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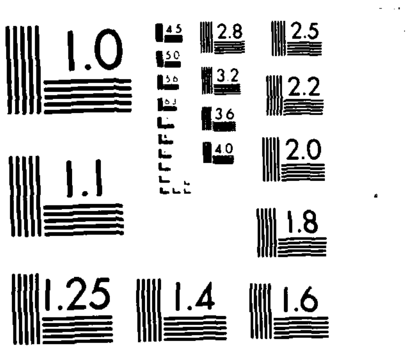

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NONPARAMETRIC RENEWAL FUNCTION ESTIMATION

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ABSTRACT

The renewal function is a basic tool used in many probabilistic models and sequential analysis. Based on a random sample of size n, a nonparametric estimator of the renewal function is introduced. Asymptotic properties of the estimator such as the almost sure consistency and local asymptotic normality are developed. A discussion of an application of the estimator is also provided.

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Key Words: U-statistics, reverse martingales, warranty analysis.

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SIGNIFICANCE AND EXPLANATION

Let $X_1, X_2, \ldots$ be identically and independently distributed with distribution function F. With $S_k = X_1 + \ldots + X_k$, let $F^{(k)}(t) = P(S_k \leq t)$ be the k-fold convolution of F for $k \geq 1$. The renewal function H is defined by

$$H(t) = \sum_{k \geq 1} F^{(k)}(t)$$

for $t \geq 0$.

The renewal function is a basic tool used in sequential analysis and used in probabilistic models arising in areas such as reliability theory, inventory theory, continuous sampling plans and warranty analysis. Use of the renewal function is becoming widespread as efficient computational techniques, which can be applied when the failure distribution is known, become available. It is surprising that, especially for small sample sizes, estimation of the key function based on available data has not been addressed directly.

Based on a random sample of size n, a nonparametric estimator of the renewal function is introduced. Various statistical properties of the estimator, such as consistency and asymptotic normality, are developed. A discussion of an application of the estimator to warranty analysis is also provided.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NONPARAMETRIC RENEWAL FUNCTION ESTIMATION

Edward W. Frees

§1. Introduction

Let $X_1, X_2, \ldots$ be identically and independently distributed random variables with distribution function F. Assume that F has positive mean μ and finite variance σ^2. With $S_k = X_1 + \ldots + X_k$, let $F^{(k)}(t) = P(S_k \leq t)$ be the k-fold convolution of F for $k \geq 1$. The renewal function H is defined by

$$H(t) = \sum_{k \geq 1} F^{(k)}(t) \tag{1.1}$$

for $t > 0$. The renewal function can be thought of as the expected number of renewals in $[0,t]$, where the number of renewals in $[0,t]$ is denoted by $N(t)$ and defined by

$$N(t) = \sum_{k \geq 1} I(S_k \leq t). \tag{1.2}$$

Here $I(\cdot)$ is the indicator function of a set. When the observations X_i are nonnegative, an equivalent definition for the number of renewals in $[0,t]$ is $N(t) = \{\sup k: \ S_k \leq t\}$. The renewal function plays an important role in many probabilistic models (cf., Feller, 1971 and Karlin and Taylor, 1975) and sequential analysis (cf., Woodroofe, 1982).

Most classical estimators of the renewal function are based on the assumption of a parametric form for F, typically an exponential or Gamma distribution. See Cox and Lewis (1966) for an early treatment of the statistical analysis of renewal processes. Most nonparametric estimators of $H(t)$ are based on a realization of the renewal process $\{X_i\}_{i=1}^{\infty}$ and on theorems which yield simple approximations of $H(t)$ for asymptotically large values of time t. For example, suppose that the nonnegative values X_i are recorded and that F has an arithmetic distribution. Recall that a distribution function is said to be arithmetic if its support is on $\{0, \pm d, \pm 2d, \ldots\}$ for some constant d. Then, the result

$$\lim_{t \to \infty} H(t) - t/\mu = (\sigma^2 + \mu - \mu^2)/(2\sigma^2)$$

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(cf., Feller, 1968, p. 341) has suggested the use of the estimator

$$\hat{H}(t) = t/\hat{\mu} + (\hat{\sigma}^2 + \hat{\mu} - \hat{\mu}^2)/(2\hat{\sigma}^2) \tag{1.3}$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are estimators of μ and σ^2 based on the data recorded up to time t. See Yang (1983) for an application of $\hat{H}(t)$ to continuous sampling plans. Another well-known example is a functional central limit theorem for $N(t)$ given by Billingsley (1968, Theorem 17.3). Here the limits are for t approaching ∞.

In this paper, estimators of $H(t)$ for a fixed time t are based on a random sample of size n, $X_1, X_2, \ldots, X_n$. Estimation of the distribution function F and linear functionals of F are problems that have been thoroughly investigated in the literature (cf., Serfling, 1980, chapters 2 and 6). Viewing $H(t)$ defined in (1.1) as merely the infinite sum of convolutions of F, it seems natural to estimate $H(t)$ based on a sum of estimators of the convolutions of F. As one would suspect, even though estimators of the type in (1.3) are based on recorded observations, they do not perform well for small (relative to μ) times t. This was pointed out by Frees (1984). In that study the author introduced several estimators, both parametric and nonparametric, of $H(t)$ for a fixed time t based on a random sample of size n. One nonparametric estimator performed particularly well in the simulation portion of that study. A variation of that estimator is now defined. Let $\{i_1, i_2, \ldots, i_k\}$ be a subset of size k of $\{1, 2, \ldots, n\}$ and let $\sum_c$ be the sum over all $\binom{n}{k}$ distinct combinations of $\{i_1, i_2, \ldots i_k\}$. Then, an unbiased estimator of $F^{(k)}(t)$ is

$$F_n^{(k)}(t) = \binom{n}{k}^{-1} \sum_c I(X_{i_1} + \ldots + X_{i_k} \leq t). \tag{1.4}$$

Let $m = m(n)$ be a positive integer depending on n such that $m \leq n$ and $m \uparrow \infty$ as $n \uparrow \infty$. Then, a nonparametric estimator of the renewal function is

$$H_n(t) = \sum_{k=1}^{m} F_n^{(k)}(t). \tag{1.5}$$

The advantages of introducing the design parameter m are discussed in §5.

