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THE CAUCHY PROBLEM FOR  $U_T = \Delta U(x)$  WHEN  $0 < M < 1$   
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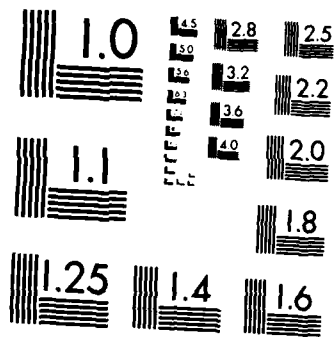
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WHEN  $0 < m < 1$

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THE CAUCHY PROBLEM FOR  $u_t = \Delta u^m$  WHEN  $0 < m < 1$

Miguel A. Herrero\*† and Michel Pierre\*\*††

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ABSTRACT

This paper deals with the Cauchy problem for the nonlinear diffusion equation  $\frac{\partial u}{\partial t} - \Delta(u|u|^{m-1}) = 0$  on  $(0, \infty) \times \mathbb{R}^N$ ,  $u(0, \cdot) = u_0$  when  $0 < m < 1$  (fast diffusion case). We prove that there exists a global time solution for any locally integrable function  $u_0$ : hence, no growth condition at infinity for  $u_0$  is required. Moreover the solution is shown to be unique in that class. Behavior at infinity of the solution and  $L_{loc}^\infty$ -regularizing effects are also examined when  $m \in (\max\{(N-2) \setminus N, 0\}, 1)$ .

AMS(MOS) Subject Classification: 35Q20, 35K55, 76X05.

Key Words: Cauchy Problem, nonlinear diffusion, initial-value problem, regularizing effects.

Work Unit Number 1 - Applied Analysis

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## SIGNIFICANCE AND EXPLANATION

This paper deals with the Cauchy problem for the nonlinear diffusion equation

$$\begin{aligned} \text{(CP)} \quad & \frac{\partial u}{\partial t} - \Delta (u|u|^{m-1}) = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^N \\ & u(0, \cdot) = u_0(\cdot) \quad \text{on } \mathbb{R}^N \end{aligned}$$

when  $0 < m < 1$  (case of a fast nonlinear diffusion).

Let us recall that the corresponding linear Cauchy problem ( $m = 1$  in the above) has a solution, at least on a small time interval, if  $u_0$  does not grow too fast at infinity, namely if

$$\exists c > 0, \int_{\mathbb{R}^N} |u_0(x)| e^{-c|x|^2} dx < \infty.$$

This condition is known to be necessary for the existence of nonnegative solutions. Similarly the above Cauchy problem is well-posed when  $m > 1$  only in a class of data  $u_0$  satisfying some adequate growth condition at infinity.

It turns out that, for  $0 < m < 1$ , no growth condition is required on  $u_0$ . We prove here that for any locally integrable  $u_0$  on  $\mathbb{R}^N$ , (CP) has a solution which is global in time. Moreover, the solution is unique in that class.

This paper fills in the missing gap in the understanding of the Cauchy problem for nonlinear diffusion equations of above type and emphasizes once more the singular structure of this model when  $0 < m < 1$ . Such situations arise in some problems in plasma physics.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the authors of this report.

THE CAUCHY PROBLEM FOR  $u_t = \Delta u^m$  WHEN  $0 < m < 1$

Miguel A. Herrero\*† and Michel Pierre\*\*††

1. INTRODUCTION :

This paper deals with the following Cauchy problem

$$(1.1) \quad \frac{\partial u}{\partial t} = \Delta(u|u|^{m-1}) \quad \text{on} \quad (0, \infty) \times \mathbb{R}^N$$

$$(1.2) \quad u(0, \cdot) = u_0$$

where

$$(1.3) \quad 0 < m < 1$$

and

$$(1.4) \quad u_0 \in L^1_{loc}(\mathbb{R}^N) .$$

Equation (1.1) has been suggested as a mathematical model for a lot of physical problems. We will not recall them here and we refer to the survey [15] where the very extensive literature on (1.1) is summarized.

Our goal is to emphasize some features of (1.1) when  $m < 1$ . Thus our main result claims that the Cauchy problem (1.1), (1.2) has a global solution in time for any  $u_0 \in L^1_{loc}(\mathbb{R}^N)$ . This is in sharp contrast with the case  $m \geq 1$  where some growth condition at infinity is required on  $u_0$  to provide even a local solution in time, namely

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• If  $m = 1$ , there exists  $c > 0$  such that

$$\int_{\mathbb{R}^N} e^{-c|x|^2} |u_0(x)| dx < \infty .$$

• If  $m > 1$

$$\sup_{R \geq 1} R^{-(N+2/(m-1))} \int_{\{x; |x| \leq R\}} u_0(x) < \infty .$$

Note that these conditions are necessary to obtain non negative solutions. The necessity is proved in [2] and the sufficiency in [8] for  $m > 1$  (see [12] for  $m = 1$ ).

Here we prove that, if  $0 < m < 1$ , there is a local estimate of the form

$$(1.5) \quad \int_{\{x; |x| \leq R\}} |u(t,x)| dx \leq F\left(t,R, \int_{\{x; |x| \leq 2R\}} |u_0(x)| dx\right),$$

where  $F$  is bounded on bounded subsets of  $\mathbb{R}^3$  (see (2.4) for its precise form).

Combined with classical monotonicity properties, this suffices to establish existence for any  $u_0 \in L^1_{loc}(\mathbb{R}^N)$  independently of the behavior of  $u_0(x)$  for  $|x|$  large. The estimate (1.5) relies essentially on an idea introduced in [3]. It has been extensively used for semi-linear problems in [3], [4] and also recently in [7]. The same kind of local estimates enables us to prove uniqueness of strong solutions for all  $0 < m < 1$  (see [8] and [10] when  $m > 1$ ).

In the case when

$$(1.6) \quad (N-2)^+/N = \max((N-2)/N, 0) < m < 1,$$

we obtain here more precise results on the solutions of (1.1).

i) We exhibit a regularizing effect from  $L^1_{loc}(\mathbb{R}^N)$  into  $L^\infty_{loc}(\mathbb{R}^N)$ , that is, an estimate of the form

$$(1.7) \quad \sup_{|x| \leq R} |u(t,x)| \leq F\left(t,R, \int_{\{x; |x| \leq 2R\}} |u_0(x)| dx\right) \quad \forall t > 0 .$$

Conditions under which solutions become bounded are also discussed.

ii) We look at the behavior for  $|x|$  large of the non-negative solutions.

The precise results and more comments are to be found in the next section. Their proofs are the content of section 3.

