

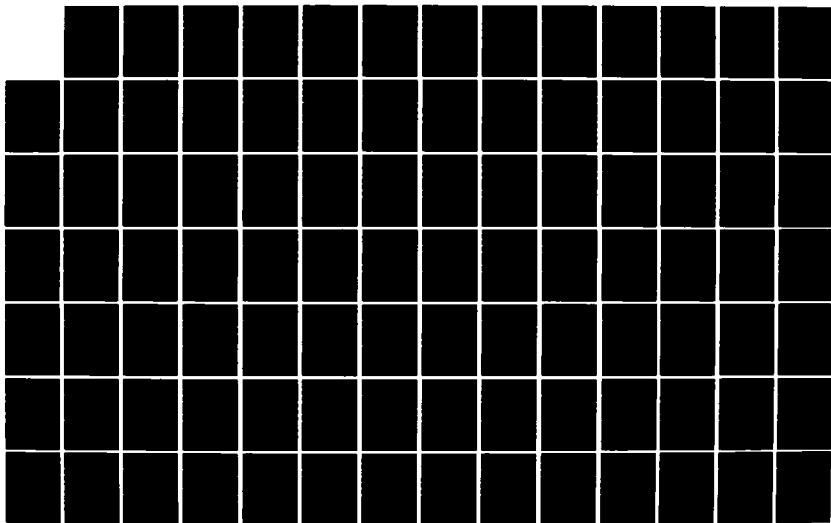
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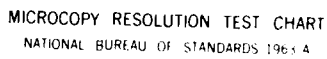
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ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF
TEST STATISTICS ASSOCIATED WITH SEVERAL
TWO PARAMETER EXPONENTIAL DISTRIBUTIONS

THESIS

David J. Lawton
Captain, USAF

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TWO PARAMETER EXPONENTIAL DISTRIBUTIONS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Applied Mathematics

David J. Lawton, B.S.
Captain, USAF

December 1984

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Preface

Many events occurring in nature have an underlying exponential distribution. This thesis obtained the distribution of three test criteria and tables of their percentage points are computed. These tables allow us to determine whether p exponential populations are equivalent, where $p > 2$. Methods were previously available to allow testing for equivalency of two populations, the test statistics presented in this paper and their distributions empower the testing of equality of more than two populations. I believe that anyone desiring to determine if up to 10 populations have the same exponential distribution will find this a valuable tool.

I am deeply indebted to my advisor Dr. B. N. Nagarsenker for his guidance and acknowledge his instruction on the theory and techniques on which this thesis is based. I also thank my readers Dr. P. B. Nagarsenker and Lt Col Thomas Lanier for their constructive remarks. The superb work of Susan Beachy, my typist reflects her patience and diligence in typing this project.

I formally thank my wife, Susan, for encouragement, forbearance, and enthusiasm to see me through this project and the AFIT program. The loving playfulness of our son Daniel has been a continuous source of refreshment and perspective. I dedicate this thesis to Jesus Christ, our faithful Saviour whose constant strength and matchless wisdom have sustained me during this endeavor.

David J. Lawton

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Notation

c. d. f.	:Cumulative distribution function
Def	:Definition
log	:Logarithm to the base e
LRT	:Likelihood-ratio test
M. G. F.	:Moment generating function
p. d. f.	:Probability density function
$\sim$	:Is distributed

Abstract

This thesis uses three criteria to test for equality of p populations with underlying two parameter exponential distributions (θ =location parameter, σ =scale parameter). These criteria use n random samples drawn from each of the p populations. The three criteria are based on the following three hypotheses:

1. $H_0 : \theta_1 = \theta_2 = \dots = \theta_p,$
 $\sigma_1 = \sigma_2 = \dots = \sigma_p$
2. $H_1 : \sigma_1 = \sigma_2 = \dots = \sigma_p$
 θ_i 's are unspecified ($i=1,2,\dots,p$)
3. $H_2 : \theta_1 = \theta_2 = \dots = \theta_p$
given that $\sigma_1 = \sigma_2 = \dots = \sigma_p$

against the general alternatives.

The asymptotic expansions of the distributions for λ_1 , λ_2 , and λ_2 are found based on the Neyman-Pearson likelihood ratio, where λ_0 , λ_1 , and λ_2 are the criteria for H_0 , H_1 , and H_2 respectively. The asymptotic expansions are computed using Bernoulli polynomials and a recursive relationship developed by Kalinin and Shalaevskii. Nine tables of percentage points are computed for each test statistic from the expansions where $p = 2(1)10$, $n = 10(1)20(5)50(10)100$, and $\alpha = .100, .050, .025, .010, .005$. These tables along with a practical illustration give the analyst a good technique that can be applied to many exponentially related situations.

ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF
TEST STATISTICS ASSOCIATED WITH SEVERAL
TWO PARAMETER EXPONENTIAL DISTRIBUTIONS

I. Introduction

Statistical techniques allow analysts to examine the characteristics of events and determine if these events can be transformed into a known model. One model useful in describing the observed variation in events occurring "randomly in time" is the exponential distribution. Some of the situations in which the exponential distribution is appropriate are the interval between successive telephone calls, the interval between consecutive accidents incurred by the same individual worker, the "time to failure" of identical electronic components, and the time interval between industrial accidents.

This paper will develop tests, their distributions, and tables of percentage points to test if several two parameter exponential populations are the same on the basis of samples of size 'n' taken from these populations using methods analogous to analysis of variance.

Thus suppose that p samples are available and that the i^{th} sample contains n observations X_{ij} , with mean $\bar{X}_i$ ($i=1,2,\dots,p; j=1,2,\dots,n$) and have been drawn from an exponential distribution with probability density given by

$$p(x, \theta_i, \sigma_i) = \begin{cases} \frac{1}{\sigma_i} \exp \left[\frac{-(x - \theta_i)}{\sigma_i} \right] & x > \theta_i, \sigma_i > 0, \theta_i > 0 \\ 0 & \text{otherwise} \\ & (i=1, 2, \dots, p) \end{cases} \quad (1.1)$$

where θ_i is the location parameter or the distance from the origin to the start of the distribution curve and σ_i is the scale parameter of the exponential distribution.

The following hypotheses and the corresponding test procedures using the likelihood ratio criteria will be considered in this thesis.

Null Hypothesis H_0 : The test of hypothesis H_0 that the p samples have been randomly drawn from the same population is equivalent to testing that the p exponential populations in equation 1.1 are identical. In other words it is desired to test the hypothesis H_0 :

$$\theta_1 = \theta_2 = \dots = \theta_p \quad (1.2)$$

$$\text{and } \sigma_1 = \sigma_2 = \dots = \sigma_p \quad (1.3)$$

against general alternatives.

Null Hypothesis H_1 : The second null hypothesis H_1 tests to determine if the p scale parameters (σ_i) are identical, but with the location parameters (θ_i) being any value whatsoever or equation 1.3 is true, while θ_i 's take on any combination of values.

Null Hypothesis H_2 : The null hypothesis H_2 tests to determine if the p populations have the same location parameter given they have the same scale parameter, or tests to determine if equation 1.2 is true given that equation 1.3 is true.

Because of the difficulties of obtaining the exact distributions of the likelihood ratio criteria for tests of hypothesis about the location and scale parameters of more than two exponential distributions, this thesis obtains an asymptotic expansion of the distributions of the three test statistics up to order $O(n^{-4})$. The expansions can therefore be used to obtain accurate approximations to the percentage points of the test statistics even for comparatively small values of n (as small as 10).

A discussion of the practical applications of the above tests will now be considered. One example involves life testing experiments where the θ_i 's represent either the guaranteed minimum operational time of an electronic part, the normal minimum interval between accidents because of increased safety consciousness/mandatory work stoppage, or some circumstance that eliminates the possibility of simultaneous events occurring. The different population θ_i 's (location parameters) can now be tested for equivalence. A second example consists of testing the same equipment on several aircraft. The interval between successive failures of this equipment in operating hours is gathered for each aircraft. A maintenance application considers whether the failure rate differs significantly among the different aircraft. One of the test statistics from the three hypotheses can accomplish this. A third example entails comparing the failure rate of several similar electronic components. A test is conducted and the failure times of each of the different samples of electronic components are recorded. To test if the components have the same average failure rate, the null hypothesis H_0 is appropriate. Null Hypothesis H_0 tests to determine if the samples

are identical with the same scale parameter (expected time between events/failure time) and the same location parameter (minimum interval between events/minimum failure time/guaranteed minimum operation time).

If different information is required on the different populations, Hypotheses H_1 or H_2 might be appropriate. Null Hypothesis H_1 tests to determine if the scale parameters of each of the samples are identical, but does not care what values the location parameter assumes. Null Hypothesis H_2 can be used to determine if one storage area/condition or installation technique had an effect on the electronic components in their minimum failure time (location parameter) when it is known that the failure rate (scale parameter) will remain constant once installed.

The criteria derived for each of the hypotheses is based on the likelihood ratio test of Neyman and Pearson. For each of the hypotheses H_0 , H_1 , and H_2 , their associated criteria are λ_0 , λ_1 , and λ_2 , respectively, and these are discussed in Chapter III.

Background

The examination of several populations with underlying exponential distributions to determine if they were equivalent was initially studied by P.V. Sukhatme in 1936 in his paper "On the Analysis of K Samples From Exponential Populations With Especial Reference to the Problem of Random Intervals" (Ref 18). This thesis is an extension of Sukhatme's 1936 paper. In 1941, E. Paulson discussed the power functions and the question of bias of some of the likelihood - ratio tests dealing with the exponential distribution (Ref 16). Maguire, Pearson, and Wynn argued that if one is trying to have the earliest detection of possible changes in the expectation of accidents it is better to use the interval

between accidents with the exponential distribution instead of the frequency of accidents in a given time period with the poisson distribution (Ref 13). Epstein and Tsao in 1953, took special cases of Sukhatme's LRT for two exponential populations and reduced these tests to equivalent tests which are expressed in terms of CHI Square and F distributions (Ref 5). Epstein and Sobel in their papers "Life Testing" and "Some Theorems Relevant to Life Testing From an Exponential Distribution" propose that the exponential distribution is appropriate in dealing with electronic components and that when life testing a sample of components, every component does not need to be destroyed to get a good estimate of the scale parameter (Ref 7 and 6). Zelen expands the life testing methods to include the analyzing of data when the data is taken over a range of different environmental conditions (Ref 21). Epstein justifies the validity of using the exponential distribution in additional papers and gives tests to determine if the underlying distribution is exponential and how to determine the validity of the parameters (Ref 3 and 4). In 1963, Hogg and Tanis demonstrated how an iterated procedure could be used to test the hypotheses considered in this thesis. This procedure allows the analyst to determine which observations caused the hypothesis to fail (Ref 9). Weinmann, Dugger, Franck, and Hewett compare the Kumar and Patel test with the Epstein and Tsao test and give suggestions on which test is more appropriate when testing for two exponential distributions for equivalent location parameters (Ref 19). Further results on distribution theory associated with two parameter exponential populations can be found in Hsieh (Ref 10) and Mann, Shaffer, and Singpurwalla (Ref 14).

Objective

The aim of this thesis is to:

1. Derive asymptotic expansions of the distributions of the three test statistics λ_0 , λ_1 , and λ_2 which are valid for moderately small values of $N(N > 10)$.
2. Prepare tables of percentage points for each test statistic for $\alpha = .10, .05, .025, .01, .005$ and for $N = 10$ to 100 .
3. Illustrate the proper technique in applying the results of this thesis to actual data.

Chapter II provides preliminary information on one and two parameter exponential distributions, the Gamma distribution, random variables drawn from exponential distributions, and the Neyman and Pearson likelihood ratio criterion. Chapter III derives the criteria for the three hypotheses while Chapter IV derives the h^{th} moment of the criteria. Using the moments of these criteria, Chapter V finds the asymptotic expansion of the distributions of the test statistics. Chapter VI uses actual data to demonstrate the practical application of the theoretical results obtained in this thesis.

II. Statistical Preliminaries

Chapter II provides preliminary information on one and two parameter exponential distributions, the relationship between each other and with the gamma distribution. In addition, the distributions of random variables drawn from exponential distributions are derived.

Two Parameter Exponential Distribution (θ, σ)

The p.d.f. of a two parameter exponential distribution with parameters θ and σ , where θ is the location parameter and σ is the scale parameter is

$$p(x, \theta, \sigma) = \begin{cases} \frac{1}{\sigma} \exp\left[-\frac{(x-\theta)}{\sigma}\right] & , x > \theta, \sigma > 0, \theta > 0 \\ 0 & , \text{otherwise} \end{cases} \quad (2.1)$$

By definition a p.d.f. equals one when evaluated over its parameter space and so.

$$\int_0^{\infty} \frac{1}{\sigma} \exp\left[-\frac{(x-\theta)}{\sigma}\right] dx = 1$$

Multiplying through by σ yields

$$\int_0^{\infty} \exp\left[-\frac{(x-\theta)}{\sigma}\right] dx = \sigma \quad (2.2)$$

One Parameter Exponential Distribution (σ)

The p.d.f. of a one parameter exponential distribution with parameter σ , where σ is the scale parameter is

$$p(x, \sigma) = \begin{cases} \frac{1}{\sigma} \exp \left[-\frac{x}{\sigma} \right] & , x > 0, \sigma > 0 \\ 0 & , \text{otherwise} \end{cases} \quad (2.3)$$

By definition of p.d.f.,

$$\int_0^{\infty} \frac{1}{\sigma} \exp \left[-\frac{x}{\sigma} \right] dx = 1$$

Multiplying through by σ

$$\int_0^{\infty} \exp \left[-\frac{x}{\sigma} \right] dx = \sigma \quad (2.4)$$

Gamma Distribution (α, β)

The p.d.f. of the gamma distribution is

$$\text{Gamma}(\alpha, \beta) = \begin{cases} \frac{x^{\alpha-1} \exp \left[-\frac{x}{\beta} \right]}{\beta^{\alpha} \Gamma(\alpha)} & , x > 0, \beta > 0, \alpha > 0 \\ 0 & , \text{otherwise} \end{cases} \quad (2.5)$$

By definition of a p.d.f.

$$\int_0^{\infty} \frac{x^{\alpha-1} \exp \left[-\frac{x}{\beta} \right]}{\beta^{\alpha} \Gamma(\alpha)} dx = 1$$

The above equation is equivalent to

$$\int_0^{\infty} x^{\alpha-1} \exp \left[-\frac{x}{\beta} \right] dx = \beta^{\alpha} \Gamma(\alpha) \quad (2.6)$$

Relationship Between Two Parameter and
One Parameter Exponential Distributions

The following theorem shows the relationship between the two parameter and one parameter exponential distribution:

Theorem 2.1: If X has an exponential p.d.f. $p(x, \theta, \sigma)$ defined in equation (2.1), then $X - \theta$ has a one parameter exponential p.d.f. $p(x, \sigma)$ given in equation (2.3).

Proof: Let $y = x - \theta$

From hypothesis $X \sim p(x, \theta, \sigma)$

where $p(x, \theta, \sigma)$ is exponential distribution described by equation 2.1.

The following lemma is required:

Lemma 2.1: Given the p.d.f. of X , then the p.d.f. of Y , where

$$g(X) = Y, \text{ is } p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \quad (\text{Ref 2:245})$$

It follows from first assumption that $x = y + \theta$

$$\text{and } \left| \frac{d}{dy} g^{-1}(y) \right| = 1$$

Therefore the p.d.f. of Y is

$$\begin{aligned} g(y) &= p(x, \theta, \sigma) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sigma} \exp \left[-\frac{(x - \theta)}{\sigma} \right] \cdot |1| \\ &= \frac{1}{\sigma} \exp \left[-\frac{(y + \theta - \theta)}{\sigma} \right] = \frac{1}{\sigma} \exp \left[-\frac{y}{\sigma} \right] \end{aligned}$$

$$g(y) = \begin{cases} \frac{1}{\sigma} \exp \left[-\frac{y}{\sigma} \right] & , y > 0, \sigma > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Relationship Between Gamma and One Parameter Exponential Distribution

A gamma distribution with $\alpha=1$ and $\beta=\sigma$ is equivalent to a one parameter distribution with parameter σ or

$$p(x, \sigma) = \text{gamma}(1, \sigma)$$

Properties of Gamma Distribution

Moment Generating Function: The moment generating function for the gamma distribution is

$$\begin{aligned} M_X(t) &= E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} p(x) dx \\ &= \int_0^{\infty} e^{tx} \frac{x^{\alpha-1} \exp\left[-\frac{x}{\beta}\right]}{\beta^{\alpha} \Gamma(\alpha)} dx = \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} \cdot \exp\left[-x \left[\frac{1}{\beta} - t\right]\right] dx \end{aligned}$$

Using equation 2.6 this is equivalent to

$$= \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\left[\frac{1}{\beta} - t\right]^{\alpha}} = \frac{1}{(1 - \beta t)^{\alpha}} = (1 - \beta t)^{-\alpha}$$

Therefore the moment generating function for gamma distribution is

$$M_X(t) = (1 - \beta t)^{-\alpha} \quad (2.8)$$

Using equations 2.7 and 2.8 the moment generating function of X where X has a one parameter exponential p.d.f. is

$$M_X(t) = (1 - \sigma t)^{-1} \quad (2.9)$$

Lemma 2.2: If $M_X(t)$ is the M.G.F. of X , then

$$M_{ax+b}(t) = e^{tb} M_X(at)$$

$$\begin{aligned} \text{Proof: } M_{ax+b}(t) &= E(e^{t(ax+b)}) = E(e^{tax+tb}) \\ &= E(e^{tax} \cdot e^{tb}) = e^{tb} \cdot E(e^{tax}) \\ &= e^{tb} \cdot M_X(at) \end{aligned}$$

The moment generating function of X where X has a two parameter exponential p.d.f. is

$$M_X(t) = e^{t\theta(1-\sigma t)^{-1}} \quad (2.10)$$

Proof: Let $Y=X-\theta$, Y therefore has a one parameter exponential distribution

$$\text{from equation 2.9 } M_Y(t) = (1-\sigma t)^{-1}$$

since $X=Y+\theta$, combine Lemma 2.2 and equation 2.9 to get

$$M_{Y+\theta}(t) = e^{t\theta(1-\sigma t)^{-1}}$$

Theorem 2.2: Suppose $X_1, X_2, \dots, X_n$ are independent and X_i has M.G.F. M_i , $i=1,2,\dots,n$. Then the M.G.F. of $a_1X_1 + a_2X_2 + \dots + a_nX_n$

$$M_{a_1X_1 + a_2X_2 + \dots + a_nX_n}(t) = M_1(a_1t) M_2(a_2t) \dots M_n(a_nt)$$

Proof: (Ref 2:258)

Theorem 2.3: If $X_1, X_2, \dots, X_n$ are independent, each with a Gamma (a_i, β) p.d.f., then the p.d.f. of $X_1 + X_2 + \dots + X_n$ is Gamma $(a_1 + a_2 + \dots + a_n, \beta)$.

Proof: From Theorem 2.2

$$\begin{aligned} M_{X_1 + X_2 + \dots + X_n}(t) &= M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t) \\ &= (1-\beta t)^{-a_1} (1-\beta t)^{-a_2} \dots (1-\beta t)^{-a_n} \\ &= (1-\beta t)^{-(a_1 + a_2 + \dots + a_n)} \end{aligned}$$

Therefore $X_1 + X_2 + \dots + X_n \sim \text{Gamma}(a_1 + a_2 + \dots + a_n, \beta)$

Theorem 2.4: If $X \sim \text{Gamma}(a, \beta)$, then $cX \sim \text{Gamma}(a, c\beta)$

Proof: If the random variable $X \sim \text{Gamma}(a, \beta)$ then it's M.G.F.

$$M_X(t) = (1 - \beta t)^{-a}$$

Applying Lemma 2.2 to equation

$$M_{cX}(t) \text{ yields } M_X(ct)$$

which is equivalent to $(1 - \beta ct)^{-a}$

and is the M.G.F. of a $\text{Gamma}(a, c\beta)$

Therefore $cX \sim \text{Gamma}(a, c\beta)$

Corollary 2.4.1: If X is a one parameter exponential distribution, then $cX \sim \text{Gamma}(1, c\sigma)$.

Distribution Function of One and Two Parameter Exponential Distributions

A One parameter exponential distribution has the following p.d.f. and c.d.f.

$$f(x) = \begin{cases} \frac{1}{\sigma} \exp\left[-\frac{x}{\sigma}\right] & , x > 0, \sigma > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx = \int_0^x \exp\left[-\frac{x}{\sigma}\right] \cdot \frac{1}{\sigma} dx = - \int_0^x \exp\left[-\frac{x}{\sigma}\right] \cdot \left[-\frac{1}{\sigma}\right] dx \\ &= -\exp\left[-\frac{x}{\sigma}\right] \Big|_0^x = 1 - \exp\left[-\frac{x}{\sigma}\right] \end{aligned}$$

A Two parameter exponential distribution has the following p.d.f. and c.d.f.

$$f(x) = \begin{cases} \frac{1}{\sigma} \exp \left[\frac{-(x-\theta)}{\sigma} \right] & , x > \theta, \theta > 0, \sigma > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx = \int_{\theta}^x \frac{1}{\sigma} \exp \left[\frac{-(x-\theta)}{\sigma} \right] dx \\ &= \int_{\theta}^x \frac{1}{\sigma} \exp \left[\frac{-x}{\sigma} \right] \cdot \exp \left[\frac{\theta}{\sigma} \right] dx = \exp \left[\frac{\theta}{\sigma} \right] \int_{\theta}^x \frac{1}{\sigma} \exp \left[\frac{-x}{\sigma} \right] dx \\ &= -\exp \left[\frac{\theta}{\sigma} \right] \cdot \left[\exp \left[\frac{-x}{\sigma} \right] \right]_{\theta}^x = -\exp \left[\frac{\theta}{\sigma} \right] \cdot \left[\exp \left[\frac{-x}{\sigma} \right] - \exp \left[\frac{-\theta}{\sigma} \right] \right] \\ &= 1 - \exp \left[\frac{-(x-\theta)}{\sigma} \right] \end{aligned}$$

THEOREM 2.5: Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ be the order statistics from a population with a p.d.f. $p(X, \theta, \sigma)$ as defined in equation (2.1), then the p.d.f. of $X_{(1)}$ is $p(X, \theta, \sigma/n)$.

$$\begin{aligned} \text{Proof: } P(X_{(1)} \leq x) &= 1 - (1 - F_X(x))^n = 1 - (1 - P(X \leq x))^n \\ &= 1 - \left[1 - \exp \left[\frac{-(x-\theta)}{\sigma} \right] \right]^n = 1 - \exp \left[\frac{-n(x-\theta)}{\sigma} \right] \end{aligned}$$

Take the derivative and the p.d.f. of $X_{(1)}$ is

$$\frac{n}{\sigma} \exp \left[\frac{-n(x-\theta)}{\sigma} \right] = \frac{1}{\sigma/n} \exp \left[\frac{-(x-\theta)}{\sigma/n} \right]$$

It therefore follows that the order statistic $X_{(1)}$ has a two parameter exponential distribution with parameters of θ and σ/n . (2.11)

THEOREM 2.6: Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics from a population with a p.d.f. $p(x, \theta)$ defined in equation (2.3), then the p.d.f. of $X_{(1)}$ is $p(X, \sigma/n)$.

$$\begin{aligned} \text{Proof: } P(X_{(1)} \leq x) &= 1 - (1 - F_X(x))^n = 1 - (1 - 1 + \exp\left[\frac{-x}{\sigma}\right])^n \\ &= 1 - \exp\left[\frac{-nx}{\sigma}\right] = 1 - \exp\left[\frac{-x}{\sigma/n}\right] \end{aligned}$$

Once again take the derivative and the p.d.f. of $X_{(1)}$ is

$$\frac{1}{\sigma/n} \exp\left[\frac{-x}{\sigma/n}\right]$$

It has been shown that $X_{(1)}$ has a one parameter exponential distribution with parameter σ/n . (2.12)

THEOREM 2.7: Let $X_1, X_2, \dots, X_n$ denote a random sample from a distribution having a p.d.f. $f(x; \theta)$, $\theta \in \Omega$, where Ω is an interval set. Let $Y_1 = u_1(X_1, X_2, \dots, X_n)$ be a sufficient statistic for θ , and let the family $\{g_1(y_1; \theta); \theta \in \Omega\}$ of probability density functions of Y_1 be complete. Let $Z = u(X_1, X_2, \dots, X_n)$ be any other statistic (not a function of Y_1 alone). If the distribution of Z does not depend upon θ , then Z is stochastically independent of the sufficient statistic Y_1 .