The estimator of $F^{(k)}(t)$, $F_n^{(k)}(t)$, is a U-statistic and thus it is easy to establish that for each $k\geq 1$ and for each $t>0$ that

$$F_n^{(k)}(t) \to F^{(k)}(t) \qquad \text{a.s.}$$

However, the almost sure (a.s.) consistency of $H_n(t)$ is surprisingly difficult to establish. We do so in §2 by establishing that $H_n(t)$ is a reverse martingale with respect to an appropriate sequence of sub σ-fields plus some negligible terms. Also in that section we prove a.s. uniform consistency, the Glivenko-Cantelli property, when the uniformity is restricted to bounded subsets of the positive real line. A counter-example is given which shows that a.s. consistency cannot hold uniformly over all of the positive real line. In §3, the asymptotic normality, when properly standardized, of $H_n(t)$ is proved via the projection technique popularized by Hajek (cf., Serfling, 1980, Chapter 9.2.5). To keep potential applications for this estimator as broad as possible, we distinguish between the usual renewal theory assumptions of nonnegative observations and the more general framework which also permits negative observations. The latter is the situation usually encountered in sequential analysis. In §4, we prove the a.s. consistency of an estimator of the asymptotic variance. This provides the important result of large sample interval estimates. We conclude in §5 by commenting on applications and by providing an example in warranty analysis.

§2. Almost Sure Consistency

Let $a \in R$ and g_a be a real valued function defined on $R^+ = [0,\infty)$ such that

$$\int_0^\infty |g_a(u)|\, d(\sum_{k\geq 1} k^a F^{(k)}(u)) < \infty. \tag{2.1}$$

In this section we establish the following result.

Theorem 2.1

Suppose that (2.1) holds. Then,

$$\lim_{n\to\infty} \int_0^\infty g_a(u)\, d(\textstyle\sum_{k=1}^m k^a\, F_n^{(k)}(u)) = \int_0^\infty g_a(u)\, d(\textstyle\sum_{k\geq 1} k^a\, F^{(k)}(u)) \quad \text{a.s.} \tag{2.2}$$

In §4 and §5, we give applications where $a\neq 0$. To provide motivation for Theorem 2.1, we consider the following corollary for the case $a=0$.

Corollary 2.1

Suppose g is a function defined on R^+ such that

$$\int_0^\infty |g(u)|\, dH(u) < \infty.$$

Then,

$$\lim_{n\to\infty} \int_0^\infty g(u)\, dH_n(u) = \int_0^\infty g(u)\, dH(u) \qquad \text{a.s.}$$

Corollary 2.1 indicates that the sequence of random measures associated with the sequence $\{H, H_n, n\geq 1\}$ possess a type of ergodic property. The statement of Corollary 2.1 is similar in flavor to the statement of the key renewal theorem of Smith (see 1958, (1.3)). Some of the applications of Smith's key renewal theorem are also present in the estimation context of Corollary 2.1. For example, since $H(t) < \infty$ for each $t>0$ when μ is positive and σ^2 is finite, we may let $g(u) = I(u\leq t)$ to get the following

Corollary 2.2.

Suppose F has positive mean μ and finite variance σ^2. Then, for each $t>0$,

$$H_n(t) \to H(t) \qquad \text{a.s.}$$

Corollary 2.1, together with some results on uniformly convergent measures due to Rao (1962), is also used to prove

<u>Theorem 2.2.</u>

Suppose F has positive mean μ and finite variance σ^2. Then, for each $t>0$,

$$\sup_{u\epsilon[0,t]} |H_n(u) - H(u)| \to 0 \qquad \text{a.s.}$$

Remarks: Note that Theorem 2.2 does not require that the support of F be on R^+ and also holds for both arithmetic and nonarithmetic distributions. The theorem is a Glivenko-Cantelli type result and is important in practice. A minor drawback of the result is that the supremum extends only over bounded intervals and not over all of R^+. That an extension of the result to R^+ does not hold in general is given by the following

<u>Example 2.1.</u>

Let $\{X_i\}_{i=1}^n$ be a random sample with d.f. F, $\mu>0$. Define $X_{nn} = \max\{X_1,\ldots,X_n\}$. Suppose F is such that for some $\epsilon>0$ and for sufficiently large N, we have $X_{nn} > \mu+\epsilon$ a.s. for all $n>N$. Since by the elementary renewal theorem, $\lim_{t\to\infty} H(t)/t = 1/\mu$, we have for all $n>N$,

$$\sup_{t\epsilon R^+} |H_n(t) - H(t)| \geq |H(n\, X_{nn}) - H_n(n\, X_{nn})|$$

$$= |H(n\, X_{nn}) - m| \sim n(X_{nn}/\mu - m/n) \to \infty \qquad \text{a.s.}\ddagger$$

The remainder of the section is devoted to the proof of Theorems 2.1 and 2.2. The technique is to show that $\int_0^\infty g(u)\, d(\sum_{k=1}^m k^a F_n^{(k)}(u))$ is a reverse martingale plus negligible terms. Reverse martingales are a natural tool in this context if we note that $F_n^{(k)}(t)$ is a U-statistic. The idea of applying reverse martingales to U-statistics is due to Berk (1966). An application of Doob's (reverse) martingale convergence theorem will then establish Theorem 2.1. Let $\{X_{1n}, X_{2n}, \ldots, X_{nn}\}$ be the order statistics associated with $\{X_1, X_2, \ldots, X_n\}$. We use $G_n = \sigma(X_{1n}, \ldots, X_{nn}, X_{n+1}, X_{n+2}, \ldots)$, $n\geq 1$, to define the sequence of nonincreasing sub σ-fields which are implicitly used in all of the following reverse martingales. We preface the proof of Theorem 2.1 with two lemmas.

