1\}^q$, there exists some $\varphi \in \Phi$ such that $f(x) = h_\varphi(x)$.

Proof. Use any mapping satisfying $\varphi(Q_j) \in Q_j \backslash Q(x)$ for all $j \in N$ such that $Q_j \backslash Q(x) \neq \emptyset$; then $(9)_\varphi$ holds as equality.||

Thus the family

$$\mathcal{U} = \{h_\varphi(x) | \varphi \in \Phi\}$$

of upper bounding functions for $f(x)$ is complete in the above defined sense.

There is actually a more general class of upper bounding linear functions for $f(x)$, namely

<u>Remark 8</u>. Let $f(x)$ be as in (1) with $a_j > 0$, $j \in N$, and let λ_{ji}, $i \in Q_j$, $j \in N$ be nonnegative numbers satisfying

$$(11) \qquad \sum_{i \in Q_j} \lambda_{ji} = 1, \qquad j \in N.$$

Define

$$h(\lambda,x) = \sum_{j \in N} a_j \left(\sum_{i \in Q_j} \lambda_{ji} x_i \right).$$

Then every $x \in \{0,1\}^q$ satisfies the inequality

$$f(x) \leq h(\lambda,x)$$

for every $\lambda \geq 0$ satisfying (11).

<u>Proof</u>. For $x \in \{0,1\}^q$,

$$h(\lambda,x) = \sum_{j \in N} a_j \left(\sum_{i \in Q_j \cap Q(x)} \lambda_{ji} \right)$$

$$\geq \sum_{j \in N | Q_j \subseteq Q(x)} a_j \left(\sum_{i \in Q_j} \lambda_{ji} \right)$$

$$= \sum_{j \in N | Q_j \subseteq Q(x)} a_j = f(x). \|$$

Like in the case of lower bounding functions, when $f(x)$ is a quadratic function, the class of upper bounding functions defined in Remark 8 specializes to the one introduced by Hammer, Hansen and Simeone [16].

The family $\mathcal{U}$ introduced earlier in this section consists of those $h(\lambda,x)$ such that

$$\lambda_{ji} = \begin{cases} 1 & \text{for } i = \varphi(j) \\ 0 & \text{for } i \neq \varphi(j). \end{cases}$$

Since the set $\mathcal{U}$ is complete and already very large (not excluding repetitions, $|\mathcal{U}| = \prod_{j\in N} |Q_j|$), we will not consider further the more general class of functions defined in Remark 8, but rather move in the opposite direction of identifying a proper subset of $\mathcal{U}$ that is complete.

For any $\varphi \in \Phi$, let $Q(\varphi)$ denote the range of φ, i.e.,

$$Q(\varphi) = \{i \in Q \mid i = \varphi(j) \text{ for some } j \in N\}.$$

We will say that a mapping $\varphi \in \Phi$ is <u>sequence-related</u> if there exists a permutation $< i_1,\ldots,i_m >$ of the elements of $Q(\varphi)$, with the property that for $k = 1,\ldots,m$, $i_k = \varphi(j)$ for <u>all</u> $j \in N$ such that $i_k \in Q_j$ but $i_\ell \notin Q_j$ for $\ell = 1,\ldots,k-1$. We will say that $s_\varphi = < i_1,\ldots,i_m >$ is <u>the sequence associated with</u> the mapping φ. To put it differently, a mapping $\varphi \in \Phi$ is sequence-related if it can be generated as follows. Define $Q = \bigcup_{j \in S} Q_j$. Starting with Q defined by $S = N$, choose some $i \in Q$ and set $i = \varphi(j)$ for all $j \in N$ such that $i \in Q$; then redefine Q by setting $S \leftarrow S\setminus\{j \in N \mid i \in Q_j\}$, and apply the procedure again, stopping when Q becomes empty.

<u>Example 2</u>. Let $Q_1 = \{1,2,3\}$, $Q_2 = \{1,4,5\}$, $Q_3 = \{2,5\}$. Then each of the mappings φ_1, φ_2, φ_3, defined as

$$\varphi_1(1) = 1, \quad \varphi_1(2) = 1, \quad \varphi_1(3) = 2$$
$$\varphi_2(1) = 1, \quad \varphi_2(2) = 5, \quad \varphi_2(3) = 5$$
$$\varphi_3(1) = 1, \quad \varphi_3(2) = 4, \quad \varphi_3(3) = 2,$$

is sequence-related, with the associated sequences <1,2>, <5,1> and <4,1,2> respectively; but the mapping

$$\varphi_4(1) = 1, \quad \varphi_4(2) = 5, \quad \varphi_4(3) = 2$$

is not sequence-related, since for any permutation of the indices 1,5,2, if i denotes the first index, then $i \in Q_j$ for some j such that $\varphi(j) \neq i$.||

Let $\Phi' = \{\varphi \in \Phi \mid \varphi \text{ is sequence-related}\}$, and

$$\mathcal{U}' = \{h_\varphi(x) \mid \varphi \in \Phi'\}.$$

It can be shown (see below) that $\mathcal{U}'$ is a complete set of upper bounding functions for f(x). However, it turns out that Φ' can be further restricted without losing completeness.

Let V be an arbitrary set with $|V| = v$, and let $\mathcal{P}$ be the set of all permutations of the elements of V. For any $S \subseteq V$, a permutation $p \in \mathcal{P}$ will be said to <u>represent</u> in $\mathcal{P}$ the 2-partition $(S, V \setminus S)$ of V, if every element of S precedes every element of $V \setminus S$ in p. In other words, the permutation $p = \langle i_1, \ldots, i_v \rangle$ represents the partition $(S, V \setminus S)$ if $i_k \in S$ and $i_\ell \in V \setminus S$ imply $k < \ell$. A set of permutations $P \subseteq \mathcal{P}$ will be called <u>representative</u> (of the 2-partitions of V), if for every $S \subseteq V$, the partition $(S, V \setminus S)$ is represented in P.