Proof: (Ref 8:390)

Corollary 2.7.1: If the M.L.E. of θ is $X_{(1)}$ and the M.L.E. of σ is $\bar{X} - X_{(1)}$, then $X_{(1)}$ and $\bar{X} - X_{(1)}$ are independent.

Theorem 2.8: The M.G.F. of $\bar{X} - X_{(1)}$ is $(1 - \sigma/n t)^{-(n-1)}$ and the p.d.f. of $\bar{X} - X_{(1)}$ is Gamma $(n-1, \sigma/n)$, where $X_i \sim p(x, \theta, \sigma)$.

Proof: Order a sample $x_1, x_2, \dots, x_n$ so that $x_{(1)} < x_{(2)} < \dots < x_{(n)}$,

therefore

$$\sum_{i=1}^n (X_{(i)}) = \sum_{i=1}^n X_i$$

and

$$\sum_{i=1}^n (X_{(i)} - \theta) = \sum_{i=1}^n (X_{(i)} - X_{(1)}) + n(X_{(1)} - \theta)$$

$$= \sum_{i=1}^n (X_i - \theta) \quad (I)$$

By Theorem 2.1 and equation 2.7 $X_i - \theta \sim \text{Gamma}(1, \sigma)$. (2.13)

Using Theorem 2.3 and that each of the X_i are independent; it follows that

$$\sum_{i=1}^n (X_i - \theta) \sim \text{Gamma}(n, \sigma) \quad (2.14)$$

Therefore, by Theorem 2.2 the M.G.F. of $\sum_{i=1}^n (X_i - \theta) = (1 - \sigma t)^{-n}$

If the order statistic $X_{(1)}$ is from a two parameter exponential population with parameters (θ, σ) , then by Theorem 2.1 $X_{(i)} - \theta$ has a one parameter exponential distribution and by equation 2.12 its parameter is (σ/n) which yields a Gamma $(1, \sigma/n)$ distribution. Therefore,

$$X_{(1)} - \theta \sim \text{Gamma}(1, \sigma/n) \quad (2.15)$$

Using Theorem 2.4 $n(X_{(1)} - \theta)$ is distributed as

$$\text{Gamma}(1, (\sigma/n) n) = \text{Gamma}(1, \sigma) \quad (2.16)$$

Therefore, the M.G.F. of $n(X_{(1)} - \theta)$ is $(1 - \sigma t)^{-1}$ (2.17)

$\sum_{i=1}^n (X_{(i)} - X_{(1)})$ and $n(X_{(1)} - \theta)$ are now shown to be independent.

$$\sum_{i=1}^n (X_{(i)} - X_{(1)}) = \sum_{i=1}^n X_i - nX_{(1)} = n [\bar{X} - X_{(1)}]$$

As a result of Corollary 2.7.1 $\bar{X} - X_{(1)}$ and $X_{(1)}$ are independent,

therefore the distributions of $\sum_{i=1}^n (X_i - X_{(1)})$ and $n(X_{(1)} - \theta)$ are

independent. (II)

Using statements (I) and (II),

$$M_{\sum_{i=1}^n (X_i - \theta)}(t) = M_{\sum_{i=1}^n (X_{(i)} - X_{(1)}) + n(X_{(1)} - \theta)}(t)$$

and by Theorem 2.3 this equals

$$M_{\sum_{i=1}^n (X_{(i)} - X_{(1)})}(t) \cdot M_{n(X_{(1)} - \theta)}(t)$$

Substituting in the known M.G.F., one has

$$(1 - \sigma t)^{-n} = M_{\sum_{i=1}^n (X_{(i)} - X_{(1)})}(t) \cdot (1 - \sigma t)^{-1}$$

and so

$$M_{\sum_{i=1}^n (X_{(i)} - X_{(1)})}(t) = \frac{(1 - \sigma t)^{-n}}{(1 - \sigma t)^{-1}} = (1 - \sigma t)^{-(n-1)}$$

Therefore, $\sum_{i=1}^n (X_{(i)} - X_{(1)}) = n(\bar{X} - X_{(1)})$ has a M.G.F. of $(1 - \sigma t)^{-(n-1)}$ and is

distributed as $\text{Gamma}(n-1, \sigma)$. Applying Theorem 2.4,

$\bar{X} - X_{(1)} \sim \text{Gamma}(n-1, \sigma/n)$ with a resulting M.G.F. of $(1 - (\sigma/n) \cdot t)^{-(n-1)}$. (2.18)

Maximum Likelihood Estimators of θ and σ for a Two Parameter Exponential Distribution

The M.L.E. of θ and σ is derived using the following two definitions.

Definition 1: The likelihood function L , with sample values $x_1, x_2, \dots, x_n$ is $L(X_1, X_2, \dots, X_n, \theta, \sigma) = f(X_1, \theta, \sigma) f(X_2, \theta, \sigma) \dots f(X_n, \theta, \sigma)$ where $f(X_i, \theta, \sigma)$ is the p.d.f. for two parameter exponential distribution (Ref:15:269).

Definition 2: The M.L.E. of (θ, σ) are based on a random sample $X_1, X_2, \dots, X_n$. The (θ, σ) that maximizes $L(X_1, X_2, \dots, X_n, \theta, \sigma)$ are the M.L.E. (θ, σ) . In other words, choose θ & σ such that $L(X_1, X_2, \dots, X_n, \theta, \sigma)$ or $\log L(X_1, X_2, \dots, X_n, \theta, \sigma)$ is maximized (Ref 2:270).

Given a sample from a two parameter exponential distribution where each observation x_i is independent, then the p.d.f. for each x_i is

$$f(x_i, \theta, \sigma) = \left[\frac{1}{\sigma} \right] \exp \left[\frac{-(x_i - \theta)}{\sigma} \right] \quad x_i > \theta, \theta > 0, \sigma > 0$$

To find the M.L.E. of θ and σ , the likelihood function is

$$L(X_1, X_2, \dots, X_n, \theta, \sigma) = \begin{cases} \frac{1}{\sigma^n} \exp \left[-\sum_{i=1}^n \frac{(x_i - \theta)}{\sigma} \right], & x_{(1)} > \theta, \sigma > 0, \theta_i > 0 \\ 0 & , X_{(1)} < \theta \end{cases}$$

Initially fix $\sigma > 0$, L is maximized by choosing θ as large as possible while assuring $(x_i - \theta) > 0$. If $\hat{\theta} = X_{(1)}$ is chosen, $\hat{\theta}$ will minimize the negative power of the function and force L to its maxima for X_i .

Therefore the M.L.E. of θ is

$$\hat{\theta} = X_{(1)} \quad (2.19)$$

To find the M.L.E. of σ , take the logarithm of L .

$$\log L(X_1, X_2, \dots, X_n, \theta, \sigma) = -n \log \sigma - \sum_{i=1}^n \frac{(x_i - \theta)}{\sigma}$$

take its derivative and set it equal to zero.

$$\frac{d \log L}{d \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^n \frac{(x_i - \theta)}{\sigma^2} = 0$$

Since $\sigma > 0$, multiply through by σ and nothing is lost in the equation.

$$n = \sum_{i=1}^n \frac{(x_i - \theta)}{\sigma}$$

$$\sigma = \frac{\sum_{i=1}^n (x_i - \theta)}{n} = \frac{\sum_{i=1}^n x_i - n\theta}{n} = \bar{X} - \theta$$

The M.L.E. of $\theta = X_{(1)}$, substituting in $X_{(1)}$ the M.L.E. of σ is

$$\hat{\sigma} = \bar{X} - X_{(1)} \quad (2.20)$$

Defining Criteria λ

Definition 3: Let $X_1, X_2, \dots, X_n$ denote n mutually stochastically independent random variables having the p.d.f. $f_i(x_i, \theta_1, \theta_2, \dots, \theta_m)$, $i=1, 2, \dots, n$. The set that consists of all parameter points $(\theta_1, \theta_2, \dots, \theta_m)$ is denoted by Λ . Let ω be a subset of the parameter space Λ . To test the hypothesis $H_0: (\theta_1, \theta_2, \dots, \theta_m) \in \omega$ against alternative hypotheses. Define the likelihood functions

$$L(\omega) = \prod_{i=1}^n f_i(X_i; \theta_1, \theta_2, \dots, \theta_m) \quad , \quad (\theta_1, \theta_2, \dots, \theta_m) \in \omega,$$

and

$$L(\Lambda) = \prod_{i=1}^n f_i(X_i; \theta_1, \theta_2, \dots, \theta_m) \quad , \quad (\theta_1, \theta_2, \dots, \theta_m) \in \Lambda$$

Let $L(\omega_0)$ and $L(\Lambda_0)$ be the maxima of these two likelihood functions.

The ratio of $L(\omega_0)$ to $L(\Lambda_0)$ is the likelihood ratio and is denoted by

$$\lambda = \frac{\max_{\theta \in \omega} L(\omega_0)}{\max_{\theta \in \Lambda} L(\Lambda_0)} \quad (2.2)$$

Let λ_0 be a positive proper fraction. The likelihood ratio test principle states that the hypothesis $H_0: (\theta_1, \theta_2, \dots, \theta_m) \in \omega$ is rejected if and only if $\lambda < \lambda_0$ (Ref 8:260, 261)

III. Derivation of the Criteria

In this chapter, the three likelihood ratio criteria λ_0 , λ_1 , λ_2 will be obtained for the three hypotheses stated in Chapter II. To get a better understanding of the overall problem, Figure 3.1 will illustrate the different notations and populations.

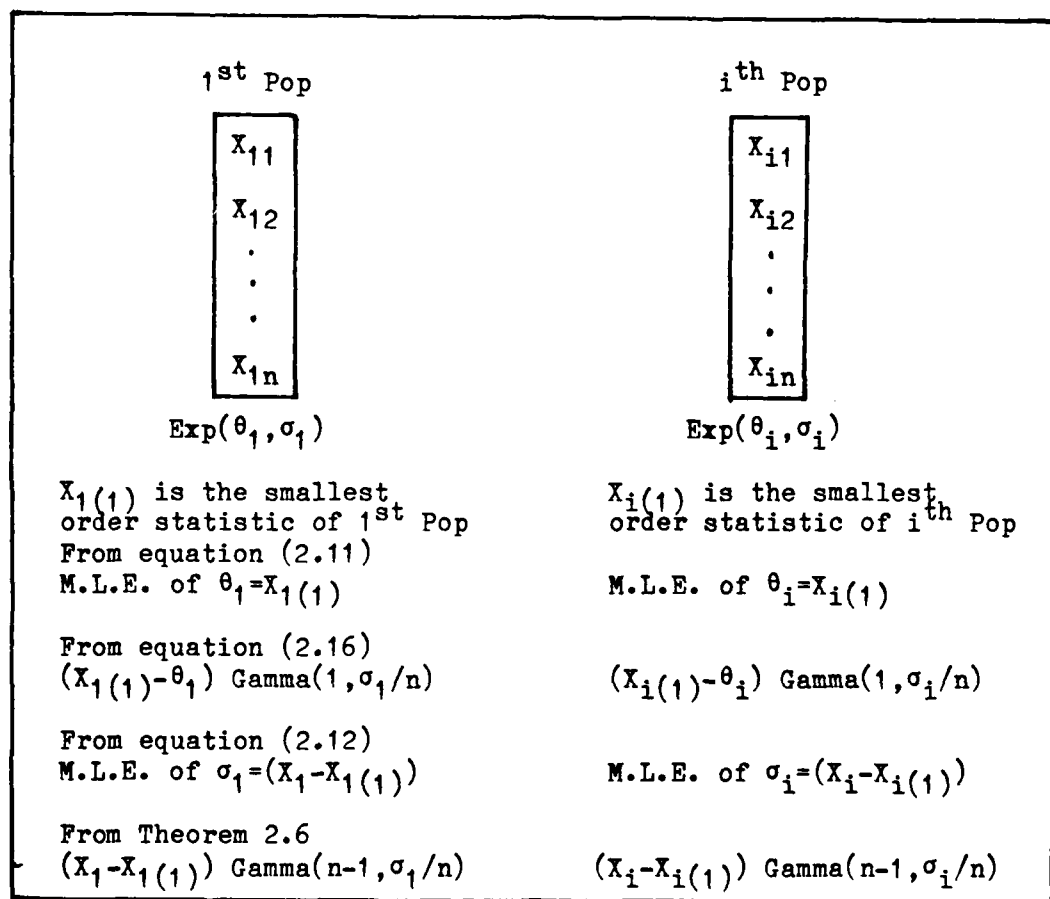


Figure 3.1. Sample Populations and Their Statistics

In Figure 3.1 "i" ranges from one to p, where p is the number of different samples. Each sample has same number of observations n, where $n > 10$.

Notations:

Let $X_{(1)}$ be the minimum of

$$X_{1(1)}, X_{2(1)}, \dots, X_{i(1)}, \dots, X_{p(1)} \quad (\text{Def 3.1})$$

$$\bar{X} = \frac{1}{p} \sum_{i=1}^p \bar{X}_i, \quad (\text{Def 3.2})$$

and

$$\bar{X}_i = \frac{n}{\sum_{j=1}^n} \frac{X_{ij}}{n} \quad (\text{Def 3.3})$$

Likelihood Functions.

Since all of the sample observations within each population are independent, the likelihood function for the first sample is

$$L_1 = f(X_{11}, \theta_1, \sigma_1) \cdot f(X_{12}, \theta_1, \sigma_1) \cdot \dots \cdot f(X_{1n}, \theta_1, \sigma_1)$$

$$= \frac{1}{(\sigma_1)^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{1j} - \theta_1)}{\sigma_1} \right\}$$

In general, the likelihood function L_i is

$$L_i = f(X_{i1}, \theta_i, \sigma_i) \cdot f(X_{i2}, \theta_i, \sigma_i) \cdot \dots \cdot f(X_{in}, \theta_i, \sigma_i)$$

$$= \frac{1}{(\sigma_i)^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \theta_j)}{\sigma_j} \right\} \quad (3.1)$$

Each of the p populations are independent and the likelihood function $L(A)$ for all np sample values is given by

$$L(\lambda) = f(X_{11}, X_{12}, \dots, X_{np}, \theta_1, \dots, \theta_p, \sigma_1, \dots, \sigma_p) \\ = L_1 \cdot L_2 \cdot \dots \cdot L_p \quad (3.2)$$

$$= \prod_{i=1}^p \frac{1}{(\sigma_i)^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \theta_i)}{\sigma_i} \right\} \quad (3.3)$$

Derivation of λ_0

The first of the three criteria λ_0 is determined using the H_0 which states that

$$\theta = \theta_1 = \theta_2 = \dots = \theta_p \quad \text{and}$$

$$\sigma = \sigma_1 = \sigma_2 = \dots = \sigma_p$$

The hypothesis tests to determine if all of the location and the scale parameters of each of the "p" populations are equal. Let $L(\omega_0)$ be the likelihood function under H_0 as given in (Def 2.3). Under H_0 , $\theta_i = \theta$ and $\sigma_i = \sigma$ the likelihood function of equation 3.6 becomes

$$L(\omega_0) = \prod_{i=1}^p \frac{1}{(\sigma)^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \theta)}{\sigma} \right\}$$

Using equations 2.19 and 2.20 and the definitions 3.1 and 3.2 $L(\omega_0)$ is a maximum when $\theta = \bar{X}_{(1)}$ and $\sigma = \bar{X} - \bar{X}_{(1)}$. Therefore the $\max_{\theta \in \omega_0} L(\omega_0)$ for equation 2.21 is

$$\prod_{i=1}^p \frac{1}{(\bar{X} - \bar{X}_{(1)})^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \bar{X}_{(1)})}{\bar{X} - \bar{X}_{(1)}} \right\}$$

$$= \prod_{i=1}^p \frac{1}{(\bar{X} - X_{(1)})^n} \exp \left[\frac{-n(\bar{X}_i - X_{(1)})}{\bar{X} - X_{(1)}} \right]$$

$$= \frac{1}{(\bar{X} - X_{(1)})^{np}} \exp \left[\frac{-n}{\bar{X} - X_{(1)}} \cdot \sum_{i=1}^p (X_i - X_{(1)}) \right]$$

using Def 3.2 yields

$$= \frac{1}{(\bar{X} - X_{(1)})^{np}} \exp \left[\frac{-n}{\bar{X} - X_{(1)}} p(\bar{X} - X_{(1)}) \right]$$

$$= \frac{1}{(\bar{X} - X_{(1)})^{np}} \exp (-np)$$

$$\max_{\theta \in \omega} L(\omega_0) = \frac{\exp (-np)}{(\bar{X} - X_{(1)})^{np}} \quad (3.4)$$

When finding the likelihood function for the denominator of the likelihood ratio test as defined in definition three of chapter 2, the maximum that $L(\Lambda)$ can achieve occurs when $\theta_i = \bar{X}_i(1)$ and $\sigma_i = \bar{X}_i - X_{(1)}$ for each population ($i=1, \dots, p$). Therefore, the $\max_{\theta \in \Lambda} L(\Lambda)$ for equation 2.1 is

$$\prod_{i=1}^p \frac{1}{(\bar{X}_i - X_{(1)})^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{1j} - X_{(1)})}{\bar{X}_i - X_{(1)}} \right\}$$

$$= \frac{1}{\prod_{i=1}^p (\bar{X}_i - X_{(1)})^n} \exp \left\{ \frac{-\sum_{i=1}^p \sum_{j=1}^n (X_{1j} - X_{(1)})}{\bar{X}_i - X_{(1)}} \right\}$$

$$\begin{aligned}
&= \frac{1}{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n} \exp \left\{ -\sum_{i=1}^p \frac{n(\bar{X}_i - X_{i(1)})}{\bar{X}_i - X_{i(1)}} \right\} \\
&= \frac{\exp(-np)}{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n} \\
\max_{\theta \in \Lambda} L(\Lambda) &= \frac{\exp(-np)}{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n} \quad (3.5)
\end{aligned}$$

From equation 2.21, criteria λ_0 which satisfies the assumptions under H_0 is

$$\begin{aligned}
\max_{\theta \in \omega} L(\omega_0) &= \frac{\exp(-np)}{(\bar{X} - X_{(1)})^{np}} \\
\max_{\theta \in \Lambda} L(\Lambda) &= \frac{\exp(-np)}{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n} \\
\frac{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n}{(\bar{X} - X_{(1)})^{np}} &= \frac{\prod_{i=1}^p l_i^n}{(l_0)^{np}} \quad (3.6)
\end{aligned}$$

$$\text{where } l_i = \bar{X}_i - X_{i(1)} \quad (3.7)$$

$$\text{and } l_0 = \bar{X} - X_{(1)} \quad (3.8)$$

Using equation 2.18 $l_1 \sim \text{Gamma}(n-1, \sigma_1/n)$

Derivation of λ_1

The second of the three criteria λ_1 is determined using the H_1 (hypothesis) which states

$$\sigma = \sigma_1 = \sigma_2 = \dots = \sigma_p \text{ and}$$

$$\theta_i \text{'s unspecified } (i=1, 2, \dots, p)$$

This hypothesis tests to determine if each of the scale parameters are equal for all "p" populations. The likelihood function, for all $\sigma_i = \sigma$, becomes

$$\begin{aligned} L(\omega_1) &= \prod_{i=1}^p \frac{1}{(\sigma)^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \theta_i)}{\sigma} \right\} \\ &= \frac{1}{(\sigma)^{np}} \exp \left\{ \frac{-\sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \theta_i)}{\sigma} \right\} \end{aligned}$$

From chapter two and equation 2.19 the M.L.E. of θ_i is $X_{i(1)}$. To determine the point at which $L(\omega_1)$ is a maximum, take the derivative of $\log L(\omega_1)$ and set it equal to zero.

$$\log L(\omega_1) = -np \log \sigma - \frac{1}{\sigma} \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \theta_i)$$

$$\frac{d \log L(\omega_1)}{d \sigma} = \frac{-np}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \theta_i)$$

Set this equation equal to 0 and solve for σ to find its maxima.

$$np = \frac{1}{\sigma} \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \theta_i)$$

$$\sigma = \frac{1}{np} \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \theta_i)$$

Substituting the M.L.E. of θ_i into the equation, the M.L.E. of σ is given by

$$= \frac{1}{np} \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - X_{i(1)}) \quad (3.9)$$

$$= \frac{1}{np} \left[\sum_{i=1}^p \sum_{j=1}^n X_{ij} - n \sum_{i=1}^p X_{i(1)} \right] = \frac{1}{p} \sum_{i=1}^p \bar{X}_i - \frac{1}{p} \sum_{i=1}^p X_{i(1)}$$

$$= \frac{1}{p} \sum_{i=1}^p (\bar{X}_i - X_{i(1)}) = \frac{1}{p} \sum_{i=1}^p \ell_i$$

where ℓ_i is defined in equation 3.7.

$$\text{Thus the M.L.E. of } \sigma = \frac{1}{p} \sum_{i=1}^p \ell_i \quad (3.10)$$

Using equations 2.19, 3.9, and 3.10 the denominator of criteria λ_1 is

$$\max_{\theta \in \omega_1} L(\omega_1) = \frac{1}{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}} \exp \left\{ \frac{- \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - X_{i(1)})}{\left[\sum_{i=1}^p \sum_{j=1}^n [X_{ij} - X_{i(1)}] \right] \cdot \frac{1}{np}} \right\}$$

$$= \exp(-np) \left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}$$

The $\max_{\theta \in \Lambda} L(\Lambda)$ for the second hypothesis is the same as the first hypothesis in determining the likelihood ratio test and is given in equation 3.5. Criteria λ_1 which satisfies the assumptions under H_1 is

$$\lambda_1 = \frac{\max_{\theta \in \omega_1} L(\omega_1)}{\max_{\theta \in \Lambda} L(\Lambda)} = \frac{\exp(-np) \left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}}{\exp(-np) \prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n} = \frac{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^n}{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}} = \frac{\prod_{i=1}^p (\ell_i)^n}{\bar{\ell}^{np}} = \lambda_1 \quad (3.11)$$

$$\text{where } \bar{\ell} = \frac{\ell_1 + \ell_2 + \dots + \ell_p}{p} \quad (3.12)$$

Note that λ_1 is a function of ℓ_i only.