We thank J.L. VAZQUEZ for some useful discussions and comments.

2. THE RESULTS :

By a solution of (1.1), we mean a function  $u$  satisfying

$$(2.1) \quad u \in C([0, \infty[; L^1_{loc}(\mathbb{R}^N))$$

$$(2.2) \quad \frac{\partial u}{\partial t} - \Delta(u |u|^{m-1}) = 0 \text{ in } \mathcal{D}'([0, \infty[ \times \mathbb{R}^N) .$$

Since  $0 < m < 1$ , (2.1) implies that  $|u|^m$  is locally integrable on  $[0, \infty[ \times \mathbb{R}^N$ . Therefore (2.2) makes sense in the space of distributions on  $]0, \infty[ \times \mathbb{R}^N$ . Throughout this paper, we shall write  $u(t, x)$  or  $u(t)$  to designate such a function.

Let  $B_R = \{x \in \mathbb{R}^N; |x| \leq R\}$ . We will write

$$\int_{B_R} f = \int_{B_R} f(x) dx .$$

THEOREM 2.1. :

Let  $u_0 \in L^1_{loc}(\mathbb{R}^N)$ . Then there exists  $u$  satisfying (2.1), (2.2) and

$$(2.3) \quad u(0, \cdot) = u_0$$

$$(2.4) \quad \forall t > 0, \forall R > 0 \quad \int_{B_R} |u(t)| \leq C \left[ \int_{B_{2R}} |u_0| + t^\alpha R^{-\gamma} \right]$$

where

$$(2.5) \quad \alpha = 1/(1-m), \quad \gamma = 2/(1-m) - N$$

and  $C = C(N, m)$ .

In the case when

$$(2.6) \quad (N-2)^+ / N < m < 1 ,$$

the solutions obtained above are locally bounded. More precisely :

THEOREM 2.2. :

Assume (2.6) holds. Let  $u_0 \in L^1_{loc}(\mathbb{R}^N)$ . Then there exists  $u$  solution of (2.1), (2.2) with  $u(0, \cdot) = u_0$  and such that  $\forall t, R > 0$

$$(2.7) \quad \sup_{x \in B_R} |u(t, x)| \leq C \left[ t^{-\theta} \left[ \int_{B_{4R}} |u_0| \right]^{2\theta/N} + (t/R^2)^\alpha \right]$$

where  $\theta^{-1} = m - 1 + 2/N$  and  $C = C(m, N)$ .

By slightly adapting the proof of Theorem 2.1 and 2.2, it will follow that if (2.6) holds, the solution obtained  $u(t)$  is bounded in  $\mathbb{R}^N$  for each  $t > 0$  if for some  $p > 0$

$$(2.8) \quad \|u_0\|_p \equiv \sup_{\xi \in \mathbb{R}^N} \int_{|x-\xi| \leq p} |u_0(x)| dx < +\infty.$$

We prove that this condition is necessary for non-negative solutions. The same result was established for  $m > 1$  in [8] and [2]; (for  $m = 1$ , it is obvious from the usual representation of the solution in terms of  $u_0$ ).

We next state a uniqueness result for (1.1), (1.2). For technical reasons, we are compelled to deal with strong solutions by which we mean a function  $u$  satisfying (2.1), (2.2) and

$$(2.9) \quad \frac{\partial u}{\partial t} \in L^1_{loc} (]0, \infty[ \times \mathbb{R}^N).$$

THEOREM 2.3. :

Let  $u, \hat{u}$  satisfy (2.1), (2.2) and (2.9) then

$$u(0, \cdot) = \hat{u}(0, \cdot) \Rightarrow u \equiv \hat{u}.$$

Remarks :

We will see that, if  $u_0 \geq 0$  and (2.6) holds, by construction  $u(t) > 0$  on  $\mathbb{R}^N$  and

$$(2.10) \quad -\frac{\theta u}{t} \leq u_t \leq \frac{\alpha u}{t}$$

for all  $t > 0$ , (this comes from the results in [1]). As a consequence, by standard regularity results  $u \in C^\infty (]0, \infty[ \times \mathbb{R}^N)$  and (2.9) is more than satisfied. If  $u$  does not have a sign, using again the arguments in [1] one can still conclude that (2.9) holds if  $u$  is continuous and  $\frac{\partial u}{\partial t}$  is a measure. Methods to prove continuity of  $u$  can be found in [11], [16]. To

prove that  $\frac{\partial u}{\partial t}$  is a measure would require a localization of the results in [6] concerning the estimate of  $\int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial t} \right|$  in terms of  $\int_{\mathbb{R}^N} |u_0|$ . We refer to [9] for uniqueness results when  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ .

Concerning the behaviour of  $u(t,x)$  for large  $|x|$  we have

THEOREM 2.4. :

Assume that (2.6) holds, and let  $u$  be a non-negative function satisfying (2.1), (2.2) and (2.9). Then if  $u \neq 0$  one has for each  $t > 0$  :

$$(2.11) \quad \liminf_{|x| \rightarrow \infty} |x|^{2\alpha} u(t,x) \geq (2m\gamma t)^\alpha$$

where  $\alpha, \gamma$  are defined in (2.5).

Remarks :

Property (2.11) is in striking contrast with the behaviour of non-negative solutions of (1.1) when  $m > 1$  : In fact in this case a compactly supported datum  $u_0$  gives raise to a (spatially) compactly supported solution  $u(t,x)$  (see [14]). However in the linear case  $m = 1$  it is well known that the minimal rate of spatial decay for non-negative, non zero solutions is precisely that of the fundamental solution ([19]). Notice that (2.11) does not involve the initial datum  $u_0$ . Its sharpness can be checked on the explicit solution given in [5]

$$(2.12) \quad U_a(t,x) = t^{-\theta} \left\{ a + (2m\gamma)^{-1} \cdot |x|^2 \cdot t^{-\frac{2\theta}{N} - \alpha} \right\}$$

where  $\theta$  is given in (2.7) and  $a$  is an arbitrary positive constant. The total mass  $M(a) = \int_{\mathbb{R}^N} U_a(t,x) dx$  is a time invariant which depends only on  $a$  for given  $m$  and  $N$ . One easily checks in (2.12) that the behaviour for large  $|x|$  does not depend on  $M$  and realizes the equality in (2.11). In the case when  $N = 1$  and  $u_0 \in L^1(\mathbb{R}^N)$  stronger results about this asymptotic behaviour have been recently obtained in [17]. In particular, equality holds in (2.11) under these assumptions when  $u_0$  is compactly supported.

