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HYDRODYNAMIC FIELD EQUATIONS IN PROPELLER COORDINATES
(U) DAVID W TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT
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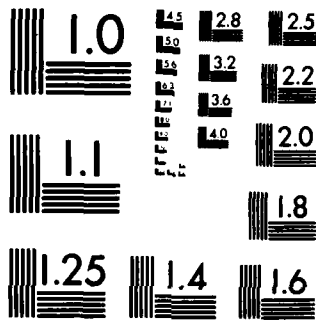
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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

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HYDRODYNAMIC FIELD EQUATIONS IN PROPELLER COORDINATES

by

John A. Fox

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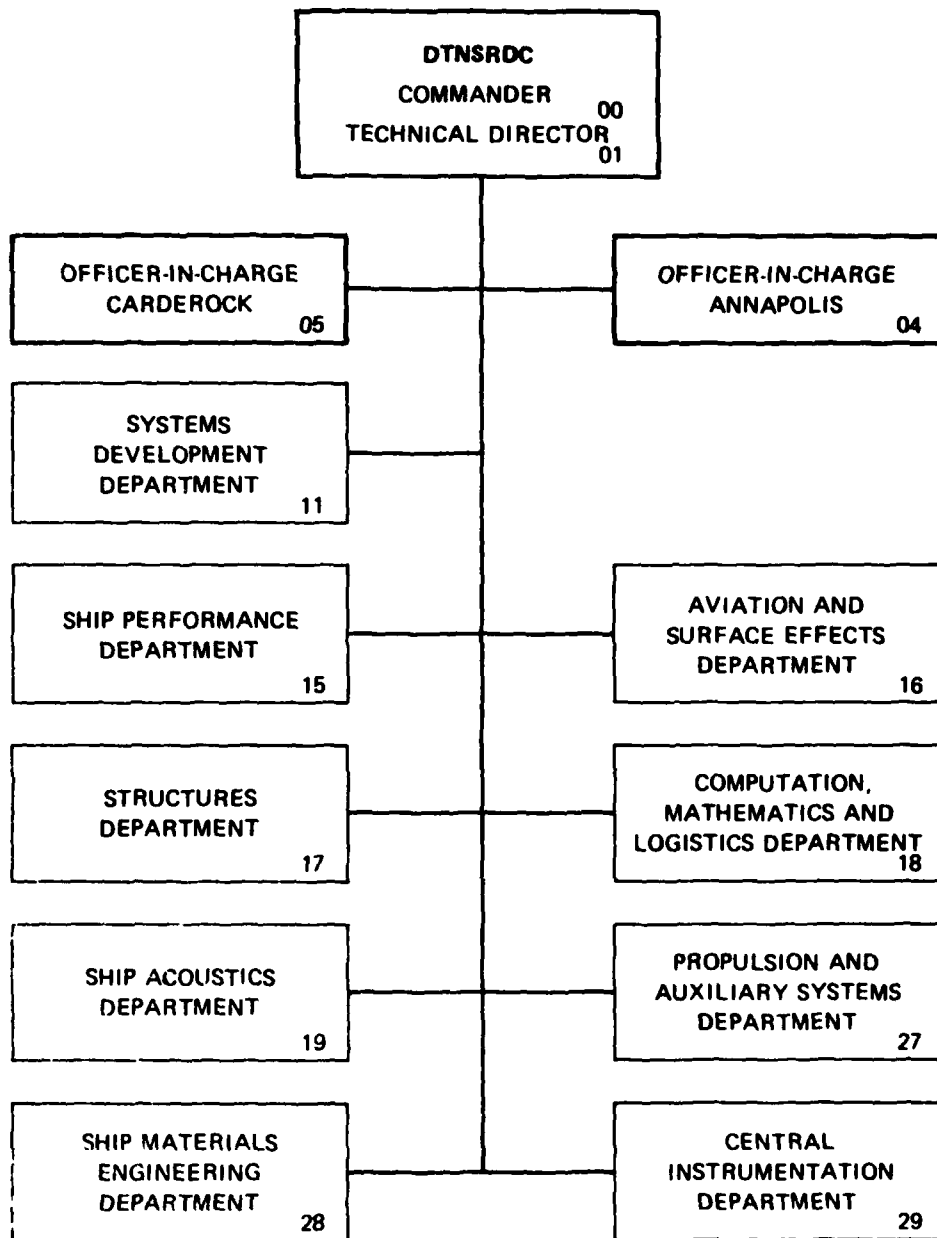
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the second coordinate system may be better suited to describe the flow at the leading edge. It is recommended that the boundary-layer limit of the field equations be determined and numerical evaluations be conducted to explore the merits of the two coordinate systems.

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PRINCIPAL NOTATION

\bar{a}	Acceleration vector
a^i	Contravariant components of \bar{a}
$\underline{a}_1, \underline{a}_2, \underline{a}_3$	Orthogonal surface-imbedded vectors such that \underline{a}_2 is along the constant fraction-of-chord lines
$\bar{a}, \bar{b}, a^\alpha, b_1$	General coordinate vectors, vector components
$\underline{c}_1, \underline{c}_2, \underline{c}_3$	Orthogonal surface-imbedded vectors such that \underline{c}_3 is along the tangent to the surface in the chordwise direction
e_{ijk}	Alternating unit tensor
$\underline{e}_r, \underline{e}_\theta$	Unit vectors in \bar{r} and $\bar{\theta}$ directions on blade
$\underline{e}_1, \underline{e}_2$	Unit vectors in helical coordinate system
F, G	Combinations of geometric variables
g_{ij}	Metric tensor
g^{ij}	Associated metric tensor
G^{ij}	Cofactor of g_{ij}
i_T	Total rake (see Figure 3)
$[ij, k]$	Christoffel symbol of the first kind (see Equation 13)
\bar{n}	Unit normal to blade surface (see Figure 4)
\bar{q}, q^1, q_1	Velocity vector and components
\bar{R}	Position vector
r	Radial coordinate
\bar{s}	Position vector of point on blade surface
\bar{U}	Far field flow velocity (ahead of propeller)
u^1, u^2, u^3	Orthogonal surface coordinates (see Figure 4)
X_1	Body force component
x^i, y^i , or x, y, z	Coordinate axes
x_R, x_C	Radial and chordwise parameters defining locations on the mean blade surface

α, β, θ

Combinations of geometric variables

$\bar{\Omega}$

Angular velocity of rotating blade

ζ

Vorticity vector

μ

Absolute viscosity

ν

Kinematic viscosity

ϕ_p

Helix pitch angle of reference blade

θ_s

Skew angle (see Figure 3)

Γ_{ij}^k

Christoffel symbol of second kind (see Equation 14)

δ_{kl}^{ij}

Generalized Kronecker Delta

ξ^1, ξ^2

Coordinates of blade-oriented coordinate system (see Reference 1 and Figure 3)

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ABSTRACT

Specific expressions for a general tensor description of the full field equations for flow about a propeller blade are given. Two orthogonal blade-surface coordinate systems are considered: one consisting of the normal and constant radius lines and one consisting of the normal and constant fraction-of-chord lines. The third direction is obtained by the cross-product of vectors in the two given directions. It is believed that the first coordinate system is appropriate to describe the flow over the majority of the blade, and the second coordinate system may be better suited to describe the flow at the leading edge. It is recommended that the boundary-layer limit of the field equations be determined and numerical evaluations be conducted to explore the merits of the two coordinate systems.

ADMINISTRATIVE INFORMATION

This investigation was conducted at DTNSRDC while the author was under Intergovernmental Personnel Act, SDN-00167-80-1PA-003. Funding was provided by DTNSRDC's Independent Research (IR) program, element 61152N, task ZR 023-01-01, initially work unit 1-1552-103, and follow-on work unit 1-1544-367.

INTRODUCTION

The various performance properties of propellers have generated a significant number of geometries for different purposes. It has been generally accepted that a blade-fixed coordinate system is most useful for general application. From this conclusion, one is, of course, faced with the necessity of adding new terms to the equations of motion dictated by the rotating coordinate system.^{1*}

The equations of motion in their general form have been considered by many and have been written down in a general way.^{1,2,3} Depending upon the problem to be solved, the simplifying assumptions made, and the technique of solution, one may find some coordinate systems have advantages over others. It is not the purpose here to pass judgement on any coordinate system but rather to suggest an approach to some hydrodynamic (fluid mechanical) problems and offer the details of appropriate coordinate systems. In at least one case, a specific coordinate system is offered that has not appeared in the literature involving propeller theory and that is believed useful to described boundary-layer flow on the blades. There are areas for which the suggested coordinates may need significant modification,

* A complete listing of references is given on page 37.

such as problems involving cavity flows. Certainly, the boundary conditions and flow surface boundaries will need attention. In some cases, these may involve an iteration to obtain a proper boundary. Whatever is needed can at least be incorporated into the current coordinate systems; furthermore, the problems themselves may suggest new geometries that may be more useful.

The notation used in the report is that of generalized tensors. The basic premise is that one can set down the momentum, continuity, and energy equations and any equations of state in this general descriptive manner. Once this is accomplished, any convenient geometry can be introduced. The selection can be made (and often is) to closely accommodate physical conditions, to permit partial uncoupling of the equations (or at least some simplification), or to describe boundary conditions rather simply. The coordinate system so chosen may not always be an orthogonal one, but in certain applications may be more convenient and perhaps more economical to solve numerically. In the following, the basic fluid-dynamics equations are introduced in this generalized tensor form without a great deal of development. Details of some of the requisites are presented in Appendix A, and the reader is referred to Reference 6 for other details. Covariant and contravariant forms of the equations are introduced. Some emphasis is placed on the contravariant form because most people feel more comfortable and have a better physical feel in that form due to the close relationship of contravariant components with physical components. Reference is made in the present report to advantages and disadvantages in particular cases of both systems.

The lengthy mathematical manipulations are not included in the text; however, in some cases, intermediate steps are included to make the procedure more easy to follow for those not familiar with the tensor notation. The tensor notation and manipulation are believed to have some advantages over the usual vector notation in two ways:

1. One knows precisely how to expand a term or operator as soon as the geometry is selected.
2. As often happens, higher rank tensors, (e.g., stresses) and their derivatives require no more expertise to handle than first rank tensors (vectors), and the concept of dyadics, which is unfamiliar territory to many, need not be employed.

The present work is intended to provide a specification of geometry immediately useful for some hydrodynamic propeller problems. In particular, the full field equations are given for flow about a propeller blade. These field equations

can be evaluated to determine which terms should be retained for boundary-layer flow on the blade surface. It is hoped that the manipulations described will provide an easy transition to other areas of endeavor as well.

ANALYSIS

TENSOR DESCRIPTION AND PHYSICAL EQUATIONS

While it is not necessary to restrict the equations of hydrodynamics to incompressible fluids, this approximation is employed here simply to address the specific purpose involved in this report, namely, problems associated with hydrodynamics and even more specifically, the hydrodynamics of propellers. The essential equations are given below in contravariant form:

Continuity Equation:

$$q^i{}_{,i} = 0 \\ = \frac{1}{\sqrt{g}} \frac{\partial [\sqrt{g} q^i]}{\partial x^i} \quad (1)$$

Equation of Motion:

$$\bar{a} + \frac{d\bar{\Omega}}{dt} \times \bar{R} + 2\bar{\Omega} \times \bar{q} + \bar{\Omega} \times [\bar{\Omega} \times \bar{R}] = \bar{F} - \frac{1}{\rho} \nabla p \quad (2)$$

where \bar{F} is the system of forces such as body forces or viscous forces. The terms containing $\bar{\Omega}$ are due to the rotating coordinate system chosen. Because $\bar{\Omega}$ is considered to be constant, the time derivative term is zero. The $2\bar{\Omega} \times \bar{q}$ term is the Coriolis acceleration due to the accelerating (rotating) system and the last term is the centripetal acceleration.⁴ The acceleration measured in the chosen coordinate system is \bar{a} , with the following components:

$$a^i = \frac{\delta q^i}{\delta t} = \frac{\partial q^i}{\partial t} + q^j{}_{,j} q^i \quad (3)$$

*The summation convention here is sum on repeated sub and superscripts.

where δ/δ_t is the intrinsic (substantial) derivative. For steady state, this reduces to

$$a^i = q^i_{,j} q^j \quad (4)$$

If one looks at the viscous description of Reference 5, it is noted that the viscous stresses are not changed because of the rotating system of coordinates. Physically, this makes sense if one recalls that all viscous stresses involve velocity gradients and not the velocities themselves.

Additional terms do arise if one considers turbulent motion and the type of term associated with perturbations about a mean flow. In the case of turbulent flow, an additional term will appear an additional term due to Reynolds' stresses. Secondary motions are not excluded in Equation (2) and may play an important part in some flows, although they normally do not become important except in the last stages of an axial flow compressor or in the root regions of a propeller.⁵

One significant term that needs some discussion is the vorticity vector.

$$\nabla \times \bar{q} = \bar{\zeta} \quad (5)$$

If there were no vorticity far ahead of the propeller, it would be necessary that \bar{U} (the flow velocity) be constant or a function of time only. If it were not, if, for example, $\bar{U} = \bar{U}(r)$, one could not even suggest that the far field is a potential field. This would cause no great problem in the previous equations because there is little to be gained by such an assumption if indeed one wants to consider the viscous terms.

For general flow of the type described here, the viscous component of \bar{F} for an incompressible fluid is accordingly

$$F_i = \mu g^{jk} q_{i,jk} \quad (6)$$

The remaining terms to be put in tensor form are $2 \bar{\Omega} \times \bar{q}$ and $\bar{\Omega} \times (\bar{\Omega} \times \bar{R})$, where \bar{R} is the position vector in the rotating system of coordinates.