<u>Example 3</u>. Representative sets of permutations for $V_1 = \{1,2,3\}$ and $V_2 = \{1,2,3,4\}$ are P_1 and P_2,

P_1:	P_2:
1 2 *	1 2 3 *
1 3 *	1 2 4 *
<u>2 3 *</u>	1 3 4 *
3 * *	<u>2 3 4 *</u>
	1 4 * *
	2 4 * *
	<u>3 4 * *</u>
	4 * * *

where a star in some $p \in P_i$ stands for an arbitrary element of V_i not yet used in p. The 2-partition represented by a permutation p consists of the nonstarred versus the starred portions of p. Thus, for example, $p = \langle 1,2,3,* \rangle$ represents the 2-partition $(\{1,2,3\}, \{4\})$; similarly, $p = \langle 2,4,*,* \rangle$ represents $(\{2,4\}, \{1,3\})$, etc.||

While the cardinality of $\mathcal{P}$ is v!, that of a representative subset $P \subset \mathcal{P}$ is only 2^{v-1}.

Consider now the set of sequence-related mappings Φ'. For any $S \subseteq Q$, we say that a mapping $\varphi \in \Phi'$, with associated sequence $s_\varphi = \langle i_1,\dots,i_m \rangle$, represents the 2-partition $(S, Q\backslash S)$, if s_φ is a subsequence of some permutation $p = \langle j_1,\dots,j_q \rangle$ of the elements of Q, that represents $(S, Q\backslash S)$. A set $\Psi \subset \Phi'$ of sequence-related mappings will be called <u>representative</u> (of the 2-partitions of Q) if every 2-partition of Q is represented in Ψ.

Now let $\Psi \subset \Phi'$ be representative, and define

$$\mathcal{U}_0 = \{h_\varphi(x) \mid \varphi \in \Psi\}.$$

<u>Theorem 4</u>. Let f(x) be as in (1), with $a_j > 0$, $j \in N$. Then $\mathcal{U}_0$ is a complete set of upper bounding functions for f(x).

<u>Proof</u>. For an arbitrary $x \in \{0,1\}^q$ let $\varphi \in \Psi$ be the mapping that represents the partition $(Q\backslash Q(x), Q(x))$ of Q (here, as before, Q(x) is the support of x). Let $s_\varphi = \langle i_1,\dots,i_m \rangle$ be the sequence associated with φ, and let $p = \langle j_1,\dots,j_q \rangle$ be a permutation of the elements of Q that represents $(Q\backslash Q(x), Q(x))$, such that s_φ is a subsequence of p.

Now if $i_m \in Q\setminus Q(x)$, then $i_\ell \in Q\setminus Q(x)$ for $\ell = 1,\ldots,m$, and $h_\varphi(x) = 0 = f(x)$. Otherwise, let h and k, respectively, be the greatest integers such that $i_h \in Q\setminus Q(x)$ and $j_k \in Q\setminus Q(x)$. Then $Q\setminus Q(x) = \{j_1,\ldots,j_k\}$, and $\{i_1,\ldots,i_h\} \subseteq Q\setminus Q(x)$, $\{i_{h+1},\ldots,i_m\} \subseteq Q(x)$. Denote

$$N_0 = \{j\in N \mid \varphi(j) \in \{i_1,\ldots,i_h\}\}$$
$$N_1 = \{j\in N \mid \varphi(j) \in \{i_{h+1},\ldots,i_m\}\}$$

Clearly, $N_0 \cup N_1 = N$. Since φ is related to the sequence s_φ, $i_\ell \notin Q_j$ for $\ell \in \{1,\ldots,h\}$ and $j\in N_1$, hence $Q_j \subseteq Q(x)$ for all $j\in N_1$. Therefore $Q_j\setminus Q(x) \neq \emptyset$ implies $j\in N_0$, which in turn implies $\varphi(j) \in Q\setminus Q(x)$; i.e., the condition of Theorem 3 holds for all $j\in N$ such that $Q_j\setminus Q(x) \neq \emptyset$. Therefore $f(x) = h_\varphi(x)$.||

Example 4. As in Example 1, let

$$f(x) = 3x_1x_2x_3 + 2x_1x_4 + x_2x_3x_4.$$

The set Φ of all mappings that associate to each $j\in\{1,2,3\}$ some $i\in Q_j$, contains $|Q_1| \times |Q_2| \times |Q_3| = 18$ elements, and the corresponding 18 upper bounding functions $h_\varphi(x)$, $\varphi \in \Phi$, happen to be pairwise distinct. However, a complete set $\mathcal{U}_0$ of upper bounding functions is defined by the representative set of sequence-related mappings associated with the set P_2 of Example 3 (where $Q = \{1,2,3,4\}$ plays the role of V_2):

$$\begin{aligned}
h_{\varphi_1}(x) &= 5x_1 + x_2 \\
h_{\varphi_2}(x) &= 5x_1 + x_3 \\
h_{\varphi_3}(x) &= 5x_1 + x_4 \\
h_{\varphi_4}(x) &= 4x_2 + 2x_4 \\
h_{\varphi_5}(x) &= 4x_3 + 2x_4 \\
h_{\varphi_6}(x) &= 3x_3 + 3x_4.
\end{aligned}$$

The mappings φ_k, $k = 1,\ldots,6$, correspond to the following permutations $p \in P_2$:

k	1	2	3	4	5	6
$p \in P$	1 2 3 * 1 2 4 *	1 3 4 *	1 4 * *	2 3 4 * 2 4 * *	3 4 * *	4 * * *