Derivation of λ_2

The third criteria λ_2 is determined using hypothesis H_2 which states that

$$\theta = \theta_1 = \theta_2 = \dots = \theta_p \quad \text{knowing that}$$

$$\sigma = \sigma_1 = \sigma_2 = \dots = \sigma_p \quad \text{is true.}$$

This hypothesis tests to determine if each of the location parameters are equal for all "p" populations, knowing that all "p" populations have the same scale parameter. The likelihood function, for all σ_i 's = σ and θ_i 's = θ , for H_2 results in the same maximum $L(\omega_2)$ as in H_0 , from equation 3.4

$$\max_{\theta \in \omega} L(\omega_2) = \frac{\exp(-np)}{(\bar{X} - X_{(1)})^{np}}$$

The denominator of the likelihood ratio test for the third hypothesis is also tied to all of the scale parameters being equal and is therefore different from the first two hypotheses. The likelihood function is

$$L(\Lambda) = \prod_{i=1}^p \frac{1}{(\sigma)^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \theta_i)}{\sigma} \right\} \quad (3.13)$$

From equations 3.9 and 3.10 the M.L.E. of σ is

$$\frac{1}{np} \sum_{i=1}^p \sum_{j=1}^n (X_{ij} - X_{i(1)}) = \frac{1}{p} \sum_{i=1}^p \ell_i$$

Applying equation 2.19 to multiple populations the M.L.E. of $\theta_i = \bar{X}_i(1)$.

Substituting the M.L.E. of the two parameters into equation 3.16 yields

$$\begin{aligned} \max_{\theta \in \Lambda_2} L(\Lambda_2) &= \frac{p}{\pi} \frac{1}{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^n} \exp \left\{ \frac{-\sum_{j=1}^n (X_{ij} - \bar{X}_i(1))}{\sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \bar{X}_i(1))} \cdot np \right\} \\ &= \frac{1}{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}} \exp \left\{ \frac{-\sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \bar{X}_i(1))}{\sum_{i=1}^p \sum_{j=1}^n (X_{ij} - \bar{X}_i(1))} \cdot np \right\} \\ &= \frac{\exp(-np)}{\left(\frac{1}{p} \sum_{i=1}^p \ell_i \right)^{np}} \end{aligned}$$

Criteria λ_2 which satisfies the assumptions under H_2 is

$$\begin{aligned} \lambda_2 &= \frac{\max_{\theta \in \omega_2} L(\omega_2)}{\max_{\theta \in \Lambda_2} L(\Lambda_2)} = \frac{\frac{\exp(-np)}{(\bar{X} - \bar{X}(1))^{np}}}{\frac{\exp(-np)}{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}}} \\ &= \frac{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}}{(\bar{X} - \bar{X}(1))^{np}} = \frac{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}}{(\ell_0)^{np}} \quad (3.14) \end{aligned}$$

From equation 3.8

$$\ell_0 = \bar{X} - \bar{X}(1) = \left(\frac{1}{p} \sum_{i=1}^p \bar{X}_i \right) - \bar{X}(1)$$

using equation 3.7

$$\bar{X}_i = \ell_i + X_{i(1)} \text{ and therefore}$$

$$\begin{aligned} \ell_0 &= \frac{1}{p} \sum_{i=1}^p (\ell_i + X_{i(1)}) - X_{(1)} \\ &= \frac{1}{p} \sum_{i=1}^p \ell_i + \frac{1}{p} \sum_{i=1}^p X_{i(1)} - X_{(1)} \end{aligned}$$

$$\text{Define } u = \frac{1}{p} \sum_{i=1}^p X_{i(1)} - X_{(1)} \quad (3.15)$$

$$\text{then } \ell_0 = \frac{1}{p} \sum_{i=1}^p \ell_i + u \quad (3.16)$$

From equations 3.14, 3.15, 3.16

$$\lambda_2 = \frac{\left[\frac{1}{p} \sum_{i=1}^p \ell_i \right]^{np}}{\left[\frac{1}{p} \sum_{i=1}^p \ell_i + u \right]^{np}} = \frac{(v)^{np}}{(v+u)^{np}} \quad (3.17)$$

$$\text{where } v = \frac{1}{p} \sum_{i=1}^p \ell_i \quad (3.18)$$

IV. Derivation of the Moments

In this chapter the h^{th} moments of criteria λ_0 , λ_1 , and λ_2 will be derived. Using these moments the sampling distribution will then be obtained in Chapter V. Prior to determining the h^{th} moment of λ_0 , a technique to obtain such moments will be explained.

A Technique to Obtain h^{th} Moments

Suppose we want to find

$$E \left\{ \frac{\prod_{i=1}^p \pi_i^{n h} l_i}{n^p h \lambda_0} \right\}$$

where $l_i > 0$, $i=1,2,\dots,p$

λ_0 is a function of $l_1, \dots, l_p$.

To find this let $f(\theta)$ be a function of θ defined by

$$f(\theta) = E \left[\left[\prod_{i=1}^p \pi_i^{n h} l_i \right] e^{\theta \lambda_0} \right]$$

$$f(\theta) = \int \dots \int_{l_1, \dots, l_p > 0} \left[\prod_{i=1}^p \pi_i^{n h} l_i \right] e^{\theta \lambda_0} p(l_1, \dots, l_p) dl_1, \dots, dl_p \quad (4.1)$$

Take the first derivative on both sides of the equation and

$$\frac{df(\theta)}{d\theta} = \int \dots \int_{l_1, \dots, l_p > 0} \left[\prod_{i=1}^p \pi_i^{n h} l_i \right] e^{\theta \lambda_0} \lambda_0 p(l_1, \dots, l_p) dl_1, \dots, dl_p \quad (4.2)$$

continuing the process the r^{th} derivative of equation 4.1 is

$$\frac{d^r f(\theta)}{d\theta^r} = \int \dots \int_{l_1, \dots, l_p > 0} \left[\prod_{i=1}^p \pi_i^{n h} l_i \right] e^{\theta \lambda_0} \lambda_0^r p(l_1, \dots, l_p) dl_1, \dots, dl_p \quad (4.3)$$

Then putting $\theta=0$ and $r=-nph$ on both sides of equation 4.3 one has

$$\left. \frac{d f(\theta)}{d\theta^r} \right|_{\substack{\theta=0 \\ r=-nph \\ l_1, \dots, l_p > 0}} = \int \dots \int_{l_1, \dots, l_p > 0} \frac{\prod_{i=1}^p l_i^{nph}}{l_0^{nph}} p(l_1, \dots, l_p) dl_1, \dots, dl_p$$

$$= E \left[\frac{\prod_{i=1}^p l_i^{nph}}{l_0^{nph}} \right] \quad (4.)$$

The validity of this operation can be justified in its similarity to uses by Wilks. (5:274)

Lemma 4.1

$$\text{Let } f(\theta) = K(1 - B_1 \theta)^{-\alpha} \quad (4.5)$$

$$\text{If } \theta=0, \text{ then } \left. \frac{d f(\theta)}{d\theta^r} \right|_{\theta=0} = K (\alpha)(\alpha+1)\dots(\alpha+(r-1)) B_1^r$$

$$= K \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} [B_1]^r \quad (4.)$$

Theorem 4.1: The h^{th} moment of λ_0 is given by

$$E(\lambda_0)^h = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot \frac{\Gamma(np-1)}{\Gamma(nph+np-1)} \cdot p^{nph}$$

Proof:

$$E(\lambda_0)^h = E \left\{ \frac{\prod_{i=1}^p l_i^{nph}}{l_0^{nph}} \right\} = E \left\{ \frac{\prod_{i=1}^p l_i^{nph}}{\left[\frac{1}{p} \prod_{i=1}^p l_i + u \right]^{nph}} \right\}$$

$$= \int \dots \int \int_{l_1, \dots, l_p > 0, u > 0} \frac{\prod_{i=1}^p l_i^{n-1}}{\left[\frac{1}{p} \sum_{i=1}^p l_i + u \right]^{nph}} p(l_1, l_2, \dots, l_p, u) dl_1 dl_2 \dots dl_p du \quad (4.7)$$

From equation 3.15, it can be shown that u is a function such that

$$u = f(x_1(1), x_2(1), \dots, x_p(1))$$

Corollary 2.7.1 states that $X_{(1)}$ and $\bar{X} - X_{(1)}$ are independent. From the corollary then, it is known that for any given sample $\bar{X}_i - X_{i(1)} = l_i$ and $\bar{X}_{i(1)}$ are independent. Since u is a function of $X_{i(1)}$, l_i and u are independent. Therefore equation 4.7 is equivalent to

$$\int \dots \int \int_{l_1, \dots, l_p > 0, u > 0} \frac{\prod_{i=1}^p l_i^{n-1}}{\left[\frac{1}{p} \sum_{i=1}^p l_i + u \right]^{nph}} p(l_1, l_2, \dots, l_p) p(u) dl_1 \dots dl_p du \quad (4.8)$$

From Chapter Two the following is true:

$$l_i \sim \text{Gamma}(n-1, \sigma/n) \quad \text{equation 2.18}$$

$$u \sim \text{Gamma}(p-1, \sigma/np) \quad (4.9)$$

Theorem 2.6 gives the distribution of $X_{i(1)}$ to be an exponential distribution with parameters θ and σ/n .

Applying Theorem 2.6 again to a sample of size p

$X_{i(1)}$ random variables yields that $u \sim \text{Gamma}(p-1, \sigma/np)$.

In order to use the technique described in the beginning of this chapter, consider a function of θ defined as follows

$$f(\theta) = E \left\{ \left[\prod_{i=1}^p \pi_{l_i}^{nh} \right] \cdot e^{\theta \cdot \left[\frac{1}{p} \sum_{i=1}^p l_i + u \right]} \right\}$$

$$= \int \dots \int \int \left[\prod_{i=1}^p \pi_{l_i}^{nh} \right] \cdot \exp \left[\frac{\theta}{p} \sum_{i=1}^p l_i + \theta \cdot u \right] p(l_1, \dots, l_p, u) dl_1 \dots dl_p du$$

$$l_1, \dots, l_p > 0, u > 0$$

each of the p populations are independent and therefore

$$= \int \dots \int \int \left[\prod_{i=1}^p \pi_{l_i}^{nh} \right] \cdot \exp \left[\frac{\theta}{p} \sum_{i=1}^p l_i \right] e^{\theta u} p(l_1) \dots p(l_p) p(u) dl_1 \dots dl_p du$$

$$l_1, \dots, l_p > 0, u > 0$$

Substituting the p.d.f. of each $p(l_i)$ and $p(u)$ results in the equation:

$$\frac{\left[\prod_{i=1}^p \pi_{l_i}^{nh} \right] \exp \left[\frac{\theta}{p} \sum_{i=1}^p l_i \right] \prod_{i=1}^p l_i^{(n-1)-1} \exp \left[-\sum_{i=1}^p \frac{l_i}{\sigma/n} \right]}{(\sigma/n)^{p(n-1)} [\Gamma(n-1)]^p (\sigma/np)^{p-1} \Gamma(p-1)}$$

$$l_1, \dots, l_p > 0, u > 0$$

$$\times e^{\theta u} u^{(p-1)-1} \exp \left[\frac{-u}{\sigma/np} \right] dl_1 \dots dl_p du$$

$$= K \cdot \int \dots \int \left[\prod_{i=1}^p \pi_{l_i}^{nh+(n-1)-1} \exp \left[\frac{\theta}{p} l_i - \frac{l_i}{\sigma/n} \right] \right] dl_1 \dots dl_p$$

$$l_1, \dots, l_p > 0$$

$$\times \int_{u>0} u^{(p-1)-1} \exp \left[\theta u - \frac{u}{\sigma/np} \right] du \quad (4.10)$$

where $K =$

$$\frac{1}{\left[\frac{\sigma}{n}\right]^{p(n-1)} \left[\Gamma(n-1)\right]^p \left[\frac{\sigma}{np}\right]^{p-1} \Gamma(p-1)}$$

Using equation 2.6 definition of the Gamma distribution and the knowledge that each ℓ_i is a random variable from the same distribution permits replacement of the product of the ℓ_i 's by raising the first integrand to the appropriate power.

Equation 4.10 makes the transformation to this equation

$$= K \cdot \left[\frac{\Gamma(nh+n-1)}{\frac{n}{\sigma} - \frac{\theta}{p}}^{nh+n-1} \right]^p \cdot \left[\frac{\Gamma(p-1)}{\frac{np}{\sigma} - \theta}^{p-1} \right] \quad (4.11)$$

Replacing K with its equation, equation 4.11 simplifies to

$$\begin{aligned} &= \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot \frac{\left[\frac{\sigma}{n}\right]^{nph}}{\left[1 - \frac{\sigma\theta}{np}\right]^{nph+np-1}} \\ &= K_1 \cdot \left[1 - \frac{\sigma\theta}{np}\right]^{-\alpha} \end{aligned} \quad (4.12)$$

$$\text{where } K_1 = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot \left[\frac{\sigma}{n}\right]^{nph}$$

$$\alpha = nph + pn - 1$$

Now according to the technique

$$E(\lambda_0) = \int_0^h \frac{d f(\theta)}{d\theta} \quad \text{with } \theta=0 \text{ and } r=-nph$$

where $f(\theta)$ is of the form described in Lemma 4.1.

So using equation 4.6, one has

$$\left. \frac{d}{d\theta^r} f(\theta) \right|_{\theta=0} = K \cdot \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \left[\frac{\sigma}{np} \right]^r$$

Substitute $\frac{\sigma}{np}$ for B_1 in equation 4.5 and $-nph$ for r in equation 4.6 and

the desired results

$$E(\lambda_0)^h = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot \frac{\Gamma(np-1)}{\Gamma(nph+np-1)} \cdot p^{nph} \quad (4.13)$$

Theorem 4.2: The h^{th} moment of λ_1 is given by

$$E(\lambda_1)^h = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot p^{nph} \cdot \frac{\Gamma(np-p)}{\Gamma(nph+np-p)}$$

Proof:

$$E(\lambda_1)^h = E \left\{ \frac{\prod_{i=1}^p (\bar{X}_i - X_{i(1)})^{nh}}{\left[\frac{1}{p} \sum_{i=1}^p l_i \right]^{nph}} \right\} = E \left[\frac{\prod_{i=1}^p (l_i)^{nh}}{\bar{l}^{nph}} \right]$$

where $\bar{l}$ is defined in equation 3.12.

$$= \int \dots \int_{l_1 \dots l_p > 0} \frac{\prod_{i=1}^p (l_i)^{nh}}{\left[\frac{1}{p} \sum_{i=1}^p l_i \right]^{nph}} p(l_1, \dots, l_p) dl_1, \dots, dl_p$$

Consider a second function defined for criteria λ_1 as follows

$$g(\theta) = E \left[\left[\prod_{i=1}^p l_i^{nh} \right] \exp \left[\frac{\theta}{p} \sum_{i=1}^p l_i \right] \right]$$

$$= \int \dots \int_{l_1 \dots l_p > 0} \left[\frac{p}{\pi} l_i^{nh} \right] \exp \left[\frac{\theta}{p} \sum_{i=1}^p l_i \right] p(l_1) \dots p(l_p) dl_1 \dots dl_p$$

Substituting the p.d.f. of each $p(l_i)$ results in the equation:

$$\int \dots \int_{l_1 \dots l_p > 0} \frac{\left[\frac{p}{\pi} l_i^{nh} \right] \exp \left[\frac{\theta}{p} \sum_{i=1}^p l_i \right] p(l_i)^{(n-1)-1} \exp \left[-\sum_{i=1}^p \frac{l_i}{\sigma/n} \right]}{\left[\frac{\sigma}{n} \right]^{p(n-1)} [\Gamma(n-1)]^p} dl_1 \dots dl_p$$

$$= K \cdot \left[\int_{l_i > 0} l_i^{nh+n-1} \exp \left[-l_i \left[\frac{1}{\sigma/n} - \frac{\theta}{p} \right] \right] dl_i \right]^p \quad (4.14)$$

$$\text{where } K = \frac{1}{\left[\frac{\sigma}{n} \right]^{p(n-1)} [\Gamma(n-1)]^p}$$

Using equation 2.6 definition of the Gamma distribution, equation 4.14 transforms to

$$K \cdot \left[\frac{\Gamma(nh+n-1)}{\left[\frac{1}{\sigma/n} - \frac{\theta}{p} \right]^{nh+n-1}} \right]^p \quad (4.15)$$

Replacing K with its equation in equation 4.14 simplifies to

$$= \frac{[\Gamma(nh+n-1)]^p \left[\frac{\sigma}{n} \right]^{nph}}{\Gamma(n-1)^p \left[1 - \frac{\theta\sigma}{np} \right]^{nph+np-p}}$$

$$g(\theta) = K_2 \left[1 - \frac{\theta\sigma}{np} \right]^{-\alpha} \quad (4.16)$$

$$\text{where } K_2 = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \left[\frac{\sigma}{n} \right]^{nph}$$

$$\alpha = nph + np - p$$

Once again it can be shown that, using same arguments as for λ_0 that

$$E(\lambda_1)^h = \frac{d}{d\theta^r} g(\theta) \Big|_{\substack{\theta=0 \\ r=-nph}}$$

Equation 4.16 is in the same form as equation 4.5, therefore

$$\frac{d}{d\theta^r} g(\theta) = K_2 \cdot \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \left[\frac{\sigma}{np} \right]^r$$

The h^{th} moment of criteria λ_1 , after substituting for K_2 , α , and r is

$$E(\lambda_1)^h = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot p^{nph} \cdot \frac{\Gamma(np-p)}{\Gamma(nph+np-p)}$$

Theorem 4.3: The h^{th} moment of λ_2 is given by

$$E(\lambda_2)^h = \frac{\Gamma(nph+np-p)}{\Gamma(nph+np-1)} \frac{\Gamma(np-1)}{\Gamma(np-p)}$$

Proof:

$$\begin{aligned} E(\lambda_2)^h &= E \left[\frac{(v)^{nph}}{(v+u)^{nph}} \right] \\ &= \int \int_{v>0 \ u>0} \frac{(v)^{nph}}{(v+u)^{nph}} p(v) p(u) dv du \end{aligned}$$

Consider a third function defined for criteria λ_2 as follows

$$\begin{aligned} h(\theta) &= E \left[v^{nph} e^{\theta(u+v)} \right] \\ &= \int \int_{v>0, u>0} v^{nph} \exp(\theta(u+v)) p(v) p(u) dv du \end{aligned} \quad (4.17)$$

The variable u is distributed Gamma $(p-1, \sigma/np)$. It is known that

$l_i \sim \text{Gamma}(n-1, \sigma/np)$. Using Theorem 2.4, $l_i/p \sim \text{Gamma}(n-1), \sigma/np$.

Since each l_i comes from a different independent sample and the criteria is testing for equivalence of samples, applying Theorem 2.3 to l_i/p will result in V being distributed Gamma $(pn-p, \sigma/np)$.

Substituting in the appropriate Gamma function for the p.d.f., Equation 4.17 becomes

$$\begin{aligned} &= \int \int_{v>0, u>0} v^{nph} \exp(\theta(v+u)) \frac{v^{pn-p-1} \exp\left[\frac{-v}{\sigma/np}\right]}{\Gamma(pn-p) \left[\frac{\sigma}{np}\right]^{pn-p}} \frac{u^{p-1-1} \exp\left[\frac{-u}{\sigma/np}\right]}{\Gamma(p-1) \left[\frac{\sigma}{np}\right]^{p-1}} dv du \\ &= K \cdot \int_{v>0} v^{nph+pn-p-1} \exp\left[-v\left[\frac{1}{\sigma/np} - \theta\right]\right] dv \int_{u>0} u^{p-1-1} \exp\left[-u\left[\frac{1}{\sigma/np} - \theta\right]\right] du \end{aligned} \quad (4.18)$$

$$\text{where } K = \frac{1}{\Gamma(pn-p) \Gamma(p-1) \left[\frac{\sigma}{np}\right]^{pn-1}}$$

Using equation 2.6 definition of the Gamma function, equation 4.18 transforms to

$$\begin{aligned}
 h(\theta) &= K \cdot \left[\frac{\Gamma(nph+np-p)}{\left[\frac{1}{\sigma/np} - \theta \right]^{nph+np-p}} \right] \left[\frac{\Gamma(p-1)}{\left[\frac{1}{\sigma/np} - \theta \right]^{p-1}} \right] \\
 &= K_3 \left[1 - \frac{\theta\sigma}{np} \right]^{-\alpha}
 \end{aligned} \tag{4.19}$$

$$\text{where } K_3 = \frac{\Gamma(nph+np-p) \left[\frac{\sigma}{np} \right]^{nph}}{\Gamma(pn-p)}$$

$$\alpha = np + np - 1$$

Just as it was true for λ_0 and λ_1 , it is so for λ_2 and

$$E(\lambda_2)^h = \left. \frac{d}{d\theta^r} h(\theta) \right|_{\substack{\theta=0 \\ r=-nph}}$$

Equation 4.19 is in the same form as equation 4.5, therefore

$$\begin{aligned}
 \frac{d}{d\theta^r} h(\theta) &= K_3 \cdot \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \left[\frac{\sigma}{np} \right]^r \\
 &= \frac{\Gamma(nph+np-p) \left[\frac{\sigma}{np} \right]^{nph}}{\Gamma(pn-p)} \frac{\Gamma(nph+np-1-nph)}{\Gamma(nph+np-1)} \left[\frac{\sigma}{np} \right]^{-nph}
 \end{aligned}$$

The h^{th} moment of λ_2 is

$$E(\lambda_2)^h = \frac{\Gamma(nph+np-p) \Gamma(np-1)}{\Gamma(nph+np-1) \Gamma(np-p)}$$

The moments of the three criteria λ_0 , λ_1 , and λ_2 do agree with those found by P.V. Sukhatme (Ref: 18:98). However, the method used by P.V. Sukhatme requires significantly more effort.