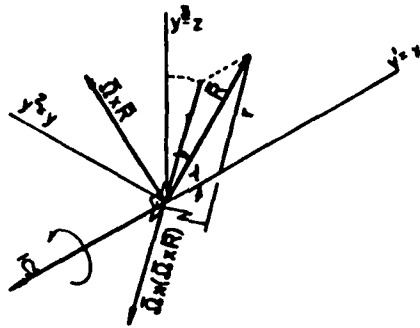


Figure 1 - Centripetal Acceleration

Hence, $\bar{\Omega} \times (\bar{\Omega} \times \bar{r})$ is the centripetal acceleration $\Omega^2 \bar{r}$ in the negative \bar{r} direction (toward the $y^1 = x$ axis of rotation). The term $\bar{\Omega} \times \bar{q}$ is found from the following figure:

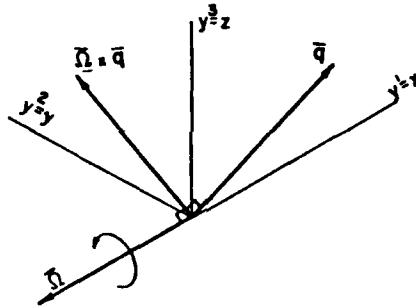


Figure 2 - Coriolis Acceleration

The components of this vector must be determined in a particular coordinate system. $\bar{\Omega}$ must be specified in the particular system also. For blade-oriented coordinates [1]:

$$\bar{\Omega} = \Omega \sin \phi_p \bar{e}_1 + \Omega \cos \phi_p \bar{e}_2 \quad (7)$$

and for both cartesian and cylindrical polar coordinates:

$$\bar{\Omega} = -\Omega \bar{i} \quad (8)$$

In expanded form (which one must use for solution, the equations of motion are

$$\begin{aligned} \frac{\partial q^i}{\partial t} + q^i \left(\frac{\partial q^i}{\partial x^i} + \Gamma_{ij}^i q^j \right) + 2\sqrt{g} g^{ij} e_{ijk} \Omega^i q^k + \delta_{pq}^{ij} \Omega_i \Omega^p R^q = \\ - \frac{1}{\rho} g^{ij} \frac{\partial p}{\partial x^i} + \nu \left[\frac{\partial^2 q^i}{\partial x^i \partial x^k} + \Gamma_{ik}^i \frac{\partial q^i}{\partial x^j} + \Gamma_{ij}^i \frac{\partial q^j}{\partial x^k} - \Gamma_{jk}^i \frac{\partial q^i}{\partial x^j} \right. \\ \left. + \left(\frac{\partial \Gamma_{ij}^i}{\partial x^k} + \Gamma_{mk}^i \Gamma_{ij}^m - \Gamma_{im}^i \Gamma_{jk}^m \right) q^j \right] + X^i \end{aligned} \quad (9a)$$

where X^i are the contravariant components of the body forces (e.g., buoyancy).

The equations of motion can be written in covariant form as:

$$\frac{\partial q_i}{\partial t} + q_{i,j} q^j = - \frac{1}{\rho} \frac{\partial p}{\partial x^i} + \nu g^{jk} q_{i,jk} + X_i - 2 e_{ijk} \Omega^j q^k - \delta_{ii}^{jk} \Omega^i \Omega_j R_k \quad (9b)$$

where

$$q_{i,jk} = \frac{\partial^2 q_i}{\partial x^j \partial x^k} - \Gamma_{ij}^\alpha \frac{\partial q_\alpha}{\partial x^k} - \Gamma_{ik}^\alpha \frac{\partial q_\alpha}{\partial x^j} - \Gamma_{jk}^\alpha \frac{\partial q_i}{\partial x^\alpha} - \left(\frac{\partial \Gamma_{ij}^\gamma}{\partial x^k} - \Gamma_{ik}^\alpha \Gamma_{\alpha j}^\gamma - \Gamma_{jk}^\alpha \Gamma_{i\alpha}^\gamma \right) q_\gamma$$

The above equations have been written for a right-handed coordinate system. A left-handed system will change some of the signs. The argument for considering the contravariant form of the differential equations (9a) as opposed to the covariant form (9b) is that only contravariant components of the velocity are involved in the equations (9a) while one has a mixed system in covariant and contravariant forms (9b). The many symbols introduced above are detailed below:

e_{ijk} is the alternating unit tensor and has the definition

$$\begin{aligned} e_{ijk} &= 0 & i = j \text{ or } j = k \text{ or } i = k \text{ or } i = j = k \\ &= 1 & i \neq j \neq k \neq i \text{ and } ijk \text{ are cyclic as } 1231 \\ &= -1 & i \neq j \neq k \neq i \text{ and } ijk \text{ are not as acyclic } 3213 \end{aligned}$$

δ_n^m is the usual Kronecker delta:

$$= 0 \quad m \neq n$$

$$= 1 \quad m = n$$

δ_{pq}^{il} has the definition

$$= 1 \quad i \neq l \text{ and } i = p \text{ and } l = q$$

$$= -1 \quad i \neq l \text{ and } i = q \text{ and } l = p$$

$$= 0 \quad \text{for all other cases.}$$

e^{ijk} is the same as e_{ijk} but with relative weight +1 instead of -1 (for e_{ijk})*

g_{ij} is the metric tensor

g^{ij} is the associated metric tensor

Γ_{jk}^i are the Christoffel symbols of the second kind. They are not tensors, and for this reason many people prefer the $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ notation instead of the Princeton notation [6].

Physical vector quantities are related to the forms as follows. The terms $\sqrt{g_{ii}}q^i$ (not summed) are the physical components of \bar{q} that are the edges of the parallelepiped, of which \bar{q} is the diagonal. The quantity $q_i/\sqrt{g_{ii}}$ (not summed) is the length of an orthogonal projection of \bar{q} on the tangent to the coordinates at a point. Because $q_r = g_{rs}q^s$, we can obtain the covariant form by first defining the contravariant form

$$q^k = \frac{Q^k}{\sqrt{g_{kk}}}$$

where Q^k are the physical components of \bar{Q} , and then deriving q_k via the above equation.

At this point one needs to develop the values of the contravariant components of specified vectors plus the Christoffel symbols indicated. One now must specify the geometry. We shall carry out some of the details that we have been indicating for the general fluid flow problem.

*See Section 41 of Sokolnikoff [6] for detailed explanation.

COORDINATE GEOMETRY

The (Y^i) Cartesian system and the $X^i:(x,r,\theta)$ system are well described in standard texts, e.g., reference [6]. The details are contained in Appendix B. However, propeller blades and the helical nature of the associated fluid motion require somewhat unique coordinate systems. One such system is shown in Figure 3.

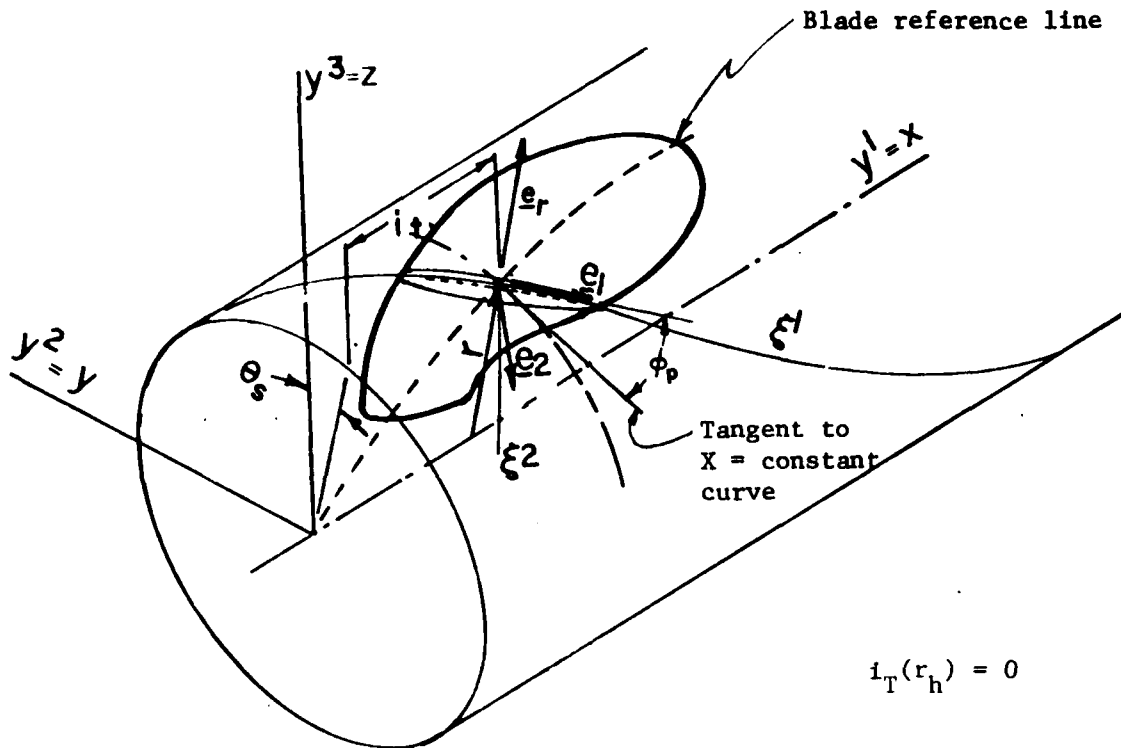


Figure 3 - Geometry of a Coordinate System Fixed to a Rotating Blade

In this geometry, the blade-reference line is any reference line one prefers. It could be a curve through the midpoint of the blade chord line (curve) on the cylinder; it could be conveniently the curve of aerodynamic centers for lifting-line calculations; or it could be the leading edge. However, the choice of reference lines should be judiciously made to avoid computational difficulties, including the introduction of some singularities (e.g., the leading edge of conventional propeller blades would have a slope of infinity referred to chordwise distance).

In Figure 3, θ_s defines the skew angle and i_T defines the total rake. ϕ_p is the pitch angle (the orientation of nose-to-tail helix at radius r), ξ^1 is defined along the helix, and ξ^2 is perpendicular to ξ^1 and lies in the cylinder defined by $r = \text{a constant}$. Of course, ξ^2 also represents a helix with a slope that is the negative reciprocal of the slope of ξ^1 at the point.

ξ^1, r, ξ^2 is a locally orthogonal set but does not represent a general orthogonal coordinate system. The direction of ξ^2 is in the binormal direction of the ξ^1 helix, and the principal normal of ξ^1 is in the negative r direction. Even though this set is not a general orthogonal curvilinear coordinate system, one still can establish the equations of motion, continuity, etc. in this coordinate system. The metrics, Christoffel symbols, etc. involve rather lengthy algebraic manipulations, which are summarized here.

Blade-Oriented Coordinates:

$$y^1 = i_T + \xi^1 \sin \phi_p - \xi^2 \cos \phi_p = i_T + \beta^*$$

$$y^2 = -r \sin \theta^*$$

$$y^3 = r \cos \theta$$

$$\frac{\partial y^1}{\partial \xi^1} = \sin \phi_p$$

$$\frac{\partial y^1}{\partial r} = i'_T + a \phi'_p = C$$

$$\frac{\partial y^1}{\partial \xi^2} = -\cos \phi_p$$

(10)

* where $\theta = \theta_b + \theta_s + \frac{\xi^1 \cos \phi_p + \xi^2 \sin \phi_p}{r} = \theta_b + \theta_s + \frac{a}{r}$

$$a = \xi^1 \cos \phi_p + \xi^2 \sin \phi_p$$

$$\beta = \xi^1 \sin \phi_p - \xi^2 \cos \phi_p$$

$$\text{and } i'_T = di_T/dr; \theta'_s = d\theta_s/dr; \theta'_p = d\theta_p/dr$$

$$\frac{\partial y^2}{\partial \xi^1} = -\cos \theta \cos \phi_p$$

$$\frac{\partial y^2}{\partial r} = -\sin \theta - r \cos \theta \left[\theta'_s - \frac{\alpha}{r^2} - \frac{\beta \phi'_p}{r} \right] = -\sin \theta - r B \cos \theta$$

$$\frac{\partial y^2}{\partial \xi^2} = -\cos \theta \sin \phi_p$$

$$\frac{\partial y^3}{\partial \xi^1} = -\sin \theta \cos \phi_p$$

$$\frac{\partial y^3}{\partial r} = \cos \theta - r \sin \theta \left\{ \theta'_s - \frac{\alpha}{r^2} - \frac{\beta \phi'_p}{r} \right\} = \cos \theta - r B \sin \theta$$

$$\frac{\partial y^3}{\partial \xi^2} = -\sin \theta \sin \phi_p$$

(10 continued)

From equations (10), the metric $g_{ij} = \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}$ summed on k can be formed:

$$g_{ij} = \begin{bmatrix} 1 & C \sin \phi_p + r B \cos \phi_p & 0 \\ C \sin \phi_p + r B \cos \phi_p & 1 + r^2 B^2 + C^2 & -C \cos \phi_p + r B \sin \phi_p \\ 0 & -C \cos \phi_p + r B \sin \phi_p & 1 \end{bmatrix} \quad (11)$$

where $C = 1 + a\phi'_p$

and $B = \theta'_p - \frac{a}{r^2} - \frac{\beta\phi'_p}{r}$

The value of $g = |g_{ij}| = 1$

For convenience, some combinations occur so often they are given another designation:

$$C\sin\phi_p + rB\cos\phi_p = K$$

$$C\cos\phi_p - rB\sin\phi_p = L$$

When these are introduced, the metric becomes:

$$g_{ij} = \begin{bmatrix} 1 & K & 0 \\ K & 1+K^2+L^2 & -L \\ 0 & -L & 1 \end{bmatrix} \quad (11a)$$

The associated metric tensor is $g_{ij} = G^{ij}/g$.

where G^{ij} is the cofactor of g^{ij} . Because $g = 1$, the G^{ij} are simply the cofactors of g_{ij} .

They are:

$$\begin{aligned} g^{11} &= 1 + C^2\sin^2\phi_p + r^2B^2\cos^2\phi_p + rBC\sin 2\phi_p \\ g^{12} &= g^{21} = -C\sin\phi_p - rB\cos\phi_p \\ g^{13} &= g^{31} = \frac{r^2B^2 - C^2}{2}\sin 2\phi_p - rBC\cos 2\phi_p \end{aligned} \quad (12)$$

$$g^{23} = g^{32} = C\cos\phi_p - rB\sin\phi_p$$

$$g^{22} = 1$$

$$g^{33} = 1 + r^2B^2\sin^2\phi_p + C^2\cos^2\phi_p - rBC\sin 2\phi_p$$

Introducing K and L as previously defined, one obtains the tensor

$$g^{ij} = \begin{bmatrix} 1+K^2 & -K & -KL \\ -K & 1 & L \\ -KL & L & 1+L^2 \end{bmatrix} \quad (12a)$$

Next, one must calculate the Christoffel symbols of the first kind:

$$[ij, k] = \frac{1}{2} \left\{ \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right\} \quad (13)$$

from which the Christoffel symbols that appear in the equations are derived.

$$\Gamma_{jk}^i = g^{i\alpha} [ij, k] \quad (14)$$

(Note that Γ_{jk}^i is not a tensor; some authors prefer the symbol $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ so as to avoid any confusion from the so-called Princeton notation).