Where a mapping φ_k corresponds to more than one $p \in P_2$ as for $k = 1,4$, this is because different permutations containing a certain subsequence $< i_1,\ldots,i_m >$ give rise to a single mapping $\varphi \in \Psi$ related to that subsequence. Thus, in the case of $k = 1$, $s_\varphi = < 1,2 >$, and in the case of $k = 4$, $s_\varphi = < 2,4 >$.||

At this point we note the existence of another class of upper bounding (affine) functions for $f(x)$, that can be derived by using the following observation. Let Q be a set whose elements are ordered in some arbitrary way, $Q = \{1,\ldots,q\}$, and let a be an arbitrary positive scalar. Denoting $\bar{x}_i = 1 - x_i$ for $i \in Q$, it can be shown that

$$(12) \qquad -a \prod_{i \in Q} x_i = a(\bar{x}_q \prod_{i=1}^{q-1} x_i + \bar{x}_{q-1} \prod_{i=1}^{q-2} x_i + \ldots + \bar{x}_2 x_1 + \bar{x}_1 - 1).$$

Note that the right hand side of (12) has $q = |Q|$ variable terms (each one containing exactly one complemented variable) and a constant term.

Thus for any $f(x)$ of the form (1) with $a_j > 0$, $j \in N$, using (12) one can write

$$-f(x) = p_\alpha(x,\bar{x}),$$

where p_α is a multilinear function of the variables x_i and their complements $\bar{x}_i$, $i \in Q_j$, $j \in N$, with coefficients $a_j > 0$ and with $|Q_j|$ terms for each $j \in N$.

The subscript α refers to the particular ordering of the sets Q_j, $j \in N$, which was used in (12) to derive $p_\alpha(x,\bar{x})$. Taking any α and applying Theorem 1, one can derive a family of lower bounding functions $g_M(x,\bar{x})$ for $p_\alpha(x,\bar{x})$, one for each set of the form $M = \bigcup_{j \in N} M_j$, where $M_j \subseteq Q_j$, $j \in N$. Substituting for $\bar{x}_i$, $i \in Q_j$, $j \in N$, one obtains the corresponding functions $\hat{g}_M(x)$ $(= g_M(x,\bar{x}))$ in the variables x_i, and by changing signs, the affine upper bounding functions $-\hat{g}_M(x)$ for $f(x)$.

For a given $f(x)$ of the form (1), there are $\pi_{j \in N} (|Q_j|!)$ different functions $p_\alpha(x,\bar{x})$ such that $p_\alpha(x,\bar{x}) = -f(x)$, and for each α, there are $\pi_{j \in N} 2^{|Q_j|}$ lower bounding functions $g_M(x,\bar{x})$ (not necessarily all distinct) for $p_\alpha(x,\bar{x})$, hence upper bounding functions $-\hat{g}_M(x)$ for $f(x)$. However, as stated in the next theorem, every such upper bounding affine function is dominated by some linear function in the class $\mathcal{U}$.

To simplify the notation, we will assume that $f(x)$ has a single term, i.e., the equation $-f(x) = p_\alpha(x,\bar{x})$ is as in (12). In view of Remark 2, this implies no loss of generality. Further, we shall let each of the $|Q| = q$ terms of $p_\alpha(x,\bar{x})$ be indexed by the index of its (unique) complemented variable.

<u>Theorem 5</u>. Let $M \subseteq Q$, $M = \{i_1,\ldots,i_m\}$, with $i_k < i_\ell$ whenever $k < \ell$. Then

$$ax_{i_m} \leq -\hat{g}_M(x)$$

for every $x \in \{0,1\}^q$.

<u>Proof</u>. Applying Theorem 1 to $p_\alpha(x,\bar{x})$, we obtain the lower bounding function

$$g_M(x,\bar{x}) = a[\sum_{k=1}^{m} (\bar{x}_{i_k} + \sum_{i=1}^{i_k - 1} x_i - (i_k - 1)) - 1]$$

or

$$\hat{g}_M(x) = -a\left[\sum_{k=1}^{m}\left(x_{i_k} - \sum_{i=1}^{i_k-1} x_i + i_k - 2\right) + 1\right].$$

Therefore

$$-\hat{g}_M(x) = ax_{i_m} + a\,\sigma(x)$$

where

$$\sigma(x) = \left(i_m - 1 - \sum_{i=1}^{i_m-1} x_i\right) + \sum_{k=1}^{m-1}\left(i_k - 2 + x_{i_k} - \sum_{i=1}^{i_k-1} x_i\right), \tag{13}$$

and to prove the theorem we have to show that $\sigma(x) \geq 0$ for all $x \in \{0,1\}^q$. Note that for $k = 1,\ldots,m-1$,

$$i_k - 2 + x_{i_k} - \sum_{i=1}^{i_k-1} x_i \begin{cases} = -1 \text{ if } x_i = 1,\ i = 1,\ldots,i_k - 1, \text{ and } x_{i_k} = 0 \\ \geq 0 \text{ otherwise} \end{cases}$$

and

$$i_m - 1 - \sum_{i=1}^{i_m-1} x_i \begin{cases} = 0 \ \text{ if } x_i = 1,\ i = 1,\ldots,i_m - 1 \\ \geq 1 \text{ otherwise} \end{cases}$$

Since $x_{i_k} = 0$ for some $k \in \{1,\ldots,m-1\}$ excludes $x_i = 1$, $i = 1,\ldots,i_\ell - 1$ for any $\ell > k$, at most one term under the summation sign in (13) can be negative, and if there exists such a term, then

$$i_m - 1 - \sum_{i=1}^{i_m-1} x_i \geq 1.$$

Thus for any $x \in \{0,1\}^q$, $\sigma(x) \geq 0$, hence $-\hat{g}_M(x) \geq ax_{i_m}$.||