Lemma 2.1

Let $g_a(\ )$ be as defined in (2.1). Then, for each $k>1$, $\int_o^\infty g_a(u)\, dF_n^{(k)}(u)$ is a reverse martingale.

Proof: It is easy to see that $\int_o^\infty g_a(u)\, dF_n^{(k)}(u)$ is G_n-measurable. Now, note that (2.1) implies that $g_a(u) \to 0$ as $u \to \infty$. Integration by parts yields

$$\int_o^\infty g_a(u)\, dF_n^{(k)}(u) = g_a(u)\, F_n^{(k)}(u)\ \Big|_o^\infty - \int_o^\infty F_n^{(k)}(u)\, dg_a(u)$$

$$= - \int_o^\infty F_n^{(k)}(u)\, dg_a(u). \qquad (2.3)$$

An application of Fubini's Theorem using (2.1) establishes that $\int_o^\infty g_a(u)\, dF_n^{(k)}(u)$ is integrable. Since $\{F_n^{(k)}(u), G_n\}$ is a reverse martingale we have $E(F_n^{(k)}(u)|G_{n+1}) = F_{n+1}^{(k)}(u)$ a.s. for each $u>0$. Thus

$$E(\int_o^\infty g_a(u)\, dF_n^{(k)}(u)|G_{n+1}) = - \int_o^\infty E(F_n^{(k)}(u)|G_{n+1})\, dg_a(u)$$

$$= - \int_o^\infty F_{n+1}^{(k+1)}(u)\, dg_a(u) = \int_o^\infty g_a(u)\, dF_{n+1}^{(k)}(u) \qquad \text{a.s.} \ddagger$$

Lemma 2.2

Let $g_a(\ )$ be as defined in (2.1). Then,

$$RM_n(a) = \int_o^\infty g_a(u)\, d(\textstyle\sum_{k=1}^n k^a\, F_n^{(k)}(u)) + \sum_{k>n} k^a \int_o^\infty g_a(u)\, dI(S_k<u)$$

$$- \int_o^\infty g_a(u)\, d(\textstyle\sum_{k\geq 1} k^a\, F^{(k)}(u))$$

is a zero mean reverse martingale.

Proof:

The proof is similar to Lemma 2.1 if we note that

$$E(I(S_k \le u)|G_n) = \begin{cases} F_n^{(k)}(u) & k \le n \\ I(S_k \le u) & k > n \end{cases} \qquad \text{a.s.} \ddagger$$

Proof of Theorem 2.1:

To prove (2.2), it is sufficient to show

$$\lim_{n\to\infty} \int_0^\infty g_a(u)\, d(\sum_{k=1}^{n} k^a F_n^{(k)}(u)) = \int_0^\infty g_a(u)\, d(\sum k^a F^{(k)}(u)) \text{ a.s.} \tag{2.4}$$

To see this, by Fatou's Lemma and similarly to (2.3), we have

$$\lim \left| \int_0^\infty g_a(u)\, d(\sum_{k=1}^{n} k^a F_n^{(k)}(u)) - \int_0^\infty g_a(u)\, d(\textstyle\sum_{k=1}^{m} k^a F_n^{(k)}(u)) \right|$$

$$= \lim \int_0^\infty \sum_{k=m+1}^{n} k^a F_n^{(k)}(u)\, d(|g_a(u)|)$$

$$\le \lim \sum_{k=m+1}^{\infty} \int_0^\infty k^a F_n^{(k)}(u)\, d(|g_a(u)|) = 0.$$

To prove (2.4), we use Lemma 2.2 and Doob's (reverse) martingale convergence theorem. Thus, there exists a random variable Z such that $\lim_{n\to\infty} RM_n = Z$ a.s. and $\lim_{n\to\infty} E|RM_n - Z| = 0$. From Fatou's Lemma,

$$0 = \lim E|RM_n - Z| \ge \lim E(RM_n - Z) \ge E \lim (RM_n - Z) = 0$$

and thus EZ=0. Further

$$Z = \lim_{n\to\infty} \int_0^\infty (\sum_{k=1}^{n} k^a F_n^{(k)}(u))\, d(-g_a(u)) - \int_0^\infty (\textstyle\sum_{k\ge 1} k^a F^{(k)}(u))\, d(-g_a(u))$$

$$= \lim \sum_{k=1}^{\infty} \int_0^{\infty} k^a (F_n^{(k)}(u) - F^{(k)}(u))\, d(-g_a(u))$$

$$\geq \sum_{k=1}^{\infty} \int_0^{\infty} k^a \lim \; (F_n^{(k)}(u) - F^{(k)}(u))\, d(-g_a(u)) = 0 \qquad \text{a.s.}$$

Since $Z \geq 0$ a.s. and $EZ=0$, we have $Z=0$ a.s. This is sufficient for (2.4) and the proof.‡

Proof of Theorem 2.2:

Let $A = \{\omega\colon\ \omega<t$ and ω is a discontinuity point of $H\}$. Since the set of discontinuity points of $F^{(k)}(\cdot)$ is countable for each $k \geq 1$, A is countable and we can let $\{a_i\}$ be some enumeration of A. Define $g(u) = \sum_{i\geq 1} I(u = a_i)$. Since $\int_0^{\infty} g(u)\, dH(u) < H(t) < \infty$, by Theorem 2.1 we have

$$\sum_{i\geq 1} \sum_{k=1}^{m} (F_n^{(k)}(a_i) - F_n^{(k)}(a_i-)) + \sum_{i\geq 1} (H(a_i) - H(a_i-)).$$

Thus, without loss of generality, we may assume that $H(u)$ is continuous for $u<t$. The result is now immediate from Theorem 2.1 and Theorems 4.2 and 6.1 of Rao (1962).‡

§3. Asymptotic Normality

Define

$$\xi_{rs}(c) = \mathrm{Cov}(F^{(s-c)}(t-(X_1 + \ldots + X_c)),\ F^{(s-c)}(t-(X_1 + \ldots + X_c))). \quad (3.1)$$

In this section we prove the following result.

