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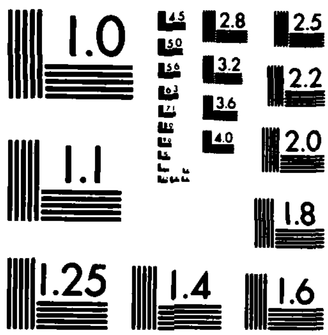

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A COLLECTION OF PROBABILITY DISTRIBUTIONS

BY

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1. INTRODUCTION

In this article we present a large number of probability density functions from 20 different families. They were found through an extensive search of the statistical literature, with the intention of listing distributions that arise in practical work. The purpose of the collection was to use the exact percentage points of a typical distribution to assess the accuracy of various methods of approximating the density using four or more moments. The methods of approximation were to fit Pearson curves or Johnson curves, using four moments in each case, or to fit Cornish-Fisher expansions, using four or more moments. Thus, we did not wish to use members of the Pearson system themselves, or members of the Johnson system; this excludes the normal, chi-squared, t, F, gamma and beta distributions.

For the purposes of the study, calculation of the exact significance points, along with at least the first four moments, was necessary. As many as three parameters could be varied for several of the families of densities, giving rise to a large number of possible individual distributions. The functions of the moments that we used to index the distributions are:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2}$$

where μ_i are the central moments of the distribution. We chose the parameters to cover a fairly broad area of the $(\sqrt{\beta_1}, \beta_2)$ plane, subject to the constraints $\sqrt{\beta_1} \leq 2$ and $\beta_2 \leq 14$. The 395 distributions, from 20 different families of distributions, are displayed in Figure 1, each family denoted by a different letter of the alphabet. A remarkable feature of the large number of distributions, which are naturally occurring densities, but a little off the mainstream of statistical work, is how many of them lie either on the line of symmetry $\sqrt{\beta_1} = 0$, or are close to the chi-squared line (see Figure 1).

The results of the comparison of approximations will be presented in a later report. In the meantime, this collection is documented as it will be useful to other workers who wish to find representative distributions with given $\sqrt{\beta_1}$, β_2 values. In section 2 the various families and types of distributions, along with parameter values and $(\sqrt{\beta_1}, \beta_2)$ values, are described. Numerical methods for computation of moments and cumulative distribution functions are discussed in section 3.

2. DESCRIPTION OF THE DISTRIBUTIONS

2.1 Noncentral Chi-squared Distributions (denoted by 'A' on Figure 1)

Let $X = \sum_{i=1}^{v} (Z_i+d_i)^2$, where the Z_i are i.i.d. (independent and identically distributed) N(0,1) and the d_i are constants. The distribution of X depends on $d_1, d_2, \ldots, d_v$ only through $\lambda = \sum_{i=1}^{v} d_i^2$ and is called the noncentral chi-squared distribution with v degrees of freedom and non-centrality parameter λ, denoted here by $\chi^2(v,\lambda)$. Cumulants of all order exist, with $\kappa_r = 2^{r-1}(v+r\lambda)(r-1)!$ (Johnson and Kotz (1970b, p. 134)). The cdf F(x) was evaluated using the algorithm of Sheil and O'Muircheartaigh (1977). Parameters of the fifteen $\chi^2(v,\lambda)$ distributions used are shown in Table 1.

Figure 1

Probability distributions, indexed by $\sqrt{\beta_1}$ and β_2

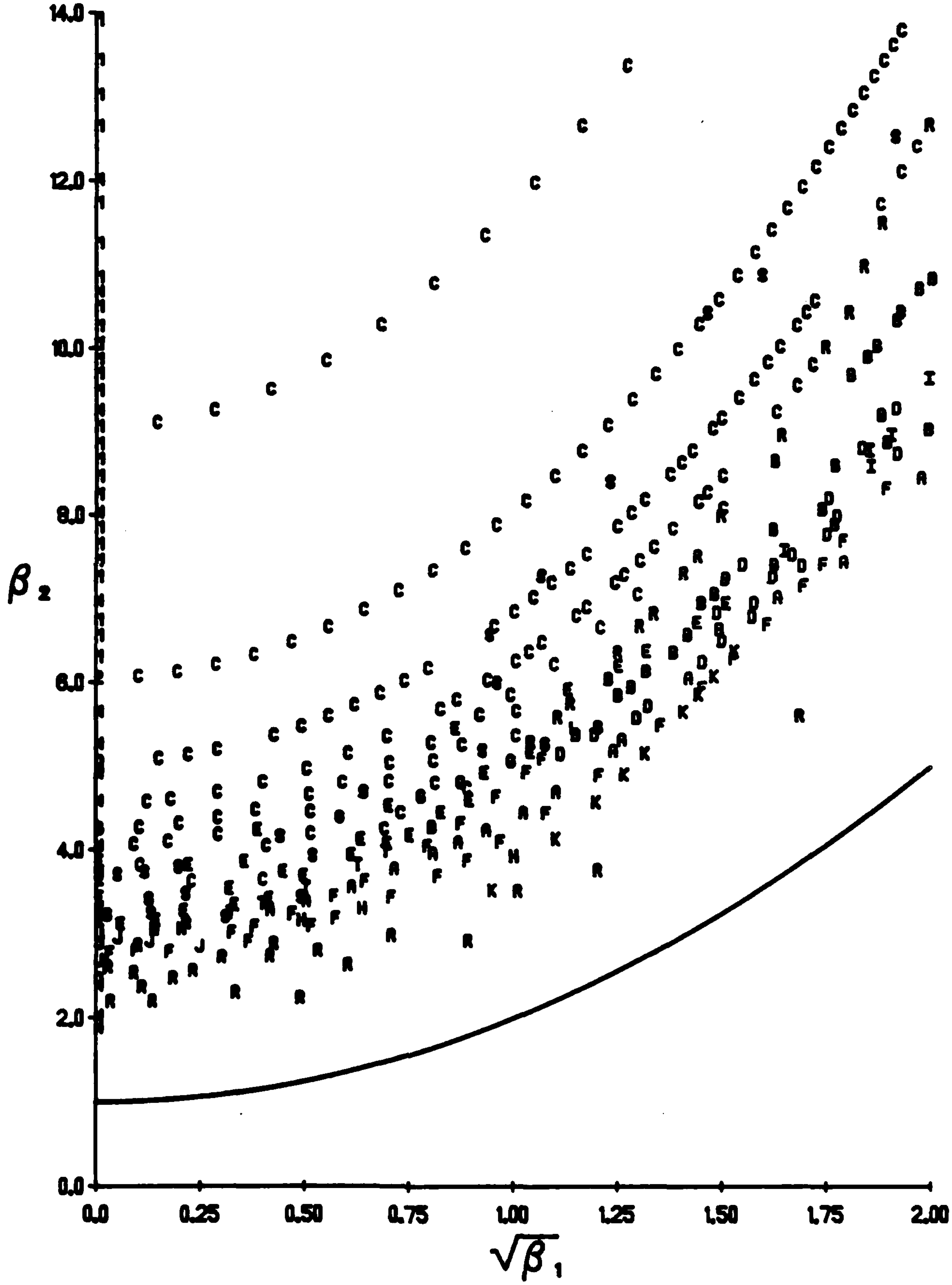

Table 1. $\chi^2(v,\lambda)$ Distributions

v	λ	$\sqrt{\beta_1}$	β_2	v	λ	$\sqrt{\beta_1}$	β_2
1.0	1.44	1.97	8.39	4.0	1.96	1.25	5.27
1.0	1.96	1.78	7.38	6.0	6.76	0.86	4.04
1.0	7.84	1.02	4.40	6.0	19.36	0.61	3.50
2.0	5.76	1.10	4.64	6.0	31.36	0.50	3.33
3.0	0.16	1.63	6.96	8.0	6.76	0.80	3.91
3.0	3.24	1.23	5.13	12.0	6.76	0.71	3.72
3.0	7.84	0.93	4.18	20.0	36.00	0.41	3.23
4.0	0.04	1.41	6.00				