The relation (12) can be used in the reverse direction too; i.e., in order to derive a set of lower bounding functions for some $f(x)$ as in (1), with $a_j > 0$, $j \in N$, one can use Theorem 3 to derive a set of upper bounding linear functions $h_\varphi(x,\bar{x})$ for $p_\alpha(x,\bar{x}) = -f(x)$, and then substitute for $\bar{x}_i$, $\forall$ i, to obtain a set of functions $\delta_\varphi(x)$, whose negatives, $-\delta_\varphi(x)$, are lower bounding functions for f(x). In this case one recovers the lower bounding function $g_N(x)$, by using the mapping φ which associates to the index set of each term of $p_\alpha(x,\bar{x})$, the index of its complemented variable. The functions $g_M(x)$, $M \subsetneq N$, can be recovered by using the same mapping for $j \in M$, while for $j \in N \setminus M$ one uses a mapping that produces a lower bounding function identically equal to zero. When the number of terms of $p_\alpha(x,\bar{x})$ corresponding to the j^{th} term of f(x) is even, this is accomplished by any mapping that produces pairwise complementary images. When it is odd, one has to use the construction of Remark 8 to find a lower bounding function that vanishes for all $x \in \{0,1\}^q$.

All other lower bounding functions that one obtains via this procedure are uninteresting because they cannot take on a positive value for any $x \in \{0,1\}^q$.

We conclude this section by combining the above results to derive a family of lower bounding functions for f(x) as in (1), with coefficients a_j of arbitrary sign. Let $N^+ = \{j \in N | a_j > 0\}$, $N^- = \{j \in N | a_j < 0\}$, and

$$f^+(x) = \sum_{j \in N^+} a_j \prod_{i \in Q_j} x_i \quad , \qquad f^-(x) = \sum_{j \in N^-} a_j \prod_{i \in Q_j} x_i \, .$$

For every $M \subseteq N^+$, let

$$g_M(x) = \sum_{i \in Q_M} \Big(\sum_{j \in M | i \in Q_j} a_j \Big) x_i - \sum_{j \in M} (|Q_j| - 1) a_j ,$$

as in Theorem 1. Let Φ^- be the family of mappings φ that associate to every $j \in N^-$ some $i \in Q_j$, and for every $\varphi \in \Phi^-$, let

$$h_\varphi^-(x) = \sum_{j\in N^-} a_j x_{\varphi(j)} .$$

The function $h_\varphi^-(x)$ differs from the function $h_\varphi(x)$ of Theorem 3 only in that here the coefficients a_j are negative.

<u>Theorem 6</u>. Let $f(x)$ be as in (1). Then every $x \in \{0,1\}^q$ satisfies

$$(14)_{M,\varphi} \qquad f(x) \geq g_M(x) + h_\varphi^-(x)$$

for every $M \subseteq N^+$ and every $\varphi \in \Phi^-$. Further, $(14)_{M,\varphi}$ holds as equality if and only if

$$(15) \qquad \{j\in N^+ \,\big|\, |Q_j \backslash Q(x)| = 0\} \subseteq M \subseteq \{j\in N^+ \,\big|\, |Q_j \backslash Q(x)| \leq 1\}$$

and for all $j\in N^-$ such that $Q_j \backslash Q(x) \neq \emptyset$,

$$(16) \qquad \varphi(j) \in Q_j \backslash Q(x).$$

<u>Proof</u>. The theorem follows directly from Theorems 1 and 3 and the fact that a vector $x \in \{0,1\}^q$ satisfies $(14)_{M,\varphi}$ with equality if and only if it satisfies with equality both

$$(17) \qquad f^+(x) \geq g_M(x)$$

and

$$(18) \qquad f^-(x) \geq h_\varphi^-(x). \|$$

Next we define a subfamily of the lower bounding functions introduced in Theorem 6, that is complete.

For every $M \subseteq N^+$, let $E(M)$ be defined as in Theorem 2, and let

$$\mathcal{L}_0^+ = \{g_{E(M)}(x) \,|\, M \subseteq N^+\}.$$

Further, let $\Psi^- \subseteq \Phi^-$ be a representative set of sequence-related mappings as defined earlier, and let

$$\mathcal{U}_0^- = \{h_\varphi^-(x) \mid \varphi \in \Psi^-\}.$$

Finally, define the set

$$\mathcal{B}_0 = \{g_{E(M)}(x) + h_\varphi^-(x) \mid M \subseteq N^+ \text{ and } \varphi \in \Psi^-\}.$$

Theorem 7. $\mathcal{B}_0$ is a complete set of lower bounding functions for f(x).

Proof. Follows from the completeness of $\mathcal{L}_0^+$ and $\mathcal{U}_0^-$.||

3. Linearizing Multilinear Inequalities in 0-1 Variables

In this section we linearize the multilinear inequality

$$(2) \qquad f(x) = \sum_{j\in N} a_j \left(\prod_{i\in Q_j} x_i \right) \le b$$

by defining a family $\mathcal{F}$ of linear inequalities, equivalent to (2) in the sense that a 0-1 vector x satisfies $f(x) \le b$ if and only if it satisfies the linear inequalities in $\mathcal{F}$.

We continue to use the notation introduced earlier. In particular, N^+, N^-, $f^+(x)$ and $f^-(x)$ are as in Theorem 6.

A set $M \subseteq N$ is said to be a cover for the inequality (2), if

$$\sum_{j\in M} |a_j| > b - \sum_{j\in N^-} a_j.$$

A cover M is said to be minimal, if T is not a cover for any $T \subsetneq M$.