Theorem 3.1

Suppose F is such that, for some $\theta > 0$,

$$\int \exp(-\theta u)\, dF(u) < 1. \tag{3.2}$$

Let m grow sufficiently quickly so that $\log n = o(m)$ and sufficiently slowly so that $m = o(n^{1/2})$. Then, for each $t > 0$

$$\sqrt{n}\,(H_n(t) - H(t)) \to_D N(0, \sigma^2)$$

where

$$\sigma^2 = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} rs\, \xi_{rs}(1).$$

Remarks: Note that Theorem 3.1 holds for F both arithmetic and nonarithmetic. Further, there is no requirement that X be a nonnegative random variable. The requirement (3.2) is needed to ensure that $P(S_k < t)$ dies out sufficiently quickly as $k \to \infty$. Since $\mu > 0$, a sufficient condition for (3.2) is that the moment generating function of X exists in a neighborhood of zero. However, such strong moment conditions are not always necessary. Consider a random variable Z whose distribution is defined by the probability density function, for some $\delta > 0$,

$$f(z) = K\, z^{-3-\delta}\, I(z > 1)$$

where K is an appropriate constant. Then, Var(Z) is finite, (3.2) is easily satisfied, and yet $E\, Z^{2+\delta} = \infty$.

Using W as the duration of a warranty, suppose we are interested in estimating the expected number of renewals (or replacements) by time W, $H(W)$, or more generally, we wish to estimate $H(t)$ for $t \leq W$. Suppose we replace the i.i.d. observations X_i in the definition of $H_n(t)$ (see (1.5)) by $X_i' = \min(X_i, W)$. A little thought leads us to the conclusion that all the results of §2 through §4 remain valid with the provision that the time t under consideration is bounded by W. This is important in commercial applications where typically the manufacturer of a product only has knowledge of the time to failure if it occurs before expiration of the warranty.

In many situations the choice of the design parameters m and m_1 is dictated by practical considerations, as in the example above. Theorem 3.1 gives some theoretical guidelines for the choice of m. However, the convolution $F^{(k)}(t)$ dies out exponentially (cf. (3.10)) as k approaches infinity, and typically m can be small compared to the sample size. A similar argument can be made for m_1. This is important since the amount of computations increases quickly as m (or m_1) increases.

called a warranty and W is the duration of the warranty. In this example, one reasonable warranty duration is the end of the early failure period and thus we give a point and interval estimate of H(20).

From Table 1, the estimate is $H_n(20) = \sum_{k=1}^{5} F_{105}^{(k)}(20) = .46194$. Frees (1984) compared this estimate with other estimators of H(20) and found it reasonable. To calculate the estimated variance of this estimator, from Table 2 we have

$$\sigma_n^2 = \sum_{r,s=1}^{4} rs\ \hat{\xi}_{rs}(1) = .75385.$$

Thus, an approximate 95% interval estimate of H(20) is $H_{105}(20) \pm 2\ \sigma_n/\sqrt{105}$ which is roughly .46 ± .17 or (.29, .63).

TABLE 1

Convolution estimates for failure of a unit of electronic ground support equipment.

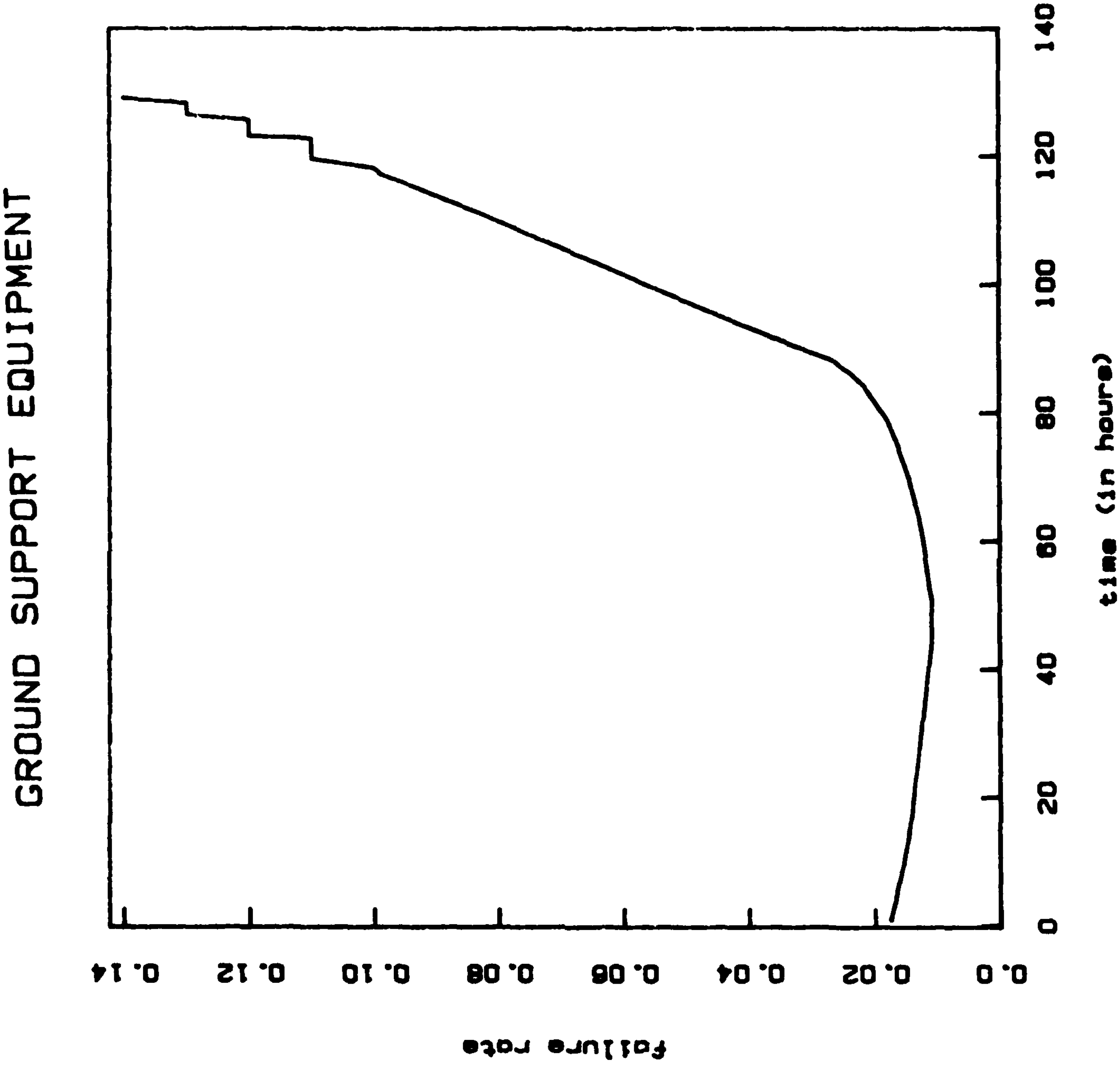
k	1	2	3	4	5	6	7	8
$F_{105}^{(k)}(20)$	.35238	.09048	.01684	.00208	.00016	0	0	0