$$\begin{aligned} [11, 1] &= 0 \\ [11, 3] &= [13, 1] = [31, 1] = 0 \\ [33, 1] &= [33, 3] = [13, 3] = [31, 3] = 0 \\ [12, 1] &= [21, 1] = 0 \\ [23, 3] &= [32, 3] = 0 \end{aligned}$$

$$[11, 2] = \frac{\partial g_{12}}{\partial x^1} = \frac{\partial}{\partial \xi^1} [C \sin \phi_p + r B \cos \phi_p] = \frac{\cos^2 \phi_p}{r}$$

$$[33, 2] = \frac{\partial g_{33}}{\partial x^3} = \frac{\partial}{\partial \xi^2} [-C \cos \phi_p + r B \sin \phi_p] = -\frac{\sin^2 \phi_p}{r}$$

$$[13, 2] = [31, 2] = \frac{1}{2} \left[\frac{\partial g_{12}}{\partial \xi^2} + \frac{\partial g_{21}}{\partial \xi^1} \right] = -\frac{1}{2} \frac{\sin 2\phi_p}{r}$$

$$[12, 3] = [21, 3] = \frac{1}{2} \left[\frac{\partial g_{23}}{\partial \xi^1} - \frac{\partial g_{12}}{\partial \xi^2} \right] = -\phi'_p$$

$$[32, 1] = [23, 1] = \frac{1}{2} \left[\frac{\partial g_{12}}{\partial \xi^2} - \frac{\partial g_{23}}{\partial \xi^1} \right] = \phi_p$$

$$\begin{aligned} [22, 1] &= \frac{\partial g_{12}}{\partial r} \frac{1}{2} \frac{\partial g_{22}}{\partial \xi^1} = 2B + r \frac{\partial B}{\partial r} + \frac{\partial C}{\partial r} \sin \phi_p = \frac{\partial K}{\partial r} - L \phi_p + B \cos \phi_p \\ &= i_T^* \sin \phi_p + (2\theta_s' + r\theta_s'') \cos \phi_p + \xi^2 \phi_p' - \xi^1 (\phi_p')^2 \end{aligned}$$

$$\begin{aligned} [12, 2] = [21, 2] &= \frac{1}{2} \frac{\partial g_{22}}{\partial \xi^1} = -B \cos \phi_p + (C \cos \phi_p - rB \sin \phi_p) \phi_p' \\ &= -B \cos \phi_p + L \phi_p' \\ &= -\theta_s' \cos \phi_p + i_T^* \phi_p' \cos \phi_p - r\theta_s' \phi_p' \sin \phi_p \\ &\quad + \frac{\xi^1 \cos^2 \phi_p + \xi^2 \sin \phi_p \cos \phi_p + \xi^1 (\phi_p')^2}{r^2} \\ &\quad + \frac{\xi^1 \sin 2\phi_p - \xi^2 \cos 2\phi_p}{r} \phi_p' \end{aligned}$$

$$\begin{aligned} [23, 2] = [32, 2] &= -B \sin \phi_p + [C \sin \phi_p + rB \cos \phi_p] \phi_p' \\ &= -B \sin \phi_p + K \phi_p' \\ &= -\theta_s' \sin \phi_p + i_T^* \phi_p' \sin \phi_p + r\theta_s' \phi_p' \cos \phi_p + \frac{\xi^1 \cos \phi_p \sin \phi_p + \xi^2 \sin^2 \phi_p}{r^2} \\ &\quad + \xi^2 (\phi_p')^2 - \frac{\xi^1 \cos 2\phi_p + \xi^2 \sin 2\phi_p}{r} \phi_p' \end{aligned}$$

$$\begin{aligned} [22, 3] &= \frac{\partial g_{22}}{\partial r} - \frac{1}{2} \frac{\partial g_{22}}{\partial \xi^2} = \left(2B + r \frac{\partial B}{\partial r} \right) \sin \phi_p + \frac{\partial C}{\partial r} \cos \phi_p \\ &= i_T^* \cos \phi_p + (2\theta_s' + r\theta_s'') \sin \phi_p - \xi^1 \phi_p'' - \xi^2 (\phi_p')^2 \\ &= -\frac{\partial L}{\partial r} - K \phi_p' + B \sin \phi_p \end{aligned}$$

$$\begin{aligned}
[22, 2] &= \frac{1}{2} \frac{\partial g_{22}}{\partial r} = r^2 B \frac{\partial B}{\partial r} + r B^2 + C \frac{\partial C}{\partial r} \\
&= K \frac{\partial K}{\partial r} + L \frac{\partial L}{\partial r} \\
&= i_T' i_T'' + r (\theta_s')^2 + r^2 \theta_s' \theta_s'' - \theta_s'' (\xi^1 \cos \phi_p + \xi^2 \sin \phi_p) - \frac{(\xi^1 \cos \phi_p + \xi^2 \sin \phi_p)^2}{r^2} \\
&\quad + \phi_p' \phi_p'' \left((\xi^1)^2 + (\xi^2)^2 \right) + \phi_p' \left\{ i_T'' (\xi^1 \cos \phi_p + \xi^2 \sin \phi_p) - r \theta_s'' (\xi^1 \sin \phi_p - \xi^2 \cos \phi_p) \right. \\
&\quad \left. + \frac{1}{r^2} \left[(\xi^1)^2 - (\xi^2)^2 \right] \sin 2\phi_p - 2\xi^1 \xi^2 \cos 2\phi_p \right\} \\
&\quad + \phi_p'' \left\{ -r \theta_s' (\xi^1 \sin \phi_p - \xi^2 \cos \phi_p) - i_T' (\xi^1 \cos \phi_p + \xi^2 \sin \phi_p) + \right. \\
&\quad \left. \frac{1}{r} \left[\frac{(\xi^1)^2 - (\xi^2)^2}{2} \sin 2\phi_p - \xi^1 \xi^2 \cos 2\phi_p \right] \right\} \\
&\quad + (\phi_p')^2 \left\{ -i_T' (\xi^1 \sin \phi_p - \xi^2 \cos \phi_p) - r \theta_s' (\xi^1 \cos \phi_p + \xi^2 \sin \phi_p) \right. \\
&\quad \left. + \frac{1}{r} \left[\left((\xi^1)^2 - (\xi^2)^2 \right) \cos 2\phi_p + 2\xi^1 \xi^2 \sin 2\phi_p \right] \right\}
\end{aligned}$$

The Christoffel symbols of the second kind are:

$$\Gamma_{11}^1 = K \frac{\cos^2 \phi_p}{r}$$

$$\Gamma_{13}^1 = \Gamma_{31}^1 = K \frac{\sin \phi_p \cos \phi_p}{r}$$

$$\Gamma_{12}^1 = \Gamma_{21}^1 = K B \cos \phi_p$$

$$\Gamma_{23}^2 = \Gamma_{32}^2 = -L B \sin \phi_p$$

$$\Gamma_{13}^2 = \Gamma_{31}^2 = -L \frac{\sin \phi_p \cos \phi_p}{r}$$

$$\Gamma_{33}^2 = -L \frac{\sin^2 \phi_p}{r}$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = -B \cos \phi_p$$

$$\Gamma_{32}^2 = \Gamma_{23}^2 = -B \sin \phi_p$$

$$\Gamma_{22}^2 = -r B^2$$

$$\Gamma_{13}^2 = \Gamma_{31}^2 = -\frac{\sin \phi_p \cos \phi_p}{r}$$

$$\Gamma_{22}^1 = \frac{\partial K}{\partial r} - L \phi_p' + B \cos \phi_p + r K B^2$$

$$\Gamma_{22}^3 = -\frac{\partial L}{\partial r} - K \phi_p' + B \sin \phi_p - r L B^2$$

$$\Gamma_{11}^3 = -L \frac{\cos^2 \phi_p}{r}$$

$$\Gamma_{11}^2 = -\frac{\cos^2 \phi_p}{r}$$

$$\Gamma_{33}^1 = K \frac{\sin^2 \phi_p}{r}$$

$$\Gamma_{33}^2 = -\frac{\sin^2 \phi_p}{r}$$

$$\Gamma_{12}^3 = \Gamma_{21}^3 = -\phi_p' - L B \cos \phi_p$$

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \phi_p' + K B \sin \phi_p$$

Looking at the equations of continuity and motion, (4) and (9a), we see that some derivatives are needed. The derivatives of the Christoffel symbols are given below.

Derivatives with respect to ξ^1 and ξ^2 or x^1 and x^3 coordinates:

$$\frac{\partial \Gamma_{11}^1}{\partial \xi^1} = \frac{\partial K}{\partial \xi^1} \frac{\cos^2 \phi_p}{r}$$

$$\frac{\partial \Gamma_{11}^1}{\partial \xi^2} = \frac{\partial K}{\partial \xi^2} \frac{\cos^2 \phi_p}{r}$$

$$\frac{\partial \Gamma_{31}^1}{\partial \xi^1} = \frac{\partial \Gamma_{13}^1}{\partial \xi^1} = \frac{\partial K}{\partial \xi^1} \frac{\sin \phi_p \cos \phi_p}{r}$$

$$\frac{\partial \Gamma_{31}^1}{\partial \xi^2} = \frac{\partial \Gamma_{13}^1}{\partial \xi^2} = \frac{\partial K}{\partial \xi^2} \frac{\sin \phi_p \cos \phi_p}{r}$$

$$\begin{aligned}
\frac{\partial \Gamma_{21}^1}{\partial \xi^1} &= \frac{\partial \Gamma_{12}^1}{\partial \xi^1} = \frac{\partial K}{\partial \xi^1} B \cos \phi_p + K \frac{\partial B}{\partial \xi^1} \cos \phi_p & \frac{\partial \Gamma_{21}^1}{\partial \xi^2} &= \frac{\partial \Gamma_{12}^1}{\partial \xi^2} = \frac{\partial K}{\partial \xi^2} B \cos \phi_p + K \frac{\partial B}{\partial \xi^2} \cos \phi_p \\
\frac{\partial \Gamma_{13}^3}{\partial \xi^1} &= \frac{\partial \Gamma_{31}^3}{\partial \xi^1} = -\frac{\partial L}{\partial \xi^1} \frac{\sin \phi_p \cos \phi_p}{r} & \frac{\partial \Gamma_{23}^3}{\partial \xi^2} &= \frac{\partial \Gamma_{32}^3}{\partial \xi^2} = -\left(\frac{\partial L}{\partial \xi^2} B + L \frac{\partial B}{\partial \xi^2}\right) \sin \phi_p \\
\frac{\partial \Gamma_{23}^3}{\partial \xi^1} &= \frac{\partial \Gamma_{32}^3}{\partial \xi^1} = -\left(\frac{\partial L}{\partial \xi^1} B + L \frac{\partial B}{\partial \xi^1}\right) \sin \phi_p & \frac{\partial \Gamma_{13}^3}{\partial \xi^2} &= \frac{\partial \Gamma_{31}^3}{\partial \xi^2} = -\frac{\partial L}{\partial \xi^2} \frac{\sin \phi_p \cos \phi_p}{r} \\
\frac{\partial \Gamma_{33}^3}{\partial \xi^1} &= -\frac{\partial L}{\partial \xi^1} \sin^2 \phi_p & \frac{\partial \Gamma_{33}^3}{\partial \xi^2} &= -\frac{\partial L}{\partial \xi^2} \sin^2 \phi_p \\
\frac{\partial \Gamma_{12}^2}{\partial \xi^1} &= \frac{\partial \Gamma_{21}^2}{\partial \xi^1} = -\frac{\partial B}{\partial \xi^1} \cos \phi_p & \frac{\partial \Gamma_{12}^2}{\partial \xi^2} &= \frac{\partial \Gamma_{21}^2}{\partial \xi^2} = -\frac{\partial B}{\partial \xi^2} \cos \phi_p \\
\frac{\partial \Gamma_{32}^2}{\partial \xi^1} &= \frac{\partial \Gamma_{23}^2}{\partial \xi^1} = -\frac{\partial B}{\partial \xi^1} \sin \phi_p & \frac{\partial \Gamma_{32}^2}{\partial \xi^2} &= \frac{\partial \Gamma_{23}^2}{\partial \xi^2} = -\frac{\partial B}{\partial \xi^2} \sin \phi_p \\
\frac{\partial \Gamma_{22}^2}{\partial \xi^1} &= \frac{\partial \Gamma_{22}^2}{\partial \xi^2} = 0 \\
\frac{\partial \Gamma_{22}^1}{\partial \xi^1} &= \frac{\partial^2 K}{\partial \xi^1 \partial r} - \frac{\partial L}{\partial \xi^1} \phi_p' + \frac{\partial B}{\partial \xi^1} \cos \phi_p + r \left\{ \frac{\partial K}{\partial \xi^1} B^2 + 2KB \frac{\partial B}{\partial \xi^1} \right\} \\
\frac{\partial \Gamma_{22}^1}{\partial \xi^2} &= \frac{\partial^2 K}{\partial \xi^2 \partial r} - \frac{\partial L}{\partial \xi^2} \phi_p' + \frac{\partial B}{\partial \xi^2} \cos \phi_p + r \left\{ \frac{\partial K}{\partial \xi^2} B^2 + 2KB \frac{\partial B}{\partial \xi^2} \right\} \\
\frac{\partial \Gamma_{22}^3}{\partial \xi^1} &= \frac{\partial^2 L}{\partial \xi^1 \partial r} - \frac{\partial K}{\partial \xi^1} \phi_p' + \frac{\partial B}{\partial \xi^1} \sin \phi_p - r \left\{ \frac{\partial L}{\partial \xi^1} B^2 + 2BL \frac{\partial B}{\partial \xi^1} \right\} \\
\frac{\partial \Gamma_{22}^3}{\partial \xi^2} &= \frac{\partial^2 L}{\partial \xi^2 \partial r} - \frac{\partial K}{\partial \xi^2} \phi_p' + \frac{\partial B}{\partial \xi^2} \sin \phi_p - r \left\{ \frac{\partial L}{\partial \xi^2} B^2 + 2BL \frac{\partial B}{\partial \xi^2} \right\} \\
\frac{\partial \Gamma_{11}^3}{\partial \xi^1} &= -\frac{\partial L}{\partial \xi^1} & \frac{\partial \Gamma_{11}^3}{\partial \xi^2} &= -\frac{\partial L}{\partial \xi^2} \\
\frac{\partial \Gamma_{11}^2}{\partial \xi^1} &= 0 & \frac{\partial \Gamma_{11}^2}{\partial \xi^2} &= 0 \\
\frac{\partial \Gamma_{33}^1}{\partial \xi^1} &= \frac{\partial K}{\partial \xi^1} \frac{\sin^2 \phi_p}{r} & \frac{\partial \Gamma_{33}^1}{\partial \xi^2} &= \frac{\partial K}{\partial \xi^2} \\
\frac{\partial \Gamma_{33}^2}{\partial \xi^1} &= 0 & \frac{\partial \Gamma_{33}^2}{\partial \xi^2} &= 0 \\
\frac{\partial \Gamma_{21}^3}{\partial \xi^1} &= \frac{\partial \Gamma_{12}^3}{\partial \xi^1} = -\left(\frac{\partial L}{\partial \xi^1} B + L \frac{\partial B}{\partial \xi^1}\right) & \frac{\partial \Gamma_{21}^3}{\partial \xi^2} &= \frac{\partial \Gamma_{12}^3}{\partial \xi^2} = -\left(\frac{\partial L}{\partial \xi^2} B + L \frac{\partial B}{\partial \xi^2}\right) \\
\frac{\partial \Gamma_{13}^2}{\partial \xi^1} &= \frac{\partial \Gamma_{31}^2}{\partial \xi^1} = 0 & \frac{\partial \Gamma_{13}^2}{\partial \xi^2} &= \frac{\partial \Gamma_{31}^2}{\partial \xi^2} = 0
\end{aligned}$$

$$\frac{\partial \Gamma_{23}^1}{\partial \xi^1} = \frac{\partial \Gamma_{32}^1}{\partial \xi^1} = \left(K \frac{\partial B}{\partial \xi^1} + B \frac{\partial K}{\partial \xi^1} \right) \quad \frac{\partial \Gamma_{23}^1}{\partial \xi^2} = \frac{\partial \Gamma_{32}^1}{\partial \xi^2} = \left(K \frac{\partial B}{\partial \xi^2} + B \frac{\partial K}{\partial \xi^2} \right)$$