2.2 Noncentral F distributions (denoted by 'B' on Figure 1)

Let Y_1 be distributed as $\chi^2(v_1,\lambda)$, Y_2 be distributed as $\chi^2(v_2,0.0)$ and suppose Y_1 and Y_2 are independent. Then $X = (Y_1/v_1)/(Y_2/v_2)$ is said to have a noncentral F-distribution with v_1, v_2 degrees of freedom and noncentrality parameter λ, denoted here by $F(v_1,v_2,\lambda)$. The first four central moments, given by Pearson and Tiku (1970), who set $\ell = \lambda/v_1$, are:

$$\mu = \frac{v_2(1+\ell)}{v_2-2} \qquad (v_2 > 2)$$

$$\mu_2 = \frac{2v_2^{\,2}(v_1+v_2-2)}{v_1(v_2-2)^2(v_2-4)}\left\{1 + 2\ell + \frac{v_1\ell^2}{v_1+v_2-2}\right\} \qquad (v_2 > 4);$$

$$\mu_3 = \frac{8v_2^{\,3}(v_1+v_2-2)(2v_1+v_2-2)}{v_1^{\,2}(v_2-2)^3(v_2-4)(v_2-6)}\left\{1+3\ell+\frac{6v_1\ell^2}{2v_1+v_2-2} + \frac{2v_1^2\ell^3}{(v_1+v_2-2)(2v_1+v_2-2)}\right\} \qquad (v_2 > 6);$$

$$\mu_4 = \frac{12v_2^4(v_1+v_2-2)}{v_1^3(v_2-2)^4(v_2-4)(v_2-6)(v_2-8)}$$

$$\Big[\{2(3v_1+v_2-2)(2v_1+v_2-2)+(v_1+v_2-2)(v_2-2)(v_1+2)\}(1+4\ell) +$$

$$2v_1(3v_1+2v_2-4)(v_2+10)\ell^2+4v_1^2(v_2+10)\ell^3+\frac{v_1^3(v_2+10)\ell^4}{(v_1+v_2-2)}\Big] \qquad (v_2 > 8).$$

The exact pdf $f(x)$ $(0 < x < \infty)$ and cdf $F(x)$ are very complicated and are not given here. Lachenbruch (1967) gives values of x_α for $\alpha = .01, .025, .05, .10, .5, .90, .95, .975, .99$, $v_1 = 1(1)\ 10, 15, 20(10)60, 120$, $v_2 = 2(2)10(10)40, 60$ and $\lambda = 1(1)10$. The parameters and significance points of the twenty-eight $F(v_1,v_2,\lambda)$ distributions used are given in Table 2.

2.3 Noncentral t distributions (denoted by 'C' on Figure 1)

Let $X = (Z+\delta)/\sqrt{U/v}$, where Z is N(0,1), U is central chi-squared with v degrees of freedom, δ is a constant and Z and U are independent. Then X has the noncentral t distribution with v degrees of freedom and noncentrality parameter δ, denoted here by $t(v,\delta)$. Hogben, Pinkham and Wilk (1961) give coefficients to find the first four central moments of $t(v,\delta)$ for $v = 2(1)\ 25(5)50, 60, 70, 80, 90, 100, 150, 200(100)1000$. These appear as Table 28 in Pearson and Hartley (1972). Merrington and Pearson (1958) give for the central moments of $t(v,\delta)$:

$$\mu = \delta\sqrt{v/2}\ \Gamma(\tfrac{1}{2}(v-1))/\Gamma(v/2), \text{ where } \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt;$$

$$\mu_2 = v(1+\delta^2)/(v-2) - \mu^2 \qquad (v > 2);$$

$$\mu_3 = \mu\left[\frac{v(2v-3+\delta^2)}{(v-2)(v-3)}\right] - 2\mu_2 \qquad (v > 3);$$

$$\mu_4 = \frac{v^2(3+6\delta^2+\delta^4)}{(v-2)(v-4)} - \mu\left[\frac{v(v+1)\delta^2+3(3v-5)}{(v-2)(v-3)} - 3\mu_2\right] \qquad (v > 4).$$

It should also be noted that $t(v,\delta)$ has range $(-\infty,\infty)$.

The cdf for integral values of v can be expressed (Owen (1962, p. 108)) as a finite sum involving the N(0,1) pdf and cdf and Owen's T-function T(h,a) (see Owen (1962, p. 184)). This function is defined by:

$$T(h,a) = \frac{1}{2\pi}\int_0^a \frac{\exp\{(-h^2/2)(1+x^2)\}}{1+x^2}\,dx \quad (-\infty<h,\ a<\infty).$$

The algorithm of Cooper (1968) for computing $\Pr(t(v,\delta) < t)$, was used with two changes. For computing the N(0,1) cdf the algorithm of Hill (1973) was used. Also, the more accurate algorithm of Young and Minder (1974) for computing T(h,a) was utilized (see also Hill (1978) and Thomas (1979)). The parameters of the 136 $t(v,\delta)$ distributions used are given in Table 3.

2.4 Quadratic Forms (denoted by 'D' on Figure 1)

Let $Z_1, Z_2, \ldots, Z_k$ be independent N(0,1) random variables and let $Q(\underset{\sim}{\lambda},k) = \sum_{i=1}^{k} \lambda_i Z_i^2$, where $\underset{\sim}{\lambda}' = (\lambda_1,\lambda_2,\ldots,\lambda_k)$ is a vector whose components are constants. The central quadratic form $Q(\underset{\sim}{\lambda},k)$ ("central" is hereafter omitted) is a simple weighted sum of independent chi-squared variables, each with one degree of freedom. Cumulants of all orders are easily computed from (Johnson and Kotz (1970b, p. 153)):