TABLE 2

Covariance estimates for failure of a unit of electronic ground support equipment

$\hat{\xi}_{rs}(1)$

r/s	1	2	3	4
1	.22821	.05860	.01091	.00135
2		.01733	.00490	.00061
3			.00355	.00048
4				.00037


GROUND SUPPORT EQUIPMENT
failure rate
0.0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0
20
40
60
80
100
120
140
time (in hours)

GRAPH 1

etric estimation of this key function based on the available data. The simulation study of Frees (1984) showed that an estimator similar to $H_n(t)$ performed well for small ($n < 30$) sample sizes. An example in warranty analysis is given below. The techniques of this paper may also be useful in sequential analysis. For example, an important parameter in sequential estimation is the mean of the renewal function,

$$\rho = \int_0^\infty u \, dH(u),$$

see Woodroofe (1982). Defining $\rho_n = \sum_{k=1}^m \int_0^\infty u \, dF_n^{(k)}(u)$, from Corollary 2.1 we have that ρ_n is an a.s. consistent estimator of ρ if $\rho < \infty$. Another example is the expected value of the first passage time

$$\tau = \inf\{n \geq 1: \quad S_n > 0\}.$$

From, for example, Woodroofe (1982, Corollary 2.4) we have $E(\tau) = \exp\{\sum_{k \geq 1} k^{-1} P(S_k \leq 0)\}$ when $\mu > 0$. Thus, by Theorem 2.1, $\tau_n = \exp\{\sum_{k=1}^m k^{-1} F_n^{(k)}(0)\}$ is an a.s. consistent estimator of $E(\tau)$. We intend to explore other applications of nonparametric renewal function estimation in sequential analysis in another paper.

To illustrate how to calculate the estimator, we used observations of the time to failure of a unit of electronic ground support equipment which were previously used by the author (Frees, 1984). The data can also be found in Juran and Gryna (1970, p. 171) and Kolb and Ross (1980, p. 170). In Graph 1 an estimate of the failure rate curve is given which suggests an early failure rate of about 20 hours. The estimate of the failure rate was based on Epanevicoch's method. The calculations were done on a VAX 11/750 owned and operated by the Department of Statistics at the University of Wisconsin-Madison. Now, it is not unusual for the manufacturer of equipment to enter into an agreement to replace the equipment for a certain length of time, say, W. This type of agreement is

To prove (4.4), we first establish an analogue to Lemma 2.2. Let

$$R_n = \sum_{r+s-1\leq n} rs\, \hat{\xi}_{rs} + \sum_{r+s-1>n} rs\, E\{I(S_r \leq t)\, I(S_{r+s-1} - S_{r-1} \leq 1)|G_n\}$$

$$- \sum_{r,s\geq 1} rs\, \xi_{rs}.$$

It is easy to see that R_n is G_n-measurable and integrable. Further, since

$$E(E\{I(S_r \leq t)\, I(S_{r+s-1} - S_{r-1} \leq t)|G_n\}|G_{n+1})$$

$$= E(I(S_r \leq t)\, I(S_{r+s-1} - S_{r-1} \leq t)|G_{n+1}) \qquad \text{a.s.}$$

and for $r+s-1 \leq n$, $E(I(S_r \leq t)\, I(S_{r+s-1} \leq t)|G_n) = \hat{\xi}_{rs}$ a.s., R_n is a zero mean reverse martingale. Now, as in the proof of Theorem 2.1, an application of Doob's (reverse) martingale convergence theorem and repeated application of Fatou's Lemma yields

$$\lim_{n\to\infty} \sum_{r+s-1\leq n} rs\, \hat{\xi}_{rs} + \sum_{r+s-1>n} rs\, E(I(S_n \leq t)\, I(S_{r+s-1} - S_{r-1} \leq t)|G_n)$$

$$= \sum_{r,s\geq 1} rs\, \xi_{rs} \qquad \text{a.s.}$$

Thus, to complete the proof, we note that

$$\lim_{n\to\infty} \sum_{r+s-1>n} E(I(S_r \leq t)\, I(S_{r+s-1} - S_{r-1} \leq t)|G_n) = 0 \qquad \text{a.s.}$$

and

$$\lim_{n\to\infty} \sum_{r+s-1\leq n} rs\, \hat{\xi}_{rs} - \sum_{r,s=1}^{m_1} rs\, \hat{\xi}_{rs} = 0 \qquad \text{a.s.}\ddagger$$