It follows from this definition that a set $M \subseteq N^+$ is a cover for the inequality

$$(2^+) \qquad f^+(x) \le b$$

if and only if

$$\sum_{j\in M} a_j > b.$$

<u>Theorem 8</u>. Let

$$\mathcal{C} = \{M \subseteq N^+ | M \text{ is a cover for } (2^+)\},$$

and let $g_M(x)$, $h_\varphi^-(x)$ and Φ^- be as in Theorem 6. Then $x \in \{0,1\}^q$ satisfies (2) if and only if it satisfies

$$(19)_{M,\varphi} \qquad g_M(x) + h_\varphi^-(x) \leq b$$

for every $M \in \mathcal{C}$ and $\varphi \in \Phi^-$.

<u>Proof</u>. From Theorem 6, if $x \in \{0,1\}^q$ satisfies (2), then

$$g_M(x) + h_\varphi^-(x) \leq f(x) \leq b$$

for every $\varphi \in \Phi^-$ and every $M \subseteq N^+$, hence every $M \in \mathcal{C}$. This proves the "only if" part of the Theorem. To prove the "if" part, suppose $f(\hat{x}) > b$ for some $\hat{x} \in \{0,1\}^q$. From Theorem 6, there exists $M_0 \subseteq N^+$ and $\varphi_0 \in \Phi^-$ such that

$$(20) \qquad g_{M_0}(\hat{x}) + h_{\varphi_0}^-(\hat{x}) = f(\hat{x}) > b,$$

i.e., $\hat{x}$ violates the inequality $(19)_{M_0,\varphi_0}$. It remains to be shown that M_0 is a cover for (2^+). We have

$$\begin{aligned} \sum_{j \in M_0} |a_j| &\geq g_{M_0}(\hat{x}) && \text{(from the definition of } g_{M_0}) \\ &> b - h_{\varphi_0}^-(\hat{x}) && \text{(from (20))} \\ &> b && \text{(since } - h_{\varphi_0}^-(\hat{x}) \geq 0), \end{aligned}$$

hence M_0 is a cover for (2^+).||

Let $\mathcal{S}$ be the system of linear inequalities $(19)_{M,\varphi}$, for all $M \in \mathcal{C}$ and $\varphi \in \Phi^-$. According to Theorem 8, the system $\mathcal{S}$ is equivalent to (has the same solution set as) the nonlinear inequality (2). As one may suspect from

Theorem 7, $\mathcal{S}$ is not a minimal set with this property. Indeed, for $M \subseteq N^+$, let $E(M)$ be defined as in Theorem 2; and let $\Psi^- \subseteq \Phi^-$ be a set of representative sequence-related mappings. We then have

Theorem 9. Theorem 8 remains true if the system $(19)_{M,\varphi}$, $\varphi \in \Phi^-$ and $M \in \mathcal{C}$ is replaced by

$$(21)_{M,\varphi} \qquad g_{E(M)}(x) - h^-_{\varphi}(x) \leq b,$$

for every $M \in \mathcal{C}$ and $\varphi \in \Psi^-$.

Proof. Along the same lines as the proof of Theorem 8, using the fact that, from Theorem 7, there exists $M_0 \subseteq N^+$ and $\varphi_0 \in \Psi^-$ such that

$$g_{E(M_0)}(\hat{x}) + h^-_{\varphi_0}(\hat{x}) = f(\hat{x}).\|$$

Since Ψ^- is a proper subset of Φ^-, and different sets $M \subseteq N^+$ under certain conditions give rise to the same set $E(M)$, the system $\mathcal{S}_0$ of linear inequalities $(21)_{M,\varphi}$, $M \in \mathcal{C}$, $\varphi \in \Psi^-$, is a proper subset of $\mathcal{S}$, and usually of much smaller cardinality.

It is sometimes desirable to use an alternative linearization of the nonlinear inequality (2), obtained by first replacing (2) with an equivalent system of (nonlinear) inequalities whose coefficients are all positive, and then linearizing this system. The first of these two steps is easily accomplished by replacing $f^-(x)$ in (2) with the family of lower bounding functions $h^-_{\varphi}(x)$, $\varphi \in \Psi^-$, and then complementing x_j, $j \in N^-$:

Theorem 10. A vector $x \in \{0,1\}^q$ satisfies (2) if and only if it satisfies

$$(22)_{\varphi} \qquad f^+(x) + \sum_{j \in N^-} |a_j| \bar{x}_{\varphi(j)} \leq b - \sum_{j \in N^-} a_j$$

for every $\varphi \in \Psi^-$.

<u>Proof</u>. The "only if" part follows from the fact that

$$(23) \qquad \sum_{j\in N^-} a_j x_{\varphi(j)} \leq f^-(x)$$

for all $\varphi\in\Psi^-$. The "if" part follows from the fact that the set of lower bounding functions $h^-_\varphi(x)$, $\varphi\in\Psi^-$, for $f^-(x)$, is complete.||

<u>Remark 9</u>. Theorem 10 remains true if Ψ^- is replaced by Φ^-.