Derivatives of Γ_{jk}^1 with respect to r (the x^2 coordinate)

$$\frac{\partial \Gamma_{11}^1}{\partial r} = \frac{\partial K}{\partial r} \frac{\cos^2 \phi_p}{r} - K \left(\frac{2 \sin \phi_p \cos \phi_p}{r} \phi_p' + \frac{\cos^2 \phi_p}{r^2} \right)$$

$$\frac{\partial \Gamma_{13}^1}{\partial r} = \frac{\partial \Gamma_{31}^1}{\partial r} = \frac{\partial K}{\partial r} \frac{\sin \phi_p \cos \phi_p}{r} + K \left(\frac{\cos^2 \phi_p - \sin^2 \phi_p}{r} \phi_p' - \frac{\sin \phi_p \cos \phi_p}{r^2} \right)$$

$$\frac{\partial \Gamma_{12}^1}{\partial r} = \frac{\partial \Gamma_{21}^1}{\partial r} = \left(\frac{\partial K}{\partial r} B + K \frac{\partial B}{\partial r} \right) \cos \phi_p - KB \sin \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{23}^3}{\partial r} = \frac{\partial \Gamma_{32}^3}{\partial r} = - \left(\frac{\partial L}{\partial r} B + L \frac{\partial B}{\partial r} \right) \sin \phi_p - BL \cos \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{31}^3}{\partial r} = \frac{\partial \Gamma_{13}^3}{\partial r} = - \frac{\partial L}{\partial r} \frac{\sin \phi_p \cos \phi_p}{r} + L \frac{\sin \phi_p \cos \phi_p}{r^2} + \frac{L}{r} (\sin^2 \phi_p - \cos^2 \phi_p) \phi_p'$$

$$\frac{\partial \Gamma_{33}^3}{\partial r} = \left(\frac{L}{r^2} - \frac{1}{r} \frac{\partial L}{\partial r} \right) \sin^2 \phi_p - \frac{2L}{r} \sin \phi_p \cos \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{12}^2}{\partial r} = \frac{\partial \Gamma_{21}^2}{\partial r} = - \frac{\partial B}{\partial r} \cos \phi_p + B \sin \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{23}^2}{\partial r} = \frac{\partial \Gamma_{32}^2}{\partial r} = - \frac{\partial B}{\partial r} \sin \phi_p - B \cos \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{22}^2}{\partial r} = -B^2 - 2rB \frac{\partial B}{\partial r}$$

$$\frac{\partial \Gamma_{21}^1}{\partial r} = \frac{\partial^2 K}{\partial r^2} - \left(\frac{\partial L}{\partial r} + B \sin \phi_p \right) \phi_p' - L \phi_p'' + \frac{\partial B}{\partial r} (\cos \phi_p + 2rKB) + KB^2 + rB^2 \frac{\partial K}{\partial r}$$

$$\frac{\partial \Gamma_{22}^3}{\partial r} = - \frac{\partial^2 L}{\partial r^2} - \left(\frac{\partial K}{\partial r} - B \cos \phi_p \right) \phi_p' - K \phi_p'' + \frac{\partial B}{\partial r} (\sin \phi_p - 2rLB) - LB^2 - rB^2 \frac{\partial K}{\partial r}$$

$$\frac{\partial \Gamma_{11}^3}{\partial r} = \left(\frac{L}{r^2} - \frac{1}{r} \frac{\partial L}{\partial r} \right) \cos^2 \phi_p + \frac{2L}{r} \cos \phi_p \sin \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{11}^2}{\partial r} = \frac{2 \cos \phi_p \sin \phi_p \phi_p'}{r} + \frac{\cos^2 \phi_p}{r^2}$$

$$\frac{\partial \Gamma_{33}^1}{\partial r} = \left(\frac{1}{r} \frac{\partial K}{\partial r} - \frac{K^2}{r^2} \right) \sin^2 \phi_p + \frac{2K \sin \phi_p \cos \phi_p \phi_p'}{r}$$

$$\frac{\partial \Gamma_{33}^2}{\partial r} = - \frac{2 \sin \phi_p \cos \phi_p \phi_p'}{r} + \frac{\sin^2 \phi_p}{r^2}$$

$$\frac{\partial \Gamma_{22}^1}{\partial r} = \frac{\partial \Gamma_{21}^2}{\partial r} = -\phi_p'' - \left(B \frac{\partial L}{\partial r} + L \frac{\partial B}{\partial r} \right) \cos \phi_p + LB \sin \phi_p \phi_p'$$

$$\frac{\partial \Gamma_{22}^2}{\partial r} = \frac{\partial \Gamma_{21}^2}{\partial r} = \frac{\sin^2 \phi_p - \cos^2 \phi_p \phi_p'}{r} + \frac{\sin \phi_p \cos \phi_p}{r^2}$$

$$\frac{\partial \Gamma_{23}^1}{\partial r} = \frac{\partial \Gamma_{32}^1}{\partial r} = \phi_p'' + \left(B \frac{\partial K}{\partial r} + K \frac{\partial B}{\partial r} \right) \sin \phi_p + KB \cos \phi_p \phi_p'$$

The following are given to express L, B, K, and associated derivatives in terms of the basic geometry:

$$B = \theta_s' - \frac{\alpha}{r^2} - \frac{\beta \phi_p'}{r}$$

$$C = i_T' + \alpha \phi_p'$$

$$\alpha = \xi^1 \cos \phi_p + \xi^2 \sin \phi_p$$

$$\beta = \xi^1 \sin \phi_p - \xi^2 \cos \phi_p$$

$$L = (i_T' + \alpha \phi_p') \cos \phi_p - (r \theta_s' - \frac{\alpha}{r} - \beta \phi_p') \sin \phi_p$$

$$K = (i_T' + \alpha \phi_p') \sin \phi_p + (r \theta_s' - \frac{\alpha}{r} - \beta \phi_p') \cos \phi_p$$

$$\frac{\partial L}{\partial \xi^1} = \phi_p' + \frac{\cos \phi_p \sin \phi_p}{r}$$

$$\frac{\partial B}{\partial \xi^1} = - \frac{\sin \phi_p \phi_p'}{r} - \frac{\cos \phi_p}{r^2}$$

$$\frac{\partial L}{\partial \xi^2} = \frac{\sin^2 \phi_p}{r}$$

$$\frac{\partial B}{\partial \xi^2} = \frac{\cos \phi_p \phi_p'}{r} - \frac{\sin \phi_p}{r^2}$$

$$\frac{\partial K}{\partial \xi^1} = - \frac{\cos^2 \phi_p}{r}$$

$$\frac{\partial B}{\partial r} = \theta_s'' + \frac{2\beta \phi_p'}{r^2} + \frac{2\alpha}{r^3} - \frac{\alpha (\phi_p')^2}{r} - \beta \phi_p''$$

$$\frac{\partial K}{\partial \xi^2} = \phi_p' - \frac{\cos \phi_p \sin \phi_p}{r}$$

$$\frac{\partial L}{\partial r} = i_T'' \sin \phi_p - \left(r \theta_s'' + \theta_s' + \frac{\alpha}{r^2} \right) \sin \phi_p - \frac{\beta \phi_p'}{r} \sin \phi_p + \xi^2 (\phi_p')^2 + \xi^1 \phi_p'' - K \phi_p'$$

$$\frac{\partial K}{\partial r} = i_T'' \sin \phi_p + \left(r \theta_s'' + \theta_s' + \frac{\alpha}{r} \right) \cos \phi_p + \frac{\beta \phi_p'}{r} \cos \phi_p - \xi^1 (\phi_p')^2 + \xi^2 \phi_p'' + L \phi_p'$$

$$\begin{aligned} \frac{\partial^2 K}{\partial r^2} = & i_T''' \sin \phi_p + \left(r \theta_s''' + 2 \theta_s'' - \frac{2\alpha}{r^2} \right) \cos \phi_p + 2 \phi_p' \left[i_T'' \cos \phi_p - (r \theta_s'' + \theta_s') \sin \phi_p \right. \\ & \left. - \frac{\alpha \sin \phi_p + \beta \cos \phi_p}{r^2} \right] + (\phi_p')^2 \left[-i_T' \sin \phi_p + r \theta_s' \cos \phi_p + \frac{2}{r} (\alpha \cos \phi_p - \beta \sin \phi_p) \right] \\ & + \xi^2 \phi_p''' + \phi_p'' \left[i_T' \cos \phi_p - \left(r \theta_s' - \frac{\alpha}{r} \right) \sin \phi_p \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial r^2} = & i_T''' \cos \phi_p - \left(r \theta_s''' + 2 \theta_s'' - \frac{2\alpha}{r^2} \right) \sin \phi_p + 2 \phi_p' \left[-i_T'' \sin \phi_p - (r \theta_s'' + \theta_s') \cos \phi_p \right. \\ & \left. + \frac{\beta \sin \phi_p - \alpha \cos \phi_p}{r} \right] + (\phi_p')^2 \left[-i_T' \cos \phi_p + r \theta_s' \sin \phi_p - \frac{2}{r} (\alpha \sin \phi_p + \beta \cos \phi_p) \right] \\ & + \xi^1 \phi_p''' + \phi_p'' \left[-i_T' \sin \phi_p + \left(r \theta_s' - \frac{\alpha}{r} \right) \cos \phi_p \right] \end{aligned}$$

$$\frac{\partial^2 L}{\partial \xi^2 \partial r} = \frac{2 \sin \phi_p \cos \phi_p \phi_p'}{r} - \frac{\sin^2 \phi_p}{r^2}$$

$$\frac{\partial^2 L}{\partial \xi^1 \partial r} = \phi_p'' + \frac{\cos^2 \phi_p - \sin^2 \phi_p \phi_p'}{r} - \frac{\sin \phi_p \cos \phi_p}{r^2}$$

$$\frac{\partial^2 K}{\partial \xi^1 \partial r} = \frac{2 \sin \phi_p \cos \phi_p \phi_p'}{r} + \frac{\cos^2 \phi_p}{r^2}$$

$$\frac{\partial^2 K}{\partial \xi^2 \partial r} = \phi_p'' - \frac{\cos^2 \phi_p \sin^2 \phi_p \phi_p'}{r} + \frac{\sin \phi_p \cos \phi_p}{r^2}$$

Some insight about the behavior of derivatives of geometric elements is given in the following:

g_{ij} and g^{ij} are dimensionless;

Christoffel symbols of the second kind are like $-(1/r)$, and differentiation gives terms like $(1/r^2)$;

higher derivatives are like $(1/r^3)$ etc;

and only second derivatives end up as derivatives of r alone.

These properties will be reflected in the boundary-layer approximation if this coordinate system is used. For example, if one were to consider a Flat Plate blade operating at zero angle of attack (something like Meyne's work [7]), ξ^2 would be the

normal to a blade which physically would have to lie on a single helicoidal sheet. Note that the leading edge does not have to be at $\theta = 0$ and no restrictions are placed on the blade shape.

In this case, one boundary is at $\xi^2 = 0$, and with the usual boundary layer assumptions, derivatives of the dependent variable with respect to ξ^2 are considered to increase the magnitude of the term. That is, the q^3 velocity in the ξ^2 direction is considered much smaller than q^1 or q^2 . Thus, to retain all elements of the continuity equation (1), differentiation with respect to ξ^2 must increase the magnitude. With this conclusion, only the term involving $\partial^2 / \partial (\xi^2)^2$ will be of prime importance in the viscous terms. However, use of the contravariant form in the pressure term with a non-orthogonal set, as we have, gives rise to some complexity. That is, $-(1/\rho) g^{11} \partial p / \partial x^1$ will have (for our case) three pressure terms in all equations. Thus, for a boundary layer similar to a flat plate, one would be advised to use the covariant form. This will not cause any real problem because one will undoubtedly substitute the physical velocity components in an appropriate manner. (Note: It is not necessary, though.)

Now: $q^\alpha = v^\alpha / \sqrt{g_{\alpha\alpha}}$ where v^α is the physical quantity.

then $q_1 = g_{1\alpha} q^\alpha$

There are some problems associated with the previous coordinate system when one seeks to use them for the general boundary-layer problem over a propeller blade. One difficulty is to establish the boundary and to represent the normal for a general warped blade. Hence, the physical arguments that are associated with the direction of differentiation along the normal are somewhat suspect. However, except in the area of the leading edge of the blade, for many blades the previous formulation will be adequate over most of the blade (this would be true for most thin blades lying close to a helicoid).

ORTHOGONAL SURFACE COORDINATES (CONSTANT FRACTION OF CHORD)

Another system that removes the restriction of the blade thickness, camber, arbitrary warp, and rake can be established. This has some improvements but is not sufficiently general for all aspects relative to the flow field. One serious problem is that the components of the flow outside the boundary layer must also be

couched in this system, and this may cause considerable difficulty at times. However, it is possible that this system may still have advantages outweighing the above problem. One distinct advantage is the anticipated ability to easily treat the leading edge problem directly, including the stagnation point region. The general transformation is summarized in the following. Figure 4 illustrates the necessary elements associated with this system.

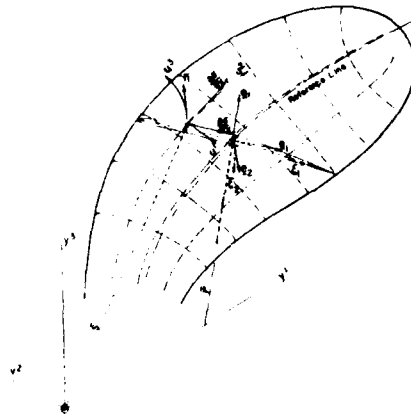


Figure 4 - Constant Fraction of Chord Blade Geometry

If \bar{S} (or \bar{S}/D in a dimensionless system) be the position vector of a point on a surface, then $d\bar{S}$ lies in the surface. If $d\bar{S} = \frac{\partial \bar{S}}{\partial z^\alpha} dz^\alpha$ represents the vector description of $d\bar{S}$ (which lies in the surface), the vectors $\partial \bar{S} / \partial z^\alpha$ lie in the surface. Furthermore, because we are dealing with a surface, only two surface coordinates are necessary to describe the surface. Thus, let

$$\begin{aligned} \bar{S} &= \bar{S}(x_R, x_c) \text{ only} \\ &= y^i \underline{b}_i \quad \text{where } \underline{b}_i \text{ are the usual cartesian unit vectors.} \end{aligned}$$

We will denote these two surface coordinates by U^α . Because x_c represents a chord-wise parameter, curves described by setting x_c equal to a constant represent constant chord curves on the surface.

Thus, one has
$$d\left\{\frac{\bar{S}}{D}\right\} = \frac{\partial\left\{\frac{\bar{S}}{D}\right\}}{\partial x_R} dx_R + \frac{\partial\left\{\frac{\bar{S}}{D}\right\}}{\partial x_c} dx_c \quad (15)$$

where $\partial\left\{\frac{\bar{S}}{D}\right\}/\partial x_R$ must be a vector lying along the curve of constant x_c . Generally,

$$\frac{dy^i}{ds} = \frac{\partial y^i}{\partial u^\alpha} \frac{du^\alpha}{ds} = \text{direction cosine of a surface curve where } \frac{du^\alpha}{ds} \text{ is a direction}$$

parameter of some sort; furthermore, it is uniquely determined once $\frac{dy^i}{ds}$ and $\frac{\partial y^i}{\partial u^\alpha}$ are known.

