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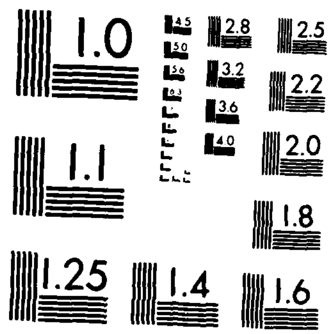
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COMPUTABLE NUMERICAL BOUNDS FOR LAGRANGE MULTIPLIERS OF STATIONARY POINTS OF NONCONVEX DIFFERENTIABLE NONLINEAR PROGRAMS

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ABSTRACT

It is shown that the satisfaction of a standard constraint qualification of mathematical programming ~~[5]~~ at a stationary point of a nonconvex differentiable nonlinear program provides explicit numerical bounds for the set of all Lagrange multipliers associated with the stationary point. Solution of a single linear program gives a sharper bound together with an achievable bound on the 1-norm of the multipliers associated with the inequality constraints. The simplicity of obtaining these bounds contrasts sharply with the intractable NP-complete problem of computing an achievable upper bound on the p-norm of the multipliers associated with the equality constraints for integer $p \geq 1$.

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SIGNIFICANCE AND EXPLANATION

The purpose of this work is to show that a fundamental regularity condition of nonlinear programming contains information which provides numerical bounds for the Lagrange multipliers of local solutions of nonlinear programs. Lagrange multipliers play a fundamental role in stability and perturbation analysis of nonlinear programs.

COMPUTABLE NUMERICAL BOUNDS FOR LAGRANGE MULTIPLIERS OF STATIONARY POINTS OF NONCONVEX DIFFERENTIABLE NONLINEAR PROGRAMS

O. L. Mangasarian

Consider the constrained optimization problem

(1) $$\text{minimize} \quad f(x) \quad \text{subject to} \quad g(x) \leq 0,\ h(x) = 0$$

where $f : R^n \to R$, $g : R^n \to R^m$ and $h : R^n \to R^k$. It is well known that if a standard constraint qualification [2, 5]

(2) $$\begin{cases} \nabla g_I(\bar{x})z \leqq -e,\ \nabla h(\bar{x})z = 0 \ \text{ for some } \ z \in R^n, \ \text{ and} \\ \text{rows of } \ \nabla h(\bar{x}) \ \text{ are linearly independent} \end{cases}$$

holds at a local solution $\bar{x}$ of (1) at which f, g and h are continuously differentiable, $I = \{i \mid g_i(\bar{x}) = 0\}$, $\nabla g(\bar{x})$, $\nabla g_I(\bar{x})$ and $\nabla h(\bar{x})$ are $m \times n$, $\bar{m} \times n$ and $k \times n$ Jacobian matrices respectively, e is a vector of ones and $\bar{m}$ is the number of elements in I, then $\bar{x}$ is a <u>stationary point</u> of (1), that is it satisfies the Karush-Kuhn-Tucker conditions [2]

(3) $$\nabla f(\bar{x}) + \bar{u}\nabla g(\bar{x}) + \bar{v}\nabla h(\bar{x}) = 0,\ \bar{u}g(\bar{x}) = 0,\ g(\bar{x}) \leqq 0,\ \bar{u} \geqq 0,\ h(\bar{x}) = 0$$

for some Lagrange multipliers $(\bar{u},\bar{v}) \in R^{m+k}$. Let $\bar{W}$ denote the set of all Lagrange multipliers which satisfy (3) for a fixed $\bar{x}$. It follows from Gauvin's theorem [1] that if $\bar{x}$ is a local solution of (1), then $\bar{W}$ is nonempty and bounded if and only if the constraint qualification (2) holds. What we would like to point out in this note is that <u>any</u> z in the set Z of points satisfying the constraint qualification (2) for a fixed $\bar{x}$ provides an explicit numerical bound for all $(\bar{u},\bar{v})$ in $\bar{W}$ as follows:

(4) $$\|\bar{u}\|_p \leqq \nabla f(\bar{x})z$$

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(5) $$\|\bar{v}\|_p \leqq \max_{j\in I}\{\|\nabla f(\bar{x})B\|_p\ ,\ \|(\nabla f(\bar{x}) + (\nabla f(\bar{x})z)\nabla g_j(\bar{x}))B\|_p\}$$

where B is the $n \times k$ matrix defined by

(6) $$B := \nabla h(\bar{x})^T(\nabla h(\bar{x})\nabla h(\bar{x})^T)^{-1}$$

and $\|\bar{u}\|_p$ denotes the p-norm $\left(\sum_{j=1}^{m} |\bar{u}_j|^p\right)^{1/p}$ for $p \in [1,\infty)$ and $\|\bar{u}\|_\infty = \max_{1\leqq j\leqq m} |\bar{u}_j|$. In particular we have the following.