Note that if we first replace (2) by the family $(22)_\varphi$, $\varphi\in\Phi^-$ and then generate the sets $\mathcal{S}_\varphi$ of linear inequalities equivalent to each inequality $(22)_\varphi$, we end up with a family $\mathcal{F} = \bigcup_{\varphi\in\Phi^-} \mathcal{S}_\varphi$ of linear inequalities that is a proper superset of the family $\mathcal{S}$ generated directly from (2). The reason for this is that, applying Theorem 8 to an inequality $(22)_\varphi$, produces a linear inequality for every cover $M\subseteq N$ for $(22)_\varphi$, hence for (2); whereas applying it to the inequality (2), produces linear inequalities only for covers $M\subseteq N^+$ for (2^+). It is easy to see that if $M_1\subseteq N$ and $M_2\subseteq N$ are covers for (2), we may have $M_1\neq M_2$, but $M_1\cap N^+ = M_2\cap N^+$. On the other hand, $M\subseteq N^+$ is a cover for (2) if and only if $M\cup N^-$ is a cover for $(22)_\varphi$, $\forall\ \varphi\in\Phi^-$. Thus the system of linear inequalities $\mathcal{S}$, obtained by applying Theorem 8 directly to (2), is the subsystem of $\mathcal{F}$ whose inequalities correspond to those covers $M\subseteq N$ such that $N^-\subseteq M$.

Next we turn to another way of using complements of the variables. By reversing the sense of the inequality in the system $(19)_{M,\varphi}$, $M\in\mathcal{C}$, $\varphi\in\Phi^-$, and complementing the variables x_i, $i\in Q_M$, an immediate strengthening of some inequalities can often be obtained. For any $\varphi\in\Phi^-$, we will again denote

$$Q_\varphi = \{i\in Q \mid i = \varphi(j) \text{ for some } j\in N^-\}.$$

<u>Theorem 11</u>. The vector $x\in\{0,1\}^q$ satisfies (2) if and only if it satisfies

$$(24)_{M,\varphi} \qquad \sum_{i\in Q_M} \alpha_i^M \bar{x}_i + \sum_{i\in Q_\varphi} \beta_i^\varphi x_i \geq \alpha_0^M$$

for every $M \in \mathcal{C}$, $\varphi \in \Phi^-$, where

$$\alpha_0^M = \sum_{j \in M} a_j - b \ (> 0),$$

$$\alpha_i^M = \min\{\alpha_0^M, \overline{\sum_{j \in M | i \in Q_j}} a_j\} \quad , \qquad i \in Q_M,$$

and

$$\beta_i^\varphi = \min\{\alpha_0^M, \overline{\sum_{j \in N^- | i = \varphi(j)}} |a_j|\}, \qquad i \in Q_\varphi .$$

Proof. Substituting for $g_M(x)$ and $h_\varphi^-(x)$ in $(19)_{M,\varphi}$ their expressions in terms of the coefficients a_j, $j \in N$, yields

$$(19')_{M,\varphi} \qquad \sum_{i \in Q_M} (\overline{\sum_{j \in M | i \in Q_j}} a_j) x_i + \sum_{j \in N^-} a_j x_{\varphi(j)} \le b + \sum_{j \in M} (|Q_j| - 1) a_j .$$

Substituting $1 - \bar{x}_i$ for x_i, $i \in Q_M$, and $-|a_j|$ for a_j, $j \in N^-$, and collecting terms for each $i \in Q_\varphi$, then changing the sign of the inequality, produces

$$(25) \qquad \sum_{i \in Q_M} (\overline{\sum_{j \in M | i \in Q_j}} a_j) \bar{x}_i + \sum_{i \in Q_\varphi} (\overline{\sum_{j \in N^- | i = \varphi(j)}} |a_j|) x_i \ge \sum_{j \in M} a_j - b,$$

since

$$\sum_{i \in Q_M} (\overline{\sum_{j \in M | i \in Q_j}} a_j) 1 = \sum_{j \in M} |Q_j| a_j .$$

Since $M \in \mathcal{C}$, the right hand side of (25) is positive.

Finally, since all coefficients of (25) are positive, each coefficient whose value exceeds that of the right hand side can be reduced to the value of the latter, without cutting off any 0-1 point x satisfying (25).||

Note that if for some $i \in Q_M$ and $j \in N^-$ we have $i = \varphi(j)$, then $(24)_{M,\varphi}$ has a term in $\bar{x}_i$ and one in x_i. Each such pair can obviously be reduced to a single term, with a corresponding adjustment of the constant on the right hand side. Since this adjustment reduces the value of the right hand side, the presence of such terms denotes a certain "weakness" of the inequality. This weakness can partly be made up by reducing those coefficients on the left hand side whose value exceeds the new value of the right hand side. To be specific, let us denote

$$Q_0^+ = \{i \in Q_M | i \neq \varphi(j),\ \forall\ j \in N^-\}$$

$$Q_1^+ = \{i \in Q_M | i = \varphi(j) \text{ for some } j \in N^-, \text{ and } \alpha_i^M \geq \beta_i^\varphi\}$$

$$Q_0^- = \{i \in Q \setminus Q_M | i = \varphi(j) \text{ for some } j \in N^-\}$$

$$Q_1^- = \{i \in Q_M | i = \varphi(j) \text{ for some } j \in N^- \text{ and } \beta_i^\varphi > \alpha_i^M\}$$

and let $Q^+ = Q_0^+ \cup Q_1^+$, $Q^- = Q_0^- \cup Q_1^-$.