Now, $\underline{a}^\alpha = \frac{\partial y^i}{\partial u^\alpha} \underline{b}_i$ where \underline{b}_i are defined as before. Selecting u^2 as the constant chord direction, one finds

$$\underline{a}_2 = \frac{\partial y^i}{\partial u^2} \underline{b}_i = \frac{\partial\left\{\frac{\bar{S}}{D}\right\}}{\partial x_R} = f_2\{x_R, x_c\} \underline{i} + g_2\{x_R, x_c\} \underline{j} + h_2\{x_R, x_c\} \underline{k} \quad (16a)$$

Now we could make \underline{a}_2 a unit vector by appropriate scaling, but there seems to be little gained by such a move; it seems better to let \underline{a}_2 be as defined, and then

$$\underline{a}_2 \cdot \underline{a}_2 = g_{22} = f_2^2 + g_2^2 + h_2^2 \quad (16b)$$

Hence,

$$|\underline{a}_2| = \sqrt{g_{22}} \quad (17a)$$

Also, we may define other vectors.

$$|\underline{a}_1| = \sqrt{g_{11}}; \quad |\underline{a}_3| = \sqrt{g_{33}} \quad (17b)$$

Thus,

$$g_{\alpha\beta} = \frac{\partial y^i}{\partial u^\alpha} \frac{\partial y^i}{\partial u^\beta} \quad \sum \text{ on } i$$

and

$$g_{22} = \left| \frac{\partial \{\bar{S}/D\}}{\partial x_R} \right|^2 \quad (18)$$

Now, because $\frac{\partial \{\bar{S}/D\}}{\partial x_C}$ also lies in the surface, a normal can be defined as

$$\bar{n} = \frac{\partial \{\bar{S}/D\}}{\partial x_C} \times \frac{\partial \{\bar{S}/D\}}{\partial x_R} = \frac{\partial \{\bar{S}/D\}}{\partial x_C} \times \underline{a}_2 = \underline{a}_3 \quad (19)$$

Again, \underline{a}_3 is not a unit vector.

$$\underline{a}_3 = f_3(x_R, x_C) \underline{i} + g_3(x_R, x_C) \underline{j} + h_3(x_R, x_C) \underline{k}$$

from which, according to our above definition, f_3 , g_3 , and h_3 are the components of \underline{a}_3 in the Cartesian system.

$$g_{33} = \underline{a}_3 \cdot \underline{a}_3 = \left| \frac{\partial \{\bar{S}/D\}}{\partial x_R} \times \frac{\partial \{\bar{S}/D\}}{\partial x_C} \right|^2 \quad (20)$$

Finally, we may take, in a right-handed manner,

$$\underline{a}_2 \times \underline{a}_3 = \underline{a}_1$$

an orthogonal vector tangent to the surface (call it the u^1 direction)

$$\underline{a}_1 \cdot \underline{a}_1 = g_{11} = \left| \frac{\partial \{\bar{S}/D\}}{\partial x_R} \times \left(\frac{\partial \{\bar{S}/D\}}{\partial x_R} \times \frac{\partial \{\bar{S}/D\}}{\partial x_C} \right) \right|^2 \quad (21)$$

Thus we know all of the elements of the metric, and the coordinate system is a general curvilinear coordinate system. The proof is quite easy; one has to take only other inner products of \underline{a}_α . The U^1 , U^2 , and U^3 were chosen in the manner described previously to comply with a right-handed system and to fit the usual U^1 , U^2 in the surface so that one can follow the development details usual to Gaussian coordinates.⁶ It might be well to note here that the development to this point applies to the suction or back side of a propeller rotating clockwise about the $y^1 = x$ trailing axis. On the face (pressure) side, \bar{n} will be the negative of the above, \underline{a}_2 will have the same description, and $\underline{a}_1 = \underline{a}_3 \times \underline{a}_2$ where \underline{a}_3 refers to the pressure side and is equal to $-\underline{a}_3$ on the suction side as described previously. Thus $\underline{a}_1 = -\underline{a}_3 \times \underline{a}_2 = \underline{a}_2 \times \underline{a}_3$ or is the same as before. However, the order of right-handedness is

$$\underline{a}_1 \rightarrow \underline{a}_3 \rightarrow \underline{a}_2 \rightarrow \underline{a}_1 \text{ when substituting in the equations.}$$

This assures that U^1 is still toward the trailing edge.

One remaining preliminary exists, to form the Christoffel symbols one needs:

$$\frac{\partial g_{ij}}{\partial u^\alpha} = \frac{\partial g_{ij}}{\partial x_R} \frac{\partial x_R}{\partial y^j} \frac{\partial y^j}{\partial u^\alpha} + \frac{\partial g_{ij}}{\partial x_c} \frac{\partial x_c}{\partial y^j} \frac{\partial y^j}{\partial u^\alpha} \quad (22)$$

In this equation, there will be no difficulty (other than algebra) in determining

$$\frac{\partial g_{ij}}{\partial x_R}, \frac{\partial g_{ij}}{\partial x_c}$$

Also, $\frac{\partial y^j}{\partial u^\alpha}$ are the f_α , g_α , and h_α previously defined. However, one will need the derivatives $\frac{\partial x_B}{\partial y^j}$ and $\frac{\partial x_c}{\partial y^j}$ in order to complete the process. The required steps are contained in the following section along with specific values of the other terms.

FORMULATION OF THE GEOMETRY FOR SURFACE COORDINATES

The blade surface is defined by the vector \bar{S}/D as:

$$\frac{\bar{S}}{D} = \left(\frac{1}{D} + \frac{c}{D}(x_c - 0.5) \sin \phi_p - \frac{E}{D} \cos \phi_p \right) \underline{i} - \frac{x_R}{2} \sin \theta \underline{j} + \frac{x}{2} \cos \theta \underline{k}$$

Let

$$G = I_T/D + c/D(x_c - 0.5)\sin\phi_p - E/D\cos\phi_p$$

and

$$G_R = \frac{\partial G}{\partial x_R} = \frac{I_T}{D} + \frac{d(c/D)}{dx_R}(x_c - 0.5)\sin\phi_p - \frac{\partial(E/D)}{\partial x_R}\cos\phi_p + \phi_p'(c/D(x_c - 0.5)\cos\phi_p + E/D\sin\phi_p)$$

With a slight modification of α and β from the previous work, one can write:

$$\alpha = c/D(x_c - 0.5)\cos\phi_p + E/D\sin\phi_p$$

$$\beta = c/D(x_c - 0.5)\sin\phi_p - E/D\cos\phi_p$$

(23)

where the following have been substituted, as in Reference 3:

$$\xi^1 = c/D(x_R)\{x_c - 0.5\}$$

$$\xi^2 = E/D(x_R x_c)$$

$$\theta = \theta_b + \theta_s + \frac{\alpha}{x_R/2}$$

Other useful formulas are:

$$\frac{\partial \theta}{\partial x_R} = \theta_s' - \frac{2\alpha}{x_R^2} - \frac{\beta\phi_p'}{x_R/2} + \left(\frac{d(c/D)}{dx_R}(x_c - 0.5)\cos\phi_p + \frac{\partial(E/D)}{\partial x_R}\sin\phi_p \right) \frac{2}{x_R}$$

where

$$\theta_s' = \frac{d\theta_s}{dx_R}$$

$$\phi_p' = \frac{d\phi_p}{dx_R}$$

$$F_R = \frac{x_R\theta_s'}{2} - \frac{\alpha}{x_R} + \frac{\partial\alpha}{\partial x_R}$$

and

$$G_R = (I_T/D)' + \frac{\partial\beta}{\partial x_R}$$

With this shorthand, one has:

$$\begin{aligned}\bar{S}/D &= G\mathbf{i} - \frac{x_R}{2} \sin\theta\mathbf{j} + \frac{x_R}{2} \cos\theta\mathbf{k} \\ \underline{a}_2 = \frac{\partial(\bar{S}/D)}{\partial x_R} &= G_R\mathbf{i} + F_R e_\theta + \frac{e_r}{2} = G_R\mathbf{i} - \left(\frac{\sin\theta}{2} + F_R \cos\theta\right)\mathbf{j} + \left(\frac{\cos\theta}{2} - F_R \sin\theta\right)\mathbf{k}\end{aligned}\quad (24)$$

where

$$\begin{aligned}e_\theta &= -\cos\theta\mathbf{j} - \sin\theta\mathbf{k} \\ e_r &= -\sin\theta\mathbf{j} + \cos\theta\mathbf{k}\end{aligned}\quad (\text{see references 1 and 3})\quad (25)$$

$$\frac{\partial(\bar{S}/D)}{\partial x_c} = \left[\left(\frac{C}{D}\right) \sin\phi_p - \frac{\partial(E/D)}{\partial x_c} \cos\phi_p \right] \mathbf{i} - \frac{x_R}{2} \frac{\partial\theta}{\partial x_c} (\cos\theta\mathbf{j} + \sin\theta\mathbf{k})$$

$$\left(\frac{C}{D}\right) \sin\phi_p - \frac{\partial(E/D)}{\partial x_c} \cos\phi_p = \frac{\partial\beta}{\partial x_c}$$

$$\frac{x_R}{2} \frac{\partial\theta}{\partial x_c} = \frac{\partial\alpha}{\partial x_c}$$

$$\frac{\partial(\bar{S}/D)}{\partial x_c} = \frac{\partial\beta}{\partial x_c} \mathbf{i} + \frac{\partial\alpha}{\partial x_c} e_\theta$$

$$\underline{a}_3 = \bar{n} = \frac{\partial(\bar{S}/D)}{\partial x_c} \times \frac{\partial(\bar{S}/D)}{\partial x_R} = \begin{vmatrix} \mathbf{i} & e_r & e_\theta \\ \frac{\partial\beta}{\partial x_c} & 0 & \frac{\partial\alpha}{\partial x_c} \\ G_R & \frac{1}{2} & F_R \end{vmatrix}$$

$$= -\frac{1}{2} \frac{\partial\alpha}{\partial x_c} \mathbf{i} + \left(G_R \frac{\partial\alpha}{\partial x_c} - F_R \frac{\partial\beta}{\partial x_c} \right) e_r + \frac{1}{2} \frac{\partial\beta}{\partial x_c} e_\theta \quad (26)$$

Finally,

$$\underline{a}_1 = \underline{a}_2 \times \underline{a}_3 = \begin{vmatrix} \mathbf{i} & e_r & e_\theta \\ G_R & \frac{1}{2} & F_R \\ -\frac{1}{2} \frac{\partial\alpha}{\partial x_c} & \left(G_R \frac{\partial\alpha}{\partial x_c} - F_R \frac{\partial\beta}{\partial x_c} \right) & \frac{1}{2} \frac{\partial\beta}{\partial x_c} \end{vmatrix}$$

$$= \left\{ \frac{1}{4} \frac{\partial\beta}{\partial x_c} + F_R \left(F_R \frac{\partial\beta}{\partial x_c} - G_R \frac{\partial\alpha}{\partial x_c} \right) \right\} \mathbf{i} - \frac{1}{2} \left\{ F_R \frac{\partial\alpha}{\partial x_c} + G_R \frac{\partial\beta}{\partial x_c} \right\} e_r$$

$$+ \left\{ G_R \left(G_R \frac{\partial\alpha}{\partial x_c} - F_R \frac{\partial\beta}{\partial x_c} \right) + \frac{1}{4} \frac{\partial\alpha}{\partial x_c} \right\} e_\theta \quad (27)$$

Of course, we need this in the cartesian y^1 system:

$$\begin{aligned} \underline{a}_1 = & \left[\frac{1}{4} \frac{\partial \beta}{\partial x_c} + F_R \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right) \right] \underline{i} \\ & + \left[\frac{1}{2} \left(G_R \frac{\partial \beta}{\partial x_c} + F_R \frac{\partial \alpha}{\partial x_c} \right) \sin \theta - \left[\frac{1}{4} \frac{\partial \alpha}{\partial x_c} + G_R \left(G_R \frac{\partial \alpha}{\partial x_c} - F_R \frac{\partial \beta}{\partial x_c} \right) \right] \cos \theta \right] \underline{j} \\ & + \left[-\frac{1}{2} \left(G_R \frac{\partial \beta}{\partial x_c} + F_R \frac{\partial \alpha}{\partial x_c} \right) \cos \theta - \left[\frac{1}{4} \frac{\partial \alpha}{\partial x_c} + G_R \left(G_R \frac{\partial \alpha}{\partial x_c} - F_R \frac{\partial \beta}{\partial x_c} \right) \right] \sin \theta \right] \underline{k} \\ \underline{a}_3 = & -\frac{1}{2} \frac{\partial \alpha}{\partial x_c} \underline{i} + \left[-\frac{1}{2} \frac{\partial \beta}{\partial x_c} \cos \theta + \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right) \sin \theta \right] \underline{j} \end{aligned} \quad (28)$$

$$+ \left[-\frac{1}{2} \frac{\partial \beta}{\partial x_c} \sin \theta - \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right) \cos \theta \right] \underline{k} \quad (29)$$

Now we can express the metric using either vector form.

$$\begin{aligned} g_{11} = \underline{a}_1 \cdot \underline{a}_1 = & \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right)^2 \left[F_R^2 + G_R^2 + \frac{1}{2} \right] \\ & + \frac{1}{16} \left[\left(\frac{\partial \beta}{\partial x_c} \right)^2 + \left(\frac{\partial \alpha}{\partial x_c} \right)^2 \right] + \frac{1}{4} \left(G_R \frac{\partial \beta}{\partial x_c} + F_R \frac{\partial \alpha}{\partial x_c} \right)^2 \end{aligned} \quad (30)$$

$$g_{22} = \underline{a}_2 \cdot \underline{a}_2 = G_R^2 + F_R^2 + \frac{1}{4}$$

$$g_{33} = \underline{a}_3 \cdot \underline{a}_3 = \frac{1}{4} \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] + \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right)^2$$

Lastly, we need $\frac{\partial x_R}{\partial y^1}$ and $\frac{\partial x_c}{\partial y^1}$.

