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"A 0.487 Throughput Limited Sensing Algorithm"

L. Georgiadis
and
P. Papantoni-Kazakos

Technical Report UCT/DEECS/TR-85-3

March 1985

Department of

Electrical Engineering and Computer Science

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The LSTFA is a refinement of the algorithm developed in (5) and it attains the same throughput as Gallager's algorithm does, without the full feedback sensing requirement in the latter. The algorithm is also easy to implement, it requires reasonable memory storage, it induces uniformly good transmission delays, and it is robust in the presence of feedback errors. In the presence of binary (collision versus noncollision) feedback, the algorithm may attain throughput 0.4493; the highest known to this point, among both full and limited sensing algorithms.

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A 0.487 THROUGHPUT LIMITED SENSING ALGORITHM

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Abstract

~~We~~ The authors consider Poisson packet traffic, accessing a single slotted channel. ~~We~~ They assume the existence of ternary feedback, per channel slot. ~~We~~ They also adopt the limited feedback sensing model, where each packet senses the feedback only while it is blocked. For the above model, ~~we~~ They develop a collision resolution protocol, with last-come first serve characteristics, and we name the protocol, LSTFA.

The LSTFA is a refinement of the algorithm developed in ~~[5]~~ another document and it attains the same throughput as Gallager's algorithm does, without the full feedback sensing requirement in the latter. The algorithm is also easy to implement, it requires reasonable memory storage, it induces uniformly good transmission delays, and it is robust in the presence of feedback errors. In the presence of binary (collision versus noncollision) feedback, the algorithm may attain throughput 0.4493; the highest known to this point, among both full and limited sensing algorithms.

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1. Introduction

Let us consider the case, where a single, slotted, channel is being accessed by a number of independent, identical, and packet-transmitting users. Let us assume that feedback broadcast, per slot, exists. The limited sensing algorithms arise, then, when the users tune to the feedback broadcast, only while they are blocked. In addition to being practically appealing, the limited sensing algorithms are also, in general, more immune to channel errors, as compared to the continuous sensing such algorithms, for the same channel and user model.

Limited sensing algorithms were first considered by Tsybakov and Vvedenskaya [2], for the Poisson user model and for ternary feedback broadcast. The algorithm in [2] induces a throughput equal to 0.384. Vvedenskaya and Tsybakov [3] developed a number of algorithms for both ternary and binary CNC (collision versus noncollision) feedbacks, and they studied the effects of feedback errors on their performance. Georgiadis et al [4] proposed and analyzed limited sensing algorithms for binary CNC and ternary feedbacks, with respective throughputs, 0.42 and 0.425. The latter authors also developed a limited sensing algorithm for ternary feedback [5], with throughput equal to 0.4566, which is basically an interrupted version of Gallager's algorithm [1]. We point out here that Ryter [6] modified the algorithm in [1], for better behavior in the presence of feedback errors. The algortihms in [1] and [6] require full feedback sensing, however.

In this paper, we consider the same model as in [5], and we propose and analyze a limited sensing algorithm that attains throughput 0.487, and induces uniformly good delay characteristics. The algorithm is also robust in the presence of feedback errors. We name the algorithm, Limited Sensing Ternary Feedback Algorithm (LSTFA). We were recently informed that Humblet [7] provided an outline of the LSTFA.

2. The Model

We assume that a single, slotted, channel is being accessed by infinitely many, identical, and independent packet-transmitting users. We model the cumulative packet arrival process as Poisson, we consider the case where the length of a single packet equals the length of a slot, and we assume that a packet transmission can only start at the beginning of some slot. We initially assume that the channel is errorless; that is, errors can occur only due to collision, where collisions correspond to simultaneous transmissions by at least two packets. We assume that a collision results in complete loss of the information included in the involved packets. Thus, retransmission is then necessary. We consider the existence of ternary feedback per slot (emptiness, versus success, versus collision), and we assume limited feedback sensing. That is, each packet tunes to the feedback only while it is blocked. As a result, only limited sensing transmission algorithms can be considered here.

Let time be measured in slot units. Let the integer, T, denote then slot indices, where slot T occupies the transmission time interval $[T, T+1)$. Let x_T denote the ternary feedback corredponding to slot T, where $x_T = 0$, $x_T = 1$, and $x_T = c$ represent respectively, empty, versus busy with a single packet, versus collision slot, T. Given T, let us consider some nontransmitted packet, that arrived in the time interval $[T'-1,T')$, where $T' \leq T$. Then, we denote by, t_a, the exact arrival instant, in $[T'-1,T')$, of the packet, and we assume that the packet observes the feedbacks, x_i; $T'-1 \leq i \leq T-1$, and it does not observe the feedbacks, x_i; $i \leq T'-2$. Thus, given T, each nontransmitted packet, that arrived in $(-\infty,T)$, has observed only part of the channel feedback history, x_i; $i \leq T-1$, and it has observed at least the feedback x_{T-1}.

3. The Algorithm

In this section, we describe a limited sensing algorithm, for the model in section 2. At each point in time, the algorithm distributes the newly arrived and the nontransmitted packets across three classes, A, B, and C. Transitions in time within or across classes and transmissions are controlled by the operations of the algorithm.