V. Distribution of the Criteria

In this chapter, the distributions of criteria λ_0 , λ_1 , and λ_2 are obtained by inverting their characteristic functions. These distributions are obtained up to an order of n^{-4} and can be used to find accurate approximations to the percentage points of the test statistics.

The Distribution of λ_0

The h^{th} moment of λ_0 from Theorem 4.1 is

$$E(\lambda_0)^h = p^{nph} \frac{\Gamma(np-1)}{\Gamma(nph+np-1)} \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \quad (5.1)$$

Let $M = -2s \log \lambda_0$ (Def 5.1)

where 's' is to be determined later for computational ease.

Define $\phi(t)$ to be the characteristic function of M. (Def 5.2)

Then

$$\phi_m(t) = E(\lambda_0^{-2sit})$$

Since equation 5.1 holds for any complex number h, substituting $-2sit$ for h gives

$$\phi_m(t) = \frac{\Gamma(np-1)}{[\Gamma(n-1)]^p} \cdot p^{-2snpit} \cdot \frac{[\Gamma(n(1-2sit)-1)]^p}{\Gamma(np(1-2sit)-1)} \quad (5.2)$$

$$= K(n,p) \cdot Y(t) \quad (\text{Def 5.3})$$

where

$$K(n,p) = \frac{\Gamma(np-1)}{[\Gamma(n-1)]^p} \quad (\text{Def 5.4})$$

$$Y(t) = p^{-2snpit} \cdot \frac{[\Gamma(n(1-2sit)-1)]^p}{\Gamma(np(1-2sit)-1)} \quad (\text{Def 5.6})$$

Also define $T = (1-2it)$

(Def 5.6)

The function $\phi_m(t)$ has been broken into two parts one independent of t and the second part $Y(t)$, which depends on t , shall be operated on first. Using Definition 5.6 and after some algebra

$$Y(t) = p^{-2snpit} \cdot \frac{[\Gamma(snT - sn + n - 1)]^p}{\Gamma(npsT - nps + np - 1)}$$

Therefore,

$$\begin{aligned} \log Y(t) &= -2snpit \log p \\ &+ p \log \Gamma(snT - sn + n - 1) - \log \Gamma(npsT - nps + np - 1) \end{aligned} \quad (5.3)$$

The expansion of several logarithmic Gamma functions will be based on the following expansion (Ref 1:204):

$$\begin{aligned} \log \Gamma(x+h) &= \frac{1}{2} \log(2\pi) + (x+h-\frac{1}{2}) \log x - x \\ &- \sum_{r=1}^m \frac{(-1)^r}{r(r+1)} \frac{B_{r+1}(h)}{x^r} + R_{m+1}(x) \end{aligned} \quad (5.4)$$

where $R_m(x)$ is the remainder such that $R_m(x) < \theta/x^m$, θ a constant and $B_r(h)$ is the Bernoulli polynomial of degree r , order 1 defined by

$$\frac{ve^{hv}}{e^v - 1} = \sum_{r=0}^{\infty} \frac{v^r B_r(h)}{r!}$$

Using the expansion of equation 5.4 on the second and third terms of equation 5.3 gives the following two expansions.

$$\log \Gamma(\text{snT} - \text{sn} + n - 1) = \frac{1}{2} \log(2\pi) + (\text{snT} - \text{sn} + n - \frac{3}{2}) \log(\text{snT})$$

$$-\text{snT} \sum_{r=1}^m \frac{(-1)^r}{r(r+1)(\text{snT})^r} B_{r+1}(-\text{sn} + n - 1) + R_{m+1} \quad (5.5)$$

$$\log \Gamma(\text{npsT} - \text{nps} + np - 1) = \frac{1}{2} \log(2\pi) + (\text{npsT} - \text{nps} + np - \frac{3}{2}) \log(\text{npsT})$$

$$-\text{npsT} \sum_{r=1}^m \frac{(-1)^r}{r(r+1)(\text{npsT})^r} B_{r+1}(-\text{nps} + np - 1) + R_{m+1}' \quad (5.6)$$

Substitute equations 5.5 and 5.6 into equation 5.3, the following results occur after some algebra.

$$\begin{aligned} \log Y(t) = & \left(\frac{3}{2} - np\right) \log p + \left[\frac{p-1}{2}\right] \cdot \log(2\pi) + \frac{3(1-p)}{2} \log(\text{snT}) \\ & + \sum_{r=1}^m \frac{(-1)^r}{r(r+1)(\text{snT})^r} \left[p^{-r} B_{r+1}(-\text{nps} + np - 1) - p B_{r+1}(-\text{sn} + n - 1) \right] \\ & + R_{m+1}'' \end{aligned} \quad (5.7)$$

Define

$$C_r = \frac{(-1)^r}{r(r+1)} \left[p^{-r} B_{r+1}(-\text{nps} + np - 1) - p B_{r+1}(-\text{sn} + n - 1) \right] \quad (\text{Def } 5.7)$$

Thus, from equation 5.7 and Definition 5.7

$$\begin{aligned} \log Y(t) = & \left(\frac{3}{2} - np\right) \log p + \left[\frac{p-1}{2}\right] \cdot \log(2\pi) + \frac{3(1-p)}{2} \log(\text{snT}) \\ & + \sum_{r=1}^m \frac{C_r}{(\text{snT})^r} + R_{m+1}'' \end{aligned} \quad (5.8)$$

Using the following relationship between C_r and D_z presented by Kalinin and Shalaevskii (Ref 11:6-11), D_z is recursively computed as follows,

$$D_z = \sum_{k=1}^z k \frac{C_k D_{z-k}}{z}, \quad D_0 = 1$$

Therefore, equation 5.8 can be transformed to

$$Y(t) = p^{\frac{3}{2}-np} (2\pi)^{(p-1)/2} (\text{sn}T)^{3(1-p)/2} \left[1 + \sum_{z=1}^{\infty} \frac{D_z}{(\text{sn}T)^z} \right] + R'_{m+1} \quad (5.9)$$

From Definition 5.4,

$$\begin{aligned} K(n,p) &= \frac{\Gamma(np-1)}{[\Gamma(n-1)]^p} \\ &= \frac{\Gamma(nps-nps+np-1)}{[\Gamma(ns-ns+n-1)]^p} \end{aligned}$$

Therefore,

$$\log K(n,p) = \log \Gamma(nps-nps+np-1) - p \log \Gamma(ns-ns+n-1) \quad (5.10)$$

By using the logarithmic expansion of equation 5.4 and some algebra, equation 5.10 becomes

$$\log K(n,p) = \left[\frac{1-p}{2} \right] \cdot \log(2\pi) + \frac{3}{2}(p-1) \cdot \log(ns) + (np-2) \cdot \log p$$

$$+ \sum_{r=1}^m \frac{A_r}{(ns)^r} + R'_{1,m+1}$$

$$\text{where } A_r = \frac{(-1)^r}{r(r+1)} [pB_{r+1}(-ns+n-1) - p^{-r}B_{r+1}(-nps+np-1)] \quad (\text{Def 5.8})$$

Once again using the Kalinin and Shalaevskii recursive relationship

$$K(n,p) = (2\pi)^{(1-p)/2} (ns)^{3(p-1)/2} \frac{np-3/2}{p} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(ns)^v} \right] + R'_{1,m+1} \quad (5.11)$$

Reuniting $Y(t)$ and $K(n,p)$ together and some algebra results in the characteristic function of M ,

$$\phi_m(t) = T^{3(1-p)/2} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(ns)^v} \right] \left[1 + \sum_{z=1}^{\infty} \frac{D_z}{(nsT)^z} \right] + R_{2,m+1}$$

Let $m=ns$ (Def 5.9)

Then up to the order m^{-4} , the asymptotic expansion of the distribution of $M = -2s \log \lambda_0$ is given by

$$\begin{aligned} \phi_m(t) = T^{\frac{3(1-p)}{2}} & \left[1 + m^{-1} \left[\frac{D_1}{T} + B_1 \right] + m^{-2} \left[\frac{D_2}{T^2} + \frac{B_1 D_1}{T} + B_2 \right] \right. \\ & + m^{-3} \left[\frac{D_3}{T^3} + \frac{B_1 D_2}{T^2} + \frac{B_2 D_1}{T} + B_3 \right] + m^{-4} \left[\frac{D_4}{T^4} + \frac{B_1 D_3}{T^3} + \frac{B_2 D_2}{T^2} \right. \\ & \left. \left. + \frac{B_3 D_1}{T} + B_4 \right] \right] + O(m^{-5}) \end{aligned} \quad (5.12)$$

The variable 's' is chosen such that $A_1=0$. This gives

$$s = \frac{(18n-13)p-13}{18np}$$

Note from definitions 5.7 and 5.8 that $A_r = -C_r$.

Using the Kalinin and Shalaevskii recursive relation, the following equations are listed.

$$B_1 = A_1$$

$$B_2 = A_2 + 1/2 A_1^2$$

$$B_3 = 1/3 (A_1 B_2 + 2A_2 B_1 + 3A_3)$$

$$B_4 = 1/4 (A_1 B_3 + 2A_2 B_2 + 3A_3 B_1 + 4A_4)$$

Since 's' is chosen such that $A_1=0$ and noting that $A_r = -C_r$,

$$B_1 = A_1 = C_1 = D_1 = 0$$

$$B_2 = A_2 = -C_2 = -D_2$$

$$B_3 = A_3 = -C_3 = -D_3$$

$$B_4 = 1/2 (A_2^2) + A_4$$

$$D_4 = 1/2 (A_2^2) - A_4$$

Finally, after some more algebra

$$\begin{aligned} \phi_m(t) = T \frac{3(1-p)}{2} [1 + m^{-2} A_2(1-T^{-2}) + m^{-3} A_3(1-T^{-3}) \\ + m^{-4} (A_2^2(1-T^{-2}) + (A_4 - 1/2(A_2^2))(1-T^{-4}))] \\ + O(m^{-5}) \end{aligned} \quad (5.13)$$

Inverting the characteristic function in equation 5.13 and noting that $(1-2it)^{-r/2}$ is the characteristic function of a CHI Square variable with r degrees of freedom, the complementary distribution function of M is given by

$$\begin{aligned} F(x) = P(M > x) = G_{3p-3}(x) + m^{-2} A_2 (G_{3p-3}(x) - G_{3p+1}(x)) \\ + m^{-3} A_3 (G_{3p-3}(x) - G_{3p+3}(x)) + m^{-4} A_2^2 (G_{3p-3}(x) - G_{3p+1}(x)) \\ + m^{-4} (A_4 - 1/2(A_2^2)) (G_{3p-3}(x) - G_{3p+5}(x)) + O(m^{-5}) \end{aligned} \quad (5.14)$$

where $G_r(X)$ is the complementary distribution function of a CHI Square variable with r degrees of freedom. The computed value of A_i ($i=2,3,4$) are now listed.

$$A_2 = \frac{-(47p^3 - 169p^2 + 169p - 47)}{432 p^2}$$

$$A_3 = \frac{218p^4 - 455p^3 + 455p - 218}{14580p^3}$$

$$A_4 = \frac{-(4723p^5 - 32487p^4 + 79430p^3 - 79430p^2 + 32487p - 4723)}{466560p^4}$$

The Distribution of λ_1

The h^{th} moment of λ_1 from Theorem 4.2 is

$$E(\lambda_1)^h = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)} \right]^p \cdot p^{nph} \cdot \frac{\Gamma(np-p)}{\Gamma(np+n-p)} \quad (5.15)$$

$$\text{Let } M_1 = -2u \log \lambda_1 \quad (\text{Def 5.10})$$

where 'u' is to be computed later for computational ease.

$$\text{Define } \phi_1(t) \text{ to be the characteristic function of } M_1. \quad (\text{Def 5.11})$$

Then

$$\begin{aligned} \phi_1(t) &= E(\lambda_1^{-2uit}) \\ &= \frac{\Gamma(np-p)}{[\Gamma(n-1)]^p} p^{-2npuit} \frac{[\Gamma(n(1-2uit)-1)]^p}{\Gamma(np(1-2uit)-p)} \end{aligned} \quad (5.16)$$

$$= K_1(n,p) \cdot Y_1(t) \quad (\text{Def 5.11})$$

where

$$K_1(n, p) = \frac{\Gamma(np-p)}{[\Gamma(n-1)]^p} \quad (\text{Def 5.12})$$

$$Y_1(t) = p^{-2npuit} \frac{[\Gamma(n(1-2uit)-1)]^p}{\Gamma(np(1-2uit)-p)} \quad (\text{Def 5.13})$$

As before $\phi_1(t)$ is broken into two parts to be operated on.

Using Definition 5.6 and after some algebra

$$Y_1(t) = p^{-2npuit} \frac{[\Gamma(unT+n-un-1)]^p}{\Gamma(nupT+np-nup-p)}$$

Therefore,

$$\begin{aligned} \log Y_1(t) &= -2npuit \log p + p \log \Gamma(unT+n-un-1) \\ &\quad - \log \Gamma(nupT+np-nup-p) \end{aligned} \quad (5.17)$$

Using the expansion of equation 5.4 on the second and third terms of equation 5.17, the following results occur after some algebra.

$$\begin{aligned} \log Y_1(t) &= (-pn+p+1/2) \log p + 1/2(p-1) \log (2\pi) \\ &\quad + 1/2(1-p) \log (unT) + \sum_{r=1}^m \frac{C_r}{(unT)^r} + R_{m+1} \end{aligned} \quad (5.18)$$

where

$$C_r = \frac{(-1)^r}{r(r+1)} [p^{-r} B_{r+1}(np-npu-p) - p B_{r+1}(n-nu-1)] \quad (\text{Def 5.14})$$

Once again using the Kalinin and Shalaevskii recursive relationship equation 5.18 can be transformed to

$$Y_1(t) = p^{-pn+p+1/2} \frac{1/2(p-1)}{(2\pi)} \frac{1/2(1-p)}{(unT)} \left[1 + \sum_{s=1}^{\infty} \frac{D_s}{(unT)^s} \right] + R'_{m+1} \quad (5.19)$$

From Definition 5.12,

$$\begin{aligned} K_1(n,p) &= \frac{\Gamma(np-p)}{[\Gamma(n-1)]^p} \\ &= \frac{\Gamma(nup+np-nup-p)}{[\Gamma(un+n-un-1)]^p} \end{aligned}$$

Therefore,

$$\log K_1(n,p) = \log \Gamma(nup+np-nup-p) - p \log \Gamma(un+n-un-1) \quad (5.20)$$

Using the logarithmic expansion of equation 5.4 and some algebra, equation 5.20 becomes

$$\log K_1(n,p) = \frac{1}{2} (1-p) \log(2\pi) + \frac{1}{2}(p-1) \log(un) + (np-p-2) \log p$$

$$+ \sum_{r=1}^m \frac{A_r}{(un)^r} + R_{1,m+1}$$

where

$$A_r = \frac{(-1)^r}{r(r+1)} [pB_{r+1}(n-un-1) - p^{-r}B_{r+1}(np-nup-p)] \quad (\text{Def 5.15})$$

Using the Kalinin and Shalaevskii recursive relationship

$$K_1(n, p) = (2\pi)^{\frac{(1-p)/2}{(un)^{\frac{(p-1)/2}{p}}}} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(un)^v} \right] + R_{1, m+1}^{(5, 21)}$$

Reuniting $Y_1(t)$ and $K_1(n, p)$ and some algebra results in the characteristic function of M_1 .

$$\phi_1(t) = T^{\frac{(1-p)}{2}} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(un)^v} \right] \left[1 + \sum_{s=1}^{\infty} \frac{D_s}{(unT)^s} \right] + R_{2, m+1}$$

Let $w=un$ (Def 5.16)

Then up to the order w^{-4} , the asymptotic expansion of the distribution M_1 $= -2u \log \lambda_1$ is given by

$$\begin{aligned} \phi_1(t) = T^{\frac{(1-p)}{2}} & \left[1 + w^{-1} \left[\frac{D_1}{T} + B_1 \right] + w^{-2} \left[\frac{D_2}{T^2} + \frac{B_1 D_1}{T} + B_2 \right] \right. \\ & + w^{-3} \left[\frac{D_3}{T^3} + \frac{B_1 D_2}{T^2} + \frac{B_2 D_1}{T} + B_3 \right] + w^{-4} \left[\frac{D_4}{T^4} + \frac{B_1 D_3}{T^3} \right. \\ & \left. \left. + \frac{B_2 D_2}{T^2} + \frac{B_3 D_1}{T} + B_4 \right] \right] + O(w^{-5}) \end{aligned} \quad (5.22)$$

The variable 'u' is now chosen such that $A_1=0$. This gives

$$u = \frac{(6n-7)p-1}{6np}$$

Note from Definitions 5.14 and 5.15 that $A_r = -C_r$. Using the previous listed Kalinin and Shalaevskii variable relationships and some algebra

$$\begin{aligned}\phi_1(t) = T \frac{(1-p)}{2} [1 + w^{-2} A_2(1-T^{-2}) + w^{-3} A_3(1-T^{-3}) \\ + w^{-4} (A_2^2(1-T^{-2}) + (A_4 - 1/2(A_2^2))(1-T^{-4}))] \\ + O(w^{-5})\end{aligned}\quad (5.23)$$

Inverting the characteristic function in equation 5.23, the complementary distribution function of M_1 is given by

$$\begin{aligned}F(x) = P(M > x) = G_{p-1}(X) + w^{-2} A_2(G_{p-1}(X) - G_{p+3}(X)) \\ + w^{-3} A_3(G_{p-1}(X) - G_{p+5}(X)) + w^{-4} A_2^2(G_{p-1}(X) - G_{p+3}(X)) \\ + w^{-4} (A_4 - 1/2(A_2^2))(G_{p-1}(X) - G_{p+7}(X)) + O(w^{-5})\end{aligned}\quad (5.24)$$

where $G_r(X)$ is the complementary distribution function of a CHI Square variable with r degrees of freedom. The computed values of A_i ($i=2,3,4$) are now listed.

$$A_2 = \frac{p^3 + p^2 - p - 1}{144 p^2}$$

$$A_3 = \frac{2p^4 - 5p^3 + 5p - 2}{1620p^3}$$

$$A_4 = \frac{-(19p^5 + 9p^4 - 10p^3 + 10p^2 - 9p - 19)}{17280 p^4}$$

The Distribution of λ_2

The h^{th} moment of λ_2 from Theorem 4.3 is

$$E(\lambda_2)^h = \frac{\Gamma(nph + np - p) \Gamma(np - 1)}{\Gamma(nph + np - 1) \Gamma(np - p)} \quad (5.25)$$

$$\text{Let } M_2 = -2b \log \lambda_2 \quad (\text{Def 5.17})$$

where 'b' is to be computed later for computational ease.

$$\text{Define } \phi_2(t) \text{ to be the characteristic function of } M_2. \quad (\text{Def 5.18})$$

Then

$$\begin{aligned} \phi_2(t) &= E(\lambda_2^{-2bit}) \\ &= \frac{\Gamma(np-1)\Gamma(np(1-2bit)-p)}{\Gamma(np-p)\Gamma(np(1-2bit)-1)} \end{aligned} \quad (5.26)$$

$$= K_2(n,p) \cdot Y_2(t) \quad (\text{Def 5.19})$$

where

$$K_2(n,p) = \frac{\Gamma(np-1)}{\Gamma(np-p)} \quad (\text{Def 5.20})$$

$$Y_2(t) = \frac{\Gamma(np(1-2bit)-p)}{\Gamma(np(1-2bit)-1)} \quad (\text{Def 5.21})$$

As before $\phi_2(t)$ is broken into two parts to be operated on.

Using Definition 5.6 and after some algebra

$$Y_2(t) = \frac{\Gamma(npbT+np-npb-p)}{\Gamma(npbT+np-npb-1)}$$

Therefore,

$$\log Y_2(t) = \log \Gamma(npbT+np-npb-p) - \log \Gamma(npbT+np-npb-1) \quad (5.27)$$

Using the expansion of equation 5.4 on the two terms of equation 5.27, the following results occur after some algebra.

$$\log Y_2(t) = (1-p) \log (npbT) + \sum_{r=1}^m \frac{C_r}{(npbT)^r} + R_{m+1} \quad (5.28)$$

where

$$C_r = \frac{(-1)^r}{r(r+1)} [B_{r+1}(np(1-b)-1) - B_{r+1}(p(n-nb-1))] \quad (\text{Def 5.22})$$

Once again using the Kalinin and Shalaevskii recursive relationship equation 5.28 can be transformed to

$$Y_2(t) = (npbT)^{1-p} \left[1 + \sum_{s=1}^{\infty} \frac{D_s}{(npbT)^s} \right] + R'_{m+1} \quad (5.29)$$

From Definition 5.20

$$\begin{aligned} K_2(n,p) &= \frac{\Gamma(np-1)}{\Gamma(np-p)} \\ &= \frac{\Gamma(npb+np(1-b)-1)}{\Gamma(npb+p(n-nb-1))} \end{aligned}$$

Therefore,

$$\log K_2(n,p) = \log \Gamma(npb+np(1-b)-1) - \log \Gamma(npb+p(n-nb-1)) \quad (5.30)$$

Using the logarithmic expansion of equation 5.4 and some algebra, equation 5.30 becomes

$$\log K_2(n,p) = (p-1)\log(npb) + \sum_{r=1}^m \frac{A_r}{(npb)^r} + R_{1,m+1}$$

where

$$A_r = \frac{(-1)}{r(r+1)} [B_{r+1}(p(n-nb-1)) - B_{r+1}(np(1-b)-1)] \quad (\text{Def 5.23})$$

Using the Kalinin and Shalaevskii recursive relationship

$$K_2(n,p) = (npb)^{p-1} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(npb)^v} \right] + R_{1,m+1} \quad (5.31)$$

Reuniting $Y_2(t)$ and $K_2(n,p)$ and some algebra results in the characteristic function of M_2 .

$$\phi_2(t) = T^{1-p} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(npb)^v} \right] \left[1 + \sum_{s=1}^{\infty} \frac{D_s}{(npbT)^s} \right] + R_{2,m+1}$$

Let $z=npb$ (Def 5.24)

Then up to the order z^{-4} , the asymptotic expansion of the distribution $M_2 = -2b \log \lambda_2$ is given by

$$\begin{aligned} \phi_2(t) = T^{1-p} & \left[1 + z^{-1} \left[\frac{D_1}{T} + B_1 \right] + z^{-2} \left[\frac{D_2}{T^2} + \frac{B_1 D_1}{T} + B_2 \right] \right. \\ & + z^{-3} \left[\frac{D_3}{T^3} + \frac{B_1 D_2}{T^2} + \frac{B_2 D_1}{T} + B_3 \right] + z^{-4} \left[\frac{D_4}{T^4} + \frac{B_1 D_3}{T^3} \right. \\ & \left. \left. + \frac{B_2 D_2}{T^2} + \frac{B_3 D_1}{T} + B_4 \right] \right] + O(z^{-5}) \end{aligned} \quad (5.32)$$

The variable 'b' is now chosen such that $A_1=0$. This gives

$$b = \frac{(2n-1)p-2}{2np}$$

Note from Definitions 5.22 and 5.23 that $A_r = -C_r$.