Now,

$$x_R = 2\sqrt{(y/D)^2 + (z/D)^2}$$

$$\begin{aligned} x_c = 0.5 + \frac{1}{(c/D)} & \left[\cos \phi_p \sqrt{(y/D)^2 + (z/D)^2} \left[\tan^{-1}(y/z) - \theta_b - \theta_s \right] \right. \\ & \left. + \frac{x-1}{D} \sin \phi_p \right] \end{aligned}$$

and

$$\frac{\partial(c/D)}{\partial y'} = \frac{d(c/D)}{dx_R} \frac{\partial x_R}{\partial y'}; \quad \frac{\partial \theta_s}{\partial y'} = \theta'_s \frac{\partial x_R}{\partial y'}; \quad \frac{\partial(i_T/D)}{\partial y'} = (i_T/D) \frac{\partial x_R}{\partial y'}$$

thus

$$\begin{aligned} \frac{\partial x_R}{\partial y'} &= 0 \\ \frac{\partial x_R}{\partial y^2} &= \frac{2y/D}{\sqrt{y^2+z^2}} = \frac{2y/D}{D \times 1/2} & \frac{\partial x_R}{\partial y/D} &= -2 \sin \theta \\ \frac{\partial x_R}{\partial y^3} &= \frac{2z/D}{\sqrt{y^2+z^2}} = \frac{2z/D}{D \times 1/2} & \frac{\partial x_R}{\partial z/D} &= 2 \cos \theta \\ \frac{\partial x_c}{\partial y'} &= \frac{1}{D} \frac{\sin \phi_p}{c/D} & \frac{\partial x_c}{\partial(x/D)} &= \frac{\sin \phi_p}{(c/D)} \\ \frac{\partial x_c}{\partial(y/D)} &= \frac{\partial x_c}{\partial(y/D)} \frac{1}{(c/D)} \left[\begin{aligned} &2\phi'_p \sin \theta (\alpha \sin \phi_p - \beta \cos \phi_p) - \frac{2\alpha}{x_R} \sin \theta \cos \phi_p \\ &- \cos \theta \cos \phi_p + x_R \sin \theta \cos \phi_p \theta'_s + 2(i_T/D) \sin \theta \sin \phi_p \\ &+ \frac{d(c/D)}{dx_R} \frac{2}{(c/D)} (\alpha \cos \phi_p + \beta \sin \phi_p) \sin \theta \end{aligned} \right] \end{aligned} \quad (31)$$

But

$$\alpha \sin \phi_p - \beta \cos \phi_p = E/D$$

$$\alpha \cos \phi_p + \beta \sin \phi_p = c/D(x_c - 0.5)$$

and if the radial variation of offset were simply $E = c(x_R)Y(x_c)^*$, then

$$\frac{d(E/D)}{dx_R} = \frac{d(E/c \cdot c/D)}{dx_R} = \frac{E}{c} \frac{d(c/D)}{dx_R}$$

and
$$\frac{\partial x_c}{\partial(y/D)} = \frac{1}{c/D} [2 \sin \theta (F_R \cos \phi_p + G_R \sin \phi_p) - \cos \theta \cos \phi_p] \quad (31 \text{ continued})$$

*This assumes that the blade section geometry is not changed radially. Many designs are such that
$$\frac{E}{D} = \frac{c(x_R)}{D} \frac{f(x_R)}{c(x_R)} Y_c(x_c) \pm \frac{t(x_R)}{c(x_R)} \frac{Y_T(x_c)}{\partial x_c}$$
 and hence, the expanded form in the bracketed terms for $\frac{\partial x_c}{\partial(y/D)}$ (involving α and β) must be used.

Finally,

$$\frac{\partial x_c}{\partial (y^3/D)} = \frac{\partial x_c}{\partial (z/D)} = \frac{1}{c/D} \left\{ \begin{array}{l} -2\phi_p' \cos\theta [\alpha \cos\phi_p - \beta \sin\phi_p] + \frac{2\alpha}{x_R} \cos\theta \cos\phi_p \\ -\sin\theta \cos\phi_p - 2[i\gamma/D]' \cos\theta \sin\phi_p - x_R \theta_s' \cos\theta \cos\phi_p \\ -\frac{d(c/D)}{dx_R} \frac{2}{(c/D)} [\alpha \cos\phi_p + \beta \sin\phi_p] \cos\theta \end{array} \right.$$

$$\frac{\partial x_c}{\partial (z/D)} = \frac{1}{(c/D)} \left[-2 \cos\theta [F_R \cos\phi_p + G_R \sin\phi_p] - \sin\theta \cos\phi_p \right] \quad (31 \text{ continued})$$

We next write down the geometrical quantities. The g_{ij} were previously written down and will not be repeated. The following is a direct result of the coordinate system.

Because

$$g = g_{11} g_{22} g_{33}$$

$$g^{ii} = \frac{G^{ii}}{g} = \frac{1}{g_{ii}}$$

$$g^{11} = \frac{1}{g_{11}} ; \quad g^{22} = \frac{1}{g_{22}} ; \quad g^{33} = \frac{1}{g_{33}}$$

The Christoffel symbols of the first kind are:

$$[ij, i] = [ji, i] = -[ii, j] = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^i}$$

$$[ii, i] = \frac{1}{2} \frac{\partial g_{ii}}{\partial x^i}$$

All others are zero.

The Christoffel symbols of the second kind are given by

$$\Gamma_{jk}^i = g^{i\alpha} [jk, \alpha]$$

Using the previous results*

$$\begin{aligned}
 \Gamma_{11}^1 &= g^{11} [11,1] = g^{11} \frac{1}{2} \frac{\partial g_{11}}{\partial x^1} \\
 \Gamma_{22}^1 &= g^{11} [22,1] = -\frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1} \\
 \Gamma_{33}^1 &= g^{11} [33,1] = -\frac{1}{2} g^{11} \frac{\partial g_{33}}{\partial x^1} \\
 \Gamma_{12}^1 &= \Gamma_{21}^1 = g^{11} [12,1] = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} \\
 \Gamma_{13}^1 &= \Gamma_{31}^1 = g^{11} [13,1] = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} \\
 \Gamma_{21}^2 &= \Gamma_{12}^2 = g^{22} [12,2] = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x^1} \\
 \Gamma_{22}^2 &= g^{22} [22,2] = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x^2} \\
 \Gamma_{23}^2 &= \Gamma_{32}^2 = g^{22} [23,2] = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x^3}
 \end{aligned}$$

$$\begin{aligned}
 \Gamma_{11}^2 &= g^{22} [11,2] = -\frac{1}{2} g^{22} \frac{\partial g_{11}}{\partial x^2} \\
 \Gamma_{11}^3 &= g^{33} [11,3] = -\frac{1}{2} g^{33} \frac{\partial g_{11}}{\partial x^3} \\
 \Gamma_{22}^3 &= g^{33} [22,3] = -\frac{1}{2} g^{33} \frac{\partial g_{22}}{\partial x^3} \\
 \Gamma_{33}^2 &= g^{22} [33,2] = -\frac{1}{2} g^{22} \frac{\partial g_{33}}{\partial x^2} \\
 \Gamma_{13}^3 &= \Gamma_{31}^3 = g^{33} [13,3] = \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial x^1} \\
 \Gamma_{23}^3 &= \Gamma_{32}^3 = g^{33} [23,3] = \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial x^2} \\
 \Gamma_{33}^3 &= g^{33} [33,3] = \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial x^3}
 \end{aligned}$$

There should be no local points that have a singularity problem for a finite thickness airfoil section, and the arguments used previously to justify only consideration of the highest normal derivative in the viscous term should still be valid. The significant single problem in this system is to provide the boundary condition at the outer limit of the boundary layer. The surface itself is the other limit with X_c going from 0 to 1 and X_R from the hub to the tip. This surface will be given by \bar{S}/D and represent (for our purposes) $u^3 = 0$. However, the outer boundary condition (at ∞ or the δ boundary layer limit) is that the velocity must approach the velocity of the inviscid flow, or at least approach it asymptotically. Furthermore, the quantities $2\bar{q} \times \bar{\Omega}$ and $\bar{\Omega} \times (\bar{\Omega} \times \bar{R})$ must be expressed in this coordinate system. Now these vectors are

$$2\bar{q} \times \bar{\Omega} = 2 \begin{vmatrix} \mathbf{i} & \mathbf{e}_r & \mathbf{e}_\theta \\ q^x & q^r & q^\theta \\ -\Omega & 0 & 0 \end{vmatrix} = (-q^\theta \Omega \mathbf{e}_r + q^r \Omega \mathbf{e}_\theta) 2$$

$$= 2 [-(q^\theta \cos \phi_p + q^r \sin \phi_p) \mathbf{e}_r + q^r \Omega \cos \phi_p \mathbf{e}_1 + q^r \Omega \sin \phi_p \mathbf{e}_2]$$

*See Appendix B for details.

and $\bar{\Omega} \times (\bar{\Omega} \times \bar{R}) = \bar{\Omega}^2 \underline{e}_r$, needs to be put into the new coordinate system. The appropriate components can be determined by taking the inner product of the given vectors with a unit vector in the appropriate direction, e.g.,

$$\bar{q} \cdot \frac{\underline{a}_\alpha}{|\underline{a}_\alpha|} = \text{physical component of } \bar{q} \text{ in direction of } \underline{a}_\alpha$$

Using the \underline{i} , \underline{e}_r and \underline{e} as perhaps the simplest description, one has

$$\bar{q} \cdot \frac{\underline{a}_2}{|\underline{a}_2|} = \frac{q^2 G_R + q^2 F_R + q^2/2}{\sqrt{g_{22}}}$$

as physical components of \bar{q}_α where $\sqrt{g_{22}} = |\underline{a}_2|$

The contravariant (needed first in any case) is

$$q^\alpha = \frac{\text{physical component of } \bar{q} \text{ in } \alpha \text{ direction}}{\sqrt{g_{\alpha\alpha}}}$$

$$\text{e.g., } q^2 = \frac{q^2 G_R + q^2 F_R + q^2/2}{g_{22}}$$

Thus every vector will need to be dotted with the unit vectors $\frac{\underline{a}_1}{g_{11}}$, $\frac{\underline{a}_2}{g_{22}}$, and $\frac{\underline{a}_3}{g_{33}}$ to convert the inviscid flow field vectors to the components needed and also to express the Coriolis and centripetal accelerations in the equations of motion. One should be careful in the process to nondimensionalize the quantities using D as the length, because all geometry was done this way.

Another observation may be made that may be helpful. Presumably, most sections will exhibit some stagnation point for only the chordwise velocity components at some point other than $x_c = 0$, probably on the pressure side of the blade for the design condition. This is usually a good starting point. There is a good chance that with the favorable pressure gradient that the flow on the suction side may be laminar to near the point of minimum pressure but turbulent elsewhere. See Reference 8, page 644. It may be possible to use an approximate technique similar to that of Wild⁹ and Nager.⁵

CHORDWISE IMBEDDED ORTHOGONAL SURFACE COORDINATES (See Figure 5)

In this coordinate system, one coordinate lies on a constant radius cylinder and is tangent to the blade section. As before, one has

$$\bar{S}/D = G_{\underline{j}} - \frac{x_B}{2} \sin \theta \underline{j} + \frac{x_B}{2} \cos \theta \underline{k}$$

Now
$$\frac{\partial \bar{S}/D}{\partial x_R} = G_{R\underline{i}} - \frac{\sin \theta}{2} \underline{j} + \frac{\cos \theta}{2} \underline{k} - F_R (\cos \theta \underline{j} + \sin \theta \underline{k}) \quad (24)$$

One base vector is tangent to the blade section. Let this vector be \underline{c}_3 :

$$\underline{c}_3 = \frac{\partial (\bar{S}/D)}{\partial x_c} = \frac{\partial \beta}{\partial x_c} \underline{i} - \cos \theta \frac{\partial \alpha}{\partial x_c} \underline{j} - \sin \theta \frac{\partial \alpha}{\partial x_c} \underline{k} \quad (32)$$

Take
$$\underline{c}_2 = \underline{n} = \frac{\partial (\bar{S}/D)}{\partial x_c} \times \frac{\partial (\bar{S}/D)}{\partial x_R} = -\frac{1}{2} \frac{\partial \alpha}{\partial x_c} \underline{i} - \left[\left(G_R \frac{\partial \alpha}{\partial x_c} - F_R \frac{\partial \beta}{\partial x_c} \right) \sin \theta + \frac{1}{2} \frac{\partial \beta}{\partial x_c} \cos \theta \right] \underline{j} + \left[-\frac{1}{2} \frac{\partial \beta}{\partial x_c} \sin \theta - \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right) \cos \theta \right] \underline{k} \quad (29)$$

and thus, one has

$$\underline{c}_1 = \underline{c}_2 \times \underline{c}_3 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\frac{1}{2} \frac{\partial \alpha}{\partial x_c} & -\left(G_R \frac{\partial \alpha}{\partial x_c} - F_R \frac{\partial \beta}{\partial x_c} \right) \sin \theta & -\frac{1}{2} \frac{\partial \beta}{\partial x_c} \sin \theta \\ \frac{\partial \beta}{\partial x_c} & -\cos \theta \frac{\partial \alpha}{\partial x_c} & -\sin \theta \frac{\partial \alpha}{\partial x_c} \\ -\frac{1}{2} \frac{\partial \beta}{\partial x_c} \cos \theta & -\left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right) \cos \theta & \end{vmatrix}$$

$$= -\frac{\partial \alpha}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \underline{i} - \left(\frac{1}{2} \left[\frac{\partial \alpha}{\partial x_c} \right] + \left[\frac{\partial \beta}{\partial x_c} \right] \right) \sin \theta + \frac{\partial \beta}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \cos \theta \underline{j} + \left(\frac{1}{2} \left[\frac{\partial \alpha}{\partial x_c} \right] + \left[\frac{\partial \beta}{\partial x_c} \right] \right) \cos \theta - \frac{\partial \beta}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \sin \theta \underline{k} \quad (33)$$

and in the cylindrical polar coordinates,

$$\underline{c}_1 = \frac{\partial \alpha}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \underline{i} + \frac{1}{2} \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] \underline{e}_r + \frac{\partial \beta}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \underline{e}_\theta \quad (33a)$$

$$\underline{c}_2 = -\frac{1}{2} \frac{\partial \alpha}{\partial x_c} \underline{i} - \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right) \underline{e}_r + \frac{1}{2} \frac{\partial \beta}{\partial x_c} \underline{e}_\theta \quad (36)$$

$$\underline{c}_3 = \frac{\partial \beta}{\partial x_c} \underline{i} + \frac{\partial \alpha}{\partial x_c} \underline{e}_\theta \quad (33b)$$

Also,

$$\left. \begin{aligned} g_{11} &= \underline{c}_1 \cdot \underline{c}_1 = \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] \left[\frac{1}{4} \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] + \left(F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right)^2 \right] \\ g_{22} &= \underline{c}_2 \cdot \underline{c}_2 = \frac{1}{4} \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] + \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right]^2 \\ g_{33} &= \underline{c}_3 \cdot \underline{c}_3 = \left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \end{aligned} \right\} \quad (34)$$

$\frac{\partial y^i}{\partial u^\alpha}$ = direction cosine referred to y^i and u^α coordinate system.