$$\kappa_r = 2^{r-1}\,(r-1)!\,\sum_{i=1}^{k} \lambda_i^{\,r}.$$

Table 2: $F(v_1, v_2, \lambda)$ Distributions

					x_α								
v_1	v_2	λ	$\sqrt{\beta_1}$	β_2	.01	.025	.05	.10	.50	.90	.95	.975	.99
1	20	6	2.00	10.77	1.25	2.16	3.16	4.57	12.3	26.8	32.8	39.1	47.7
1	30	10	1.48	7.00	4.33	5.94	7.56	9.72	20.4	37.9	44.6	51.4	60.5
1	60	2	1.99	8.97	0.0136	0.062	0.209	0.542	4.03	11.2	14.1	16.9	20.5
2	20	8	1.86	9.97	1.5928	2.2569	2.9410	3.8704	8.7473	17.5839	21.2603	25.0338	30.3137
2	40	2	1.89	8.82	0.071	0.170	0.322	0.592	2.56	6.47	8.04	9.60	11.7
2	40	4	1.62	7.35	0.350	0.657	1.02	1.55	4.57	9.82	11.9	13.9	16.5
2	60	2	1.76	7.83	0.073	0.171	0.322	0.593	2.55	6.32	7.80	9.24	11.10
3	20	6	1.91	10.28	0.7696	1.1252	1.4981	2.0173	4.8103	9.9596	12.1121	14.3355	17.4184
3	20	8	1.84	9.84	1.2501	1.7132	2.1874	2.8291	6.1809	12.2381	14.7599	17.3532	20.9402
3	30	2	1.87	9.14	0.135	0.248	0.388	0.612	2.07	4.92	6.09	7.26	8.84
3	40	2	1.73	8.02	0.136	0.249	0.390	0.614	2.06	4.80	5.89	6.97	8.40
3	40	10	1.28	5.89	1.89	2.47	3.05	3.80	7.44	13.1	15.2	17.3	20.0
4	20	4	1.96	10.65	0.4015	0.5999	0.8160	1.1237	2.8460	6.1102	7.4851	8.9065	10.8876
4	30	2	1.76	8.54	0.188	0.299	0.430	0.626	1.81	4.06	4.98	5.90	7.14
4	40	6	1.38	6.31	0.738	1.03	1.33	1.74	3.81	7.17	8.43	9.68	11.30
4	60	2	1.49	6.58	0.189	0.304	0.436	0.633	1.79	3.85	4.64	5.41	6.42
5	20	4	1.92	10.37	0.4166	0.5924	0.7792	1.0417	2.4869	5.2027	6.3466	7.5208	9.1646
5	20	8	1.80	9.62	0.9748	1.2766	1.5828	1.9944	4.1271	7.9654	9.5612	11.2005	13.4867
5	30	8	1.44	6.90	1.01	1.32	1.63	2.04	4.09	7.46	8.77	10.1	11.8
6	30	2	1.62	7.77	0.260	0.366	0.482	0.645	1.56	3.21	3.87	4.53	5.43
7	30	4	1.50	7.18	0.454	0.602	0.756	0.966	2.05	3.92	4.65	5.38	6.37
7	40	2	1.41	6.52	0.291	0.395	0.505	0.658	1.47	2.87	3.42	3.95	4.66
7	40	8	1.24	5.80	0.910	1.15	1.39	1.70	3.20	5.54	6.42	7.28	8.40
7	60	4	1.20	5.41	0.468	0.621	0.777	0.987	2.03	3.68	4.28	4.86	5.61
8	40	4	1.31	6.08	0.471	0.612	0.756	0.948	1.92	3.49	4.09	4.68	5.46
60	30	10	1.22	5.99	0.664	0.741	0.814	0.909	1.35	2.04	2.31	2.57	2.93
120	20	10	1.62	8.59	0.574	0.640	0.704	0.788	1.20	1.92	2.21	2.51	2.93
120	60	10	0.80	4.22	0.706	0.764	0.818	0.885	1.17	1.57	1.71	1.84	2.01

Owen (1962, pp. 182-183, 205-206) gives cumulative probabilities and selected significance points for the cases $k = 2,3$. These are abridged versions of the reuslts in Grad and Solomon (1955) and Solomon (1960). Johnson and Kotz (1968) give significance points x_α for $\alpha = .01, .025, .05, .10, .25, .50, .75, .90, .95, .975, .99, .995$, $k = 4,5$. Owen choses $\underset{\sim}{\lambda}$ so that $\sum_{i=1}^{k} \lambda_i^2 = 1$, while Johnson and Kotz have $\sum_{i=1}^{k} \lambda_i^2 = k$. Several additional $Q(\underset{\sim}{\lambda},k)$ distributions were also considered, with $k = 5,6,7, 8,9,10$ and 12. Cumulative probabilities were computed using the algorithm of Sheil and O'Muircheartaigh (1977). Twenty distributions from the $Q(\underset{\sim}{\lambda},k)$ family were used, with parameters as given in Table 5.

2.5 z Distributions (denoted by 'E' on Figure 1)

Let $z = \frac{1}{2} \ell n\, F$, where F has the (central) F distribution with v_1, v_2 degrees of freedom. Although z, used by Fisher (1924) in place of F for purposes of approximation and tabulation, is a random variable, we shall adhere to the convention of denoting it by a small letter. This distribution will be noted here by $z(v_1,v_2)$. Cumulants of all orders are finite and are given by Johnson and Kotz (1970b, p.78) as follows:

$$\kappa_1 = \frac{1}{2} [\ell n(v_2/v_1) + \psi(v_1/2) - \psi(v_2/2)];$$

$$\kappa_r = 2^{-r} [\psi^{(r-1)}(v_1/2) + (-1)^r \psi^{(r-1)}(v_2/2)], \ (r \geq 2).$$

Computation of $\psi(x)$ and $\psi^{(s)}(x)$, the digamma function and its derivatives, is discussed in section 3.1.

Table 3: t(v,δ) Distributions

v	δ	$\sqrt{\beta_1}$	β_2
5	0.1	0.14	9.05
5	0.2	0.28	9.20
5	0.3	0.41	9.45
5	0.4	0.54	9.79
5	0.5	0.68	10.22
5	0.6	0.80	10.72
5	0.7	0.93	11.29
5	0.8	1.04	11.93
5	0.9	1.16	12.60
5	1.0	1.27	13.32
6	0.1	0.09	6.02
6	0.2	0.19	6.07
6	0.3	0.28	6.16
6	0.4	0.37	6.28
6	0.5	0.46	6.43
6	0.6	0.55	6.61
6	0.7	0.64	6.81
6	0.8	0.72	7.04
6	0.9	0.80	7.29
6	1.0	0.88	7.55
6	1.1	0.95	7.83
6	1.2	1.02	8.12
6	1.3	1.09	8.42
6	1.4	1.16	8.72
6	1.5	1.22	9.02
6	1.6	1.28	9.33
6	1.7	1.33	9.63
6	1.8	1.39	9.94
6	1.9	1.44	10.23
6	2.0	1.49	10.52
6	2.1	1.53	10.81
6	2.2	1.57	11.09
6	2.3	1.61	11.36
6	2.4	1.65	11.62
6	2.5	1.68	11.87
6	2.6	1.72	12.11
6	2.7	1.75	12.34
6	2.8	1.78	12.57
6	2.9	1.81	12.78
6	3.0	1.83	12.99
6	3.1	1.86	13.19
6	3.2	1.88	13.38
6	3.3	1.90	13.56
6	3.4	1.92	13.73
7	0.2	0.14	5.04
7	0.3	0.21	5.08
7	0.4	0.28	5.14
7	0.6	0.42	5.31
7	0.7	0.49	5.42
7	0.8	0.55	5.54
7	0.9	0.61	5.67
7	1.0	0.67	5.81
7	1.1	0.73	5.96
7	1.2	0.79	6.12
7	1.5	0.95	6.62
7	1.6	1.00	6.79
7	1.7	1.04	6.96
7	1.8	1.09	7.14
7	1.9	1.13	7.31
7	2.0	1.17	7.48
7	2.2	1.24	7.81
7	2.3	1.28	7.97
7	2.4	1.31	8.13
7	2.6	1.37	8.43
7	2.7	1.40	8.58
7	2.8	1.42	8.72
7	3.0	1.47	8.99
7	3.1	1.49	9.11
7	3.3	1.53	9.35
7	3.5	1.57	9.57
7	3.7	1.60	9.77
7	3.9	1.63	9.96
7	4.2	1.67	10.22
7	4.4	1.69	10.37
7	4.6	1.71	10.51
7	7.5	1.87	11.66
7	10.0	1.92	12.05
7	15.0	1.96	12.35
8	0.2	0.11	4.52
8	0.3	0.17	4.55
8	0.5	0.28	4.63
8	0.7	0.39	4.76
8	0.9	0.50	4.91
8	1.1	0.60	5.10
8	1.3	0.69	5.30
8	1.6	0.82	5.63
8	1.7	0.86	5.75
8	1.9	0.93	5.97
8	2.1	1.00	6.20
8	2.2	1.03	6.32
8	2.3	1.06	6.43
8	2.6	1.14	6.75
8	2.7	1.17	6.85
8	3.0	1.24	7.14
8	3.1	1.26	7.23
8	3.3	1.30	7.40
8	3.5	1.33	7.56
8	3.8	1.38	7.79
8	4.3	1.44	8.11
8	4.5	1.46	8.22
8	4.9	1.49	8.42
8	7.5	1.62	9.18
8	10.0	1.67	9.50
8	15.0	1.71	9.74
9	0.2	0.10	4.21
9	0.4	0.19	4.26
9	0.6	0.28	4.33
9	0.8	0.38	4.43
9	1.1	0.50	4.61
9	1.3	0.58	4.75
9	1.6	0.70	4.98
9	1.9	0.80	5.23
9	2.3	0.91	5.56
9	2.6	0.99	5.80
9	3.1	1.09	6.17
9	3.8	1.20	6.61
9	4.6	1.29	7.00
9	10.0	1.50	8.03
10	0.2	0.08	4.01
10	0.4	0.17	4.04
10	0.7	0.29	4.13
10	1.3	0.51	4.40
10	1.9	0.70	4.76
10	2.3	0.80	5.02
10	2.6	0.87	5.21
10	3.3	1.00	5.61
12	0.3	0.10	3.76
12	1.3	0.40	3.99
12	1.7	0.51	4.14
12	3.2	0.81	4.75
12	5.0	1.00	5.31
14	3.5	0.72	4.39
16	1.0	0.22	3.57
16	4.0	0.68	4.20
16	7.5	0.88	4.69
20	2.5	0.39	3.60