Using the previously listed Kalinin and Shalaevskii variable relationships and some algebra

$$\begin{aligned} \phi_2(t) = & T^{1-p} [1+z^{-2} A_2(1-T^{-2}) + z^{-3} A_3(1-T^{-3}) \\ & + z^{-4} (A_2^2(1-T^{-2}) + (A_4 - 1/2(A_2^2))(1-T^{-4}))] \\ & + O(z^{-5}) \end{aligned} \quad (5.33)$$

Inverting the characteristic function in equation 5.33, the complementary distribution function of M_2 is given by

$$\begin{aligned} F(x) = P(M_2 > x) = & G_{2p-2}(X) + z^{-2} A_2(G_{2p-2}(X) - G_{2p+2}(X)) \\ & + z^{-3} A_3(G_{2p-2}(X) - G_{2p+4}(X)) + z^{-4} A_2^2(G_{2p-2}(X) - G_{2p+2}(X)) \\ & + z^{-4} (A_4 - 1/2(A_2^2))(G_{2p-2}(X) - G_{2p+6}(X)) + O(z^{-5}) \end{aligned} \quad (5.34)$$

where $G_r(X)$ is the complementary distribution function of a CHI Square variable with r degrees of freedom. The computed values of A_i ($i=2,3,4$) are now listed.

$$A_2 = \frac{-(p^3 - 3p^2 + 2p)}{24}$$

$$A_3 = 0$$

$$A_4 = \frac{-(3p^5 - 15p^4 + 20p^3 - 8p)}{960}$$

Computation of Percentage Points

The percentage points of L_i and M_i ($i=0,1,2$) have been computed using the Newton-Secant method for finding zeros of a function. The arithmetic is carried using Double Precision on a CDC Cyber 750 system at Wright-Patterson AFB, OH. A subroutine of International Mathematical and Statistical Libraries (IMSL) was used in finding the percentage points. The computation of the various coefficients A_i was done using the MACSYMA system of Massachusetts Institute of Technology. The percentage points are computed for $p = 2(1)10$, $n = 10(1)20(5)50(10)100$, and $\alpha = .100, .050, .025, .010, .005$.

A good approximation to the asymptotic expansion of the test statistic distributions is the first term of the expansion. This approximation is accurate to three significant figures as a result of the computational technique of solving variables s, u, b so that $A_1 = 0$ and the characteristics of the expansions. The first term of the expansion is a CHI-Square distribution and therefore its' percentage points are readily available. To use this approximation the correct percentage points for M_0 are found by looking up a CHI-Square percentage point for the appropriate alpha with $3p-3$ degrees of freedom, for M_1 and M_2 percentage points are found by using $p-1$ and $2p-2$ degrees of freedom respectively.

VI. Test Statistics and Practical Illustrations

In this chapter the tests considered in this thesis will be applied to a set of aircraft life data. First a test is performed to determine if the underlying distribution of the samples is exponentially distributed. The procedure chosen is the Bartlett Test. After checking to assure that the distributions are exponential, the three hypotheses stated in Chapter I are tested using the test statistics L_i and M_i ($i=0,1,2$). The data is a sample of the intervals between successive failures (given in operating hours) of airconditioning equipment on four Boeing 720 aircraft (Ref 17:376). The data is given in Table 6.1.

TABLE 6.1

INTERVAL BETWEEN FAILURES OF AIRCRAFT AIRCONDITIONING

7909 (1)	7912 (2)	7913 (3)	7914 (4)
90	23	97	50
10	261	51	44
60	87	11	102
186	7	4	72
61	120	141	22
49	14	18	39
14	62	142	3
24	47	68	15
56	225	77	197
20	71	80	188
79	246	1	79
84	21	16	88
44	42	106	46
59	20	206	5
29	5	82	5
118	12	54	36
25	120	31	22
156	11	216	139
310	3	46	210
76	14	111	97
26	71	39	30
44	11	63	23
23	14	18	13
62	11	191	14

Bartlett's Test is used to determine if each of the samples failure intervals are exponentially distributed. The Bartlett Test statistic (Ref 12: 238,239) is

$$B_r = \frac{2r \left[\log \left[\frac{tr}{r} \right] - \frac{1}{r} \sum_{i=1}^r \log X_i \right]}{1 + ((r+1)/6r)}$$

where X_i is time to failure (interval), r is number of failures and

$t_r = \sum_{i=1}^r X_i$. The statistic B_r is CHI-Square distributed with $r-1$

degrees of freedom when the sample being tested has an underlying exponential distribution. A two-tailed CHI-Square test is used for this test. The statistics B_{24} for each aircraft are given in Table 6.2.

TABLE 6.2

B ₂₄ STATISTICS				
Airplane	7909 (1)	7912 (2)	7913 (3)	7914 (4)
B ₂₄	13.487	29.546	20.510	22.180

In order for these four samples to have an underlying exponential distribution with a 95% confidence interval they must fall between 11.688 and 38.076. All four statistics are within that region and are therefore exponentially distributed. One can also use the procedures given in Epstein (Ref 3) for testing whether the four populations are exponentially distributed.

Test Statistics for H_0

To determine if the airconditioners are failing after the same interval or they have identical location and scale parameters, the first hypothesis H_0 is used. H_0 states

$$\theta_1 = \theta_2 = \dots = \theta_p \text{ and } \sigma_1 = \sigma_2 = \dots = \sigma_p$$

against the alternative that one of the parameters is not equal.

$$\begin{aligned} \text{A. Let } L_0 &= \lambda_0^{\frac{1}{np}} \\ &= \frac{\left[\sum_{i=1}^p \frac{1}{n} (\bar{X}_i - X_{i(1)}) \right] \frac{1}{p}}{\bar{X} - X_{(1)}} \end{aligned} \quad (\text{Def 6.1})$$

where $\bar{X}_i$ is the i^{th} sample mean, $X_{i(1)}$ is the minimum observation in the i^{th} sample, $\bar{X}$ is the grand mean of all observations, $X_{(1)}$ is the minimum observation of all the observations and p is the number of populations being tested. These definitions will be used for all test statistics.

$$\text{B. } M_0 = -2s \log \lambda_0 \text{ from Definition 5.1}$$

Where

$$s = \frac{(18n - 13)p - 13}{18np}$$

$$\lambda_0 = \frac{\sum_{i=1}^p \frac{1}{n} (\bar{X}_i - X_{i(1)})^n}{(\bar{X} - X_{(1)})^{np}}$$

Decision Rule for H_0 : H_0 is rejected if $L_0 \leq L_T$ or if $M_0 > M_T$, where L_T and M_T are values obtained from Tables in the Appendix corresponding to L_0 and M_0 . L_0 and M_0 are equivalent test statistics.

In the example of aircraft airconditioning equipment, the calculations for L_0 are now shown.

$$\begin{aligned}\bar{X}_1 &= 71.0417 & \bar{X}_2 &= 63.250 & \bar{X}_3 &= 77.875 & \bar{X}_4 &= 64.125 \\ X_1(1) &= 10 & X_2(1) &= 3 & X_3(1) &= 1 & X_4(1) &= 3 \\ \bar{X} &= 69.0729 & X_{(1)} &= 1 & p &= 4\end{aligned}$$

$$\begin{aligned}L_0 &= \frac{[(61.0417)(60.250)(76.875)(61.125)]^{1/4}}{68.0729} = \frac{64.475818}{68.0729} \\ &= .947158\end{aligned}$$

From Table A3 after interpolating

$$L_{T, \alpha=.025} = .901399$$

Since $L_0 > L_T$, by the decision rule, accept the null hypothesis. For this example $M_0 = 10.0314$

$$M_{T, \alpha=.025} = 19.0257$$

The same result is obtained, since $M_0 < M_T$, by the decision rule, accept the null hypothesis.

Test Statistics for H_1

If the question is to determine if the scale parameters (mean/variance) are equal and the location parameter is not significant, the second hypothesis H_1 should be used. H_1 states $\sigma_1 = \sigma_2 = \dots = \sigma_p$ and θ_i 's are unspecified ($i=1,2,\dots,p$) against alternative that one of the scale parameters is not equal to the others.

$$\begin{aligned}
 \text{A. Let } L_1 &= \lambda_1 \\
 &= \frac{\left[\frac{1}{np} \sum_{i=1}^p (\bar{X}_i - X_{i(1)}) \right]^{1/p}}{\frac{1}{p} \sum_{i=1}^p (\bar{X}_i - X_{i(1)})} \quad (\text{Def 6.2})
 \end{aligned}$$

$$\text{B. } M_1 = -2u \log \lambda_1 \text{ from Definition 5.10}$$

Where

$$u = \frac{(6n-7)p-1}{6np}$$

$$\lambda_1 = \frac{\sum_{i=1}^p (\bar{X}_i - X_{i(1)})^n}{\left[\frac{1}{p} \sum_{i=1}^p (\bar{X}_i - X_{i(1)}) \right]^{np}}$$

Decision Rule for H_1 : H_1 is rejected if $L_1 < L_T$ or if $M_1 > M_T$, where L_T and M_T are values obtained from Tables in the Appendix corresponding to L_1 and M_1 . L_1 and M_1 are equivalent test statistics.

In the example of aircraft airconditioning equipment, L_1 is shown.

$$\begin{aligned}
 L_1 &= \frac{[(61.0417)(60.250)(76.875)(61.125)]^{1/4}}{1/4 (61.0417 + 60.250 + 76.875 + 61.125)} = \frac{64.475818}{64.822925} \\
 &= .994645
 \end{aligned}$$

From Table A12 after interpolating

$$L_{T, \alpha=.025} = .949605$$

Since $L_1 > L_T$, by the decision rule, accept the null hypothesis. For this example $M_1 = .978963$

$$M_{T, \alpha=.025} = 9.34725$$

The same result is obtained, since $M_1 < M_T$, by the decision rule accept the null hypothesis.

Test Statistics for H_2

The third hypothesis H_2 is used when it is known that all scale parameters (mean/variance) are equal and testing to determine if the location parameters (guaranteed life/minimum failure time) are equal.

H_2 states

$$\theta_1 = \theta_2 = \dots = \theta_p, \text{ given that } \sigma_1 = \sigma_2 = \dots = \sigma_p$$

against alternative that one of the location parameters is not equal to the others.

$$\begin{aligned} \text{A. Let } L_2 &= \lambda_2 \\ &= \frac{\frac{1}{p} \sum_{i=1}^p (\bar{X}_i - X_{i(1)})}{\bar{X} - X_{(1)}} \end{aligned} \quad (\text{Def 6.3})$$

B. $M_2 = -2b \log \lambda_2$ from Definition 5.17.

Where

$$b = \frac{(2n-1)p-2}{2np}$$

$$\lambda_2 = \frac{\left[\frac{1}{p} \sum_{i=1}^p (\bar{X}_i - X_{i(1)}) \right]^{np}}{[\bar{X} - X_{(1)}]^{np}}$$

Decision Rule for H_2 : H_2 is rejected if $L_2 > L_T$ or if $M_2 > M_T$, where L_T and M_T are values obtained from Tables in the Appendix corresponding to L_2 and M_2 . L_2 and M_2 are equivalent test statistics.

In the example of aircraft airconditioning equipment, L_2 is now shown.

$$L_2 = \frac{1/4 (61.0417 + 60.250 + 76.875 + 61.125)}{68.0729} = \frac{64.822925}{68.0729}$$

$$= .952257$$

From Table A21 after interpolating

$$L_T, \alpha=.025 = .924660$$

Since $L_2 > L_T$, by the decision rule, accept the null hypothesis that all of the location parameters are equal. For this example $M_2 = 9.09918$

$$M_T, \alpha=.025 = 14.45098$$

The same result is obtained, since $M_2 < M_T$, by the decision rule, accept the null hypothesis.

VII. Conclusion

In this thesis three hypotheses connected with the testing of equality of several exponential populations are considered. The asymptotic expansions of the distributions are found based on the Neyman-Pearson likelihood ratio criteria for λ_0 , λ_1 , and λ_2 . Tables of percentage points are computed from these expansions for each test statistic distribution using the Newton-Secant method for finding zeros of a function. The CHI-Square distribution with the appropriate degrees of freedom is found to be a good approximation to the percentage points for M_0 , M_1 , and M_2 . A practical illustration in Chapter VI combines definitions, test statistics, and percentage point tables to make this thesis a product that can be applied to many situations in both the Air Force and Industry.

The situations where these tests can be applied include the time interval between failure of parts on aircraft, time to failure of identical electronic components, the interval between consecutive accidents incurred by the same individual, and other life testing experiments involving the exponential distribution.

In order to make these tests more useful to the analyst further study should be continued in developing tests and tables for testing samples with unequal sizes. Since the test statistics are based on the likelihood ratio test it is known that the tests are uniformly most powerful. In order to find the power of the test statistics, further

research should be conducted to find the non-central distributions of the criteria. This additional research would enhance further applications of tests of hypothesis connected with exponential populations.

APPENDIX

TABLES

Table A1 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

$P = 2$

		$\alpha = .100$	.050	.025	.010	.005
M_0	N					
	10	6.25069	7.81371	9.34702	11.34292	12.83572
	11	6.25083	7.81392	9.34730	11.34332	12.83622
	12	6.25093	7.81407	9.34751	11.34361	12.83658
	13	6.25101	7.81418	9.34766	11.34382	12.83685
	14	6.25107	7.81426	9.34778	11.34399	12.83706
	15	6.25111	7.81433	9.34787	11.34411	12.83722
	16	6.25115	7.81439	9.34794	11.34422	12.83735
	17	6.25118	7.81443	9.34800	11.34430	12.83745
	18	6.25121	7.81446	9.34805	11.34437	12.83754
	19	6.25123	7.81449	9.34809	11.34442	12.83761
	20	6.25124	7.81452	9.34812	11.34447	12.83767
	25	6.25130	7.81460	9.34823	11.34462	12.83786
	30	6.25133	7.81464	9.34829	11.34470	12.83795
	35	6.25135	7.81467	9.34832	11.34475	12.83801
	40	6.25136	7.81468	9.34834	11.34478	12.83805
	45	6.25136	7.81469	9.34835	11.34480	12.83807
	50	6.25137	7.81470	9.34836	11.34481	12.83809
	60	6.25137	7.81471	9.34838	11.34483	12.83811
	70	6.25138	7.81471	9.34838	11.34484	12.83812
	80	6.25138	7.81472	9.34839	11.34485	12.83813
	90	6.25138	7.81472	9.34839	11.34485	12.83814
	100	6.25138	7.81472	9.34839	11.34485	12.83814
L_0	10	.839245	.803261	.769460	.727584	.697760
	11	.854205	.821199	.790060	.751288	.723538
	12	.866623	.836149	.807296	.771223	.745301
	13	.877095	.848798	.821926	.788217	.763911
	14	.886044	.859638	.834498	.802871	.780002
	15	.893780	.869030	.845417	.815637	.794050
	16	.900534	.877246	.854987	.826855	.806420
	17	.906480	.884494	.863443	.836790	.817393
	18	.911757	.890934	.870969	.845649	.827193
	19	.916470	.896695	.877711	.853599	.835998
	20	.920705	.901878	.883783	.860771	.843951
	25	.936745	.921562	.906906	.888176	.874421
	30	.947389	.934670	.922359	.906576	.894947
	35	.954967	.944026	.933414	.919779	.909710
	40	.960637	.951038	.941714	.929714	.920838
	45	.965039	.956489	.948175	.937460	.929525
	50	.968556	.960848	.953346	.943669	.936494
	60	.973822	.967384	.961109	.953001	.946982
	70	.977578	.972050	.966657	.959681	.954497
	80	.980391	.975548	.970820	.964699	.960146
	90	.982577	.978268	.974058	.968606	.964548
	100	.984325	.980443	.976650	.971734	.968074

Table A2 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$ $P = 3$

		$\alpha = .100$	.050	.025	.010	.005
M_0	N					
	10	10.64976	12.59824	14.45765	16.82245	18.55997
	11	10.64882	12.59701	14.45613	16.82052	18.55771
	12	10.64811	12.59610	14.45499	16.81907	18.55602
	13	10.64757	12.59540	14.45412	16.81796	18.55471
	14	10.64715	12.59485	14.45344	16.81709	18.55369
	15	10.64681	12.59441	14.45289	16.81639	18.55287
	16	10.64653	12.59405	14.45245	16.81583	18.55221
	17	10.64631	12.59376	14.45208	16.81536	18.55166
	18	10.64612	12.59352	14.45178	16.81497	18.55121
	19	10.64596	12.59331	14.45153	16.81465	18.55083
	20	10.64583	12.59314	14.45131	16.81437	18.55050
	25	10.64539	12.59257	14.45060	16.81346	18.54943
	30	10.64516	12.59226	14.45022	16.81297	18.54885
	35	10.64502	12.59208	14.44999	16.81268	18.54851
	40	10.64493	12.59196	14.44984	16.81249	18.54829
	45	10.64487	12.59188	14.44974	16.81237	18.54814
	50	10.64482	12.59183	14.44967	16.81228	18.54803
	60	10.64477	12.59175	14.44958	16.81216	18.54790
	70	10.64473	12.59171	14.44953	16.81209	18.54781
	80	10.64471	12.59168	14.44949	16.81204	18.54776
	90	10.64470	12.59166	14.44947	16.81201	18.54772
	100	10.64469	12.59165	14.44945	16.81199	18.54770
L_0	10	.821676	.792673	.765951	.733264	.710139
	11	.837926	.811253	.786591	.756307	.734802
	12	.851468	.826786	.803900	.775708	.755626
	13	.862925	.839963	.818621	.792261	.773437
	14	.872743	.851281	.831292	.806549	.788840
	15	.881251	.861106	.842313	.819004	.802290
	16	.888693	.869715	.851984	.829956	.814136
	17	.895258	.877321	.860540	.839662	.824647
	18	.901093	.884089	.868163	.848323	.834036
	19	.906312	.890150	.874996	.856097	.842473
	20	.911009	.895609	.881157	.863115	.850095
	25	.928846	.916390	.904659	.889959	.879311
	30	.940728	.930273	.920405	.908007	.899005
	35	.949211	.940204	.931689	.920972	.913177
	40	.955569	.947659	.940172	.930736	.923864
	45	.960513	.953461	.946781	.938353	.932208
	50	.964467	.958106	.952075	.944461	.938905
	60	.970396	.965077	.960028	.953647	.948985
	70	.974629	.970059	.965718	.960226	.956210
	80	.977803	.973797	.969990	.965169	.961643
	90	.980272	.976706	.973315	.969020	.965877
	100	.982246	.979033	.975977	.972104	.969269

Table A3 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

P = 4

		$\alpha=.100$	.050	.025	.010	.005
M_0	N					
	10	14.69559	16.93387	19.04070	21.68812	23.61477
	11	14.69338	16.93112	19.03740	21.68405	23.61010
	12	14.69173	16.92906	19.03493	21.68101	23.60661
	13	14.69047	16.92749	19.03303	21.67867	23.60393
	14	14.68949	16.92626	19.03155	21.67684	23.60183
	15	14.68870	16.92528	19.03037	21.67538	23.60015
	16	14.68806	16.92448	19.02941	21.67420	23.59879
	17	14.68754	16.92383	19.02862	21.67323	23.59767
	18	14.68710	16.92328	19.02796	21.67242	23.59674
	19	14.68673	16.92282	19.02741	21.67173	23.59596
	20	14.68642	16.92244	19.02694	21.67116	23.59529
	25	14.68540	16.92116	19.02540	21.66925	23.59310
	30	14.68486	16.92048	19.02458	21.66824	23.59193
	35	14.68453	16.92007	19.02409	21.66763	23.59124
	40	14.68432	16.91981	19.02378	21.66724	23.59079
	45	14.68418	16.91963	19.02356	21.66698	23.59048
	50	14.68408	16.91951	19.02341	21.66679	23.59026
	60	14.68395	16.91934	19.02321	21.66654	23.58998
	70	14.68387	16.91925	19.02309	21.66640	23.58981
	80	14.68382	16.91918	19.02302	21.66630	23.58970
	90	14.68379	16.91914	19.02296	21.66624	23.58963
	100	14.68376	16.91911	19.02293	21.66619	23.58958
L_0	10	.817157	.792408	.769797	.742298	.722905
	11	.833686	.810908	.790037	.764571	.746556
	12	.847479	.826389	.807017	.783318	.766511
	13	.859164	.839532	.821464	.799311	.783567
	14	.869187	.850829	.833904	.813114	.798312
	15	.877881	.860642	.844727	.825146	.811182
	16	.885492	.869246	.854228	.835725	.822514
	17	.892210	.876850	.862635	.845101	.832566
	18	.898185	.883620	.870127	.853466	.841543
	19	.903532	.889685	.876845	.860975	.849608
	20	.908347	.895149	.882903	.867754	.856893
	25	.926651	.915967	.906023	.893682	.884808
	30	.938863	.929890	.921522	.911116	.903617
	35	.947590	.939856	.932634	.923640	.917149
	40	.954137	.947342	.940991	.933071	.927351
	45	.959230	.953171	.947503	.940430	.935316
	50	.963305	.957838	.952721	.946331	.941708
	60	.969419	.964846	.960562	.955206	.951328
	70	.973786	.969856	.966172	.961563	.958223
	80	.977062	.973616	.970385	.966340	.963407
	90	.979610	.976543	.973665	.970060	.967446
	100	.981649	.978885	.976290	.973041	.970688

Table A4 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$ $P = 5$

		$\alpha = .100$	.050	.025	.010	.005
M_0	N					
	10	18.56820	21.04900	23.36372	26.24959	28.33646
	11	18.56471	21.04477	23.35873	26.24358	28.32966
	12	18.56211	21.04161	23.35500	26.23910	28.32459
	13	18.56012	21.03918	23.35215	26.23565	28.32069
	14	18.55856	21.03729	23.34991	26.23296	28.31764
	15	18.55732	21.03577	23.34813	26.23081	28.31520
	16	18.55631	21.03455	23.34668	26.22906	28.31323
	17	18.55548	21.03354	23.34549	26.22763	28.31160
	18	18.55479	21.03270	23.34450	26.22643	28.31025
	19	18.55421	21.03200	23.34367	26.22543	28.30911
	20	18.55372	21.03140	23.34296	26.22458	28.30815
	25	18.55211	21.02943	23.34064	26.22177	28.30497
	30	18.55125	21.02838	23.33940	26.22027	28.30327
	35	18.55073	21.02776	23.33866	26.21938	28.30226
	40	18.55040	21.02736	23.33819	26.21881	28.30161
	45	18.55018	21.02708	23.33786	26.21842	28.30116
	50	18.55002	21.02689	23.33763	26.21814	28.30085
	60	18.54981	21.02664	23.33733	26.21778	28.30044
	70	18.54969	21.02648	23.33715	26.21756	28.30019
	80	18.54961	21.02639	23.33704	26.21742	28.30003
	90	18.54955	21.02632	23.33696	26.21732	28.29992
	100	18.54951	21.02627	23.33690	26.21726	28.29985
L_0	10	.816032	.794165	.774291	.750208	.733261
	11	.832598	.812468	.794126	.771836	.756110
	12	.846432	.827789	.810766	.790033	.775372
	13	.858157	.840801	.824925	.805552	.791827
	14	.868221	.851988	.837117	.818941	.806044
	15	.876953	.861709	.847725	.830610	.818450
	16	.884601	.870233	.857038	.840869	.829368
	17	.891354	.877768	.865279	.849958	.839050
	18	.897362	.884477	.872623	.858068	.847695
	19	.902740	.890488	.879208	.865346	.855460
	20	.907583	.895906	.885147	.871916	.862473
	25	.926007	.916550	.907814	.897041	.889331
	30	.938308	.930364	.923013	.913930	.907419
	35	.947103	.940254	.933910	.926062	.920428
	40	.953703	.947686	.942106	.935197	.930232
	45	.958839	.953473	.948494	.942323	.937886
	50	.962949	.958107	.953612	.948038	.944027
	60	.969118	.965067	.961303	.956632	.953200
	70	.973525	.970044	.966807	.962787	.959891
	80	.976832	.973779	.970940	.967412	.964870
	90	.979404	.976686	.974158	.971015	.968749
	100	.981462	.979013	.976734	.973900	.971856