If $\underline{c}_\alpha = f_\alpha \underline{i} + g_\alpha \underline{j} + h_\alpha \underline{k}$

then

$$\begin{aligned} f_1 &= -\frac{\partial \alpha}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \\ g_1 &= - \left\{ \frac{1}{2} \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] \sin \theta + \frac{\partial \beta}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \cos \theta \right\} \\ h_1 &= \frac{1}{2} \left[\left(\frac{\partial \alpha}{\partial x_c} \right)^2 + \left(\frac{\partial \beta}{\partial x_c} \right)^2 \right] \cos \theta - \frac{\partial \beta}{\partial x_c} \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \sin \theta \end{aligned} \quad (35a)$$

$$f_2 = -\frac{1}{2} \frac{\partial \alpha}{\partial x_c}$$

$$g_2 = -\frac{1}{2} \frac{\partial \beta}{\partial x_c} \cos \theta + \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \sin \theta \quad (35b)$$

$$h_2 = -\frac{1}{2} \frac{\partial \beta}{\partial x_c} \sin \theta - \left[F_R \frac{\partial \beta}{\partial x_c} - G_R \frac{\partial \alpha}{\partial x_c} \right] \cos \theta$$

$$f_3 = \frac{\partial \beta}{\partial x_c}$$

$$g_3 = -\frac{\partial \alpha}{\partial x_c} \cos \theta \quad (35c)$$

$$h_3 = -\frac{\partial \alpha}{\partial x_c} \sin \theta$$

In addition to the above, terms like $\frac{\partial x_c}{\partial y^i}$, $\frac{\partial x_R}{\partial y^i}$ are needed. These are the same terms previously defined. The form of the Christoffel symbols is again the same; the difference will be in the definition of g_{ij} and g^{ij} , which obviously differ from the previously defined coordinate system.

Because the g_{ij} and g^{ij} are functions of x_c and x_R

$$\frac{\partial g_{ij}}{\partial x^a} = \frac{\partial g_{ij}}{\partial x_R} \frac{\partial x_R}{\partial y^k} \frac{\partial y^k}{\partial x^a} + \frac{\partial g_{ij}}{\partial x_c} \frac{\partial x_c}{\partial y^k} \frac{\partial y^k}{\partial x^a}$$

The $\frac{\partial y^k}{\partial x^a}$ are the $\frac{\partial y^k}{\partial u^a} = f_a, g_a, h_a$ given on page 34 for this coordinate system. The $\frac{\partial x_R}{\partial y^k}$ and $\frac{\partial x_c}{\partial y^k}$ are given in the previous coordinate system. The $\frac{\partial g_{ij}}{\partial x^a}$ and g_{ij}, g^{ij} will involve the same derivatives in the present system (here) as in the previous surface system. Again, because $\frac{\partial \bar{S}/\partial y^i}{\partial x_R}$ and $\frac{\partial \bar{S}/\partial y^i}{\partial x_c}$ have the same directions as before, \bar{n} , to be the outward normal, will be the negative of equation (29) page 27 and to continue to have \underline{c} the same (outward), we will have to reorder the coordinates for the pressure side as follows:

$$\bar{n}_F = -\bar{n}_B \text{ [back or suction side]}$$

$$\underline{c}_1 = \underline{c}_3 \times \underline{c}_2 = \underline{c}_3 \times \bar{n}_F = -\underline{c}_3 \times \bar{n}_B = \bar{n}_B \times \underline{c}_3$$

so that \underline{c}_1 is defined as before. The right-handed system will now require that the order is

$$\underline{c}_3 \rightarrow \bar{n}_F \rightarrow \underline{c}_1 \rightarrow \underline{c}_3 \quad \text{pressure side}$$

and $\underline{c}_1 \rightarrow \bar{n}_S \rightarrow \underline{c}_3 \rightarrow \underline{c}_1 \quad \text{suction side}$

and $\bar{n}_P = \bar{n}_F = -\bar{n}_B = -\bar{n}_S$

The appearance of the coordinate system is shown in Figure 5. The boundary condition at the outer edge of the boundary layer here may be very close to the specification that the x_c direction is a streamline. If this is true, there may be some simplification.

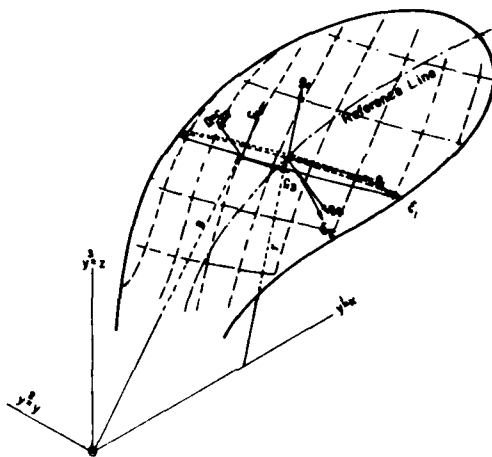


Figure 5 - Chordwise Imbedded Coordinates

The equations are now complete. Appropriate geometric configurations can now be selected and individual terms examined to derive a set of equations appropriate for the boundary layer flow.

FINAL REMARKS

One should be able to perform boundary-layer calculations with the coordinate systems developed. A zero pressure gradient solution may be made for a skewed

blade similar to Meyne.⁷ However, the real blade is, of course, the necessary problem. The leading-edge region involving stagnation point flow in the u^1 direction (perpendicular to the constant chord line) probably should be approached first. One usually can expect some uncoupling of the equations in this region, and a laminar solution may be both amenable and practical because there are favorable pressure gradients in the vicinity of the stagnation point, which may be particularly strong on the suction side. Whether a series or a polynomial approximation is used may not make too much difference. The ensuing laminar solution on the rest of the blade may be academic and of little practical use, but some model for the transition region and turbulent region beyond can be applied.

In the case of turbulence, one must add the Reynolds stress terms that appear as a result of the momentum terms in equations (9a) and (9b). Because q^i and q_j do not appear anywhere else as nonlinear terms, this is the only addition necessary. The choice of coordinates for the turbulent boundary layer problem may be the latter surface coordinate set ($x_R = \text{a constant}$) because physical evidence seems to indicate the flow is essentially in that direction. It may be that the original coordinate set may be used with the appropriate pressure gradient included. This would be convenient because no component construction would be needed for either the pressure gradient or the matching fluid dynamic velocities as one proceeds to the limit of the boundary layer. The appropriateness of this latter possibility may have to be determined by two calculations using the surface coordinates in one computation and the references surface coordinates in the other.

so that \underline{c}_1 is defined as before. The right-handed system will now require that the order is

$$\underline{c}_3 \rightarrow \bar{n}_F \rightarrow \underline{c}_1 \rightarrow \underline{c}_3 \quad \text{pressure side}$$

and $\underline{c}_1 \rightarrow \bar{n}_S \rightarrow \underline{c}_3 \rightarrow \underline{c}_1 \quad \text{suction side}$

and $\bar{n}_p = \bar{n}_F = -\bar{n}_B = -\bar{n}_S$

The appearance of the coordinate system is shown in Figure 5. The boundary condition at the outer edge of the boundary layer here may be very close to the specification that the x_c direction is a streamline. If this is true, there may be some simplification.

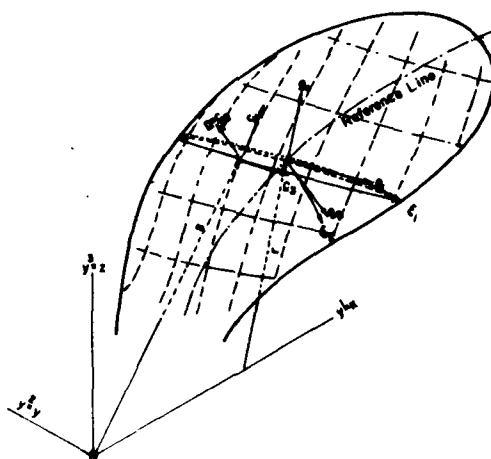


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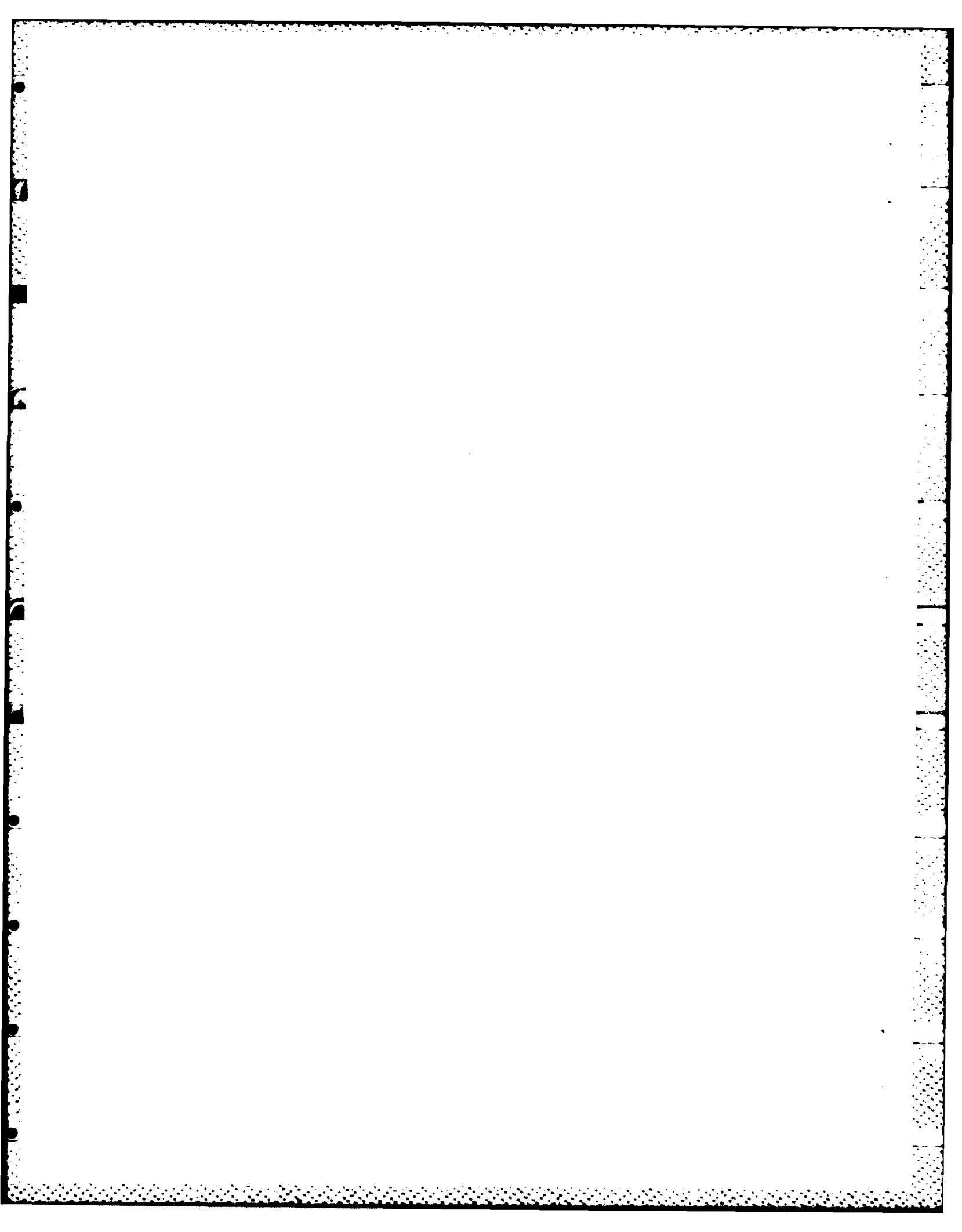
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REFERENCES

1. Brockett, T., "Propeller Perturbation Problems," Naval Ship Research and Development Center Report 3880 (1972).
2. Kerwin, J.E., "Computer Techniques for Propeller Blade Section Design," Second LIPS Propeller Symposium (1973).
3. Brockett, T., "Analytic Specification of Propeller Blade Surface Geometry," Report DTNSRDC/SPD-699-01 (1976).
4. Kochin, N.E. et al., "Theoretical Hydromechanics," Interscience Publishers, Inc., N.Y., pp. 52-53 (1964).
5. Mager, A., "Generalization of Boundary-Layer-Momentum-Integral Equations to Three-Dimensional Flows Including Those of Rotating Systems," NACA-TR 1069 (1952).
6. Sokolnikoff, I.S., "Tensor Analysis Theory and Applications to Geometry and Mechanics of Continua," John Wiley & Sons, Inc., N.Y. (1964).
7. Meyne, K., "Investigation of Propeller Boundary-Layer Flow and Friction Effect on Propeller Characteristics," Jahrbuch der Schiffbautechnischen Gesellschaft, pp. 317-399 (1972).
8. Schlichting, H., "Boundary-Layer Theory," McGraw-Hill Book Co., N.Y. (1968).
9. Wild, J.M., "The Boundary Layer of Yawed Infinite Wings," Journal of Aerospace Science, N.Y., Vol. 16, pp. 41-45 (1949).
10. Springer, C.E., "Tensor and Vector Analysis with Applications to Differential Geometry," The Ronald Press Co., N.Y. (1962).



Appendix A

Some Tensor Concepts and Identities

We define a tensor by the way the quantity transforms, for example, $B(i,x) = \partial y^a / \partial x^i A(a,y)$ covariant transformation. Sometimes $B(i,x) = \partial x^i / \partial y^a A(a,y)$ contravariant transformation.

This does not make much sense unless one looks at some examples. Obviously, a scalar, f say, is the same in any system, but look at the derivative. $\partial f / \partial x^i = \partial f / \partial y^a \cdot \partial y^a / \partial x^i$ This is a covariant transformation. We use F_i to represent a first rank tensor that transforms in a covariant manner and F^i for a first rank tensor that transforms in a contravariant manner. As one might expect, the $\partial f / \partial x^i$ is in some way related to a vector in the x^i direction, which is true. The precise meaning of F_i , F^i in terms of the usual vector components will be discussed shortly.

First, however, one must understand that higher rank tensors than the first exist. Typically, in fluids one may get to third rank tensors, in solids all the way to fourth. Tensors may be mixed, e.g., $R^i_{jkl}(y)$ is a fourth ranked tensor and transforms

$$R^i_{jkl}(y) = \frac{\partial y^i}{\partial x^a} \frac{\partial x^p}{\partial y^j} \frac{\partial x^q}{\partial y^k} \frac{\partial x^r}{\partial y^l} R^a_{pqr}(x)$$

Note that the transformation is by products of first partial derivatives. Furthermore, one should sum on the repeated sub and superscript and a superscript in the denominator can be considered as a subscript in the numerator. Coordinates are labeled with superscripts because they transform in a contravariant manner.