3. THE PROOFS :

Lemma 3.1. :

Let  $u, \hat{u}$  satisfy (2.1), (2.2) with  $u \geq \hat{u}$ . Then, for all  $R > 0$  and  $t, s \geq 0$

$$(3.1) \quad \int_{B_R} [u(t) - \hat{u}(t)] \leq C \left[ \int_{B_{2R}} [u(s) - \hat{u}(s)] + |t-s|^\alpha R^{-\gamma} \right]$$

where  $\alpha, \gamma$  are given in (2.5) and  $C = C(m, N)$ .

Remark :

Actually, this Lemma proves that any non-negative solution of (2.1), (2.2) satisfies the local estimate (2.4) and not only the one we will construct below. Moreover, the proof we are going to give could be applied as well to solutions of the same equation in open cylinders containing  $[t, s] \times \overline{B_{2R}}$ . Existence on the whole space is not required. Note that in view of the explicit solutions (2.12), exponents  $\alpha, \gamma$  in (3.1) are sharp.

Proof :

From (2.1), (2.2) applied to  $u$  and  $\hat{u}$ , we have for any  $\psi \in C_0^\infty(\mathbb{R}^N)$  and  $\alpha \in C_0^\infty(0, \infty)$

$$-\int_0^\infty \int_{\mathbb{R}^N} \alpha' \psi (u - \hat{u}) = \int_0^\infty \int_{\mathbb{R}^N} \alpha \Delta \psi (u |u|^{m-1} - \hat{u} |\hat{u}|^{m-1})$$

which implies

$$(3.2) \quad \frac{d}{dt} \int_{\mathbb{R}^N} \psi (u(t) - \hat{u}(t)) = \int_{\mathbb{R}^N} \Delta \psi (u |u|^{m-1} - \hat{u} |\hat{u}|^{m-1})$$

in  $\mathcal{D}'(0, \infty)$  and therefore in  $L^1_{loc}(0, \infty)$  as well. Since

$$(r|r|^{m-1} - s|s|^{m-1}) \leq 2^{1-m}(r-s)^m \quad \forall r \geq s,$$

(3.2) implies

$$\left| \frac{d}{dt} \int \psi (u(t) - \hat{u}(t)) \right| < 2^{1-m} \int |\Delta \psi| (u - \hat{u})^m.$$

We set  $v = u - \hat{u}$ . By Hölder's inequality, we obtain

$$(3.3) \quad \left| \frac{d}{dt} \int \psi v(t) \right| \leq C(\psi) \left[ \int \psi v(t) \right]^m$$

where

$$C(\psi) = \left[ 2 \int_{\mathbb{R}^N} |\Delta \psi|^\alpha \psi^{-\alpha m} \right]^{1-m}.$$

By integrating the differential inequality (3.3), we are led to

$$(3.4) \quad \forall s, t \geq 0 \quad \left[ \int \psi v(t) \right]^{1-m} \leq \left[ \int \psi v(s) \right]^{1-m} + (1-m) C(\psi) |t-s| .$$

If one can choose  $\psi \in C_0^\infty(\mathbb{R}^N)$  such that

$$(3.5) \quad 0 \leq \psi \leq 1, \quad \psi = 1 \text{ on } B_R, \quad \psi = 0 \text{ outside } B_{2R}$$

and

$$(3.6) \quad C(\psi) \leq C(m, N) R^{-\gamma(1-m)}$$

one obtains the desired estimate (3.1). By setting  $\psi(x) = \psi_0(x/R)$ , since by change of variable

$$C(\psi) = R^{-2+N(1-m)} C(\psi_0) = R^{-\gamma(1-m)} C(\psi_0),$$

we are reduced to the case  $R = 1$ . We then choose for instance  $\psi_0 = \varphi^k$  where  $k$  is an integer  $\geq 2\alpha$  and  $\varphi \in C_0^\infty(\mathbb{R}^N)$  satisfies (3.6) with  $R = 1$ . Then we verify

$$C(\psi_0) \leq C'(m, N) \int (|\Delta \varphi|^\alpha + |\nabla \varphi|^{2\alpha}) \leq C(m, N) .$$

Proof of Theorem 2.1. :

The starting point is the following classical result (see [15]) : for any  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , there exists  $u \in C([0, \infty[; L^1(\mathbb{R}^N)) \cap L^\infty([0, \infty[ \times \mathbb{R}^N)$  solution of (2.2) with  $u(0, \cdot) = u_0$ . Moreover the mapping  $u_0 \mapsto u$  is non-decreasing.

Now, let  $u_0 \in L_{loc}^1(\mathbb{R}^N)$  and  $u_0^+ = \sup(u_0, 0)$ ,  $u_0^- = \sup(-u_0, 0)$ . We denote by  $(v^n)$  and  $(v_p)$  two sequences of non-negative functions of  $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  increasing to  $u_0^+$  and  $u_0^-$  respectively.

Let  $p$  be fixed and let us denote by  $u^n$  (resp.  $w^n$ ) the solution of (2.1), (2.2) with  $u^n(0, \cdot) = v^n - v_p$  (resp.  $w^n(0, \cdot) = v^n$ ). By monotonicity  $u^n \leq w^n$ , which implies  $(u^n)^+ \leq w^n$  since  $w^n \geq 0$ . By Lemma 3.1 applied with  $u = w^n$  and  $\hat{u} \equiv 0$ , we have for all  $R, t > 0$

$$(3.7) \quad \int_{B_R} (u^n)^+(t) \leq \int_{B_R} w^n(t) \leq C \left[ \int_{B_{2R}} u_0^+ + t^\alpha R^{-\gamma} \right] .$$

Since  $n \mapsto u^n$  is non-decreasing, it follows that, when  $n \uparrow \infty$ ,  $u^n(t)$  increases for all  $t$  to some function  $u_p(t) \in L^1_{loc}(\mathbb{R}^N)$ . Obviously  $u_p$  is also solution of (2.2) and satisfies

$$(3.8) \quad \int_{B_R} (u_p)^+(t) \leq C \left[ \int_{B_{2R}} u_0^+ + t^\alpha R^{-\gamma} \right].$$

It remains to show that  $u_p$  satisfies (2.1). We remark that, for all  $s, t \geq 0$

$$(3.9) \quad \int_{B_R} |u_p(t) - u_p(s)| \leq \int_{B_R} [u_p(t) - u^n(t)] + \int_{B_R} |u^n(t) - u^n(s)| + \int_{B_R} [u_p(s) - u^n(s)].$$

By Lemma 3.1 applied with  $u$  replaced by  $u^k$  and  $\hat{u}$  by  $u^n$  ( $k \geq n$ ) and after letting  $k$  tend to  $\infty$ , we obtain

$$(3.10) \quad \int_{B_R} [u_p(t) - u^n(t)] \leq C \left[ \int_{B_{2R}} [u_p(s) - u^n(s)] + |t-s|^\alpha R^{-\gamma} \right].$$

By (3.9), (3.10) and the fact that  $u^n \in C([0, \infty[; L^1(\mathbb{R}^N))$

$$\limsup_{t \rightarrow s} \int_{B_R} |u_p(t) - u_p(s)| \leq (1+C) \int_{B_{2R}} [u_p(s) - u^n(s)].$$

We obtain the continuity of  $u_p(t)$  in  $L^1_{loc}(\mathbb{R}^N)$  by letting  $n$  go to  $\infty$  in this inequality.