Percentage points and cumulative probabilities of the F distribution can be used, since $\Pr(z(v_1,v_2) < z) = \Pr(F < \exp(2z))$, where F has the F distribution with v_1,v_2 degrees of freedom. Thirty-one $z(v_1,v_2)$ distributions were used, with parameters as given in Table 4.

Table 4: $z(v_1,v_2)$ distributions

v_1	v_2	$\sqrt{\beta_1}$	β_2	v_1	v_2	$\sqrt{\beta_1}$	β_2	v_1	v_2	$\sqrt{\beta_1}$	β_2
1	2	0.85	5.40	3	4	0.21	3.76	5	10	0.31	3.48
1	3	1.13	5.87	3	5	0.35	3.80	5	20	0.49	3.64
1	4	1.25	6.14	3	10	0.63	4.07	6	6	0.00	3.38
1	5	1.31	6.32	3	30	0.82	4.40	10	10	0.00	3.22
1	10	1.43	6.66	3	100	0.89	4.55	10	20	0.20	3.22
1	30	1.50	6.89	4	4	0.00	3.59	10	40	0.32	3.29
2	3	0.38	4.19	4	10	0.44	3.69	10	100	0.44	3.37
2	5	0.69	4.47	4	20	0.61	3.88	20	20	0.00	3.10
2	10	0.92	4.87	4	40	0.69	4.02	20	40	0.14	3.11
2	20	1.03	5.12	4	100	0.74	4.12	30	40	0.05	3.06
3	3	0.00	3.81								

Table 5: Q(λ,k) Distributions

k	λ												$\sqrt{\beta_1}$	β_2
3	0.4	0.3	0.3										1.68	7.34
4	1.8	1.5	0.4	0.3									1.91	8.68
4	1.5	1.5	0.8	0.2									1.74	7.71
4	1.5	1.0	0.8	0.7									1.61	7.20
4	1.2	1.2	1.2	0.4									1.57	6.73
4	1.2	1.2	0.8	0.8									1.49	6.44
4	1.2	1.0	0.9	0.9									1.45	6.19
5	1.8	1.8	0.6	0.5	0.3								1.77	7.93
5	1.2	1.2	0.9	0.9	0.8								1.32	5.67
5	1.2	1.0	1.0	0.9	0.9								1.29	5.52
5	0.4	0.2	0.2	0.1	0.1								1.75	8.15
6	0.3	0.3	0.2	0.1	0.05	0.05							1.57	6.89
6	0.4	0.2	0.1	0.1	0.1	0.1							1.83	8.75
7	0.3	0.2	0.1	0.1	0.1	0.1	0.1						1.48	6.78
8	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1					1.19	5.33
8	0.3	0.3	0.1	0.1	0.05	0.05	0.05	0.05					1.66	7.47
8	0.4	0.2	0.1	0.1	0.05	0.05	0.05	0.05					1.91	9.22
9	0.3	0.1	0.1	0.1	0.1	0.1	0.1	0.05	0.05				1.54	7.35
10	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.05	0.05			1.11	5.10
12	0.2	0.1	0.1	0.1	0.1	0.1	0.05	0.05	0.05	0.05	0.05	0.05	1.14	5.33

2.6 Weibull Distributions (denoted by 'F' on Figure 1)

If the random variable X has pdf:

$$f(x) = c\alpha^{-1}[(x-\theta)/\alpha]^{c-1} \exp[-((x-\theta)/\alpha)^c] \quad \theta<x<\infty, \quad c,\alpha>0,$$

X is said to have the Weibull distribution with parameters c, α and θ (see e.g. Johnson and Kotz (1970a, p. 250)). We here consider the standard form, denoted by Weib (c), of the Weibull distribution, obtained by setting $\alpha=1$, $\theta=0$, having pdf:

$$f(x) = cx^{c-1} \exp(-x^c) \quad 0<x<\infty,$$

and cdf:

$$F(x) = 1 - \exp[-x^c] \quad 0<x<\infty.$$

The Weib (c) distribution has moments about the origin given by:

$$\mu_r' = \Gamma(\frac{r}{c} + 1),$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Computer evaluation of the gamma function is discussed in Section 3.1. The parameter values for the 30 Weib (c) distributions used are displayed in Table 6.

Table 6: Weib (c) Distributions

c	$\sqrt{\beta_1}$	β_2	c	$\sqrt{\beta_1}$	β_2	c	$\sqrt{\beta_1}$	β_2
1.04	1.89	8.26	1.60	0.96	4.04	5.50	0.32	2.96
1.08	1.78	7.64	1.68	0.88	3.82	6.00	0.37	3.04
1.10	1.73	7.36	1.76	0.81	3.64	7.00	0.46	3.19
1.12	1.69	7.10	1.90	0.70	3.38	8.50	0.56	3.39
1.16	1.60	6.64	2.10	0.57	3.13	10.00	0.64	3.57
1.20	1.52	6.24	2.20	0.51	3.04	15.00	0.79	4.00
1.24	1.45	5.88	2.50	0.36	2.86	20.00	0.87	4.27
1.30	1.35	5.43	3.00	0.17	2.73	30.00	0.95	4.58
1.40	1.20	4.84	3.50	0.03	2.71	50.00	1.02	4.88
1.50	1.07	4.39	4.00	0.09	2.75	75.00	1.06	5.04

2.7 Generalized Logistic Distributions (denoted by 'G' on Figure 1)

If the random variable X has the pdf:

$$f(x) = p \exp(x) \, [1+\exp(x)]^{-(p+1)} \qquad -\infty<x<\infty, \quad p>1,$$

X has the generalized logistic distribution with parameter p, denoted by GL(p). This distribution was studied by Dubey (1969) and has cdf:

$$F(x) = 1 - (1+\exp(x))^{-p} \qquad -\infty<x<\infty.$$

Note that the logistic distribution is obtained by setting p equal to one. The distribution has finite cumulants of all orders (see Johnson and Kotz (1970b, p. 18)) given by:

$$\kappa_1 = \psi(1) - \psi(p);$$

$$\kappa_r = \psi^{(r-1)}(1) + (-1)^r \psi^{(r-1)}(p), \qquad (r\geq 2).$$

Six GL(p) distributions were used, with parameters as given in Table 7.