§5. Concluding Remarks

The renewal function arises in a wide variety of applications of probabilistic models such as in reliability theory, inventory theory and continuous sampling plans. In this paper we have presented the asymptotic theory of nonparam-

$$\sigma_n^2 = \sum_{r,s=1}^{m_1} \hat{\xi}_{rs} - \left(\sum_{k=1}^{m_1} k\, F_n^{(k)}(t)\right)^2, \tag{4.3}$$

we have

$$\lim_{n\to\infty} \sigma_n^2 = \sigma^2 \qquad \text{a.s.}$$

Remarks: For a general F, by Theorem 3 of Baum and Katz (1965), $E\,|X|^3 < \infty$ is a sufficient condition for (4.2). Further, from (3.10), it is easy to see that (3.2) is sufficient for (4.2). Thus, we have

Corollary 4.1.

Assume that the conditions of Theorem 3.1 hold and let σ_n^2 be defined as in (4.3). Then,

$$\sqrt{n}/\sigma_n\,(H_n(t) - H(t)) \to_D N(0,1)$$

and thus, for $\alpha \in (1/2,1)$,

$$\lim_{n\to\infty} P(H_n(t) - z_{\alpha/2}\,\sigma_n/\sqrt{n} \le H(t) \le H_n(t) + z_{\alpha/2}\,\sigma_n/\sqrt{n}) = 1-\alpha$$

where $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution.

Proof of Theorem 4.1:

From Theorem 2.1, with a=1 and $g_a(u) = I(u \le t)$, we have

$$\sum_{k=1}^{m_1} k\, F_n^{(k)}(t) \to \sum_{k\ge 1} k\, F^{(k)}(t) \qquad \text{a.s.}$$

Thus, we need only show that

$$\sum_{r,s=1}^{m_1} rs\, \hat{\xi}_{rs} \to \sum_{r,s\ge 1} rs\, \xi_{rs} \qquad \text{a.s.} \tag{4.4}$$

This result and Theorem 3.1 will immediately provide a confidence interval for $H(t)$.

Let $\xi_{rs} = E(F^{(r-1)}(t-X)\ F^{(s-1)}(t-X))$. We wish to estimate

$$\sigma^2 = \textstyle\sum_{r,s\geq 1} rs\ \xi_{rs}(1)$$

$$= \textstyle\sum_{r,s\geq 1} rs\ \{E\ (F^{(r-1)}(t-X)\ F^{(s-1)}\ (t-X)) - F^{(r)}(t)\ F^{(s)}(t)\}$$

$$= \textstyle\sum_{r,s\geq 1} rs\ \xi_{rs} - (\sum_{r\geq 1} r\ F^{(r)}(t))^2.$$

To define an estimator of ξ_{rs}, let $\{i_1, i_2, \ldots, i_{r+s-1}\}$ be a subset of $\{1, 2, \ldots, n\}$, not necessarily ordered. Let $\sum_p$ denote the summation over all n! permutations of subsets of size $r+s-1$ from $\{1, 2, \ldots, n\}$. An unbiased estimator of ξ_{rs} is

$$\hat{\xi}_{rs} = (n-r-s+1)!/n! \textstyle\sum_p I(X_{i_1} + X_{i_2} + \ldots + X_{i_r} \le t)$$

$$I(X_{i_1} + X_{i_{r+1}} + \ldots + X_{i_{r+s-1}} \le t). \tag{4.1}$$

In this section, we prove the following result.

<u>Theorem 4.1</u>

Assume that

$$\textstyle\sum_{k\geq 1} k\ F^{(k)}(t) < \infty. \tag{4.2}$$

Let $m_1 = m_1(n)$ be a positive integer depending on n such that $m_1 < n$ and $m_1 \uparrow \infty$ as $n \uparrow \infty$. Then, with

To prove (3.12), note that

$$n \binom{n}{r}^{-1} \binom{s}{c} \binom{n-s}{r-c}$$

$$= r!\ s!\ (n-r)!\ (n-s)!\ \{(r-c)!\ (s-c)!\ c!\ (n-r-s+c)!\ (n-1)!\}^{-1}$$

$$= A(n,r,s) \binom{r}{c} \binom{s}{c}\ c!\ (n-r-s+1)!\ /\ (n-r-s+c)!.$$

Note that $c!\ (n-s-r+1)!\ /\ (n-r-s+c)! = O(n^{-1})$,

$\sum_{c=1}^{\min(r,s)} \binom{r}{c} \binom{s}{c} = \binom{r+s}{r}$, $\quad \sup \quad A(n,r,s) \le 1 + o(1)$, and

$|\ \xi_{rs}(c)\ | \le F^{(r)}(t)\ F^{(s)}(t)$. Thus,

$$|\ \sum_{r,s=1}^{m} n \binom{n}{r}^{-1} \sum_{c=2}^{s} \binom{s}{c} \binom{n-s}{r-c} \xi_{rs}(c)\ |$$

$$\le O(n^{-1}) \sum_{r,s=1}^{m} F^{(r)}(t)\ F^{(s)}(t) \sum_{c=2}^{s} \binom{r}{c} \binom{s}{c}$$

$$\le O(n^{-1}) \sum_{r,s=1}^{m} (E\ e^{-\theta X})^{r+s} \binom{r+s}{r} = O(n^{-1})$$

by (3.10) and the fact that for $p<1$,

$$\sum_{r,s=1}^{\infty} p^{r+s} \binom{r+s}{r} < \infty. \ddagger$$

§4. Interval Estimates

A local asymptotic normality result such as Theorem 3.1 is appealing because it gives information about the rate of convergence of $H_n(t)$ to $H(t)$. However, in applications it is also desirable to give interval estimates of $H(t)$. In this section we present an estimator of σ^2 and prove its almost sure consistency. In the proof the reverse martingale technique of §2 is utilized.