Now we let  $p$  tend to  $\infty$  and by a monotone (decreasing) process exactly similar to the previous one, we obtain a solution  $u$  to (2.1), (2.2) with  $u(0, \cdot) = u_0$ . As in (3.8), we have

$$\int_{B_R} u^-(t) \leq C \left[ \int_{B_{2R}} u_0^- + t^\alpha R^{-\gamma} \right].$$

This together with (3.8) and  $u \leq u_p$  yields the estimate (2.4).

The proof of Theorem 2.2 is based on the result of Theorem 2.1 and the deep pointwise estimate established by Aronson-Bénilan [1], namely:

- [11] DI BENEDETTO E., Continuity of weak solutions to a general porous medium equation. Indiana Univ. Math. J. Vol. 32, n° 1 (1983), 83-118.
- [12] FRIEDMAN A., Partial differential equations of parabolic type. Prentice Hall, Englewood, Cliffs N.J. (1964).
- [13] KATO T., Schrödinger operators with singular potentials. Israel J. of Math. 13 (1972), 135-148.
- [14] OLEINIK O.A., A.S. KALASHNIKOV and C. YU LIN, The Cauchy problem and boundary problems for equations of the type of unsteady filtration. Izv. Akad. Nauk. SSSR, Ser. Mat. 22 (1958), 667-704.
- [15] PELETIER L.A., The porous medium equation. Application of non-linear analysis in the Physical Sciences (H. Amann et al. Editors) Pitman, London (1981), 229-241.
- [16] SACKS P., Continuity of solutions of degenerate parabolic equations. Thesis, Univ. of Wisconsin, Madison, U.S.A. (1981).
- [17] VAZQUEZ J.L., Behaviour of the velocity of one dimensional flows in porous media. To appear in Trans. A.M.S.
- [18] VERON L., Effets régularisants de semi-groupes non-linéaires dans des espaces de Banach. Ann. Fac. Sci. Toulouse 1 (1979), 171-200.
- [19] WATSON N.A., The rate of spatial decay of non-negative solutions of linear parabolic equations. Arch. Rat. Mech. Anal. 68, 2 (1978), 121-125.

REFERENCES

- [1] ARONSON D.G. and Ph. BENILAN, Régularité des solutions de l'équation des milieux poreux dans  $\mathbb{R}^N$ .  
C.R. Acad. Sci. Paris, Série A, 288 (1979), 103-105.
- [2] ARONSON D.G. and L.A. CAFFARELLI, The initial trace of a solution of the porous medium equation.  
Trans. A.M.S. 280, 1 (1983), 351-66.
- [3] BARAS P. and M. PIERRE, Singularités éliminables pour des équations semi-linéaires.  
Ann. Inst. Fourier, Grenoble (1984).
- [4] BARAS P. and M. PIERRE, Problèmes paraboliques semi-linéaires avec données mesures.  
To appear in Applicable Analysis.
- [5] BARENBLATT G.I., On some unsteady motions of a liquid or a gas in a porous medium.  
Prikl. Mat. Mekh. 16 (1952), 67-78 (Russian).
- [6] BENILAN Ph. and M.G. CRANDALL, Regularizing effects of homogeneous evolution equations. Contributions to Analysis and Geometry, D.N. Clark and al. edit., J. Hopkins Univ. Press, Baltimore, Md. (1981), 23-30
- [7] BREZIS H., Semilinear equations in  $\mathbb{R}^N$  without conditions at infinity.  
To appear.
- [8] BENILAN Ph., M.G. CRANDALL and M. PIERRE, Solutions of the porous medium equation in  $\mathbb{R}^N$  under optimal conditions on initial values.  
Indiana Un. Math. J. 33 (1984), 51-87.
- [9] BREZIS H. and M.G. CRANDALL, Uniqueness of solutions of the initial value problem for  $u_t - \Delta \phi(u) = 0$ .  
J. Math. Pures et Appl. 56 (1979), 153-163.
- [10] DAHLBERG B.J. and C.E. KENIG, Non-negative solutions of the porous medium equation.  
To appear.

Proof of Lemma 3.4. :

Let  $\psi \in C_0^\infty(\mathbb{R}^N)$ ,  $\psi \geq 0$ . By Kato's inequality

$$\frac{\partial}{\partial t}(v-w)^+ \leq \Delta(v^m - w^m)^+ \text{ on } S$$

so that, thanks to (3.35)

$$\frac{\partial}{\partial t} \int_{|x| \geq 1} (v-w)^+ \psi \leq \int_{|x| \geq 1} (v^m - w^m)^+ \Delta \psi \leq C(\psi) \left[ \int_{|x| \geq 1} |v-w|^m \right]^m$$

where again

$$C(\psi) = 2 \left[ \int_{\mathbb{R}^N} |\Delta \psi|^\alpha \psi^{-\alpha m} \right]^{1-m}.$$

Repeating the arguments in Lemma 3.1 and using (3.36), we deduce

$$\forall t \in (\tau, T), \quad \forall R > 1 \quad \int_{1 < |x| < R} (v-w)^+(t) \leq C(t-\tau)^\alpha R^{-\gamma}.$$

We let  $R$  tend to  $\infty$  to conclude.