Table 7: GL(p) Distributions

p	$\sqrt{\beta_1}$	β_2
1	0.00	4.20
2	0.58	4.33
3	0.77	4.59
4	0.87	4.76
7	0.99	5.01
15	1.07	5.21

2.8 Chi Distributions (denoted by 'H' on Figure 1)

The random variable X is said to have the chi distribution with v degrees of freedom, denoted by $\chi(v)$, if X^2 is (central) chi-squared with v degrees of freedom. Johnson and Welch (1939) give formulae for the first six cumulants of $\chi(v)$, as follow:

$$\kappa_1 = \sqrt{2}\,\Gamma((v+1)/2)/\Gamma(v/2);$$

$$\kappa_2 = v - \kappa_1^2;$$

$$\kappa_3 = \kappa_1\,\alpha;$$

$$\kappa_4 = \frac{1}{2} - (2v-1)\alpha - 1.5\,\alpha^2;$$

$$\kappa_5 = \kappa_1(-2\kappa_4 + 3\alpha^2);$$

$$\kappa_6 = 2(2v-1)\kappa_4 + 3\alpha - 12(2v-1)\alpha^2 - 15\alpha^3;$$

where $\alpha = 1 - 2\kappa_2$. $\chi(v)$ has the pdf:

$$f(x) = [2^{v/2-1}\Gamma(v/2)]^{-1}\, x^{v-1} \exp(-\tfrac{1}{2}x^2) \qquad 0<x<\infty,$$

and cumulative probabilities can be obtained from cumulative chi-square probabilities using $\Pr(\chi(v)<x) = \Pr(\sqrt{Y} < x) = \Pr(Y < x^2)$, where Y is chi-squared with v degrees of freedom. The parameters of the five $\chi(v)$ distributions used are shown in Table 8.

Table 8: $\chi(v)$ distributions

v	$\sqrt{\beta_1}$	β_2
1	1.00	3.87
2	0.63	3.25
3	0.49	3.11
14	0.20	3.00
30	0.13	3.00

2.9 EDF Statistics for Goodness-of-Fit (denoted by 'I' on Figure 1)

We here consider the asymptotic distributions of the Cramér-von Mises statistic W^2, The Watson statistic U^2 and the Anderson-Darling statistic A^2. These statistics measure the discrepancy between the empirical distribution function $F_n(x)$ and the hypothesized cdf $F(x)$, where $F(x)$ may contain unspecified parameters. Four cases are considered here:

Case 1. $F(x)$ is the normal distribution with σ^2 known, μ estimated by the sample mean $\bar{x}$.

Case 2. $F(x)$ is the normal distribution with μ known, σ^2 estimated by $\frac{1}{n}\Sigma(x_i-\mu)^2$.

Case 3. $F(x)$ is the normal distribution with μ and σ^2 estimated by $\bar{x}$ and $s^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2$.

Case 4. $F(x) = 1 - \exp(-\theta x)$, $x \geq 0$ (the exponential distribution) with θ estimated by $1/\bar{x}$.

For all these cases, the asymptotic distributions can be expressed as an infinite sum of weighted chi-squared variables each with one degree of freedom.

Exact means for all four cases of W^2, U^2 and A^2 and exact variances for W^2 and U^2 are given by Stephens (1976). Weights were obtained from Stephens (1976) and the method of Imhof (1961) was used for computing cumulative probabilities. The percentage points obtained were compared with those given by Durbin, Knott and Taylor (1975), who independently used the same method. Higher cumulants were calculated from the weights, as in section 2.4. Five distributions were used: W^2, case 1; U^2, cases 2 and 4; A^2, cases 1 and 3. The parameters and significance points for these distributions are given in Table 9.

Table 9: Goodness-of-Fit Statistics

Statistic	$\sqrt{\beta_1}$	β_2	α 0.85	0.90	0.95	0.975	0.99	0.995	0.9975
W^2, case 1	1.85	8.53	0.1165	0.1344	0.1653	0.1965	0.2381	0.2698	0.3017
U^2, case 2	1.99	9.58	0.1052	0.1218	0.1507	0.1804	0.2208	0.2519	0.2836
U^2, case 4	1.90	8.90	0.1116	0.1289	0.1588	0.1892	0.2300	0.2613	0.2930
A^2, case 1	1.85	8.72	0.7819	0.8937	1.0874	1.2847	1.5510	1.7561	1.9640
A^2, case 3	1.64	7.52	0.5610	0.6318	0.7528	0.8742	1.0359	1.1592	1.2833

2.10 Thickened Range from a Uniform Distribution (denoted by 'J' on Figure 1)

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered observations from a sample of size n from the uniform distribution with pdf $f(x) = 1$, $0<x<1$, and let the "thickened" range W_n be defined by (David (1970, p. 146):

$$W_n = (X_{(n)}-X_{(1)}) + (X_{(n-1)}-X_{(2)}) + \ldots + (X_{(n-p+1)}-X_{(p)}),$$

where $p = [\frac{n}{2}]$, the greatest integer $\le \frac{n}{2}$. The cdf of W_n is given by Stephens (1972)):

$$F(w_n) = 1 - \sum_{i=1}^{p} i^{-n}[(K_i+L_i)<i-w_n>^n+nL_i w_n<i-w_n>^{n-1}], \quad 0<w_n\le p,$$

where $<x> = x$ for $x > 0$, $<x> = 0$ for $x \le 0$. If $n = 2p+1$,

$$L_i = \prod_{\substack{r=1\\ r\neq i}}^{p} \left(\frac{i}{i-r}\right)^2, \quad i = 1,2,\ldots,p;$$

$$K_i = -2L_i \sum_{\substack{r=1\\ r\neq i}}^{p} \left(\frac{r}{i-r}\right), \quad i = 1,2,\ldots,p,$$

while if $n = 2p$, we have:

$$L_i = \frac{i-p}{i} \prod_{\substack{r=1\\ r\neq i}}^{p} \left(\frac{i}{i-r}\right)^2, \quad i = 1,2,\ldots,(p-1);$$

$$L_p = 0;$$

$$K_i = -L_i\left[\sum_{\substack{r=1\\ r\neq i}}^{p} \left(\frac{2r}{i-r}\right) - \frac{p}{i-p}\right], \quad i = 1,2,\ldots,(p-1);$$

$$K_p = \prod_{r=1}^{p-1} \left(\frac{p}{p-r}\right)^2.$$

Formulae for the first four central moments of W_n are similarly given in Stephens (1972). Six W_n distributions were used in this study, with parameters given in Table 10.