Table A5 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

P = 6

		$\alpha=.100$	.050	.025	.010	.005
M_0	N					
	10	22.33283	25.02648	27.52407	30.62023	32.84873
	11	22.32808	25.02082	27.51749	30.61244	32.84002
	12	22.32454	25.01659	27.51258	30.60662	32.83350
	13	22.32183	25.01335	27.50882	30.60216	32.82850
	14	22.31970	25.01081	27.50587	30.59866	32.82458
	15	22.31800	25.00879	27.50351	30.59587	32.82145
	16	22.31663	25.00715	27.50161	30.59361	32.81892
	17	22.31550	25.00580	27.50004	30.59174	32.81683
	18	22.31456	25.00468	27.49873	30.59020	32.81509
	19	22.31377	25.00373	27.49764	30.58889	32.81363
	20	22.31310	25.00293	27.49670	30.58779	32.81240
	25	22.31090	25.00030	27.49364	30.58415	32.80832
	30	22.30972	24.99889	27.49200	30.58221	32.80614
	35	22.30902	24.99805	27.49103	30.58105	32.80484
	40	22.30857	24.99751	27.49040	30.58030	32.80400
	45	22.30826	24.99715	27.48997	30.57979	32.80343
	50	22.30805	24.99689	27.48967	30.57943	32.80302
	60	22.30776	24.99655	27.48928	30.57896	32.80250
	70	22.30759	24.99635	27.48904	30.57868	32.80218
	80	22.30748	24.99621	27.48889	30.57850	32.80198
	90	22.30741	24.99612	27.48878	30.57838	32.80184
	100	22.30736	24.99606	27.48871	30.57829	32.80174
L_0	10	.816090	.796329	.778433	.756807	.741614
	11	.832615	.814423	.797911	.777905	.763818
	12	.846420	.829572	.814250	.795649	.782525
	13	.858125	.842438	.828151	.810776	.798497
	14	.868174	.853502	.840120	.823824	.812291
	15	.876895	.863116	.850534	.835192	.824322
	16	.884535	.871547	.859676	.845185	.834908
	17	.891282	.879001	.867765	.854038	.844293
	18	.897285	.885638	.874974	.861934	.852670
	19	.902660	.891585	.881438	.869021	.860193
	20	.907501	.896945	.887268	.875417	.866987
	25	.925924	.917373	.909517	.899872	.892994
	30	.938228	.931045	.924434	.916306	.910500
	35	.947028	.940835	.935130	.928107	.923085
	40	.953633	.948192	.943174	.936993	.932569
	45	.958774	.953921	.949444	.943924	.939971
	50	.962889	.958510	.954468	.949481	.945909
	60	.969064	.965401	.962017	.957839	.954843
	70	.973478	.970329	.967419	.963824	.961245
	80	.976789	.974028	.971476	.968321	.966057
	90	.979366	.976908	.974634	.971824	.969806
	100	.981427	.979212	.977163	.974629	.972809

Table A6 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

P = 7

		$\alpha = .100$	.050	.025	.010	.005
M_0	N					
	10	26.02186	28.90750	31.57028	34.85673	37.21360
	11	26.01587	28.90046	31.56220	34.84727	37.20310
	12	26.01140	28.89520	31.55615	34.84020	37.19524
	13	26.00798	28.89117	31.55152	34.83478	37.18922
	14	26.00530	28.88801	31.54789	34.83052	37.18449
	15	26.00316	28.88549	31.54500	34.82713	37.18072
	16	26.00142	28.88345	31.54265	34.82438	37.17766
	17	26.00000	28.88177	31.54072	34.82212	37.17515
	18	25.99881	28.88037	31.53911	34.82024	37.17306
	19	25.99781	28.87919	31.53776	34.81865	37.17130
	20	25.99697	28.87820	31.53661	34.81731	37.16980
	25	25.99418	28.87492	31.53284	34.81289	37.16488
	30	25.99270	28.87316	31.53082	34.81052	37.16226
	35	25.99181	28.87212	31.52962	34.80911	37.16069
	40	25.99124	28.87145	31.52885	34.80821	37.15968
	45	25.99086	28.87099	31.52833	34.80759	37.15899
	50	25.99058	28.87067	31.52795	34.80715	37.15851
	60	25.99022	28.87024	31.52747	34.80658	37.15787
	70	25.99001	28.86999	31.52717	34.80624	37.15749
	80	25.98987	28.86983	31.52699	34.80602	37.15725
	90	25.98978	28.86972	31.52686	34.80587	37.15708
	100	25.98971	28.86964	31.52677	34.80576	37.15696
L_0	10	.816611	.798470	.782087	.762330	.748469
	11	.833069	.816369	.801255	.782988	.770145
	12	.846821	.831354	.817333	.800355	.788397
	13	.858482	.844083	.831010	.815156	.803973
	14	.868496	.855027	.842786	.827919	.817420
	15	.877187	.864538	.853029	.839037	.829145
	16	.884802	.872880	.862022	.848808	.839458
	17	.891528	.880254	.869979	.857463	.848599
	18	.897513	.886821	.877069	.865181	.856757
	19	.902872	.892705	.883426	.872108	.864082
	20	.907699	.898008	.889159	.878359	.870695
	25	.926071	.918222	.911039	.902252	.896004
	30	.938344	.931750	.925707	.918304	.913032
	35	.947124	.941439	.936224	.929829	.925270
	40	.953715	.948719	.944133	.938505	.934490
	45	.958845	.954390	.950298	.945272	.941685
	50	.962951	.958931	.955237	.950698	.947456
	60	.969115	.965752	.962659	.958856	.956138
	70	.973520	.970629	.967970	.964698	.962359
	80	.976826	.974291	.971958	.969087	.967034
	90	.979398	.977141	.975063	.972506	.970676
	100	.981456	.979422	.977549	.975243	.973593

Table A7 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

$P = 8$

		$\alpha = .100$	.050	.025	.010	.005
M_0	N					
	10	29.65415	32.71608	35.53070	38.99227	41.46741
	11	29.64695	32.70770	35.52116	38.98121	41.45522
	12	29.64157	32.70143	35.51403	38.97295	41.44610
	13	29.63745	32.69663	35.50856	38.96661	41.43910
	14	29.63422	32.69287	35.50428	38.96165	41.43362
	15	29.63164	32.68987	35.50086	38.95768	41.42925
	16	29.62955	32.68743	35.49809	38.95447	41.42570
	17	29.62783	32.68543	35.49581	38.95182	41.42278
	18	29.62640	32.68377	35.49391	38.94962	41.42035
	19	29.62520	32.68237	35.49232	38.94777	41.41830
	20	29.62418	32.68118	35.49096	38.94620	41.41657
	25	29.62083	32.67727	35.48651	38.94103	41.41086
	30	29.61904	32.67518	35.48413	38.93827	41.40781
	35	29.61797	32.67394	35.48271	38.93663	41.40599
	40	29.61729	32.67314	35.48180	38.93557	41.40482
	45	29.61682	32.67259	35.48118	38.93484	41.40402
	50	29.61649	32.67220	35.48073	38.93433	41.40345
	60	29.61605	32.67170	35.48016	38.93366	41.40271
	70	29.61580	32.67140	35.47982	38.93327	41.40227
	80	29.61563	32.67120	35.47959	38.93301	41.40199
	90	29.61552	32.67107	35.47944	38.93283	41.40179
	100	29.61543	32.67098	35.47933	38.93271	41.40165
L_0	10	.817317	.800468	.785288	.767012	.754205
	11	.833699	.818190	.804189	.787298	.775439
	12	.847389	.833027	.820040	.804346	.793309
	13	.859000	.845629	.833522	.818871	.808554
	14	.868970	.856464	.845128	.831393	.821710
	15	.877625	.865881	.855224	.842299	.833178
	16	.885208	.874139	.864086	.851883	.843263
	17	.891907	.881440	.871927	.860370	.852201
	18	.897868	.887941	.878914	.867938	.860175
	19	.903206	.893767	.885178	.874729	.867334
	20	.908014	.899017	.890826	.880857	.873797
	25	.926316	.919029	.912382	.904274	.898521
	30	.938545	.932423	.926832	.920002	.915150
	35	.947294	.942016	.937191	.931292	.927098
	40	.953862	.949224	.944982	.939791	.936097
	45	.958975	.954839	.951053	.946418	.943119
	50	.963067	.959335	.955918	.951732	.948751
	60	.969210	.966088	.963227	.959721	.957222
	70	.973602	.970918	.968458	.965441	.963290
	80	.976897	.974543	.972386	.969739	.967851
	90	.979460	.977365	.975444	.973086	.971404
	100	.981512	.979624	.977892	.975766	.974249

Table A8 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

$P = 9$

		$\alpha=.100$	.050	.025	.010	.005
M_0	N					
	10	33.24183	36.46767	39.42357	43.04824	45.63364
	11	33.23343	36.45797	39.41262	43.03566	45.61983
	12	33.22715	36.45073	39.40444	43.02625	45.60951
	13	33.22234	36.44518	39.39816	43.01904	45.60159
	14	33.21857	36.44082	39.39325	43.01338	45.59538
	15	33.21557	36.43735	39.38932	43.00887	45.59043
	16	33.21313	36.43453	39.38614	43.00521	45.58641
	17	33.21112	36.43222	39.38352	43.00220	45.58310
	18	33.20946	36.43029	39.38134	42.99969	45.58035
	19	33.20805	36.42867	39.37951	42.99759	45.57804
	20	33.20686	36.42730	39.37796	42.99580	45.57607
	25	33.20295	36.42278	39.37284	42.98991	45.56960
	30	33.20086	36.42036	39.37011	42.98677	45.56615
	35	33.19961	36.41892	39.36848	42.98489	45.56409
	40	33.19881	36.41799	39.36743	42.98369	45.56276
	45	33.19826	36.41736	39.36672	42.98286	45.56186
	50	33.19788	36.41691	39.36621	42.98228	45.56121
	60	33.19737	36.41633	39.36555	42.98152	45.56038
	70	33.19707	36.41598	39.36516	42.98107	45.55988
	80	33.19688	36.41576	39.36490	42.98077	45.55956
	90	33.19674	36.41560	39.36473	42.98057	45.55934
	100	33.19665	36.41549	39.36460	42.98043	45.55918
L_0	10	.818085	.802299	.788101	.771034	.759087
	11	.834391	.819861	.806769	.791001	.779943
	12	.848018	.834563	.822422	.807776	.797489
	13	.859576	.847050	.835734	.822064	.812451
	14	.869502	.857787	.847192	.834380	.825360
	15	.878118	.867117	.857159	.845104	.836610
	16	.885668	.875300	.865906	.854526	.846501
	17	.892338	.882534	.873645	.862869	.855265
	18	.898273	.888975	.880541	.870308	.863083
	19	.903588	.894748	.886723	.876983	.870101
	20	.908376	.899950	.892298	.883004	.876436
	25	.926602	.919777	.913569	.906013	.900663
	30	.938781	.93048	.927826	.921463	.916951
	35	.947494	.942552	.938046	.932551	.928652
	40	.954036	.949693	.945732	.940897	.937464
	45	.959129	.955256	.951721	.947405	.944339
	50	.963206	.959711	.956520	.952622	.949852
	60	.969325	.966402	.963730	.960466	.958144
	70	.973700	.971187	.968890	.966081	.964083
	80	.976982	.974779	.972764	.970300	.968546
	90	.979536	.977574	.975780	.973585	.972023
	100	.981580	.979812	.978195	.976216	.974807

Table A9 Percentage Points of L_0 and $M_0 = -2s \log \lambda_0$

P = 10

		$\alpha=.100$	.050	.025	.010	.005
M_0	N					
	10	36.79324	40.17289	43.26148	47.03941	49.72850
	11	36.78366	40.16192	43.24915	47.02535	49.71314
	12	36.77650	40.15371	43.23995	47.01484	49.70166
	13	36.77101	40.14742	43.23288	47.00678	49.69285
	14	36.76671	40.14250	43.22735	47.00046	49.68594
	15	36.76327	40.13856	43.22293	46.99542	49.68043
	16	36.76049	40.13537	43.21935	46.99132	49.67595
	17	36.75820	40.13275	43.21640	46.98796	49.67228
	18	36.75630	40.13057	43.21395	46.98516	49.66921
	19	36.75470	40.12873	43.21189	46.98281	49.66664
	20	36.75334	40.12718	43.21014	46.98081	49.66445
	25	36.74887	40.12205	43.20438	46.97423	49.65726
	30	36.74648	40.11931	43.20130	46.97071	49.65341
	35	36.74506	40.11768	43.19947	46.96861	49.65112
	40	36.74415	40.11663	43.19829	46.96726	49.64964
	45	36.74352	40.11592	43.19749	46.96635	49.64864
	50	36.74308	40.11541	43.19691	46.96569	49.64792
	60	36.74250	40.11475	43.19617	46.96484	49.64699
	70	36.74216	40.11435	43.19573	46.96433	49.64644
	80	36.74194	40.11410	43.19544	46.96401	49.64608
	90	36.74179	40.11392	43.19524	46.96378	49.64583
	100	36.74168	40.11380	43.19510	46.96362	49.64566
L_0	10	.818860	.803965	.790591	.774533	.763303
	11	.835092	.821384	.809054	.794223	.783833
	12	.848658	.835965	.824532	.810760	.801098
	13	.860164	.848349	.837693	.824842	.815816
	14	.870046	.858996	.849021	.836978	.828511
	15	.878625	.868248	.858873	.847544	.839572
	16	.886142	.876362	.867520	.856826	.849296
	17	.892782	.883535	.875169	.865044	.857909
	18	.898692	.889922	.881984	.872371	.865593
	19	.903984	.895646	.888094	.878944	.872489
	20	.908751	.900804	.893603	.884874	.878713
	25	.926900	.920464	.914622	.907527	.902511
	30	.939028	.933621	.928708	.922734	.918506
	35	.947705	.943044	.938806	.933647	.929994
	40	.954220	.950125	.946398	.941860	.938643
	45	.959292	.955640	.952315	.948264	.945391
	50	.963352	.960057	.957055	.953397	.950802
	60	.969447	.966690	.964178	.961114	.958939
	70	.973804	.971434	.969274	.966638	.964767
	80	.977073	.974995	.973101	.970788	.969146
	90	.979617	.977767	.976080	.974020	.972557
	100	.981653	.979986	.978465	.976608	.975289

Table A10 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$ $P = 2$

		$\alpha = .100$	.050	.025	.010	.005
M_1	N					
	10	2.70345	3.83790	5.01843	6.62625	7.86785
	11	2.70386	3.83859	5.01949	6.62792	7.87009
	12	2.70416	3.83910	5.02026	6.62916	7.87174
	13	2.70438	3.83948	5.02085	6.63009	7.87299
	14	2.70456	3.83978	5.02131	6.63081	7.87396
	15	2.70469	3.84001	5.02167	6.63138	7.87473
	16	2.70481	3.84020	5.02196	6.63184	7.87534
	17	2.70490	3.84036	5.02220	6.63222	7.87585
	18	2.70497	3.84048	5.02239	6.63253	7.87626
	19	2.70503	3.84059	5.02256	6.63279	7.87661
	20	2.70509	3.84068	5.02269	6.63301	7.87690
	25	2.70526	3.84097	5.02314	6.63372	7.87786
	30	2.70535	3.84113	5.02338	6.63409	7.87836
	35	2.70540	3.84122	5.02352	6.63431	7.87866
	40	2.70544	3.84128	5.02361	6.63445	7.87884
	45	2.70546	3.84132	5.02367	6.63455	7.87897
	50	2.70548	3.84134	5.02371	6.63462	7.87906
	60	2.70550	3.84138	5.02377	6.63470	7.87918
	70	2.70551	3.84140	5.02380	6.63476	7.87925
	80	2.70552	3.84142	5.02382	6.63479	7.87930
	90	2.70552	3.84142	5.02383	6.63481	7.87933
	100	2.70553	3.84143	5.02384	6.63483	7.87935
L_1	10	.925666	.896144	.866422	.827521	.798679
	11	.933019	.906263	.879234	.843710	.817261
	12	.939049	.914588	.889807	.857129	.832715
	13	.944084	.921557	.898682	.868432	.845768
	14	.948351	.927475	.906235	.878082	.856937
	15	.952013	.932563	.912741	.886415	.866601
	16	.955191	.936985	.918404	.893683	.875045
	17	.957974	.940863	.923377	.900079	.882485
	18	.960431	.944291	.927780	.905749	.889091
	19	.962618	.947344	.931704	.910811	.894994
	20	.964575	.950080	.935224	.915358	.900302
	25	.971925	.960375	.948498	.932554	.920420
	30	.976750	.967151	.957259	.943945	.933787
	35	.980159	.971948	.963473	.952045	.943310
	40	.982697	.975522	.968109	.958100	.950439
	45	.984659	.978289	.971702	.962798	.955976
	50	.986222	.980494	.974566	.966549	.960400
	60	.988553	.983787	.978849	.972162	.967027
	70	.990210	.986128	.981898	.976162	.971755
	80	.991448	.987879	.984178	.979157	.975297
	90	.992408	.989237	.985948	.981484	.978049
	100	.993174	.990322	.987362	.983343	.980250

Table A11 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$ $P = 3$

		$\alpha = .100$	.050	.025	.010	.005
M_1	N					
	10	4.60199	5.98666	7.37102	9.20057	10.58421
	11	4.60261	5.98759	7.37232	9.20246	10.58662
	12	4.60306	5.98827	7.37329	9.20385	10.58839
	13	4.60340	5.98879	7.37401	9.20491	10.58973
	14	4.60367	5.98920	7.37458	9.20573	10.59077
	15	4.60388	5.98951	7.37502	9.20637	10.59159
	16	4.60405	5.98977	7.37538	9.20689	10.59225
	17	4.60419	5.98998	7.37567	9.20731	10.59279
	18	4.60430	5.99015	7.37591	9.20766	10.59323
	19	4.60440	5.99029	7.37612	9.20796	10.59360
	20	4.60448	5.99041	7.37629	9.20820	10.59392
	25	4.60474	5.99081	7.37684	9.20901	10.59494
	30	4.60487	5.99102	7.37713	9.20943	10.59548
	35	4.60496	5.99114	7.37730	9.20968	10.59580
	40	4.60501	5.99122	7.37741	9.20984	10.59600
	45	4.60504	5.99127	7.37749	9.20995	10.59614
	50	4.60507	5.99131	7.37754	9.21002	10.59623
	60	4.60510	5.99136	7.37761	9.21012	10.59636
	70	4.60512	5.99139	7.37765	9.21018	10.59643
	80	4.60513	5.99141	7.37768	9.21022	10.59648
	90	4.60514	5.99142	7.37769	9.21025	10.59651
	100	4.60515	5.99143	7.37771	9.21026	10.59654
L_1	10	.916329	.892552	.869397	.839714	.817940
	11	.924545	.902974	.881911	.854824	.834891
	12	.931293	.911556	.892240	.867337	.848965
	13	.936934	.918745	.900911	.877869	.860833
	14	.941720	.924854	.908291	.886854	.870976
	15	.945830	.930109	.914650	.894610	.879744
	16	.949400	.934678	.920185	.901372	.887398
	17	.952528	.938686	.925046	.907320	.894138
	18	.955292	.942231	.929349	.912592	.900117
	19	.957752	.945389	.933186	.917296	.905457
	20	.959956	.948219	.936627	.921521	.910257
	25	.968239	.958878	.949607	.937490	.928427
	30	.973683	.965898	.958175	.948061	.940481
	35	.977534	.970871	.964253	.955575	.949061
	40	.980402	.974579	.968789	.961189	.955480
	45	.982621	.977449	.972304	.965544	.960462
	50	.984388	.979737	.975107	.969021	.964442
	60	.987027	.983155	.979298	.974222	.970401
	70	.988903	.985586	.982281	.977929	.974649
	80	.990304	.987404	.984513	.980703	.977831
	90	.991392	.988815	.986245	.982858	.980303
	100	.992260	.989942	.987629	.984580	.982280

Table A12 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$

P = 4

		$\alpha = .100$	.050	.025	.010	.005
M_1	N					
	10	6.24741	7.80906	9.34081	11.33436	12.82518
	11	6.24818	7.81016	9.34228	11.33640	12.82770
	12	6.24875	7.81097	9.34337	11.33790	12.82955
	13	6.24918	7.81158	9.34419	11.33903	12.83095
	14	6.24951	7.81206	9.34482	11.33991	12.83203
	15	6.24978	7.81243	9.34532	11.34061	12.83289
	16	6.24999	7.81273	9.34573	11.34116	12.83358
	17	6.25016	7.81298	9.34606	11.34162	12.83414
	18	6.25030	7.81318	9.34633	11.34200	12.83461
	19	6.25042	7.81335	9.34656	11.34231	12.83500
	20	6.25052	7.81349	9.34675	11.34258	12.83533
	25	6.25085	7.81396	9.34737	11.34344	12.83639
	30	6.25102	7.81420	9.34770	11.34389	12.83695
	35	6.25112	7.81435	9.34789	11.34416	12.83728
	40	6.25119	7.81444	9.34802	11.34433	12.83749
	45	6.25123	7.81450	9.34810	11.34445	12.83764
	50	6.25126	7.81455	9.34816	11.34453	12.83774
	60	6.25130	7.81460	9.34824	11.34463	12.83787
	70	6.25132	7.81464	9.34828	11.34470	12.83795
	80	6.25134	7.81466	9.34831	11.34474	12.83800
	90	6.25135	7.81467	9.34833	11.34477	12.83803
	100	6.25136	7.81468	9.34834	11.34478	12.83805
L_1	10	.915005	.894913	.875633	.851163	.833311
	11	.923334	.905105	.887574	.865265	.848948
	12	.930178	.913498	.897426	.876931	.861910
	13	.935901	.920528	.905693	.886742	.872828
	14	.940757	.926502	.912728	.895106	.882149
	15	.944930	.931641	.918787	.902321	.890199
	16	.948554	.936109	.924060	.908609	.897221
	17	.951730	.940029	.928691	.914137	.903400
	18	.954538	.943497	.932790	.919034	.908879
	19	.957036	.946585	.936443	.923404	.913771
	20	.959275	.949353	.939720	.927327	.918164
	25	.967692	.959777	.952076	.942143	.934782
	30	.973226	.966643	.960229	.951943	.945793
	35	.977142	.971508	.966012	.958905	.953623
	40	.980058	.975134	.970327	.964105	.959477
	45	.982315	.977941	.973669	.968137	.964019
	50	.984112	.980179	.976335	.971355	.967646
	60	.986797	.983522	.980321	.976168	.973074
	70	.988705	.985901	.983157	.979597	.976942
	80	.990132	.987679	.985279	.982163	.979839
	90	.991238	.989059	.986926	.984156	.982089
	100	.992121	.990161	.988241	.985748	.983887