Fix  $0 < \tau < T$ . Since  $u(t,x)$  is continuous and positive in  $S$ , there exists  $\delta = \delta(\tau, T) > 0$  such that

$$(3.38) \quad \delta = \min u(t,x) \quad \tau \leq t \leq T, \quad |x| \leq 1.$$

We now select  $\mu > 0$  such that

$$(3.39) \quad U_\mu(t-\tau, x) \leq \delta \quad \text{when } \tau \leq t \leq T, \quad |x| \geq 1/2.$$

To this aim, we need according to the definition of  $U_\mu$

$$(3.40) \quad \mu^{2\alpha} (t-\tau)^{-\theta} \left\{ 1 + b\mu^2 |x|^2 (t-\tau)^{-2\theta/N} \right\}^{-\alpha} \leq \delta$$

or

$$\delta^{m-1} \leq \mu^{-2} (t-\tau)^{\theta(1-m)} + b |x|^2 (t-\tau)^{-1}$$

for  $\tau \leq t \leq T$  and  $|x| \geq 1/2$  which is implied by

$$(3.41) \quad \delta^{m-1} \leq \mu^{-2} (t-\tau)^{\theta(1-m)} + b(t-\tau)^{-1}/4.$$

But this function of  $t$  is bounded below by  $C b^{N(1-m)/2} \mu^{-N/\theta}$  where  $C = C(m, N) > 0$ . Thus (3.41) is satisfied if we choose  $\mu$  such that

$$\mu \leq C b^{\theta(1-m)/2} \delta^{(1-m)\theta/N}.$$

Since  $U_\mu(t-\tau, x) = 0$  for  $t = \tau$ ,  $|x| \geq 1$ , by (3.39), (3.38) and Lemma 3.4

$$U_\mu(t-\tau, x) \leq u(t,x) \quad \forall \tau < t < T, \quad |x| \geq 1$$

whence

$$(3.42) \quad \liminf_{|x| \rightarrow \infty} |x|^{2\alpha} u(t,x) \geq \liminf_{|x| \rightarrow \infty} |x|^{2\alpha} U_\mu(t-\tau, x) = \left[ \frac{2mN(t-\tau)}{(1-m)\theta} \right]^\alpha.$$

Since the right-hand side of (3.42) does not depend on  $\mu$ , we can let  $\tau$  tend to 0 and  $T \rightarrow \infty$ , so that (2.11) holds (note that  $\gamma = N/(1-m)\theta$ ).

Proof of Theorem 2.4. :

The idea is to show that any non-negative solution of (2.1), (2.2), (2.9) is, for  $|x|$  large, bigger than some of the similarity solutions

$$U_{\mu}(t,x) = \mu^{2\alpha} U(t,\mu x) \quad (\alpha = 1/(1-m))$$

where  $U(t,x) \equiv U_1(t,x)$  i.e. (see (2.12))

$$U(t,x) = t^{-\theta} \left( 1 + b |x|^2 t^{-2\theta/N} \right)^{-\alpha}$$

with  $\theta^{-1} = m - 1 + 2/N$ ,  $b = (2m\gamma)^{-1}$ ,  $\gamma$  and  $\alpha$  as in (2.5).

By the uniqueness result of Theorem 2.3,  $u$  is necessarily the solution obtained in Theorem 2.2. One knows that, if  $u_0 \geq 0$ ,  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and  $m > (N-2)^+/N$ , then the corresponding solution is positive for all  $t > 0$  (see [1]). Hence, by construction,  $u > 0$  for all  $t > 0$ . Since it is also bounded and satisfies

$$u_t = \operatorname{div} \left( u^{-(1-m)} \nabla u \right)$$

by classical arguments  $u$  is  $C^\infty$  on  $(0, \infty) \times \mathbb{R}^N$  (see [12]).

Now we use the following comparison Lemma

Lemma 3.4. :

Let  $0 \leq \tau < T$ ,  $S = ]\tau, T[ \times \{x \in \mathbb{R}^N; |x| > 1\}$ . Assume  $v, w$  are non-negative and  $C^\infty$  on a neighborhood of  $\bar{S}$  and satisfy

$$(3.34) \quad \frac{\partial v}{\partial t} = \Delta v^m, \quad \frac{\partial w}{\partial t} = \Delta w^m \quad \text{on } S$$

$$(3.35) \quad v(t,x) \leq w(t,x) \quad \tau < t < T, \quad |x| = 1$$

$$(3.36) \quad v(\tau,x) \leq w(\tau,x) \quad |x| \geq 1.$$

Then

$$(3.37) \quad v \leq w \quad \text{in } S.$$

Let us assume this lemma and continue.

$$\sup_{x \in B_R(\xi)} u(t, x) \leq C \left[ t^{-\theta} \left( \int_{B_{4R}(\xi)} u_0 \right)^{\frac{2\theta}{N}} + (t/R^2)^{\frac{1}{1-m}} \right].$$

Now the result follows by choosing  $R = 1/4$  and letting  $\xi$  vary over  $\mathbb{R}^N$ .

As to the necessity, we use the two-sided estimate (3.1) with  $\hat{u} = 0$  and  $u(t, x)$  replaced again by  $u(t, x+\xi)$  to get

$$\int_{B_1(\xi)} u_0 \leq C \left[ \int_{B_2(\xi)} u(t) + t^{1/(1-m)} \right]$$

which yields, if for instance  $u(1) \in L^\infty(\mathbb{R}^N)$  :

$$\sup_{\xi \in \mathbb{R}^N} \int_{B_1(\xi)} u_0 \leq C \left[ \|u(1)\|_\infty + 1 \right].$$

Remark :

When  $u_0 \geq 0$  and  $m \geq 1$  condition (2.8) is known to be necessary and sufficient for solutions of (1.1), (1.2) to be globally bounded at each positive time. For  $m > 1$ , sufficiency has been proved in [8] whereas necessity follows from the Harnack type inequality obtained in [2]. When  $m = 1$ , the result follows easily from the usual representation :

$$u(t, x) = \frac{C}{t^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \geq \frac{C}{e t^{N/2}} \int_{|x-y|^2 \leq 4t} u_0(y) dy.$$

Note that proposition 3.2 contains as a particular case the well-known regularizing effect from  $L^1$  into  $L^\infty$  (see for instance [18]).

Integrating in time in (3.31) leads to :

$$|u(t) - \hat{u}(t)| \leq \Delta w(t, x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \quad \text{for each } t > 0 .$$

Thus  $w$  is subharmonic and therefore for all  $\xi \in \mathbb{R}^N$

$$(3.33) \quad w(t, \xi) \leq \frac{C}{R^N} \int_{B_R(\xi)} w(t, x) dx \quad \forall R > 0$$

for some  $C = C(N)$ . The uniqueness results now follows by noticing that, as a consequence of (3.32), the averages on the right-hand side of (3.33) tend to zero as  $R \rightarrow \infty$ . In fact, denoting by  $C$  some constant depending only on  $m$  and  $N$ , one has :

$$\begin{aligned} \int_{B_R(\xi)} w(t, x) dx &\leq C \int_0^t \int_{B_R(\xi)} |u - \hat{u}|^m \leq C \int_0^t R^{N(1-m)} \left( \int_{B_R(\xi)} |u - \hat{u}|(s) \right)^m ds \\ &\leq C R^{N(1-m)} \int_0^t (R^{N-2\alpha} s^\alpha)^m ds \leq C t^{\frac{1}{1-m}} R^{N - \frac{2m}{1-m}} . \end{aligned}$$

We now discuss briefly about boundedness of solutions.