Table 10: W_n distributions

n	$\sqrt{\beta_1}$	β_2	n	$\sqrt{\beta_1}$	β_2
4	0.24	2.79	9	0.00	2.83
5	0.00	2.63	10	0.04	2.90
6	0.12	2.86	19	0.00	2.93

2.11 Resultants of Random Unit Vectors in Three Dimensions (denoted by 'K' on Figure 1)

Suppose n unit vectors are uniformly distributed in three dimensions; that is, if Op_i is a typical vector, the origin is fixed at O and p_i moves uniformly on the surface of the sphere with center O and radius 1. Let X be the vector sum of the n vectors. The cdf of X is given in Stephens (1964). Cumulants of $Z = X^2$ are given by Solomon and Stephens (1975), as follows:

$$\kappa_1 = n;$$

$$\kappa_2 = \frac{2}{3}(n^2-n);$$

$$\kappa_3 = \frac{8}{9}\,n(n-1)(n-2);$$

$$\kappa_4 = \frac{16}{45}(5n^4-30n^3+52n^2-27n).$$

The random variable X has range (0,n) and approximate significance points x_α of X are related to those for Z by $x_\alpha = \sqrt{z_\alpha}$. This follows from the relationship $\alpha = \Pr(X<x_\alpha) = \Pr(\sqrt{Z}<x_\alpha) = \Pr(Z<x_\alpha^2)$. Distributions of vector resultants, denoted by RUV(n), are considered for 9 values of n, with parameters given in Table 11.

Table 11: RUV(n) distributions

n	$\sqrt{\beta_1}$	β_2	n	$\sqrt{\beta_1}$	β_2
4	0.94	3.47	11	1.40	5.60
5	1.10	4.08	13	1.44	5.81
6	1.19	4.52	16	1.48	6.02
7	1.26	4.85	23	1.52	6.32
8	1.31	5.10			

2.12 Extreme Value Distribution (denoted by 'L' on Figure 1)

The random variable X has the Type I extreme value distribution if X has pdf:

$$f(x) = \exp(-x - \exp(-x)), \quad -\infty<x<\infty. \quad \text{(Johnson and Kotz 1970a, p.272).}$$

The cdf is then:

$$F(x) = \exp(- \exp(-x)), \quad -\infty<x<\infty,$$

and the cumulants are given by:

$$\kappa_1 = -\psi(1);$$
$$\kappa_r = (-1)^r \psi^{(r-1)}(1), \quad r\geq 2. \quad \text{(Johnson and Kotz (1970a, p.278).}$$

Here we are considering only one distribution, with $\sqrt{\beta_1} = 1.14$ and $\beta_2 = 5.40$.

2.13 Compound Laplace Distributions (denoted by 'M' on Figure 1)

The random variable X with pdf:

$$f(x) = \frac{\alpha}{2} (1 + |x|)^{-(\alpha+1)} \quad -\infty<x<\infty, \ \alpha>0,$$

has the (standardized) compound Laplace distribution (Johnson and Kotz (1970a, p.32)). This distribution will be denoted here by L(α). The L(α) distributions are symmetric about zero and moments of order α or greater do not exist. For r even and less than α,

$$\mu_r = \alpha \sum_{j=0}^{r} (-1)^j \binom{r}{j} (\alpha+j-r)^{-1} \quad \text{(Johnson and Kotz (1970b, p.32)).}$$

The cdf, as given by Johnson and Kotz, was found to be incorrect.

The correct cdf is:

$$\text{for } x \le 0, \quad F(x) = \int_{-\infty}^{x} \frac{\alpha}{2}(1-t)^{-(\alpha+1)}\,dt,$$

$$= \frac{1}{2}(1-x)^{-\alpha};$$

$$\text{for } x > 0, \quad F(x) = \frac{1}{2} + \int_{0}^{x} \frac{\alpha}{2}(1+t)^{-(\alpha+1)}\,dt,$$

$$= 1 - \frac{1}{2}(1+x)^{-\alpha}.$$

The parameters of the 26 L(α) distributions used are displayed in Table 12.

Table 12: L(α) Distributions

α	$\sqrt{\beta_1}$	β_2	α	$\sqrt{\beta_1}$	β_2
7.4	0.00	13.86	11.4	0.00	9.44
7.6	0.00	13.39	12.4	0.00	9.01
7.8	0.00	12.97	13.0	0.00	8.80
8.0	0.00	12.60	13.6	0.00	8.62
8.4	0.00	11.96	14.4	0.00	8.41
8.6	0.00	11.68	15.4	0.00	8.19
9.0	0.00	11.20	17.8	0.00	7.80
9.4	0.00	10.79	19.5	0.00	7.60
9.8	0.00	10.44	21.6	0.00	7.40
10.1	0.00	10.21	24.4	0.00	7.20
10.4	0.00	10.00	39.0	0.00	6.70
10.7	0.00	9.81	52.0	0.00	6.51
11.0	0.00	9.64	100.0	0.00	6.25

2.14 Subbotin Distributions (denoted by 'N' on Figure 1)

The random variable X has a Subbotin distribution with parameter δ, denoted by $S(\delta)$, if X has pdf:

$$f(x) = [2^{\delta/2+1}\Gamma(\tfrac{\delta}{2}+1)]^{-1} \exp(-\tfrac{1}{2}|x|^{2/\delta}), \quad -\infty<x<\infty, \quad \delta>0.$$

This is the standardized form of the pdf given by Johnson and Kotz (1970b, p.33). This distribution is symmetric about zero and has finite moments of all positive orders, with

$$\mu_r = 2^{\frac{1}{2}r\delta}\Gamma(\tfrac{(r+1)\delta}{2})/\Gamma(\tfrac{\delta}{2}) \quad (r \text{ even}).$$

In order to make percentage point comparisons, the cdf of $S(\delta)$ was derived, as follows:

$$\text{for } x>0, \quad F(x) = k\int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^{2/\delta})dt, \quad k = [2^{\frac{\delta}{2}+1}\,\Gamma(\tfrac{\delta}{2}+1)]^{-1},$$

$$= \frac{1}{2} + k\int_0^x \exp(-\tfrac{1}{2}\,t^{2/\delta})dt,$$

$$= \frac{1}{2} + 2^{\frac{\delta}{2}-1}\,\delta k \int_0^{\frac{1}{2}x^{2/\delta}} y^{\frac{\delta}{2}-1}\, e^{-y}\, dy,$$

$$= \frac{1}{2} + \frac{1}{2}\,P(\tfrac{\delta}{2}, \tfrac{1}{2}\,x^{2/\delta}),$$

where $P(a,x) = [\Gamma(a)]^{-1}\int_0^x t^{a-1}\, e^{-t}\, dt$ is the incomplete gamma function ratio (see e.g. Abramowitz and Stegun (1965, p.260)).

For $x<0$, $F(x) = 1 - F(-x)$.

Thirteen $S(\delta)$ distributions were used, with parameters as shown in Table 13.

Table 13: $S(\delta)$ distributions

δ	$\sqrt{\beta_1}$	β_2	δ	$\sqrt{\beta_1}$	β_2
0.1	0.0	1.82	1.8	0.0	5.21
0.3	0.0	1.97	1.9	0.0	5.59
0.6	0.0	2.32	2.2	0.0	6.92
1.3	0.0	3.68	2.4	0.0	7.97
1.42	0.0	3.99	2.6	0.0	9.20
1.6	0.0	4.53	2.8	0.0	10.62
1.7	0.0	4.86			

2.15 Hyperbolic Secant Distribution (denoted by 'O' on Figure 1

The random variable X has the hyperbolic secant distribution (Johnson and Kotz (1970b, pp.15-16)) if X has the pdf:

$$f(x) = \pi^{-1} \operatorname{sech}(x), \quad -\infty<x<\infty.$$

This distribution has cdf:

$$F(x) = \frac{1}{2} + \pi^{-1} \tan^{-1}(\sinh(x)) \quad -\infty<x<\infty.$$

(Note that the formula given for F(x) in Johnson and Kotz (1970b, p.15) is in error). The distribution is symmetric about zero and central moments of even order are given by:

$$\mu_r = \frac{4}{\pi} \Gamma(r+1)\, \beta(r+1),$$

where the function $\beta(n) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-n}$ is tabulated in Abramowitz and Stegun (1965, p.812) to 18 decimal places for $n = 1(1)38$. For this distribution, $\sqrt{\beta_1}=0$, $\beta_2=5$.