Table A13 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$ $P = 5$

		$\alpha = .100$	.050	.025	.010	.005
M_1	N					
	10	7.77478	9.48133	11.13497	13.26554	14.84673
	11	7.77568	9.48258	11.13658	13.26771	14.84936
	12	7.77635	9.48349	11.13777	13.26931	14.85129
	13	7.77685	9.48418	11.13867	13.27051	14.85275
	14	7.77724	9.48471	11.13936	13.27144	14.85388
	15	7.77755	9.48514	11.13991	13.27218	14.85478
	16	7.77780	9.48548	11.14036	13.27277	14.85550
	17	7.77800	9.48575	11.14072	13.27326	14.85608
	18	7.77817	9.48598	11.14101	13.27366	14.85657
	19	7.77831	9.48617	11.14126	13.27399	14.85697
	20	7.77842	9.48633	11.14147	13.27427	14.85731
	25	7.77881	9.48686	11.14216	13.27519	14.85842
	30	7.77901	9.48714	11.14252	13.27567	14.85901
	35	7.77913	9.48730	11.14273	13.27595	14.85935
	40	7.77920	9.48740	11.14286	13.27614	14.85957
	45	7.77925	9.48747	11.14295	13.27626	14.85972
	50	7.77929	9.48752	11.14302	13.27635	14.85982
	60	7.77934	9.48759	11.14310	13.27646	14.85996
	70	7.77937	9.48763	11.14315	13.27652	14.86004
	80	7.77938	9.48765	11.14318	13.27657	14.86009
	90	7.77940	9.48767	11.14321	13.27660	14.86013
	100	7.77940	9.48768	11.14322	13.27662	14.86015
L_1	10	.915441	.897859	.881145	.860067	.844751
	11	.923722	.907773	.892581	.873380	.859397
	12	.930528	.915935	.902012	.884384	.871524
	13	.936219	.922771	.909923	.893631	.881729
	14	.941049	.928580	.916653	.901511	.890434
	15	.945200	.933576	.922448	.908305	.897948
	16	.948804	.937920	.927491	.914223	.904498
	17	.951964	.941730	.931918	.919424	.910259
	18	.954757	.945100	.935836	.924031	.915365
	19	.957243	.948102	.939327	.928140	.919922
	20	.959470	.950793	.942459	.931827	.924014
	25	.967844	.960923	.954263	.945749	.939479
	30	.973351	.967595	.962050	.954950	.949715
	35	.977248	.972321	.967571	.961483	.956990
	40	.980150	.975845	.971690	.966362	.962426
	45	.982396	.978572	.974880	.970144	.966643
	50	.984185	.980746	.977425	.973161	.970009
	60	.986857	.983994	.981228	.977674	.975045
	70	.988757	.986305	.983934	.980888	.978633
	80	.990176	.988032	.985958	.983293	.981319
	90	.991278	.989373	.987530	.985160	.983405
	100	.992157	.990443	.988785	.986652	.985072

Table A14 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$

P = 6

		$\alpha = .100$	.050	.025	.010	.005
M_1	N					
	10	9.23107	11.06343	12.82351	15.07448	16.73551
	11	9.23210	11.06481	12.82526	15.07677	16.73825
	12	9.23285	11.06582	12.82654	15.07846	16.74026
	13	9.23342	11.06658	12.82751	15.07973	16.74179
	14	9.23386	11.06717	12.82826	15.08072	16.74296
	15	9.23421	11.06764	12.82886	15.08150	16.74390
	16	9.23449	11.06801	12.82934	15.08212	16.74464
	17	9.23472	11.06832	12.82972	15.08263	16.74525
	18	9.23491	11.06857	12.83005	15.08305	16.74576
	19	9.23507	11.06878	12.83032	15.08341	16.74618
	20	9.23520	11.06896	12.83054	15.08371	16.74654
	25	9.23564	11.06954	12.83128	15.08467	16.74769
	30	9.23587	11.06984	12.83167	15.08518	16.74830
	35	9.23600	11.07002	12.83190	15.08548	16.74866
	40	9.23609	11.07014	12.83204	15.08567	16.74889
	45	9.23615	11.07022	12.83214	15.08580	16.74904
	50	9.23619	11.07027	12.83221	15.08589	16.74915
	60	9.23624	11.07034	12.83230	15.08601	16.74929
	70	9.23627	11.07038	12.83236	15.08608	16.74938
	80	9.23629	11.07041	12.83239	15.08613	16.74943
	90	9.23631	11.07043	12.83242	15.08616	16.74947
	100	9.23632	11.07044	12.83243	15.08618	16.74949
L_1	10	.916347	.900594	.885717	.867048	.853525
	11	.924539	.910251	.896734	.879738	.867403
	12	.931271	.918200	.905816	.890220	.878884
	13	.936902	.924856	.913431	.899024	.888539
	14	.941680	.930512	.919909	.906523	.896770
	15	.945785	.935376	.925485	.912986	.903870
	16	.949351	.939604	.930335	.918613	.910058
	17	.952477	.943313	.934594	.923558	.915497
	18	.955240	.946594	.938361	.927937	.920317
	19	.957699	.949515	.941719	.931841	.924617
	20	.959902	.952134	.944730	.935344	.928478
	25	.968187	.961991	.956077	.948565	.943059
	30	.973635	.968483	.963559	.957298	.952703
	35	.977490	.973081	.968863	.963496	.959554
	40	.980361	.976508	.972820	.968123	.964672
	45	.982583	.979161	.975884	.971710	.968640
	50	.984353	.981276	.978328	.974570	.971807
	60	.986997	.984435	.981980	.978849	.976544
	70	.988876	.986682	.984578	.981895	.979919
	80	.990281	.988362	.986522	.984174	.982444
	90	.991370	.989666	.988031	.985943	.984406
	100	.992240	.990707	.989235	.987357	.985973

Table A15 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$ $P = 7$

		$\alpha=.100$	.050	.025	.010	.005
M_1	N					
	10	10.63876	12.58389	14.43974	16.79948	18.53292
	11	10.63990	12.58539	14.44161	16.80190	18.53577
	12	10.64074	12.58649	14.44299	16.80367	18.53787
	13	10.64138	12.58732	14.44403	16.80501	18.53945
	14	10.64187	12.58796	14.44484	16.80605	18.54068
	15	10.64226	12.58847	14.44547	16.80687	18.54165
	16	10.64257	12.58888	14.44598	16.80753	18.54243
	17	10.64282	12.58921	14.44640	16.80806	18.54306
	18	10.64303	12.58949	14.44675	16.80851	18.54359
	19	10.64321	12.58972	14.44703	16.80888	18.54402
	20	10.64336	12.58991	14.44728	16.80919	18.54439
	25	10.64384	12.59054	14.44807	16.81021	18.54560
	30	10.64410	12.59088	14.44848	16.81075	18.54623
	35	10.64425	12.59107	14.44873	16.81106	18.54660
	40	10.64434	12.59120	14.44889	16.81126	18.54684
	45	10.64441	12.59128	14.44899	16.81140	18.54700
	50	10.64445	12.59134	14.44907	16.81150	18.54711
	60	10.64451	12.59142	14.44916	16.81162	18.54726
	70	10.64455	12.59146	14.44922	16.81169	18.54735
	80	10.64457	12.59149	14.44926	16.81174	18.54740
	90	10.64458	12.59151	14.44928	16.81177	18.54744
	100	10.64460	12.59153	14.44930	16.81180	18.54747
L_1	10	.917356	.903001	.889515	.872658	.860478
	11	.925450	.912433	.900183	.884844	.873743
	12	.932102	.920194	.908975	.894905	.884709
	13	.937664	.926693	.916344	.903352	.893925
	14	.942385	.932214	.922611	.910543	.901778
	15	.946442	.936962	.928005	.916739	.908550
	16	.949964	.941089	.932697	.922134	.914450
	17	.953053	.944708	.936815	.926872	.919635
	18	.955782	.947909	.940458	.931067	.924227
	19	.958212	.950760	.943704	.934807	.928324
	20	.960388	.953315	.946615	.938162	.932001
	25	.968573	.962933	.957582	.950820	.945883
	30	.973955	.969265	.964811	.959177	.955059
	35	.977763	.973750	.969936	.965107	.961575
	40	.980600	.977093	.973757	.969533	.966441
	45	.982795	.979680	.976717	.972962	.970213
	50	.984544	.981742	.979077	.975698	.973223
	60	.987155	.984823	.982604	.979788	.977725
	70	.989011	.987014	.985113	.982700	.980931
	80	.990399	.988653	.986989	.984878	.983330
	90	.991475	.989924	.988446	.986570	.985193
	100	.992335	.990939	.989609	.987920	.986682

Table A16 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$

P = 8

		$\alpha=.100$	.050	.025	.010	.005
M_1	N					
	10	12.01058	14.05884	16.00250	18.46228	20.26249
	11	12.01184	14.06045	16.00450	18.46482	20.26546
	12	12.01276	14.06164	16.00597	18.46668	20.26764
	13	12.01346	14.06254	16.00707	18.46809	20.26929
	14	12.01400	14.06323	16.00793	18.46918	20.27056
	15	12.01442	14.06378	16.00861	18.47004	20.27157
	16	12.01476	14.06422	16.00915	18.47073	20.27238
	17	12.01504	14.06458	16.00960	18.47129	20.27304
	18	12.01528	14.06488	16.00997	18.47176	20.27358
	19	12.01547	14.06512	16.01027	18.47215	20.27404
	20	12.01563	14.06533	16.01053	18.47247	20.27443
	25	12.01616	14.06601	16.01137	18.47354	20.27568
	30	12.01644	14.06637	16.01182	18.47410	20.27633
	35	12.01660	14.06658	16.01208	18.47443	20.27672
	40	12.01671	14.06672	16.01224	18.47465	20.27697
	45	12.01678	14.06681	16.01236	18.47479	20.27713
	50	12.01683	14.06687	16.01244	18.47489	20.27725
	60	12.01689	14.06696	16.01254	18.47502	20.27740
	70	12.01693	14.06701	16.01260	18.47510	20.27749
	80	12.01696	14.06704	16.01264	18.47515	20.27755
	90	12.01697	14.06706	16.01267	18.47518	20.27759
	100	12.01699	14.06708	16.01268	18.47521	20.27762
L_1	10	.918346	.905102	.892711	.877272	.866143
	11	.926345	.914336	.903084	.889042	.878904
	12	.932918	.921935	.911631	.898756	.889448
	13	.938415	.928296	.918794	.906908	.898306
	14	.943079	.933699	.924884	.913846	.905851
	15	.947087	.938346	.930124	.919822	.912355
	16	.950568	.942384	.934682	.925024	.918019
	17	.953620	.945926	.938682	.929592	.922996
	18	.956316	.949058	.942220	.933636	.927404
	19	.958717	.951847	.945372	.937241	.931334
	20	.960867	.954346	.948199	.940475	.934862
	25	.968954	.963755	.958846	.952670	.948174
	30	.974271	.969948	.965864	.960718	.956970
	35	.978033	.974334	.970837	.966428	.963214
	40	.980835	.977603	.974545	.970688	.967875
	45	.983004	.980133	.977417	.973989	.971488
	50	.984731	.982150	.979706	.976622	.974371
	60	.987311	.985162	.983128	.980558	.978682
	70	.989145	.987305	.985562	.983360	.981751
	80	.990516	.988907	.987382	.985456	.984048
	90	.991579	.990149	.988795	.987083	.985831
	100	.992428	.991142	.989923	.988382	.987256

Table A17 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$

P = 9

		$\alpha=.100$	.050	.025	.010	.005
M_1	N					
	10	13.35456	15.49842	17.52368	20.07661	21.93912
	11	13.35592	15.50015	17.52579	20.07926	21.94221
	12	13.35692	15.50142	17.52735	20.08121	21.94447
	13	13.35768	15.50238	17.52852	20.08268	21.94618
	14	13.35827	15.50313	17.52943	20.08382	21.94750
	15	13.35873	15.50371	17.53015	20.08472	21.94855
	16	13.35910	15.50419	17.53072	20.08544	21.94939
	17	13.35940	15.50457	17.53120	20.08603	21.95008
	18	13.35965	15.50489	17.53158	20.08652	21.95064
	19	13.35986	15.50515	17.53191	20.08693	21.95112
	20	13.36004	15.50538	17.53218	20.08727	21.95151
	25	13.36062	15.50611	17.53307	20.08839	21.95281
	30	13.36092	15.50649	17.53354	20.08898	21.95349
	35	13.36110	15.50672	17.53382	20.08932	21.95390
	40	13.36121	15.50686	17.53399	20.08954	21.95415
	45	13.36129	15.50696	17.53411	20.08969	21.95433
	50	13.36134	15.50703	17.53420	20.08980	21.95445
	60	13.36141	15.50712	17.53431	20.08993	21.95461
	70	13.36145	15.50717	17.53437	20.09002	21.95470
	80	13.36148	15.50720	17.53441	20.09007	21.95476
	90	13.36150	15.50723	17.53444	20.09010	21.95480
	100	13.36151	15.50724	17.53446	20.09013	21.95483
L_1	10	.919277	.906940	.895437	.881145	.870862
	11	.927187	.916002	.905559	.892565	.883202
	12	.933687	.923457	.913897	.901985	.893393
	13	.939122	.929693	.920883	.909889	.901951
	14	.943733	.934999	.926821	.916614	.909238
	15	.947696	.939557	.931931	.922406	.915518
	16	.951138	.943517	.936374	.927446	.920986
	17	.954154	.946991	.940273	.931872	.925790
	18	.956820	.950062	.943722	.935789	.930043
	19	.959193	.952798	.946794	.939281	.933836
	20	.961319	.955249	.949549	.942412	.937239
	25	.969313	.964474	.959924	.954219	.950077
	30	.974569	.970546	.966760	.962009	.958557
	35	.978288	.974845	.971604	.967533	.964574
	40	.981058	.978050	.975216	.971655	.969066
	45	.983201	.980530	.978013	.974849	.972547
	50	.984909	.982507	.980242	.977395	.975324
	60	.987458	.985459	.983574	.981202	.979476
	70	.989271	.987559	.985944	.983912	.982432
	80	.990626	.989129	.987716	.985939	.984644
	90	.991677	.990347	.989092	.987512	.986361
	100	.992516	.991319	.990190	.988769	.987732

Table A18 Percentage Points of L_1 and $M_1 = -2u \log \lambda_1$

P = 10

		$\alpha=.100$	.050	.025	.010	.005
M_1	N					
	10	14.67611	16.90951	19.01131	21.65177	23.57293
	11	14.67758	16.91135	19.01354	21.65454	23.57613
	12	14.67865	16.91271	19.01518	21.65657	23.57848
	13	14.67947	16.91373	19.01641	21.65811	23.58025
	14	14.68010	16.91452	19.01737	21.65930	23.58163
	15	14.68060	16.91515	19.01813	21.66024	23.58271
	16	14.68100	16.91565	19.01874	21.66100	23.58358
	17	14.68133	16.91606	19.01923	21.66161	23.58429
	18	14.68160	16.91640	19.01964	21.66212	23.58488
	19	14.68182	16.91668	19.01999	21.66254	23.58537
	20	14.68201	16.91692	19.02028	21.66290	23.58578
	25	14.68263	16.91770	19.02122	21.66407	23.58713
	30	14.68296	16.91810	19.02171	21.66468	23.58784
	35	14.68315	16.91834	19.02200	21.66504	23.58825
	40	14.68327	16.91850	19.02219	21.66527	23.58852
	45	14.68336	16.91860	19.02231	21.66543	23.58870
	50	14.68341	16.91867	19.02240	21.66554	23.58883
	60	14.68349	16.91877	19.02252	21.66568	23.58899
	70	14.68354	16.91883	19.02258	21.66577	23.58909
	80	14.68356	16.91886	19.02263	21.66582	23.58915
	90	14.68358	16.91889	19.02266	21.66586	23.58919
	100	14.68360	16.91890	19.02268	21.66588	23.58922
L_1	10	.920140	.908559	.897794	.884450	.874867
	11	.927968	.917470	.907698	.895570	.886848
	12	.934399	.924799	.915855	.904740	.896738
	13	.939776	.930934	.922688	.912432	.905041
	14	.944340	.936144	.928495	.918975	.912109
	15	.948260	.940623	.933492	.924609	.918199
	16	.951665	.944516	.937836	.929511	.923500
	17	.954650	.947929	.941648	.933815	.928156
	18	.957287	.950947	.945019	.937624	.932279
	19	.959635	.953635	.948023	.941018	.935954
	20	.961738	.956043	.950715	.944062	.939251
	25	.969646	.965107	.960854	.955538	.951688
	30	.974845	.971072	.967534	.963107	.959899
	35	.978524	.975296	.972267	.968475	.965725
	40	.981264	.978443	.975795	.972479	.970072
	45	.983384	.980879	.978527	.975580	.973442
	50	.985073	.982821	.980705	.978054	.976129
	60	.987595	.985720	.983959	.981751	.980147
	70	.989388	.987783	.986274	.984382	.983007
	80	.990728	.989324	.988005	.986350	.985147
	90	.991768	.990521	.989348	.987877	.986808
	100	.992598	.991476	.990421	.989097	.988135

Table A19 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

P = 2

		$\alpha = .100$	.050	.025	.010	.005
M_2	N					
	10	4.60517	5.99147	7.37776	9.21034	10.59664
	11	4.60517	5.99147	7.37776	9.21034	10.59664
	12	4.60517	5.99147	7.37776	9.21034	10.59664
	13	4.60517	5.99147	7.37776	9.21034	10.59664
	14	4.60517	5.99147	7.37776	9.21034	10.59664
	15	4.60517	5.99147	7.37776	9.21034	10.59664
	16	4.60517	5.99147	7.37776	9.21034	10.59664
	17	4.60517	5.99147	7.37776	9.21034	10.59664
	18	4.60517	5.99147	7.37776	9.21034	10.59664
	19	4.60517	5.99147	7.37776	9.21034	10.59664
	20	4.60517	5.99147	7.37776	9.21034	10.59664
	25	4.60517	5.99147	7.37776	9.21034	10.59664
	30	4.60517	5.99147	7.37776	9.21034	10.59664
	35	4.60517	5.99147	7.37776	9.21034	10.59664
	40	4.60517	5.99147	7.37776	9.21034	10.59664
	45	4.60517	5.99147	7.37776	9.21034	10.59664
	50	4.60517	5.99147	7.37776	9.21034	10.59664
	60	4.60517	5.99147	7.37776	9.21034	10.59664
	70	4.60517	5.99147	7.37776	9.21034	10.59664
	80	4.60517	5.99147	7.37776	9.21034	10.59664
	90	4.60517	5.99147	7.37776	9.21034	10.59664
	100	4.60517	5.99147	7.37776	9.21034	10.59664
L_2	10	.879923	.846682	.814698	.774264	.745015
	11	.891251	.860892	.831567	.794328	.767270
	12	.900628	.872695	.845627	.811131	.785973
	13	.908518	.882654	.857526	.825404	.801907
	14	.915247	.891170	.867725	.837678	.815641
	15	.921055	.898534	.876564	.848343	.827600
	16	.926119	.904966	.884297	.857696	.838106
	17	.930572	.910632	.891119	.865964	.847408
	18	.934519	.915660	.897182	.873326	.855702
	19	.938042	.920153	.902606	.879923	.863142
	20	.941205	.924192	.907487	.885867	.869854
	25	.953162	.939497	.926027	.908518	.895492
	30	.961078	.949661	.938379	.923671	.912698
	35	.966705	.956901	.947197	.934519	.925042
	40	.970911	.962321	.953808	.942668	.934329
	45	.974174	.966531	.958947	.949014	.941569
	50	.976778	.969894	.963058	.954095	.947371
	60	.980676	.974932	.969222	.961725	.956092
	70	.983453	.978526	.973623	.967180	.962334
	80	.985532	.981218	.976923	.971274	.967022
	90	.987147	.983311	.979489	.974460	.970673
	100	.988438	.984984	.981542	.977010	.973596

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ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF TEST
STATISTICS ASSOCIATED W. (U) AIR FORCE INST OF TECH
WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI... D J LAWTON
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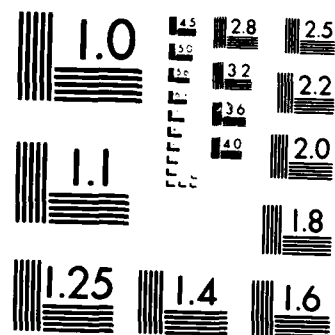
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MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