Proposition 3.2. :

Assume (2.6) holds. Then the solution obtained in Theorem (2.2) is bounded for all  $t > 0$  if :

$$\|u_0\|_p = \sup_{\xi \in \mathbb{R}^N} \int_{B_p(\xi)} |u_0(x)| dx < +\infty .$$

Moreover this condition is necessary if  $u_0 \geq 0$  and  $0 < m < 1$ .

Proof :

We can obviously reduce ourselves to the case  $p = 1$ .

We begin with the sufficiency part. Since the solution with initial value  $|u_0|$  bounds the absolute value of the solution with initial value  $u_0$ , it suffices to deal with non-negative solutions and without loss of generality we can assume  $p = 1$ . Let  $\xi \in \mathbb{R}^N$  be arbitrary. We apply Theorem 2.2 to  $u(t, x + \xi)$  for fixed  $t > 0$  to obtain :

Proof of Theorem 2.3. :

Let  $u, \hat{u}$  satisfy (2.1), (2.2), (2.9). Thanks to the regularity assumption (2.9), one can apply Kato's inequality [13]

$$(3.30) \quad -\Delta |u|^{m-1} - \hat{u} |u|^{m-1} \leq -\text{sign}(u-\hat{u}) \Delta(u |u|^{m-1} - \hat{u} |\hat{u}|^{m-1}) .$$

Hence, using (2.2) again, we have

$$(3.31) \quad \frac{\partial}{\partial t} |u-\hat{u}| - \Delta |u|^{m-1} - \hat{u} |\hat{u}|^{m-1} \leq 0 \quad \text{in } \mathcal{D}'(0, \infty) \times \mathbb{R}^N .$$

For any  $\psi \in C_0^\infty(\mathbb{R}^N)$ ,  $0 \leq \psi \leq 1$ , we then obtain

$$\begin{aligned} \frac{d}{dt} \int \psi |u-\hat{u}| &\leq \int |\Delta\psi| |u|^{m-1} - \hat{u} |\hat{u}|^{m-1}|(t) \leq C \int |\Delta\psi| |u-\hat{u}|^m(t) \\ &\leq C(\psi) \left[ \int \psi |u-\hat{u}|(t) \right]^m \end{aligned}$$

where

$$C(\psi) = \left[ 2 \int_{\mathbb{R}^N} |\Delta\psi|^\alpha \psi^{-\alpha m} \right]^{1-m} .$$

Therefore, we are led to a differential inequality of type (3.3) except that it is unilateral here. As in the proof of Lemma 3.1, we deduce

$$\int_{B_R} |u-\hat{u}|(t) \leq C \left[ \int_{B_{2R}} |u-\hat{u}|(0) + R^N (t/R^2)^{\frac{1}{1-m}} \right] .$$

Applying this estimate to  $u(t, x+\xi)$ ,  $\hat{u}(t, x+\xi)$  where  $\xi \in \mathbb{R}^N$  is fixed but otherwise arbitrary and writing  $B_R(\xi) = \{x \in \mathbb{R}^N : |x-\xi| < R\}$  we get :

$$(3.32) \quad \int_{B_R(\xi)} |u-\hat{u}|(t) \leq C \left[ \int_{B_{2R}(\xi)} |u-\hat{u}|(0) + R^N (t/R^2)^{\frac{1}{1-m}} \right] .$$

Assume now that  $u(0) = \hat{u}(0)$ . Then if  $N < \frac{2}{1-m}$  (i.e.,  $m > \frac{(N-2)^+}{N}$ ) uniqueness follows by just letting  $R \rightarrow \infty$  for each fixed  $t > 0$  in (3.32). Otherwise we can argue as follows. Set :

$$w(t, x) = \int_0^t |u|^{m-1} - \hat{u} |u|^{m-1}(s, x) ds .$$

Proof of Theorem 2.2. :

Let  $u_0 \in L^1_{loc}(\mathbb{R}^N)$ . By Theorem 2.1, we know there exists a solution  $u$  of (2.1), (2.2) with  $u(0, \cdot) = u_0$ . This solution has been constructed as the monotone limit of solutions  $u_p^n$  in  $C([0, \infty[; L^1(\mathbb{R}^N)) \cap L^\infty((0, \infty) \times \mathbb{R}^N)$  with  $u_p^n(0, \cdot) = v^n - v_p \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,  $v^n$  increasing to  $u_0^+$  and  $v_p$  to  $u_0^-$ . Moreover  $|u_p^n(t)|$  is bounded above by the non-negative solution with initial datum  $v^n + v_p$ , which increases to  $|u_0|$ .

According to these remarks, it is sufficient to prove (2.7) when  $u_0 \geq 0$  and  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ .

If  $u$  is a solution of (2.2), then so is  $w(t, x) = R^{2\alpha} T^{-\alpha} u(Tt, Rx)$  for any  $R, T > 0$  and  $\alpha = 1/(1-m)$ . By homogeneity, we check that (2.7) is nothing but the same inequality applied to  $w$  at  $t = 1$ .

Now  $w$  satisfies the estimate (3.12). In particular

$$-\Delta w^m(1) \leq Aw(1)$$

so that by Lemma 3.3

$$(3.28) \quad \sup_{|x| \leq 1} w(1, x) \leq C \left[ A^\theta \left[ \int_{|x| \leq 2} w(1) \right]^{2\theta/N} + \int_{|x| \leq 2} w(1) \right]$$

where we again denote by  $C$  any constant depending only on  $m$  and  $N$ .

Now by (2.4) :

$$(3.29) \quad \int_{|x| \leq 2} w(1) \leq C \left[ \int_{|x| \leq 4} w(0) + 1 \right].$$

Finally (3.28), (3.29) yield

$$\sup_{|x| \leq 1} w(1, x) \leq C \left[ A^\theta \left[ \int_{|x| \leq 4} w(0) \right]^{2\theta/N} + \int_{|x| \leq 4} w(0) + 1 \right].$$

Going back to the original variables this reads

$$\sup_{x \in B_R} |u(t, x)| \leq C \left[ t^{-\theta} \left( \int_{B_{4R}} |u_0| \right)^{2\theta/N} + R^{-N} \int_{B_{4R}} |u_0| + (t/R^2)^\alpha \right].$$

This together with the next consequence of Young's inequality

$$R^{-N} \int_{B_{4R}} |u_0| \leq C(m, N) \left[ t^{-\theta} \left( \int_{B_{4R}} |u_0| \right)^{2\theta/N} + (t/R^2)^\alpha \right],$$

provides the desired estimate (2.7).