Lemma 3.3

Under the assumptions and notation of Theorem 3.1,

$$\sum_{r,s=1}^{m} \{n \binom{n}{r}^{-1} \sum_{c=1}^{r} \binom{s}{c} \binom{n-s}{r-c} \xi_{rs}(c) - rs\, \xi_{rs}(1)\} \to 0.$$

Proof:

Sufficient for the proof is

$$n \sum_{r,s=1}^{m} \binom{n}{r}^{-1} \sum_{c=2}^{s} \binom{s}{c} \binom{n-s}{r-c} \xi_{rs}(c) \to 0 \tag{3.12}$$

and

$$\sum_{r,s=1}^{m} (n \binom{n}{r}^{-1} \binom{s}{1} \binom{n-s}{r-1} - rs)\, \xi_{rs}(1) \to 0. \tag{3.13}$$

To prove (3.13), by Lemma 3.2, we have

$$n \binom{n}{r}^{-1} s \binom{n-s}{r-1} - rs = rs\, ((n-s)!\, (n-r)!\, \{(n-1)!\, (n-r-s+1)!\}^{-1} - 1)$$

$$= rs\, (A(n,r,s) - 1).$$

Thus, by Fatou's Lemma,

$$\lim \left| \sum_{r,s} (n \binom{n}{r}^{-1} s \binom{n-s}{r-1} - rs)\, \xi_{rs}(1) \right|$$

$$< \sum_{r,s} rs \left| \xi_{rs}(1) \right| \lim \sup_{r,s<m} \left| A(n,r,s) - 1 \right| = 0.$$

Proof:

Recall the inequality (cf., Feller, 1968, p. 54),

$$\sqrt{2\pi}\ (n + 1/2)^{n+1/2} \exp\{-(n + 1/2) - 1/24/n + 1/2)^{-1}\}$$

$$< n! < \sqrt{2\pi}\ (n + 1/2)^{n+1/2} \exp\{-(n + 1/2)\}.$$

Thus,

$$A(n,r,s) < \frac{(n - r + 1/2)^{n-r+1/2}\ (n - s + 1/2)^{n-s+1/2}\ \{1 + o(1)\}}{(n - 1/2)^{n-1/2}\ (n - r - s + 3/2)^{n-r-s+3/2}}.$$

Define $n_1 = n-1/2$ and $n_2 = n-r-s+3/2$. Then,

$$A(n,r,s) < (1 - (r-1)/n_1)^{n_1}\ (1 + (r-1)/n_2)^{n_2}$$

$$(1 + (r-s)/(n-r+1/2))^{r}\ \{1 + o(1)\}.$$

Thus,

$$\sup_{r,s} \log A(n,r,s) < \sup_{r,s} \{n_1 \log(1 - (r-1)/n_1)$$

$$+ n_2 \log(1 + (r-1)/n_2) + k \log(1 + (r-s)/(n-r+1/2))\} + o(1)$$

$$= \sup_{s} -(r-1) + o(1) + (r-1) + o(1) = o(1).$$

Thus, $\lim \sup_{r,s} A(n,r,s) < 1$. The inequality in the opposite direction is similar.‡

Thus, from (3.2),

$$\sqrt{n}\,(H(t) - H^*(t)) = \sqrt{n}\,\textstyle\sum_{k>m} F^{(k)}(t)$$

$$< \sqrt{n}\, e^{\theta t}(E(e^{-\theta X}))^m\,(1-E(e^{-\theta X}))^{-1} \to 0$$

by an easy application of L'Hôspital's rule since log n = o(m). Thus, sufficient for the proof of the lemma is

$$\sqrt{n}\,(\hat{H}_n(t) - H^*(t)) \to_D N(0,\ \sigma^2). \tag{3.11}$$

To prove (3.11), from (3.5), define

$$U_{nj} = n^{-1} \textstyle\sum_{k=1}^{m} (F^{(k-1)}(t-X_j) - F^{(k)}(t)).$$

Now $\{U_{nj};\ j=1,\ldots,n,\ n>1\}$ is a double array of random variables that are independent within rows. Now $E\,U_{nj} = 0$ and, by (3.6),

$\mathrm{Var}(\sum_{j=1}^{n} U_{nj}) = n^{-1} \sum_{r,s=1}^{m} rs\ \xi_{rs}(1)$. Because of the strong moment conditions, it is easy to check that the usual uniform asymptotic negligibility and Lindeberg conditions hold (cf., Serfling, 1980, §1.9.3).‡

<u>Lemma 3.2</u>

Define $A(n,r,s) = (n-r)!\,(n-s)!\,\{(n-1)!\,(n-r-s+1)!\}^{-1}$. If $r=o(n^{1/2})$ and $s=o(n^{1/2})$, then

$$\lim_{n\to\infty}\ \sup_{r,s}\ A(n,r,s) = 1.$$

Hence, from (1.5) and (3.5),

$$\mathrm{Cov}(H_n(t),\ \hat{H}_n(t)) = \textstyle\sum_{r,s=1}^{m}\ \mathrm{Cov}(F_n^{(r)}(t),\ s/n \displaystyle\sum_{j=1}^{n} F^{(s-1)}(t-X_j))$$

$$= \textstyle\sum_{r,s=1}^{m}\ s\ \mathrm{Cov}(F_n^{(r)}(t),\ F^{(s-1)}(t-X_1))$$

$$= \textstyle\sum_{r,s=1}^{m}\ rs/n\ \xi_{rs}(1). \tag{3.8}$$

Thus, from (3.6) - (3.8),

$$n\ E(H_n(t) - \hat{H}_n(t))^2 = n\{\mathrm{Var}(H_n(t)) + \mathrm{Var}(\hat{H}_n(t)) - 2\ \mathrm{Cov}(H_n(t),\ \hat{H}_n(t))\}$$

$$= \textstyle\sum_{r,s=1}^{m}\ \{n \binom{n}{r}^{-1} \displaystyle\sum_{c=1}^{r} \binom{s}{c}\binom{n-r}{s-c}\ \xi_{rs}(c) - rs\ \xi_{rs}(1)\}. \tag{3.9}$$