Differentiation

If we differentiate $\partial f / \partial x^i$ above, we get

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial y^a \partial y^b} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} + \frac{\partial f}{\partial y^a} \frac{\partial^2 y^a}{\partial x^i \partial x^j}$$

and hence the expression does not transform as a tensor. To overcome this problem, proceed as follows:

$$\frac{\partial B_i(y)}{\partial y^j} = \frac{\partial x^a}{\partial y^j} \frac{\partial x^p}{\partial y^i} \frac{\partial A_a}{\partial x^p} + \frac{\partial^2 x^a}{\partial y^j \partial y^i} A_a$$

If we look at the Christoffel symbols, one can devise the following:

$$\frac{\partial^2 x^a}{\partial y^i \partial y^j} = \gamma^k_{ij} \frac{\partial x^a}{\partial y^k} - \gamma^a_{br} \frac{\partial x^b}{\partial y^i} \frac{\partial x^r}{\partial y^j}$$

Inserting this in the top equation gives

$$\frac{\partial B_i(y)}{\partial y^j} = \frac{\partial x^a}{\partial y^j} \frac{\partial x^p}{\partial y^i} \frac{\partial A_a}{\partial x^p} + \gamma^k_{ij} \frac{\partial x^a}{\partial y^k} A_a - \gamma^a_{br} \frac{\partial x^b}{\partial y^i} \frac{\partial x^r}{\partial y^j} A_a$$

now,

$$\frac{\partial x^a}{\partial y^k} A_a = B_k(y)$$

thus,

$$\frac{\partial B_i(y)}{\partial y^j} - \Gamma_{ij}^k B_k = \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \left(\frac{\partial A_a}{\partial x^b} - \Gamma_{ab}^c A_c \right)$$

which transforms as a tensor.

We designate $B_{i,j} = \partial B_i / \partial y^j - \Gamma_{ij}^k B_k$ as covariant differentiation.

If the vector was a contravariant component, then $B^i{}_{,j} = \partial B^i / \partial y^j + \Gamma_{jk}^i B^k$. Let the physical component of \bar{A} be A_p and be expressed in contravariant form by $A^i_p = \sqrt{g_{ii}} A^i$ (not summed on i). The contravariant components of \bar{A} are $A^i = A^i_p \sqrt{g_{ii}}$ (not summed on i) where A^i_p , the physical components, represent the sides of a parallelepiped, of which \bar{A} is the diagonal. Thus if $\bar{q} (u, v_r, v_\theta)$, where u, v_r , and v_θ are the physical components, then

$$\left. \begin{aligned} q^1 &= \frac{u}{\sqrt{g_{11}}} = u \\ q^2 &= \frac{v_r}{\sqrt{g_{22}}} = v_r \\ q^3 &= \frac{v_\theta}{\sqrt{g_{33}}} = \frac{v_\theta}{r} \end{aligned} \right\} \text{in cylindrical coordinates.}$$

The covariant component of A^i is obtained in a similar manner:

$$A_i = g_{ij} A^j = \sqrt{g_{ii}} A^i_p \text{ (not summed on i)}$$

The curl of a vector \bar{Q} represented by $\nabla \times \bar{Q}$ has components as follows. (Note: we use e-system here.)*

$$K^i = \frac{e^{ijk}}{\sqrt{g}} q_{j,k}$$

* e^{ijk} is 1 if $i \neq j \neq k \neq i$ and $i \rightarrow j \rightarrow k$ is cyclic in order, e.g., 123

= -1 if $i \neq j \neq k \neq i$ and $i \rightarrow j \rightarrow k$ is acyclic in order

= 0 if $i = j$ or $j = k$ or $i = k$

e_{ijk} is the same.

where

$$q_{i,k} = \frac{\partial q_i}{\partial x^k} - \Gamma_{jk}^i q_j$$

The vector or cross product of two vectors is

$$\bar{H} = \bar{\Omega} \times \bar{q}$$

and has components $H^i = e^{ijk} \Omega_{jk}$

or $H_i = e_{ijk} \Omega^{jk}$

Of course, $H^i = g^{ij} H_j$, so if contravariant forms of the vector are known, one may write H_j instead of H^i .

Among other useful things concerning the use of the e system is:

$$\underline{e}_i \times \underline{e}_j = e_{ijk} \underline{e}_k$$

One literally has the choice of using covariant or contravariant components to determine the products formed by various vectors. The only reason for using contravariant forms is if one works with the equations of motion and continuity in this form; then it is very easy to convert to the physical form. Neither form is really more appropriate than the other.

Often the equations are manipulated in one form and then converted to physical components before actually solving the equations. This is perhaps most common, but not necessarily most efficient.

Continuing, we have a triple vector product involved in our equations because of the rotating coordinate system. (Note that we have introduced the general Kroenecker delta δ_{pq}^{ij}).

$$\bar{\Omega} \times (\bar{\Omega} \times \bar{R}) \quad \text{for which}$$

$$K^i = \delta_{pq}^{ij} \Omega_j \Omega^p R^q$$

In the x, r, θ system,

$$-\Omega = \Omega^1 \text{ only, because } g_{11} = 1$$

$$\text{and } -\Omega = \Omega_1$$

\bar{R} is the general position vector and, because of the rotation only about x , only the \underline{e}_r component of \bar{R} is important, although both R_x and R_r (but not R_θ) exist.

$$*\delta_{pqr}^{ijl} = 0 \text{ unless } \begin{matrix} i \neq j \neq l \neq i \\ p \neq q \neq r \neq p \end{matrix}$$

= ± 1 i, j, k are repeated in any order in $p, q,$ and r . If the number of permutations (one at a time) is even to put pqr in the same order as ijl , the plus is used; if this number of permutations is odd, the minus is used.

$$\bar{R} = R^x e_x + R^a e_r$$

(Note: g_{22} is also 1)

$$K^1 = \delta_{1q}^{11} \quad \Omega_1 \Omega^1 R^q = \Omega^2 R^1 \delta_{11}^{11} = 0$$

$$K^2 = \delta_{1q}^{21} \quad \Omega_1 \Omega^1 R^q$$

$q = 2$ and one permutation is needed to get symbols in the same order.

$$\therefore K^2 = - \Omega^2 R^2 = - \Omega^2 r$$

For further information one should refer to [6] or [10].

Appendix B
Christoffel Symbols for Blade Surface Coordinates

Some of the details to carry out the determination of the Christoffel symbols in the blade surface coordinate systems are carried out here. As shown in equation (22) repeated here,

$$\frac{\partial g_{ii}}{\partial u^a} = \frac{\partial g_{ii}}{\partial x_r} \frac{\partial x_r}{\partial y^i} \frac{\partial y^i}{\partial u^a} + \frac{\partial g_{ii}}{\partial x_c} \frac{\partial x_c}{\partial y^i} \frac{\partial y^i}{\partial u^a} \quad (22)$$

It is necessary to differentiate F_r , G_r , $\partial\alpha/\partial x_c$, and $\partial\beta/\partial x_c$ which are contained in the g_{ii} (equations (30)). The $\partial x_r/\partial y^i$ and $\partial x_c/\partial y^i$ are given in equations (31) and the $\partial y^i/\partial u^a$ are the \underline{i} , \underline{j} , and \underline{k} components in equations (24), (28), and (29); that is,

$$\underline{a}_i = \underline{b}_j = \frac{\partial y^1}{\partial u^a} \underline{i} = \frac{\partial y^2}{\partial u^a} \underline{j} + \frac{\partial y^3}{\partial u^a} \underline{k} \quad (B1)$$

$$= f_{\alpha} \underline{i} + g_{\alpha} \underline{j} + h_{\alpha} \underline{k}$$

$$f_i = \frac{1}{4} \frac{\partial \beta}{\partial x_c} + F_r \left(F_r \frac{\partial \beta}{\partial x_c} \right) - G_r \frac{\partial \alpha}{\partial x_c}$$

$$g_i = \frac{1}{2} \left(G_r \frac{\partial \beta}{\partial x_c} + F_r \frac{\partial \alpha}{\partial x_c} \right) \sin\theta - \left[\frac{1}{4} \frac{\partial \alpha}{\partial x_c} + G_r \left(G_r \frac{\partial \alpha}{\partial x_c} - F_r \frac{\partial \beta}{\partial x_c} \right) \right] \cos\theta$$

$$h_i = \frac{1}{2} \left(G_r \frac{\partial \beta}{\partial x_c} + F_r \frac{\partial \alpha}{\partial x_c} \right) \cos\theta - \left[\frac{1}{4} \frac{\partial \alpha}{\partial x_c} + G_r \left(G_r \frac{\partial \alpha}{\partial x_c} - F_r \frac{\partial \beta}{\partial x_c} \right) \right] \sin\theta \quad (B2)$$

$$f_2 = G_r$$

$$g_2 = -F_r \cos\theta - \frac{\sin\theta}{2}$$

$$h_2 = -F_r \sin\theta + \frac{\cos\theta}{2}$$

$$f_3 = -\frac{1}{2} \frac{\partial \alpha}{\partial x_c}$$

$$g_3 = -\frac{1}{2} \frac{\partial \beta}{\partial x_c} \cos\theta + \left(F_r \frac{\partial \beta}{\partial x_c} - G_r \frac{\partial \alpha}{\partial x_c} \right) \sin\theta$$

$$h_3 = -\frac{1}{2} \frac{\partial \beta}{\partial x_c} \sin\theta - \left(F_r \frac{\partial \beta}{\partial x_c} - G_r \frac{\partial \alpha}{\partial x_c} \right) \cos\theta$$

$$\frac{\partial \alpha}{\partial x_c} = \frac{c}{D} \cos \phi_p = \frac{\partial(E/D)}{\partial x_c} \sin \phi_p \text{ or } \frac{c}{D} \left[\cos \phi_p + \frac{d(E/c)}{dx_c} \sin \phi_p \right]^*$$

$$\frac{\partial \beta}{\partial x_c} = \frac{c}{D} \sin \phi_p - \frac{\partial(E/D)}{\partial x_c} \cos \phi_p \text{ or } \frac{c}{D} \left[\sin \phi_p - \frac{d(E/c)}{dx_c} \cos \phi_p \right]$$

$$G_R = \left(\frac{i_t}{D} \right)' + \frac{\partial \beta}{\partial x_R}$$

$$F_R = \frac{x_R}{2} \theta_s' - \frac{\alpha}{x_R} + \frac{\partial \alpha}{\partial x_R}$$

$$\frac{\partial G_R}{\partial x_R} = \left(\frac{i_t}{D} \right)' + \frac{\partial^2 \beta}{\partial x_R^2}$$

$$\frac{\partial F_R}{\partial x_R} = \frac{1}{2} \theta_s' + \frac{x_R}{2} \theta_s'' + \frac{\alpha}{x_R^2} - \frac{1}{x_R} \frac{\partial \alpha}{\partial x_R} + \frac{\partial^2 \alpha}{\partial x_R^2}$$

$$\frac{\partial G_R}{\partial x_c} = \frac{\partial^2 \beta}{\partial x_R \partial x_c}$$

$$\frac{\partial F_R}{\partial x_c} = -\frac{1}{x_R} \frac{\partial \alpha}{\partial x_c} + \frac{\partial^2 \alpha}{\partial x_R \partial x_c}$$

$$\frac{\partial \beta}{\partial x_R} = \frac{d(c/D)}{dx_R} (x_c - 0.5) \sin \phi_p - \frac{\partial(E/D)}{\partial x_R} \cos \phi_p + \alpha \phi_p'$$

(B3)

$$\frac{\partial^2 \beta}{\partial x_R^2} = \frac{d(c/D)}{dx_R^2} (x_c - 0.5) \sin \phi_p - \frac{\partial^2(E/D)}{\partial x_R^2} \cos \phi_p$$

$$+ 2 \left[\frac{d(c/D)}{dx_R} (x_c - 0.5) \cos \phi_p + \frac{\partial(E/D)}{\partial x_R} \sin \phi_p \right] \phi_p'$$

$$- \beta (\phi_p')^2 + \alpha \phi_p''$$

$$\frac{\partial^2 \beta}{\partial x_R \partial x_c} = \frac{d(c/D)}{dx_R} \sin \phi_p - \frac{\partial^2(E/D)}{\partial x_R \partial x_c} \cos \phi_p + \phi_p' \frac{\partial \alpha}{\partial x_c}$$

$$\frac{\partial^2 \beta}{\partial x_c^2} = -\frac{\partial^2(E/D)}{\partial x_c^2} \cos \phi_p$$

$$\frac{\partial \alpha}{\partial x_R} = \frac{d(c/D)}{dx_R} (x_c - 0.5) \cos \phi_p + \frac{\partial(E/D)}{\partial x_R} \sin \phi_p - \beta \phi_p'$$

*Recall that it may be convenient and applicable in a few cases (where E/c does not vary radially) to write $E/D = E/c (x_c) \cdot c/D (x_R)$.

$$\frac{\partial^2 \alpha}{\partial x_R^2} = \frac{d(c/D)}{dx_R^2} (x_c - 0.5) \cos \phi_p = \frac{\partial^2 (E/D)}{\partial x_R^2} \sin \phi_p$$

$$- 2 \left[\frac{d(c/D)}{dx_R} (x_c - 0.5) \sin \phi_p - \frac{\partial (E/D)}{\partial x_R} \cos \phi_p \right] \phi_p' \quad (B3)$$

$$- \alpha (\phi_p')^2 - \beta \phi_p''$$

$$\frac{\partial^2 \alpha}{\partial x_R \partial x_c} = \frac{d(c/D)}{dx_R} \cos \phi_p + \frac{\partial^2 (E/D)}{\partial x_R \partial x_c} \sin \phi_p - \phi_p' \frac{\partial \beta}{\partial x_c}$$

$$\frac{\partial^2 \alpha}{\partial x_c^2} = \frac{\partial^2 (E/D)}{\partial x_c^2} \sin \phi_p$$

$$\alpha = \frac{c}{D} (x_c - 0.5) \cos \phi_p + \frac{E}{D} \sin \phi_p$$

$$\beta = \frac{c}{D} (x_c - 0.5) \sin \phi_p - \frac{E}{D} \cos \phi_p$$

Equations (B2) and (B3), combined with (30) and (31), will allow one to complete the determination of the Christoffel symbols, and thus the algebraic form of the equation to be solved.

Appendix C
Cylindrical Polar Coordinates

A common appearing coordinate system is given in the literature and listed below for completeness.

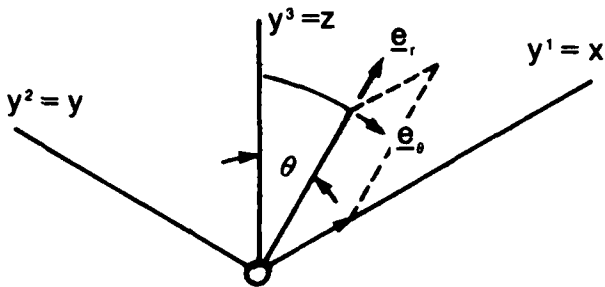


Figure 6 - Cylindrical Polar Coordinates

Referring to the figure

$$\begin{aligned} x^1 &= x, x^2 = r, x^3 = \theta \\ y^1 &= x^1 = x \\ y^2 &= y = -r \sin\theta \\ y^3 &= z = r \cos\theta \end{aligned}$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix}; g = r^2$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{r^2} \end{bmatrix}$$

The only nonzero Christoffel symbols of the first kind are:

$$\begin{aligned} [32,2] &= [23,2] = -[33,2] \\ &= r \end{aligned}$$

The Christoffel symbols of the second kind are:

$$\begin{aligned} \Gamma_{ij}^k &= g^{k\alpha} [ij, \alpha] \\ \Gamma_{32}^3 &= \Gamma_{23}^3 = g^{33} [23,3] = [1/r^2]r = 1/r \\ \Gamma_{33}^2 &= g^{22} [33,2] = -r \end{aligned}$$

All others are zero.

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