<u>Theorem 12</u>. For any $x \in \{0,1\}^q$, the inequality $(24)_{M,\varphi}$ implies

$$(26)_{M,\varphi} \qquad \sum_{i \in Q^+} \gamma_i \bar{x}_i + \sum_{i \in Q^-} \gamma_i x_i \geq \gamma_0,$$

where

$$\gamma_0 = \max\{0,\ \alpha_0^M - \sum_{i \in Q_1^+} \beta_i^\varphi - \sum_{i \in Q_1^-} \alpha_i^M\}$$

and

$$\gamma_i = \min\{\gamma_0,\ \gamma_i'\} \quad , \quad i \in Q^+ \cup Q^-,$$

with

$$\gamma_i' = \begin{cases} \alpha_i^M & i \in Q_0^+ \\ \alpha_i^M - \beta_i^\varphi & i \in Q_1^+ \\ \beta_i^\varphi & i \in Q_0^- \\ \beta_i^\varphi - \alpha_i^M & i \in Q_1^- \end{cases}$$

<u>Proof</u>. For $i \in Q_1^+$,

$$\alpha_i^M \bar{x}_i + \beta_i^\varphi x_i = (\alpha_i^M - \beta_i^\varphi)\bar{x}_i + \beta_i^\varphi$$

while for $i \in Q_1^-$,

$$\alpha_i^M \bar{x}_i + \beta_i^{\varphi} x_i = (\beta_i^{\varphi} - \alpha_i^M) x_i + \alpha_i^M .$$

Substituting these expressions into $(24)_{M,\varphi}$ yields

$$\sum_{i \in Q^+} \gamma_i' \bar{x}_i + \sum_{i \in Q^-} \gamma_i' x_i \geq \gamma_0$$

and replacing γ_i' with $\min\{\gamma_0, \gamma_i'\}$ produces $(26)_{M,\varphi}$.||

For $M \subseteq N^+$, denote $C(M) = M \cup N^-$. Note that $C(M)$ is a cover for (2) if and only if M is a cover for (2^+). If $C(M)$ is a minimal cover for (2), then M is a minimal cover for (2^+); but the converse is not true. On the other hand, if M is a minimal cover for (2^+) and $|a_j| \geq \min_{i \in M} a_i$ for all $j \in N^-$, then $C(M)$ is a minimal cover for (2).

<u>Theorem 13</u>. Let $C(M)$ be a minimal cover for (2). Then $\alpha_i^M = \alpha_0^M$, $\forall\, i \in Q_M$, $\beta_i^{\varphi} = \alpha_0^M$, $\forall\, i \in Q_{\varphi}$, and $\alpha_0^M > 0$. Further, if $\varphi(j) \in Q \setminus Q_M$, $\forall\, j \in N^-$ (i.e., $Q_M \cap Q_{\varphi} = \emptyset$), then $(26)_{M,\varphi}$ is the same as $(24)_{M,\varphi}$ which can be written as

$$(27)_{M,\varphi} \qquad \sum_{i \in Q_M} \bar{x}_i + \sum_{i \in Q_{\varphi}} x_i \geq 1.$$

<u>Proof</u>. Let $C(M)$ be a minimal cover for (2). Then

$$(28) \qquad \sum_{j \in C(M)} |a_j| \leq b - \sum_{j \in N^-} a_j + |a_k| \quad , \quad \forall\, k \in M \cup N^-$$

and therefore, for any $i \in Q_M$,

$$\sum_{j \in M | i \in Q_j} a_j \geq \min_{k \in M} |a_k|$$

$$\geq \sum_{j \in M} a_j - b = \alpha_0^M \qquad \text{(from (28))},$$

which proves that $\alpha_i^M = \alpha_0^M$, $i \in Q_M$.

Also, for any $i \in Q_\varphi$ and $k \in N^-$ such that $i = \varphi(k)$, from (28) we have

$$\sum_{j \in N^- | i = \varphi(j)} |a_j| \geq |a_k| \geq \sum_{i \in M} a_i - b = \alpha_0^M,$$

i.e., $\beta_i^\varphi = \alpha_0^M$, $\forall\, j \in N^-$, where $\alpha_0^M > 0$ since M is a cover for (2^+).

Further if $\varphi(j) \in Q \setminus Q_M$, $\forall\, j \in N^-$, then $Q_1^+ \cup Q_1^- = \emptyset$, and $\gamma_i = \alpha_i^M$, $i \in Q^+$, $\gamma_i = \beta_i^\varphi$, $i \in Q^-$ in $(26)_{M,\varphi}$. Thus, in this case $(26)_{M,\varphi}$ is the same as $(24)_{M,\varphi}$, and the latter can be written as $(27)_{M,\varphi}$. ||

Remark 10. If C(M) is a minimal cover for (2) and $\varphi(j) \in Q_M$ for some $j \in N^-$, the inequality $(26)_{M,\varphi}$ is vacuous.

Proof. If C(M) is a minimal cover for (2), then $\alpha_i^M = \alpha_0^M$, $i \in Q_M$, and $\beta_i^\varphi = \alpha_0^M$, $i \in Q_\varphi$. If, in addition, $\varphi(j_*) \in Q_M$ for some $j_* \in N^-$, then $\gamma_0 = 0$ and hence $\gamma_i = 0$, $i \in Q^+ \cup Q^-$, i.e., $(26)_{M,\varphi}$ is vacuous. ||

Example 5. Consider the inequality

$$f(x) = 3x_1x_2x_3 + 3x_2x_4 + 4x_1x_4 - 2x_1x_5x_6 - x_3x_5 - 4x_1x_3x_6 \leq 2,$$

and let the sets Q_i, $i = 1,\ldots,6$, be numbered from left to right. Choosing $M = \{1,2,3\}$ and φ such that $\varphi(4) = 1$, $\varphi(5) = \varphi(6) = 3$, one obtains the inequality (of the form $(24)_{M,\varphi}$)

$$7\bar{x}_1 + 6\bar{x}_2 + 3\bar{x}_3 + 7\bar{x}_4 + 2x_1 + 5x_3 \geq 8.$$

After reducing the terms involving the pairs $(\bar{x}_1, x_1)$ and $(\bar{x}_3, x_3)$, one obtains

$$5\bar{x}_1 + 6\bar{x}_2 + 7\bar{x}_4 + 2x_3 \geq 3$$

which in turn implies the inequality (of the form $(26)_{M,\varphi}$)