We now present a series of lemmas which, when taken together, provide a proof of Theorem 3.1.

Lemma 3.1

Under the assumptions and notation of Theorem 3.1,

$$\sqrt{n}\ (\hat{H}_n(t) - H(t)) \to_D N(0,\ \sigma^2).$$

Proof:

By the Markov inequality, for $\theta>0$, we have

$$F^{(k)}(t) = P(-\theta\ S_k > -\theta\ t) < e^{\theta t}\ (E\ e^{-\theta X})^k. \tag{3.10}$$

$$E\, I(X_{a_1} + X_{a_2} + \dots + X_{a_r} < t)\, I(X_{b_1} + X_{b_2} + \dots + X_{b_s} < t)$$

$$- F^{(r)}(t)\, F^{(s)}(t) = \xi_{rs}(c).$$

Thus,

$$\mathrm{Cov}(F_n^{(r)}(t),\, F_n^{(s)}(t))$$

$$= \binom{n}{r}^{-1} \binom{n}{s}^{-1} E \sum_c \sum_c \{[I(X_{a_1} + \dots + X_{a_r} < t) - F^{(r)}(t)]$$

$$[I(X_{b_1} + \dots + X_{b_s} < t) - F^{(s)}(t)]\}$$

$$= \binom{n}{r}^{-1} \sum_{c=o}^{r} \binom{s}{c} \binom{n-s}{r-c} \xi_{rs}(c)$$

since the number of distinct choices for two subsets of size r and s, respectively, having exactly c elements in common is $\binom{n}{s}\binom{s}{c}\binom{n-s}{r-c}$. Thus,

$$\mathrm{Var}(H_n(t)) = \sum_{r,s=1}^{m} \binom{n}{r}^{-1} \sum_{c=1}^{r} \binom{s}{c} \binom{n-s}{r-c} \xi_{rs}(c). \tag{3.7}$$

To calculate $\mathrm{Cov}(H_n(t), \hat{H}_n(t))$, we first examine the covariance between $F_n^{(r)}(t)$ and $F^{(s-1)}(t-X_1)$. Now,

$$\mathrm{Cov}(F_n^{(r)}(t),\, F^{(s-1)}(t-X_1))$$

$$= \binom{n}{r}^{-1} \{\binom{n-1}{r-1} \mathrm{Cov}(F^{(r-1)}(t-X_1),\, F^{(s-1)}(t-X_1))\}$$

$$= r/n\; \xi_{rs}(1).$$

From the definition of $F_n^{(k)}(t)$ in (1.4), an easy calculation shows that

$$E(F_n^{(k)}(t)|X_1) = \binom{n}{k}^{-1}\{\binom{n-1}{k-1} F^{(k-1)}(t-X_1) + \binom{n-1}{k} F^{(k)}(t)\}$$

$$= (k/n) F^{(k-1)}(t-X_1) + (1-k/n) F^{(k)}(t). \tag{3.3}$$

Define a truncated version of $H(t)$, $H^*(t) = \sum_{k=1}^{m} F^{(k)}(t)$. We define the projection of $H_n(t)$ on $H^*(t)$ by

$$\hat{H}_n(t) = \sum_{j=1}^{n} E(H_n(t)|X_j) - (n-1) H^*(t). \tag{3.4}$$

From (3.3), we have

$$\hat{H}_n(t) - H^*(t) = n^{-1} \sum_{j=1}^{n} \sum_{k=1}^{m} k\{F^{(k-1)}(t-X_j) - F^{(k)}(t)\}. \tag{3.5}$$

The idea of projecting $H_n(t)$ onto the original independent observations is due to Hoeffding. Since $\hat{H}_n(t)$ is just the sum of n independent random variables, the usual theorems for double arrays of independent random variables are used to obtain a limiting asymptotic distribution for $\hat{H}_n(t)$. We then show that the moments of $H_n(t) - \hat{H}_n(t)$ are small in the appropriate sense to get an identical asymptotic distribution for $H_n(t)$.

From (3.1) and (3.5), we have

$$\mathrm{Var}(\hat{H}_n(t)) = n^{-1} \textstyle\sum_{r=1}^{m} \sum_{s=1}^{m} rs\ \mathrm{Cov}(F^{(r-1)}(t-X), F^{(s-1)}(t-X))$$

$$= n^{-1} \textstyle\sum_{r,s=1}^{m} rs\ \xi_{rs}(1). \tag{3.6}$$

To calculate $\mathrm{Var}(H_n(t))$, we first examine the covariance between $F_n^{(r)}(t)$ and $F_n^{(s)}(t)$. Let $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_s\}$ be two subsets of $\{1, 2, \ldots, n\}$ that have $c \leq \min(r,s)$ elements in common. Then,

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The renewal function is a basic tool used in many probabilistic models and sequential analysis. Based on a random sample of size n, a nonparametric estimator of the renewal function is introduced. Asymptotic properties of the estimator such as the almost sure consistency and local asymptotic normality are developed. A discussion of an application of the estimator is also provided.

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