If  $N = 2$ , for all  $w \in C_0^\infty(\mathbb{R}^N)$  we have by Sobolev's inequality

$$\int w^2 \leq C \left[ \int |\nabla w|^2 \right].$$

Hence, for  $w > 0$  and all  $\delta > 0$

$$\int w^{2\delta} \leq C \left[ \int |\nabla w^\delta|^2 \right] \leq C\delta^2 \left[ \int w^{\delta-1} |\nabla w|^2 \right] \leq C\delta^2 \left[ \int w^{2(\delta-1)} \right] \left[ \int |\nabla w|^2 \right].$$

For  $w$  with support in  $B_2$ , we then have by Hölder's inequality

$$\int w^{2\delta} \leq C\delta^2 \left[ \int w^{2\delta} \right]^{(\delta-1)/\delta} \int |\nabla w|^2,$$

which implies

$$(3.27) \quad \left[ \int w^{2\delta} \right]^{1/\delta} \leq C\delta^2 \int |\nabla w|^2.$$

This inequality allows us to argue as when  $N \geq 3$ . Here we set

$$p_0 = 1, \quad p_{k+1} = 2p_k + m - 1$$

$$b_k = \max \left\{ \int_{B_k} v^{p_k, 1} \right\}.$$

By (3.15) and (3.27) applied with  $w = v^{(m+p_k-1)/2}$ ,  $k \geq 1$  and  $\delta = \delta_k = (2p_k + m - 1)/(p_k + m - 1)$ , we obtain as above

$$b_{k+1} \leq C^{k+1} (\Lambda + 1)^2 b_k^2.$$

We check independently that this also holds for  $k = 0$ . After iterating this estimate, we deduce

$$\limsup_{k \rightarrow \infty} b_{k+1}^{1/p_{k+1}} \leq C(\Lambda + 1)^\theta b_0^{\theta'}$$

where

$$\theta = \lim_{k \rightarrow \infty} \frac{2 + 2^2 + \dots + 2^{k+1}}{p_{k+1}} = \frac{2}{m}, \quad \theta' = \lim_{k \rightarrow \infty} \frac{2^{k+1}}{p_{k+1}} = \frac{1}{m}.$$

We finish by homogeneity as for  $N \geq 3$ .

The case  $N = 1$  is easier and left to the reader.

Now, let  $\lambda = \int_{B_2} v$  and  $v = \lambda \hat{v}$  so that  $1 = \int_{B_1} \hat{v}$ . We verify that  $\hat{v}$  satisfies the estimate

$$-\Delta \hat{v}^m \leq \Lambda \lambda^{1-m} \hat{v}.$$

Applying (3.25) with  $v$  replaced by  $\hat{v}$  and  $\Lambda$  by  $\Lambda \lambda^{1-m}$  yields

$$\|\hat{v}\|_{L^\infty(B_1)} \leq C(\Lambda \lambda^{1-m} + 1)^\theta \leq C(\Lambda^\theta \lambda^{(1-m)\theta + 1})$$

which implies

$$\|v\|_{L^\infty(B_1)} \leq C \left[ \Lambda^\theta \left[ \int_{B_2} v \right]^{(1-m)\theta + 1} + \int_{B_2} v \right].$$

This is the desired estimate (3.14) since  $(1-m)\theta + 1 = 2\theta/N$ .

To complete the proof when  $N \geq 3$ , we need to establish (3.22) for  $k = 0$ . For this, we use Hölder's inequality and (3.18) with  $p = p_1 = sm$  (recall that  $p_2 = s(sm+m-1)$ ):

$$\begin{aligned} \int v^{p_1} \psi^2 &= \int v^{s/(s+1)} \psi^{2/(s+1)} v^{p_2/(s+1)} \psi^{2s/(s+1)} \\ &\leq \left[ \int v \psi^{2/s} \right]^{s/(s+1)} \left[ \int v^{p_2} \psi^{2s} \right]^{1/(s+1)} \\ &\leq C \left[ \int v \psi^{2/s} \right]^{s/(s+1)} \left[ \Lambda \int v^{p_1} \psi^2 + \left[ \int v^{p_1} \psi^2 \right]^{p_2/sp_1} C(\psi) \right]^{s/(s+1)}. \end{aligned}$$

We deduce that

$$(3.26) \quad \left[ \max \left\{ \int v^{p_1} \psi^2, 1 \right\} \right]^{1/(s+1)} \leq C(\Lambda + C(\psi))^{s/(s+1)} \left[ \int v \psi^{2/s} \right]^{s/(s+1)}.$$

We now choose  $\psi = \psi_0^{p_1/(1-m)}$  with  $\psi_0 \in C_0^\infty$ ,  $0 \leq \psi_0 \leq 1$ ,  $\psi_0 = 1$  on  $B_{3/2}$  and  $\psi_0 = 0$  outside  $B_2$  to obtain

$$\max \left\{ \int_{B_{3/2}} v^{p_1}, 1 \right\} \leq C(\Lambda + 1)^s \left[ \int_{B_2} v \right]^s$$

which is (3.22) with  $k = 0$ .

Now, for any  $k \geq 0$ , we set

$$(3.19) \quad \begin{cases} B_k = \{x \in \mathbb{R}^N ; |x| \leq 1 + 2^{-k}\} \\ p_{k+1} = s p_k - (1-m)s, \quad p_0 = 1 \\ a_k = \int_{B_k} v^{p_k} \end{cases}$$

For  $k \geq 1$ , we choose in (3.18)  $p = p_k$  and  $\psi = \psi_k^{p/(1-m)}$  where  $\psi_k \in C_0^\infty$ ,  $0 \leq \psi_k \leq 1$ ,  $\psi_k = 1$  on  $B_{k+1}$  and  $\psi_k = 0$  outside  $B_k$  so that  $\|\nabla \psi_k\|_\infty \leq C 2^k$  and  $\|\Delta \psi_k\|_\infty \leq C 4^k$ . This implies :

$$(3.20) \quad C(\psi) \leq C \left[ \int |\Delta \psi_k|^{\theta_k} + |\nabla \psi_k|^{2\theta_k} \right]^{1/\theta_k} \leq C 4^k.$$