$$3\bar{x}_1 + 3\bar{x}_2 + 3\bar{x}_4 + 2x_3 \geq 3.$$

Now let $M = \{2\}$ and φ be as above; then $C(M)$ is a minimal cover for $f(x) \leq 2$, and $\varphi(j) \in Q \backslash M$, $\forall\, j \in N^-$. The corresponding inequality

$$\bar{x}_2 + \bar{x}_4 + x_1 + x_3 \geq 1$$

is of the form $(27)_{M,\varphi}$.||

Thus, when $M \subseteq N^+$ and $\varphi \in \Psi^-$ are chosen such that $C(M)$ is a minimal cover for (2) and $\varphi(j) \in Q \backslash Q_M$ for all $j \in N^-$, $(26)_{M,\varphi}$ takes on the form $(27)_{M,\varphi}$ of a generalized covering inequality. While the generalized covering inequalities $(27)_{M,\varphi}$ all correspond to minimal covers $M \subseteq N^+$, the remaining generalized covering inequalities implied by (2), corresponding to minimal covers $M \not\subseteq N^+$, can be derived by applying Theorem 11 to the family of inequalities $(22)_\varphi$, $\varphi \in \Psi^-$ (with M a minimal cover for $(22)_\varphi$) rather than directly to (2). The set of all generalized covering inequalities corresponding to minimal covers for (2) has been shown by Granot and Hammer [14] to be equivalent to (in the sense of having the same 0-1 solutions as) the nonlinear inequality (2). However, when Theorem 11 is applied to the inequality (2) with covers M that are not minimal, it produces linear inequalities that are not of the generalized covering type.

These other inequalities give rise to alternative linearizations of (2). In fact, these linearizations are typically more compact, i.e., involving fewer inequalities, than the family of generalized covering inequalities. This is illustrated in the following example.

Example 6. Consider the inequality

$$10x_1x_9 + 9x_2x_8 + 8x_3x_7 + 8x_1x_6 + 8x_3x_4 + 5x_2x_5 \leq 20. \tag{29}$$

There are 20 minimal covers M for (29), and the corresponding sets Q_M are shown in the table below. Each of these minimal covers gives rise to a set covering inequality and all twenty of them are required to linearize (29).

No.	M	Q_M	Implied by
1	1,2,3	1 2 3 7 8 9	(A)
2	1,2,4	1 2 6 8 9	(C)
3	1,2,5	1 2 3 4 8 9	(A)
4	1,2,6	1 2 5 8 9	(C)
5	1,3,4	1 3 6 7 9	(D)
6	1,3,5	1 3 4 7 9	(D)
7	1,3,6	1 2 3 5 7 9	(A)
8	1,4,5	1 3 4 6 9	(D)
9	1,4,6	1 2 5 6 9	(C)
10	1,5,6	1 2 3 4 5 9	(A)
11	2,3,4	1 2 3 6 7 8	(A)
12	2,3,5	2 3 4 7 8	(B)
13	2,3,6	2 3 5 7 8	(B)
14	2,4,5	1 2 3 4 6 8	(A)
15	2,4,6	1 2 5 6 8	(C)
16	2,5,6	2 3 4 5 8	(B)
17	3,4,5	1 3 4 6 7	(D)
18	3,4,6	1 2 3 5 6 7	(A)
19	3,5,6	2 3 4 5 7	(B)
20	4,5,6	1 2 3 4 5 6	(A)

Now, applying Theorem 11 to inequality (29) with $M_1 = \{1,2,3,4,5,6\}$ we obtain the linear inequality $(24)_{M_1}$

(A) $$18\bar{x}_1 + 14\bar{x}_2 + 16\bar{x}_3 + 8\bar{x}_4 + 5\bar{x}_5 + 8\bar{x}_6 + 8\bar{x}_7 + 9\bar{x}_8 + 10\bar{x}_9 \geq 28$$

Repeating this procedure with the sets $M_2 = \{2,3,5,6\}$, $M_3 = \{1,2,4,6\}$, and $M_4 = \{1,3,4,5\}$, we obtain the linear inequalities

(B) $$10\bar{x}_2 + 10\bar{x}_3 + 8\bar{x}_4 + 5\bar{x}_5 + 8\bar{x}_7 + 9\bar{x}_8 \geq 10$$

(C) $$12\bar{x}_1 + 12\bar{x}_2 + 5\bar{x}_5 + 8\bar{x}_6 + 9\bar{x}_8 + 10\bar{x}_9 \geq 12$$

(D) $$14\bar{x}_1 + 14\bar{x}_3 + 8\bar{x}_4 + 8\bar{x}_6 + 8\bar{x}_7 + 10\bar{x}_9 \geq 14$$

respectively. One can verify that each of the 4 inequalities (A)-(D) implies several of the set covering inequalities, as shown in the last column of the table, and together they are equivalent to the 20 set covering inequalities, i.e., to the nonlinear inequality (29).||

In the above example, none of the four covers used to obtain the inequalities (A)-(D), was minimal. This suggests that an inequality obtained from a minimal cover could be strengthened by extending the cover. While th... was shown to be always true in the linear case [4], it turns out not to be always true in the case discussed here. In a companion paper [7], we examine dominance relations between the linear inequalities of the family defined by Theorem 11, and give necessary and sufficient conditions for the extension of a minimal cover to produce a strengthening of the associated inequality. Based on these conditions, we have developed and implemented a class of algorithms for solving nonlinear 0-1 programs, that start by generating linear inequalities from minimal covers, and then successively extend the covers to strengthen the inequalities whenever the above mentioned conditions are met. In [7] we describe several variants of this class of algorithms and discuss our computational experience on nonlinear 0-1 programs with up to 20 constraints, 50 variables, and 60 terms per constraint.

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