2.16 Laplace Distribution (denoted by 'P' on Figure 1)

The random variable X has the standard form of the Laplace distribution (also known as the double exponential distribution) if:

$$f(x) = \frac{1}{2} \exp(-|x|); \quad -\infty<x<\infty.$$

The cdf of this distribution is:

$$F(x) = \frac{1}{2}\exp(x) \qquad x \le 0,$$

$$1 - \frac{1}{2}\exp(-x) \qquad x > 0.$$

The distribution is symmetric about zero and central moments of even order are given by:

$$\mu_r = r! \qquad \text{(Johnson and Kotz (1970b, p.23))}.$$

Thus $\sqrt{\beta_1} = 0$ and $\beta_2 = 6$.

2.17 Cosine Distribution (denoted by 'Q' on Figure 1)

The random variable X has the cosine distribution if X has the pdf:

$$f(x) = (2\pi)^{-1}(1 + \cos x), \ -\pi < x < \pi.$$

The distribution was studied by Raab and Green (1961) who suggest it as a possible substitute for the normal distribution. The cdf is given by:

$$F(x) = \frac{1}{2} + (2\pi)^{-1}(x + \sin x), \ -\pi < x < \pi.$$

This distribution is symmetric about zero, odd-order moments are zero and, for r even:

$$\mu_r' = E(X^r) = (2\pi)^{-1}\int_{-\pi}^{\pi}(x^r + x^r\cos x)\,dx,$$

$$= \frac{\pi^r}{r+1} + (2\pi)^{-1}\int_{-\pi}^{\pi} x^r \cos x\,dx.$$

Since $\int x^n \cos x\,dx = x^n \sin x + n\,x^{n-1}\cos x - n(n-1)\int x^{n-2}\cdot \cos x\,dx$, and using $\int_{-\pi}^{\pi} x^2\cos x\,dx = -4\pi$, moments can be easily computed for $r = 2, 4, 6, \ldots$, successively. For this distribution, $\sqrt{\beta_1} = 0$ and $\beta_2 = 2.41$.

2.18 Generalized Gamma Distributions (denoted by 'R' on Figure 1)

The random variable X has the generalized gamma distribution (standard form) if X has pdf:

$$f(x) = |p|[\Gamma(v)]^{-1} x^{pv-1} \exp(-x^p), \quad 0 < x < \infty, \ p \neq 0, \ v > 0.$$

This distribution was studied by Stacy and Mihram (1965) and will be denoted here by GG(p,v). The Weibull and chi distributions used in this study, as well as exponential, gamma and chi-squared distributions, are special cases of this distribution. Moments are obtainable from:

$$\mu_r' = E(X^r) = \Gamma[(pv + r)/p]/\Gamma(v) \qquad \frac{r}{p} > -v.$$

Cumulative probabilities can be computed from the cdf:

$$F(x) = P(v,x^p) \qquad p > 0,$$
$$1 - P(v,x^p) \qquad p < 0,$$

where P(a,x) is the incomplete gamma function ratio, defined in section 2.14. Thirty-six GG(p,v) distributions were used, with parameter values given in Table 14.

Table 14: GG(p,v) distributions

p	v	$\sqrt{\beta_1}$	β_2	p	v	$\sqrt{\beta_1}$	β_2
-7.0	1.5	1.64	8.90	3.0	0.4	0.52	2.74
-7.0	2.5	1.10	5.53	3.0	0.5	0.41	2.68
-6.0	1.5	1.80	10.37	3.0	1.5	0.09	2.81
-6.0	2.0	1.40	7.25	4.0	0.1	1.20	3.70
-6.0	3.0	1.04	5.24	4.0	0.2	0.60	2.57
-5.0	2.0	1.30	6.62	4.0	0.4	0.18	2.42
-5.0	3.0	1.13	5.70	4.0	0.5	0.08	2.48
-4.0	2.0	1.83	10.92	4.0	0.6	0.02	2.55
-4.0	2.5	1.49	7.94	5.0	0.1	0.88	2.87
-3.0	2.5	1.88	11.43	5.0	0.2	0.33	2.25
-3.0	4.0	1.24	6.31	6.0	0.2	0.13	2.14
-2.0	3.5	1.99	12.60	6.0	0.3	0.10	2.31
-2.0	4.0	1.74	9.96	6.0	0.4	0.23	2.50
-2.0	5.0	1.43	7.45	6.0	0.5	0.29	2.67
-2.0	5.5	1.33	6.77	6.0	5.5	0.28	3.07
3.0	0.1	1.68	5.55	7.0	0.1	0.48	2.18
3.0	0.2	1.00	3.46	7.0	0.2	0.03	2.14
3.0	0.3	0.70	2.92	7.0	0.5	0.42	2.83

2.19 Burr Distributions (denoted by 'S' on Figure 1)

As an alternative to fitting a theoretical pdf to data and integrating to obtain cumulative probabilities, Burr (1942) suggests fitting a theoretical cdf. Among several possibilities, he considers the cdf:

$$F(x) = 1 - (1 + x^c)^{-k}, \; 0 < x < \infty, \; c,k \geq 1.$$

This distribution, denoted here by Burr (c,k), has pdf:

$$f(x) = ck\, x^{c-1}(1 + x^c)^{-(k+1)} \qquad 0 < x < \infty.$$

Burr (1942) considers cumulative moments, M_j, defined by:

$$M_j = \int_0^\infty x^j(1 - F(x))dx - \int_{-\infty}^0 x^j F(x)dx.$$

For the Burr (c,k) distribution, $M_j = \Gamma(\frac{j+1}{c})\Gamma(k - \frac{j+1}{c})/[c\Gamma(k)]$, $j < ck - 1$. The first four central moments can then be obtained using the relations (Burr (1942, p.224)):

$$\mu = M_0;$$

$$\mu_2 = 2M_1 - M_0^2;$$

$$\mu_3 = 3M_2 - 6M_1M_0 + 2M_0^3;$$

$$\mu_4 = 4M_3 - 12M_2M_0 + 12M_1M_0^2 - 3M_0^4.$$

Twenty Burr (c,k) distributions are used here, with parameters displayed in Table 15.