Hence by (3.18), (3.20) and the fact that  $p_k \leq s^k$ , for all  $k \geq 1$

$$(3.21) \quad a_{k+1} \leq C^{k+1} \left( \Lambda^s a_k^s + a_k^{p_{k+1}/p_k} \right).$$

Since  $p_{k+1}/p_k < s$ , the sequence  $b_k = \max(a_k, 1)$  satisfies

$$(3.22) \quad \forall k \geq 1 \quad b_{k+1} \leq C^{k+1} (\Lambda+1)^s b_k^s \quad (\text{we impose } C \geq 1).$$

We will prove below that this inequality also holds for  $k = 0$ . Let us assume it and continue. By induction

$$(3.23) \quad b_{k+1} \leq C^{k+1+s+\dots+s^k} (\Lambda+1)^{s+s^2+\dots+s^{k+1}} b_0^{s^{k+1}}.$$

But, from (3.19) we also obtain by induction

$$p_{k+1} = s^{k+1} - (1-m)(s+s^2+\dots+s^{k+1}).$$

Hence passing to the limit in (3.23) yields

$$(3.24) \quad \limsup_{k \rightarrow \infty} (b_{k+1})^{1/p_{k+1}} \leq C(\Lambda+1)^\theta b_0^{2\theta/N}$$

where  $\theta^{-1} = m-1 + 2/N$ , that is

$$(3.25) \quad \max\left(\|v\|_{L^\infty(B_1)}, 1\right) \leq C(\Lambda+1)^\theta \max\left(\int_{B_2} v, 1\right)^{2\theta/N}.$$

Proof :

It is sufficient to prove the estimate (3.14) for  $R = 1$ . Indeed, if  $v$  satisfies (3.13), then so does  $w$  defined by

$$w(x) = R^{2/(1-m)} v(Rx) .$$

Now (3.14) is nothing but the same estimate applied to  $w$  with  $R = 1$ .

Let  $\psi \in C_0^\infty(\mathbb{R}^N)$ ,  $0 \leq \psi \leq 1$ . Multiplying (3.13) by  $v^{p-1} \psi^2$  where  $p > 1$  gives after integration by parts

$$\int \nabla v^m \nabla (v^{p-1} \psi^2) \leq \Lambda \int v^p \psi^2 .$$

We deduce

$$(3.15) \quad \frac{4(p-1)m}{(m+p-1)^2} \int |\nabla v^{(m+p-1)/2}|^2 \psi^2 \leq \Lambda \int v^p \psi^2 + \frac{m}{m+p-1} \int v^{m+p-1} |\Delta \psi^2| .$$

Now we assume  $N \geq 3$  and we denote by  $C$  any constant depending only on  $N$  and  $m$ . Using that

$$(3.16) \quad |\nabla (v^{(m+p-1)/2} \psi)|^2 \leq C |\nabla v^{(m+p-1)/2}|^2 \psi^2 + C v^{m+p-1} |\nabla \psi|^2 ,$$

by Sobolev's imbedding, we have with  $s = N/(N-2)$

$$(3.17) \quad \left[ \int v^{s(m+p-1)} \psi^{2s} \right]^{1/s} \leq C \left[ \frac{(m+p-1)^2}{(p-1)m} \Lambda \int v^p \psi^2 + \frac{m+p-1}{p-1} \int v^{m+p-1} |\Delta \psi^2| \right] .$$

By Hölder's inequality, for all  $p \geq sm > 1$  (by 2.6), we have

$$(3.18) \quad \left[ \int v^{s(m+p-1)} \psi^{2s} \right]^{1/s} \leq C \left[ p \Lambda \int v^p \psi^2 + \left[ \int v^p \psi^2 \right]^{p/m} C(\psi) \right]$$

where

$$C(\psi) = \left[ \int \frac{|\Delta \psi^2|^{p/(1-m)}}{\psi^{2(m+p-1)/(1-m)}} \right]^{(1-m)/p} .$$

This inequality shows that  $v$  can be estimated in  $L_{loc}^{s(m+p-1)}$  in terms of bounds of  $v$  in  $L_{loc}^p$ . Repeating this estimate will provide an estimate of  $v$  in  $L_{loc}^\infty$  in terms of an  $L^{sm}$ -norm of  $v$ . We will indicate below how one can start from an  $L^1$ -norm of  $v$ .

Lemma 3.2. :

Assume (2.6) holds. Let  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,  $u_0 \geq 0$  and let  $u$  be the non-negative solution of  $u(0) = u_0$  and

$$(3.11) \quad u \in C([0, \infty[; L^1(\mathbb{R}^N)) \quad , \quad \frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in} \quad \mathcal{D}'((0, \infty) \times \mathbb{R}^N) \quad .$$

Then

$$(3.12) \quad \Delta u^m \geq -\theta u/t \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N$$

with  $\theta = N/(2+N(m-1))$  .

Remark :

Estimate (3.12) can be understood as a strong pointwise inequality since  $u > 0$  and  $u \in C^\infty((0, \infty) \times \mathbb{R}^N)$  as proved in [1].

Now, the  $L_{loc}^\infty$ -regularising effect will be a consequence of (2.4), (3.12) and the next lemma, which is of independent interest.

Lemma 3.3. :

Let  $v$  be non-negative, smooth and satisfying

$$(3.13) \quad -\Delta v^m \leq \Lambda v \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N)$$

where  $(N-2)^+/N < m < 1$  and  $\Lambda > 0$  . Then, for all  $R > 0$

$$(3.14) \quad \|v\|_{L^\infty(B_R)} \leq C \left[ \Lambda^\theta \left[ \int_{B_{2R}} v \right]^{2\theta/N} + R^{-N} \int_{B_{2R}} v \right]$$

where  $\theta^{-1} = m - 1 + 2/N$  and  $C = C(m, N)$  .

Remark :

The method for obtaining  $L^\infty$ -estimates by using the pointwise estimate (3.12) has already been used in [8] where a result similar to Lemma 3.3 is stated (see Proposition 1.3) to treat the case when  $m > 1$  . However the local character of (3.14) (which has not been explicitly established in [8]) does not carry over to the solution of (3.11) if  $m > 1$  . Obviously, this is due to the lack of local estimates like (2.4) in that case.

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20. ABSTRACT (cont.)

for  $u_0$  is required. Moreover the solution is shown to be unique in that class. Behavior at infinity of the solution and  $L_{loc}^\infty$ -regularizing effects are also examined when  $m \in (\max\{N-2, 0\}, 1)$ .

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