1. Theorem. Let $\bar{x}$ be a stationary point of (1). The corresponding nonempty set of all Lagrange multipliers $\bar{W}$ satisfying the Karush-Kuhn-Tucker conditions (3) is bounded if and only if the constraint qualification (2) holds, in which case each $(\bar{u},\bar{v})$ in $\bar{W}$ is bounded by (4) - (5) for $p \in [1,\infty]$.

Proof. The nonempty set $\bar{W}$ is bounded if and only if there exists no (u_I,v) satisfying

(7) $$u_I\nabla g_I(\bar{x}) + v\nabla h(\bar{x}) = 0,\ u_I \geqq 0,\ (u_I,v) \neq 0$$

which by a theorem of the alternative [3, Theorem 1(i') & (iii)], is equivalent to the constraint qualification (2). Hence for such a case we have for $(\bar{u},\bar{v}) \in \bar{W}$ and $p \in [1,\infty]$ that

(8) $$\|\bar{u}\|_p \leqq \|\bar{u}\|_1 \leqq \max_{(u_I,v)\in R^{\bar{m}+k}} \{eu_I \mid u_I\nabla g_I(\bar{x}) + v\nabla h(\bar{x}) + \nabla f(\bar{x}) = 0,\ u_I \geqq 0\}$$

(8a) $$= \min_{z\in R^n} \{\nabla f(\bar{x})z \mid \nabla g_I(\bar{x})z \leqq -e,\ \nabla h(\bar{x})z = 0\}$$

(By linear programming duality)

$$\leqq \nabla f(\bar{x})z \quad \text{for} \quad z \in Z$$

which establishes (4).

Now, for any $(\bar{u},\bar{v}) \in \bar{W}$, $z \in Z$ and $p \in [1,\infty]$ we have that

(9) $$\|\bar{v}\|_p \leqq \max_{v,u_I} \{\|v\|_p \mid -v\nabla h(\bar{x}) = \nabla f(\bar{x}) + u_I\nabla g_I(\bar{x}),\ u_I \geqq 0\}$$

$$\leqq \max_{v,u_I} \{\|v\|_p \mid v = -(\nabla f(\bar{x}) + u_I\nabla g_I(\bar{x}))B,\ u_I \geqq 0,\ eu_I \leqq \nabla f(\bar{x})z\}$$

$$= \max_{u_I}\{\|(\nabla f(\bar{x}) + u_I \nabla g_I(\bar{x}))B\|_p \mid u_I \geqq 0,\ eu_I \leqq \nabla f(\bar{x})z\}$$

$$= \max_{j \in I}\{\|\nabla f(\bar{x})B\|_p\ ,\ \|(\nabla f(\bar{x}) + (\nabla f(\bar{x})z)\nabla g_j(\bar{x}))B\|_p\}$$

where the last equality follows from the fact that the maximum of a continuous convex function on a bounded polyhedral set is attained at a vertex [7, Corollary 32.3.4]. This establishes the bound (5).

□

2. Corollary. The bounds (4) - (5) of Theorem 1 can be sharpened by replacing z by $\bar{z}$ where $\bar{z}$ is a solution of the solvable linear program (8a).

We note that the bound (4) with $p = 1$ and $z = \bar{z}$, where $\bar{z}$ is a solution of (8a) is implicitly given in the elegant proof of Gauvin [1] which characterizes the nonemptiness and boundedness of $\bar{W}$ for a local solution $\bar{x}$ of (1) by the satisfaction of the constraint qualification (2).

It is interesting to note that the first part of the constraint qualification (2) (existence of z) gives an achievable bound on $\|\bar{u}\|_1$, whereas the second part of (2) (linear independence of the rows of $\nabla h(\bar{x})$) gives a bound on $\|\bar{v}\|_p$, which is not necessarily achievable. It is however possible (but impractical for large k) to compute $\max_{(u,v)\in\bar{W}} \|v\|_\infty$ by solving $2k$ linear programs: $\max_{1\leqq i\leqq k}\ \max_{(u,v)\in\bar{W}} \pm v_i$. However to obtain $\max_{(u,v)\in\bar{W}} \|v\|_1$ one is faced with the essentially impossible task (even for a moderate-sized $k \geqq 15$) of solving 2^k linear programs: $\max_{c\in C}\ \max_{(u,v)\in\bar{W}} cv$, where C is the set of 2^k vertices of the cube $\{y \mid y \in R^k,\ -e \leqq y \leqq e\}$. In fact for integer $p \geqq 1$ the problem $\max_{(u,v)\in\bar{W}} \|v\|_p$ has been shown to be an intractable NP-complete problem [6]. We finally note that the methods of [4] could also be used to obtain the bounds of this work.

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