Table 15: Burr (c,k) distributions

c	k	$\sqrt{\beta_1}$	β_2	c	k	$\sqrt{\beta_1}$	β_2
2	3	1.91	12.46	6	4	0.02	3.17
3	2	1.59	10.81	7	1	1.46	10.36
3	3	0.92	5.13	7	8	0.30	3.14
3	6	0.48	3.38	8	1	1.22	8.34
4	2	0.96	5.94	8	2	0.19	3.74
4	3	0.51	3.87	9	1	1.06	7.22
5	2	0.64	4.63	9	2	0.11	3.67
5	4	0.12	3.19	10	1	0.94	6.51
6	2	0.43	4.11	10	2	0.04	3.65
6	3	0.12	3.36	10	3	0.21	3.42

2.20 Log-gamma Distributions (denoted by 'T' on Figure 1)

Let Y have a gamma distribution with pdf:

$$f(y) = [\Gamma(\alpha)]^{-1} y^{\alpha-1} e^{-y} \quad 0 < y < \infty, \ \alpha > 0,$$

and let $X = \ell n\, Y$. It can easily be shown that X has pdf:

$$f(x) = [\Gamma(\alpha)]^{-1} \exp(\alpha x) \exp(-\exp(x)), \quad -\infty < x < \infty.$$

This distribution was considered by Olshen (1938). To evaluate the moments of X, with distribution denoted by LG(α), we first obtain the moment-generating function:

$$\begin{aligned} M_X(t) &= E[\exp(tX)], \\ &= [\Gamma(\alpha)]^{-1} \int_{-\infty}^{\infty} \exp[x(\alpha + t)] \exp(-\exp(x)dx, \\ &= [\Gamma(\alpha)]^{-1} \int_{0}^{\infty} z^{\alpha+t-1} \exp(-z)dz, \text{ where } z = \exp(x), \\ &= \Gamma(\alpha + t)/\Gamma(\alpha). \end{aligned}$$

The cumulant-generating function is then:

$$\begin{aligned} K_X(t) &= \ell n[M_X(t)], \\ &= \ell n\Gamma(\alpha + t) - \ell n\Gamma(\alpha). \end{aligned}$$

Thus, the LG(α) distribution has cumulants given by:

$$\kappa_1 = \psi(\alpha);$$
$$\kappa_r = \psi^{(-1)}(\alpha); \quad r \geq 2.$$

Cumulative probabilities can be evaluated using the relationship $Pr(X < x) = Pr(\ell n(Y) < x) = Pr(Y < \exp(x))$, where Y has the above gamma distribution. Five LG(α) distributions were used in this study, with parameters as displayed in Table 16.

Table 16: LG(α) Distributions

α	$\sqrt{\beta_1}$	β_2
2.5	0.69	3.93
3.0	0.62	3.76
4.5	0.50	3.49
7.0	0.39	3.31
10.5	0.32	3.20

3. COMPUTATIONAL METHODS

3.1 Computation of Moments and Cumulative Distribution Functions

In addition to the computational algorithms described in Section 2, evaluation of several auxiliary mathematical functions was required for the accurate computation of moments and cumulative distribution functions. The function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ was evaluated using the algorithm of Pike and Hill (1966). This function is necessary in computing the moments of the noncentral t, Weibull, chi, Subbotin, hyperbolic secant, generalized gamma and Burr distributions.

For computing the moments of the z, generalized logistic, extreme value and log-gamma distributions, evaluation of the derivatives of $\ln[\Gamma(x)]$ was required. The digamma (psi) function is defined by $\psi(x) = \frac{d}{dx} \{\ln[\Gamma(x)]\} = \Gamma'(x)/\Gamma(x)$. Similarly, $\psi^{(s)}(x) = \frac{d^s}{dx^s} \{\psi(x)\}$ is called the trigamma, tetragamma, pentagamma, hexagamma, ... function for $s = 1,2,3,4,\ldots$ For the distributions described here, $\psi(x)$ is required for integer and half-integer values only; thus the following formulae (Abramowitz and Stegun (1965, p.258)) are sufficient:

$$\psi(1) = -\gamma;$$

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}, \quad n=2,3,4,\ldots ;$$

$$\psi(\tfrac{1}{2}) = -\gamma - 2 \ln 2 ;$$

$$\psi(n+\tfrac{1}{2}) = -\gamma - 2 \ln 2 + 2(1 + \tfrac{1}{3} + \ldots + \tfrac{1}{2n-1}), \quad n=1,2,3,\ldots$$

Here $\gamma = 0.5772156649\ldots$ is Euler's constant, given to 25 decimal places in Abramowitz and Stegun (1965, p.3).

Arbitrary derivatives of $\psi(x)$ required here for integral values of x only, were computed using the following formula from Abramowitz and Stegun (1965, p.260):

$$\psi^{(m)}(1) = (-1)^{m+1} m!\ \zeta(m+1), \quad m=1,2,3,\ldots ;$$

$$\psi^{(m)}(n+1) = (-1)^{m} m!\ [-\zeta(m+1) + 1 + \frac{1}{2^{m+1}} + \ldots + \frac{1}{n^{m+1}}], \quad n=1,2,3,\ldots, \quad m=1,2,3,\ldots .$$

The Riemann zeta function $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is tabulated for $n=2(1)42$ in Abramowitz and Stegun (1965, p.811).

The incomplete gamma function ratio, $P(a,x) = [\Gamma(a)]^{-1} \int_0^x t^{a-1} e^{-t}\, dt$, required for the computation of the Subbotin and generalized gamma cumulative distribution functions, was evaluated using the algorithm of Bhattacharjee (1970). Chi-squared and gamma probabilities, required for computation of chi and log-gamma cumulative probabilities, were also computed using this algorithm. The algorithm of Majumder and Bhattacharjee (1973) for computing $I_x(a,b)$, the incomplete beta function ratio, was used for computing cumulative probabilities of the z distribution.

3.2 Interpolation for Cumulative Probabilities

Although tabulated significance points are available for the noncentral F distributions and the goodness-of-fit statistics, direct computation of cumulative probabilities is difficult. Thus inverse interpolation for cumulative probabilities is required. Given m values α_i, in ascending order, and the corresponding significance points x_i, $\hat{\alpha}$, the approximate value of α corresponding to an intermediate value x, can be computed using the n-point Lagrangian interpolation formula for unequally spaced abscissa values:

$$\hat{\alpha} = \sum_{i=k}^{k+n-1} \ell_i(x)\cdot\alpha_i ,$$

where $\ell_i(x) = \dfrac{(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_{n-1})(x-x_n)}{(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_{n-1})(x_i-x_n)}$.

Here k is chosen to determine which known values will be used in the interpolation; e.g., if $n=4$, k might be chosen so that two values of x_i lie on either side of x.

Since interpolation using the above formula was found to be occasionally inaccurate, regardless of the values chosen for n and k, the work of Pearson (1968) (see also Pearson and Hartley (1972, pp.139-141) was consulted. For the inverse problem of finding x for a given α, where the x_i values are known at standard significance levels x_i, Pearson suggested use of the logit transformation $\gamma_i = \ell n(\alpha_i/(1-\alpha_i))$. He concluded that the logit transformation, in conjunction with the Lagrangian interpolation formula, led to quite accurate results.

This method was applied here to the problem of determining α for a given x. The formula $\gamma = \sum_{i=k}^{k+n-1} \ell_i(x)\ \gamma_i$ was used; then α was determined by $\alpha = e^{\gamma}/(1+e^{\gamma})$. The case $n=4$ was found to be sufficiently accurate; where possible, k was chosen so that two of the x_i's were less than x and two greater than x.

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A large number of probability distributions, with at least the first four moments known and methods of obtaining cumulative distribution functions or percentage points available, is presented. They do not include the common statistical distributions, such as the normal, chi-squared, t, F, gamma, beta and Johnson distributions. There are 395 distributions, from 20 different families; the parameters were chosen to cover a large area of the $(\sqrt{\beta_1}, \beta_2)$ plane. The purpose of making the collection was to use the distributions to assess the accuracy of various methods of approximating densities, and they should be useful in similar studies.

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