Table A20 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

$P = 3$

		$\alpha=.100$	.050	.025	.010	.005
M_2	N					
	10	7.78239	9.49178	11.14855	13.28376	14.86880
	11	7.78184	9.49102	11.14757	13.28244	14.86720
	12	7.78143	9.49046	11.14683	13.28146	14.86601
	13	7.78112	9.49003	11.14628	13.28071	14.86511
	14	7.78087	9.48969	11.14584	13.28012	14.86440
	15	7.78068	9.48942	11.14549	13.27966	14.86384
	16	7.78052	9.48921	11.14521	13.27928	14.86338
	17	7.78039	9.48903	11.14498	13.27897	14.86301
	18	7.78028	9.48888	11.14479	13.27872	14.86269
	19	7.78019	9.48876	11.14463	13.27850	14.86243
	20	7.78012	9.48866	11.14449	13.27832	14.86221
	25	7.77987	9.48831	11.14404	13.27772	14.86149
	30	7.77973	9.48813	11.14381	13.27740	14.86110
	35	7.77965	9.48802	11.14367	13.27721	14.86087
	40	7.77960	9.48795	11.14358	13.27709	14.86073
	45	7.77957	9.48790	11.14351	13.27701	14.86063
	50	7.77954	9.48787	11.14347	13.27695	14.86056
	60	7.77951	9.48783	11.14341	13.27687	14.86046
	70	7.77949	9.48780	11.14338	13.27683	14.86041
	80	7.77948	9.48778	11.14336	13.27680	14.86037
	90	7.77947	9.48777	11.14334	13.27678	14.86035
	100	7.77947	9.48776	11.14333	13.27676	14.86033
L_2	10	.868057	.841493	.816522	.785431	.763119
	11	.880231	.855910	.832979	.804331	.783703
	12	.890350	.867926	.846732	.820181	.801011
	13	.898894	.878095	.858397	.833662	.815763
	14	.906203	.886812	.868414	.845267	.828485
	15	.912527	.894366	.877110	.855361	.839567
	16	.918053	.900976	.884728	.864222	.849308
	17	.922922	.906808	.891458	.872060	.857935
	18	.927246	.911991	.897446	.879045	.865630
	19	.931110	.916629	.902809	.885307	.872535
	20	.934585	.920802	.907639	.890953	.878766
	25	.947760	.936658	.926024	.912497	.902584
	30	.956518	.947226	.938306	.926936	.918585
	35	.962762	.954772	.947092	.937286	.930073
	40	.967437	.960430	.953687	.945068	.938721
	45	.971070	.964830	.958821	.951132	.945466
	50	.973973	.968349	.962930	.955991	.950873
	60	.978324	.973628	.969098	.963291	.959004
	70	.981429	.977397	.973506	.968514	.964825
	80	.983756	.980224	.976813	.972436	.969199
	90	.985564	.982422	.979387	.975489	.972606
	100	.987010	.984181	.981446	.977933	.975334

Table A21 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

P = 4

		$\alpha = .100$	.050	.025	.010	.005
M_2	N					
	10	10.65069	12.59949	14.45926	16.82460	18.56258
	11	10.64956	12.59802	14.45742	16.82224	18.55980
	12	10.64873	12.59693	14.45605	16.82048	18.55772
	13	10.64809	12.59609	14.45501	16.81914	18.55613
	14	10.64759	12.59544	14.45419	16.81808	18.55489
	15	10.64719	12.59491	14.45354	16.81725	18.55390
	16	10.64686	12.59449	14.45301	16.81657	18.55310
	17	10.64660	12.59415	14.45258	16.81601	18.55244
	18	10.64638	12.59386	14.45222	16.81555	18.55189
	19	10.64619	12.59362	14.45191	16.81516	18.55144
	20	10.64604	12.59341	14.45166	16.81483	18.55105
	25	10.64552	12.59274	14.45081	16.81374	18.54977
	30	10.64525	12.59238	14.45036	16.81316	18.54908
	35	10.64508	12.59216	14.45010	16.81282	18.54868
	40	10.64498	12.59203	14.44992	16.81260	18.54842
	45	10.64491	12.59193	14.44981	16.81245	18.54824
	50	10.64485	12.59187	14.44972	16.81234	18.54811
	60	10.64479	12.59178	14.44962	16.81220	18.54795
	70	10.64475	12.59173	14.44955	16.81212	18.54785
	80	10.64472	12.59170	14.44951	16.81207	18.54779
	90	10.64471	12.59167	14.44948	16.81203	18.54775
	100	10.64469	12.59166	14.44946	16.81200	18.54771
L_2	10	.865950	.843443	.822509	.796634	.778142
	11	.878207	.857586	.838358	.814526	.797448
	12	.888412	.869388	.851613	.829531	.813673
	13	.897041	.879386	.862862	.842296	.827498
	14	.904431	.887963	.872528	.853285	.839418
	15	.910832	.895403	.880923	.862845	.849801
	16	.916430	.901916	.888281	.871237	.858924
	17	.921367	.907667	.894783	.878663	.867005
	18	.925753	.912781	.900571	.885280	.874211
	19	.929676	.917358	.905756	.891213	.880677
	20	.933205	.921479	.910427	.896563	.886511
	25	.946605	.937151	.928218	.916981	.908812
	30	.955527	.947609	.940114	.930669	.923791
	35	.961894	.955083	.948629	.940484	.934545
	40	.966667	.960691	.955024	.947865	.942639
	45	.970377	.965055	.960003	.953617	.948953
	50	.973344	.968546	.963990	.958227	.954014
	60	.977793	.973785	.969976	.965153	.961625
	70	.980969	.977528	.974255	.970109	.967074
	80	.983350	.980335	.977467	.973831	.971169
	90	.985202	.982519	.979966	.976729	.974357
	100	.986683	.984267	.981966	.979049	.976911

Table A22 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$ $P = 5$

		$\alpha=.100$	.050	.025	.010	.005
M_2	N					
	10	13.37060	15.51876	17.54852	20.10773	21.97526
	11	13.36893	15.51664	17.54593	20.10449	21.97150
	12	13.36768	15.51506	17.54401	20.10208	21.96870
	13	13.36673	15.51385	17.54253	20.10023	21.96656
	14	13.36598	15.51291	17.54137	20.09878	21.96488
	15	13.36538	15.51215	17.54045	20.09763	21.96354
	16	13.36490	15.51154	17.53970	20.09669	21.96245
	17	13.36450	15.51104	17.53909	20.09593	21.96156
	18	13.36417	15.51062	17.53858	20.09529	21.96082
	19	13.36390	15.51027	17.53815	20.09475	21.96019
	20	13.36366	15.50997	17.53779	20.09429	21.95967
	25	13.36289	15.50899	17.53659	20.09280	21.95793
	30	13.36248	15.50847	17.53595	20.09200	21.95700
	35	13.36223	15.50815	17.53557	20.09152	21.95645
	40	13.36207	15.50795	17.53533	20.09121	21.95609
	45	13.36196	15.50782	17.53516	20.09101	21.95585
	50	13.36189	15.50772	17.53504	20.09086	21.95568
	60	13.36179	15.50759	17.53489	20.09067	21.95545
	70	13.36173	15.50752	17.53480	20.09055	21.95532
	80	13.36169	15.50747	17.53474	20.09048	21.95523
	90	13.36166	15.50744	17.53470	20.09043	21.95518
	100	13.36165	15.50741	17.53467	20.09039	21.95513
L_2	10	.866087	.846311	.828040	.805564	.789549
	11	.878275	.860151	.843371	.822679	.807901
	12	.888431	.871707	.856195	.837031	.823318
	13	.897024	.881501	.867082	.849237	.836449
	14	.904389	.889907	.876437	.859746	.847767
	15	.910771	.897200	.884564	.868886	.857623
	16	.916354	.903587	.891688	.876910	.866281
	17	.921280	.909228	.897985	.884009	.873949
	18	.925659	.914245	.903590	.890335	.880786
	19	.929576	.918737	.908612	.896007	.886919
	20	.933101	.922782	.913137	.901121	.892454
	25	.946493	.938171	.930375	.920640	.913601
	30	.955419	.948446	.941906	.933725	.927800
	35	.961792	.955794	.950161	.943107	.937993
	40	.966571	.961308	.956362	.950162	.945664
	45	.970288	.965599	.961190	.955661	.951646
	50	.973261	.969034	.965057	.960067	.956442
	60	.977719	.974188	.970863	.966688	.963653
	70	.980904	.977871	.975015	.971426	.968815
	80	.983292	.980635	.978131	.974983	.972693
	90	.985149	.982784	.980556	.977754	.975714
	100	.986634	.984505	.982497	.979971	.978133

Table A23 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

$P = 6$

		$\alpha = .100$	.050	.025	.010	.005
M_2	N					
	10	15.99908	18.32180	20.50088	23.23099	25.21310
	11	15.99689	18.31908	20.49762	23.22699	25.20851
	12	15.99525	18.31705	20.49518	23.22399	25.20508
	13	15.99399	18.31549	20.49331	23.22170	25.20245
	14	15.99301	18.31427	20.49185	23.21990	25.20038
	15	15.99222	18.31329	20.49068	23.21846	25.19874
	16	15.99159	18.31250	20.48973	23.21730	25.19741
	17	15.99106	18.31185	20.48895	23.21635	25.19631
	18	15.99063	18.31131	20.48831	23.21555	25.19540
	19	15.99026	18.31086	20.48776	23.21488	25.19463
	20	15.98995	18.31048	20.48730	23.21431	25.19398
	25	15.98893	18.30921	20.48578	23.21245	25.19184
	30	15.98838	18.30853	20.48497	23.21145	25.19070
	35	15.98806	18.30813	20.48448	23.21086	25.19002
	40	15.98785	18.30787	20.48417	23.21047	25.18958
	45	15.98771	18.30769	20.48396	23.21021	25.18928
	50	15.98761	18.30757	20.48381	23.21003	25.18907
	60	15.98747	18.30740	20.48362	23.20979	25.18880
	70	15.98740	18.30731	20.48350	23.20965	25.18863
	80	15.98734	18.30724	20.48342	23.20955	25.18852
	90	15.98731	18.30720	20.48337	23.20949	25.18845
	100	15.98728	18.30717	20.48333	23.20944	25.18840
L_2	10	.866885	.849092	.832732	.812679	.798423
	11	.878968	.862660	.847637	.829183	.816038
	12	.889041	.873992	.860106	.843020	.830829
	13	.897567	.883598	.870691	.854787	.843423
	14	.904877	.891844	.879788	.864915	.854275
	15	.911214	.899000	.887691	.873724	.863722
	16	.916759	.905268	.894619	.881455	.872020
	17	.921652	.910804	.900743	.888296	.879367
	18	.926003	.915729	.906194	.894390	.885917
	19	.929896	.920138	.911079	.899855	.891793
	20	.933399	.924110	.915479	.904782	.897094
	25	.946716	.939223	.932248	.923583	.917343
	30	.955595	.949317	.943465	.936186	.930936
	35	.961937	.956536	.951496	.945221	.940691
	40	.966695	.961955	.957530	.952015	.948031
	45	.970395	.966172	.962228	.957310	.953755
	50	.973355	.969548	.965991	.961552	.958343
	60	.977796	.974615	.971641	.967928	.965241
	70	.980968	.978236	.975681	.972489	.970179
	80	.983347	.980953	.978714	.975915	.973888
	90	.985197	.983067	.981074	.978582	.976777
	100	.986677	.984759	.982963	.980717	.979090

Table A24 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$ $P = 7$

		$\alpha = .100$	.050	.025	.010	.005
M_2	N					
	10	18.56403	21.04398	23.35785	26.24260	28.32863
	11	18.56133	21.04068	23.35395	26.23789	28.32327
	12	18.55931	21.03822	23.35104	26.23436	28.31927
	13	18.55776	21.03633	23.34880	26.23166	28.31620
	14	18.55655	21.03485	23.34705	26.22954	28.31379
	15	18.55558	21.03367	23.34566	26.22785	28.31187
	16	18.55479	21.03271	23.34452	26.22648	28.31032
	17	18.55415	21.03192	23.34359	26.22535	28.30904
	18	18.55361	21.03127	23.34282	26.22441	28.30797
	19	18.55316	21.03072	23.34216	26.22362	28.30707
	20	18.55277	21.03025	23.34161	26.22295	28.30631
	25	18.55151	21.02871	23.33979	26.22075	28.30381
	30	18.55084	21.02789	23.33881	26.21957	28.30247
	35	18.55044	21.02740	23.33823	26.21887	28.30168
	40	18.55018	21.02708	23.33786	26.21841	28.30116
	45	18.55000	21.02687	23.33761	26.21811	28.30081
	50	18.54988	21.02671	23.33743	26.21789	28.30056
	60	18.54971	21.02651	23.33719	26.21760	28.30024
	70	18.54962	21.02640	23.33705	26.21743	28.30005
	80	18.54955	21.02632	23.33696	26.21732	28.29992
	90	18.54951	21.02627	23.33690	26.21725	28.29984
	100	18.54948	21.02623	23.33685	26.21719	28.29978
L_2	10	.867873	.851598	.836688	.818465	.805535
	11	.879845	.864929	.851240	.834476	.822561
	12	.889830	.876064	.863413	.847897	.836852
	13	.898283	.885505	.873747	.859308	.849016
	14	.905532	.893610	.882629	.869128	.859495
	15	.911817	.900644	.890344	.877668	.868615
	16	.917318	.906806	.897108	.885163	.876625
	17	.922173	.912249	.903086	.891793	.883715
	18	.926490	.917091	.908409	.897700	.890036
	19	.930353	.921427	.913177	.902996	.895705
	20	.933830	.925332	.917474	.907771	.900819
	25	.947050	.940195	.933845	.925988	.920349
	30	.955868	.950124	.944797	.938198	.933455
	35	.962168	.957226	.952638	.946950	.942858
	40	.966894	.962557	.958529	.953531	.949934
	45	.970571	.966707	.963117	.958660	.955450
	50	.973512	.970029	.966791	.962769	.959871
	60	.977925	.975015	.972307	.968943	.966518
	70	.981078	.978578	.976252	.973361	.971275
	80	.983442	.981252	.979214	.976678	.974849
	90	.985282	.983333	.981518	.979261	.977632
	100	.986753	.984998	.983363	.981329	.979861

Table A25 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

P = 8

		$\alpha = .100$	.050	.025	.010	.005
M_2	N					
	10	21.08152	23.70571	26.14343	29.17052	31.35233
	11	21.07833	23.70187	26.13893	29.16514	31.34628
	12	21.07594	23.69899	26.13557	29.16112	31.34174
	13	21.07411	23.69679	26.13299	29.15803	31.33827
	14	21.07267	23.69506	26.13097	29.15561	31.33554
	15	21.07153	23.69368	26.12935	29.15368	31.33337
	16	21.07060	23.69256	26.12804	29.15211	31.33160
	17	21.06983	23.69164	26.12697	29.15082	31.33015
	18	21.06920	23.69088	26.12607	29.14975	31.32894
	19	21.06866	23.69023	26.12531	29.14885	31.32793
	20	21.06821	23.68968	26.12467	29.14808	31.32706
	25	21.06671	23.68788	26.12257	29.14556	31.32422
	30	21.06591	23.68692	26.12144	29.14422	31.32271
	35	21.06544	23.68635	26.12077	29.14341	31.32180
	40	21.06513	23.68598	26.12033	29.14290	31.32122
	45	21.06492	23.68572	26.12004	29.14254	31.32082
	50	21.06477	23.68554	26.11983	29.14229	31.32054
	60	21.06458	23.68531	26.11956	29.14197	31.32017
	70	21.06446	23.68517	26.11939	29.14177	31.31995
	80	21.06439	23.68508	26.11929	29.14165	31.31981
	90	21.06434	23.68502	26.11922	29.14156	31.31971
	100	21.06430	23.68498	26.11917	29.14150	31.31964
L_2	10	.868886	.853817	.840054	.823271	.811383
	11	.880753	.866943	.854308	.838875	.827925
	12	.890652	.877907	.866232	.851951	.841805
	13	.899034	.887204	.876354	.863067	.853616
	14	.906222	.895185	.885053	.872632	.863788
	15	.912456	.902113	.892610	.880949	.872640
	16	.917913	.908181	.899234	.888248	.880413
	17	.922729	.913542	.905090	.894704	.887292
	18	.927012	.918311	.910302	.900455	.893424
	19	.930845	.922582	.914972	.905611	.898923
	20	.934296	.926429	.919180	.910259	.903883
	25	.947416	.941070	.935213	.927992	.922822
	30	.956169	.950851	.945939	.939874	.935528
	35	.962423	.957848	.953618	.948391	.944643
	40	.967116	.963101	.959387	.954795	.951499
	45	.970767	.967190	.963880	.959785	.956845
	50	.973688	.970463	.967478	.963783	.961129
	60	.978071	.975376	.972880	.969790	.967569
	70	.981202	.978888	.976744	.974088	.972178
	80	.983551	.981523	.979644	.977315	.975640
	90	.985378	.983573	.981901	.979828	.978336
	100	.986839	.985214	.983707	.981839	.980495

Table A26 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

P = 9

		$\alpha=.100$	.050	.025	.010	.005
M_2	N					
	10	23.56183	26.32005	28.87299	32.03266	34.30383
	11	23.55816	26.31568	28.86792	32.02665	34.29710
	12	23.55542	26.31241	28.86413	32.02216	34.29208
	13	23.55331	26.30990	28.86122	32.01871	34.28822
	14	23.55166	26.30794	28.85893	32.01601	34.28519
	15	23.55034	26.30636	28.85711	32.01385	34.28277
	16	23.54927	26.30509	28.85563	32.01210	34.28081
	17	23.54839	26.30404	28.85441	32.01066	34.27920
	18	23.54765	26.30317	28.85340	32.00946	34.27786
	19	23.54704	26.30243	28.85255	32.00845	34.27673
	20	23.54651	26.30181	28.85183	32.00759	34.27577
	25	23.54479	26.29975	28.84944	32.00477	34.27261
	30	23.54387	26.29866	28.84817	32.00326	34.27092
	35	23.54332	26.29800	28.84741	32.00236	34.26991
	40	23.54296	26.29758	28.84692	32.00178	34.26927
	45	23.54272	26.29729	28.84659	32.00139	34.26882
	50	23.54255	26.29709	28.84635	32.00111	34.26851
	60	23.54233	26.29682	28.84604	32.00074	34.26810
	70	23.54220	26.29666	28.84586	32.00053	34.26786
	80	23.54211	26.29656	28.84574	32.00038	34.26770
	90	23.54205	26.29649	28.84566	32.00029	34.26759
	100	23.54201	26.29644	28.84560	32.00022	34.26751
L_2	10	.869863	.855782	.842951	.827338	.816294
	11	.881633	.868728	.856952	.842598	.832429
	12	.891451	.879543	.868662	.855383	.845964
	13	.899766	.888712	.878603	.866250	.857479
	14	.906899	.896586	.887145	.875599	.867394
	15	.913083	.903419	.894565	.883728	.876020
	16	.918498	.909405	.901070	.890861	.883594
	17	.923278	.914694	.906820	.897169	.890297
	18	.927528	.919398	.911938	.902789	.896270
	19	.931332	.923612	.916523	.907826	.901626
	20	.934757	.927406	.920655	.912368	.906457
	25	.947780	.941851	.936397	.929690	.924900
	30	.956470	.951502	.946927	.941296	.937270
	35	.962680	.958405	.954466	.949614	.946141
	40	.967339	.963588	.960130	.955867	.952815
	45	.970965	.967623	.964541	.960740	.958017
	50	.973866	.970853	.968073	.964643	.962186
	60	.978218	.975701	.973377	.970508	.968452
	70	.981328	.979166	.977169	.974705	.972937
	80	.983660	.981766	.980016	.977855	.976305
	90	.985475	.983789	.982232	.980308	.978928
	100	.986927	.985409	.984005	.982272	.981028

Table A27 Percentage Points of L_2 and $M_2 = -2b \log \lambda_2$

P = 10

		$\alpha=.100$	.050	.025	.010	.005
M_2	N					
	10	26.01200	28.89595	31.55707	34.84134	37.19658
	11	26.00786	28.89107	31.55144	34.83474	37.18922
	12	26.00477	28.88741	31.54723	34.82980	37.18372
	13	26.00239	28.88461	31.54400	34.82600	37.17949
	14	26.00053	28.88240	31.54147	34.82303	37.17618
	15	25.99904	28.88065	31.53944	34.82065	37.17353
	16	25.99783	28.87922	31.53780	34.81872	37.17139
	17	25.99683	28.87805	31.53645	34.81713	37.16962
	18	25.99601	28.87707	31.53533	34.81581	37.16815
	19	25.99531	28.87625	31.53438	34.81470	37.16691
	20	25.99472	28.87555	31.53358	34.81376	37.16586
	25	25.99277	28.87325	31.53093	34.81065	37.16240
	30	25.99173	28.87202	31.52951	34.80898	37.16055
	35	25.99111	28.87129	31.52867	34.80799	37.15944
	40	25.99071	28.87081	31.52812	34.80735	37.15873
	45	25.99043	28.87049	31.52775	34.80692	37.15825
	50	25.99024	28.87026	31.52749	34.80661	37.15790
	60	25.98999	28.86997	31.52715	34.80621	37.15746
	70	25.98984	28.86979	31.52694	34.80597	37.15719
	80	25.98974	28.86967	31.52681	34.80581	37.15701
	90	25.98967	28.86959	31.52672	34.80570	37.15689
	100	25.98963	28.86954	31.52665	34.80563	37.15681
L_2	10	.870784	.857528	.845475	.830833	.820490
	11	.882464	.870316	.859255	.845798	.836278
	12	.892208	.880999	.870781	.858334	.849518
	13	.900461	.890057	.880563	.868987	.860779
	14	.907541	.897834	.888970	.878151	.870474
	15	.913680	.904584	.896271	.886118	.878908
	16	.919056	.910498	.902673	.893108	.886312
	17	.923801	.915722	.908330	.899290	.892863
	18	.928021	.920369	.913366	.904796	.898701
	19	.931798	.924532	.917878	.909732	.903936
	20	.935198	.928280	.921943	.914181	.908656
	25	.948130	.942550	.937431	.931151	.926675
	30	.956759	.952084	.947791	.942519	.938757
	35	.962927	.958904	.955208	.950666	.947422
	40	.967555	.964025	.960780	.956790	.953939
	45	.971156	.968011	.965119	.961561	.959019
	50	.974037	.971202	.968593	.965384	.963089
	60	.978360	.975992	.973811	.971127	.969207
	70	.981450	.979415	.977542	.975235	.973585
	80	.983767	.981984	.980343	.978320	.976873
	90	.985570	.983983	.982522	.980722	.979433
	100	.987012	.985583	.984267	.982645	.981483

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This thesis uses three criteria to test for equality of p populations with underlying two parameter exponential distributions (θ =location parameter, σ =scale parameter). These criteria use n random samples drawn from each of the p populations. The three criteria are based on the following three hypotheses:

1. $H_0 : \theta_1 = \theta_2 = \dots = \theta_p,$
 $\sigma_1 = \sigma_2 = \dots = \sigma_p$
2. $H_1 : \sigma_1 = \sigma_2 = \dots = \sigma_p$
 θ_i 's are unspecified ($i=1,2,\dots,p$)
3. $H_2 : \theta_1 = \theta_2 = \dots = \theta_p$
 given that $\sigma_1 = \sigma_2 = \dots = \sigma_p$

against the general alternatives.

The asymptotic expansions of the distributions for λ_1 , λ_2 , and λ_3 are found based on the Neyman-Pearson likelihood ratio, where λ_0 , λ_1 , and λ_2 are the criteria for H_0 , H_1 , and H_2 respectively. The asymptotic expansions are computed using Bernoulli polynomials and a recursive relationship developed by Kalinin and Shalaevskii. Nine tables of percentage points are computed for each test statistic from the expansions where $p = 2(1)10$, $n = 10(1)20(5)50(10)100$, and $\alpha = .100, .050, .025, .010, .005$. These tables along with a practical illustration give the analyst a good technique that can be applied to many exponentially related situations.

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