Class A contains those packets which cannot yet decide whether or not some collision resolution is in process. Class B contains those packets which know that some collision resolution is in process, but do not know the time when it started. Class C contains those packets which know that some collision resolution is in process, as well as the time when it started. All packets in class C can simultaneously decide which arrival intervals will be chosen for transmission, while packets in classes A and B can not. Packets in classes A and B act essentially identically. This will be evident from the description of the algorithm in this section.

Each nontransmitted packet follows the rules of the algorithm independently, utilizing a set of parameters, R, Δ, L_A, T_1, T_g, and ℓ. Among those, parameters, R and Δ are subject to optimization for the satisfaction of the desirable throughput versus expected delay tradeoff, they are selected a priori, and they are system parameters. Parameters L_A, T_1, T_g, and ℓ are recursively updated, following the rules of the algorithm. Upon arrival, each packet initiates the algorithm independently, following the rules below.

a. Initialization

Let a packet arrive at the time instant t_a, where $t_a \varepsilon [T'-1,T')$. The packet observes then the feedback $x_{T'-1}$, and continuously observes all feedbacks from this point on, until it is successfully transmitted. At T' the packet moves to class 1 below, with initial values $T = T'-t_a$, and $L_A = 0$.

b. Class 1

All packets in class 1 act as follows:

1.1 If $x_T = 1$, set $T \to T+1$, and ,

1.1.1 If $x_T = 0$ or $x_T = 1$, move to class 2, with

$L_A = 1$

1.1.2 If $x_T = c$, move to step 1.2

1.2 If $x_T = c$, set $L_A = 0$, $T \rightarrow T + 1$, and,

1.2.1 If $x_T = 1$, move to step 1.1

1.2.2 If $x_T = c$, move to step 1.2

1.2.3 If $x_T = 0$, move to step 1.3

1.3 If $x_T = 0$, set $L_A \rightarrow L_A + 1$, set $T \rightarrow T + 1$, and,

1.3.1 If $x_T = c$, move to step 1.2

1.3.2 If $x_T = 1$, move to step 1.1

1.3.3 If $x_T = 0$ and $L_A < R + 1$, move to step 1.3

1.3.4 If $x_T = 0$ and $L_A = R + 1$, move to class 2, with $L_A = L_A - 1$

C. <u>Class 2</u>

All packets in class 2 act as follows:

Start with, $T_g = T$, and,

2. $\ell = \Delta$, $T_1 = T - L_A$

Then,

2.1 If $T_g - (T - T_1) \leq \ell$

2.1a) Set $T_g \rightarrow T_{g+1}$, set $T \rightarrow T + 1$, and TRANSMIT

2.1.1 If $x_T = 1$, the packet is successfully transmitted

2.1.2 If $x_T = c$, set $L_A = 0$,

2.1.2. a) Set $\ell \rightarrow \ell/2$, and,

2.1.2.1 If $T_g - (T - T_1) \leq \ell$, move to step 2.1a)

2.1.2.2 If $T_g - (T - T_1) > \ell$, set $T_g \rightarrow T_g+1$, set $T \rightarrow T+1$, and,

2.1.2.2.1 If $x_T = c$, move to step 2.2.3.

2.1.2.2.2 If $x_T = 0$, set $L_A \rightarrow L_A+1$, and,

If $L_A < R$, move to step 2.1.2.a).

If $L_A = R$, move to step 2.1.a).

2.1.2.2.3 If $x_T = 1$, move to step 2.1.a).

2.2 If $T_g - (T - T_1) > \ell$, set $T_g \rightarrow T_g + 1$, set $T \rightarrow T+1$, and,

2.2.1 If $x_T = 0$, set $T_g \rightarrow T_g - \ell$, $L_A \rightarrow L_A + 1$, and,

2.2.1.1 If $L_A < R + 1$, move to step 2.

2.2.1.2 If $L_A = R + 1$, set $L_A \rightarrow L_A - 1$, and move to step 2.

2.2.2 If $x_T = 1$, set $T_g \rightarrow T_g - \ell$,

2.2.2 a) Set $L_A \rightarrow L_A + 1$, set $\ell = \Delta$, set $T_1 = T - L_A$, and,

2.2.2.1 If $T_g - (T - T_1) \leq \ell$, move to step 2.1.a).

2.2.2.2 If $T_g - (T - T_1) > \ell$, set $T_g \rightarrow T_g + 1$, set $T \rightarrow T + 1$, and,

2.2.2.2.1 If $x_T = c$, move to step 2.2.3

2.2.2.2.2 If $x_T = 0$, or $x_T = 1$, set $L_A = 1$, and move to step 2.

2.2.3 If $x_T = c$, set $L_A = 0$,

2.2.3 a) Set $\ell \rightarrow \ell/2$

2.2.3 b) Set $T_g \rightarrow T_g + 1$, set $T \rightarrow T + 1$, and,

2.2.3.1 If $x_T = c$, move to step 2.2.3 a)

2.2.3.2 If $x_T = 0$, set $L_A \rightarrow L_A + 1$, set $T_g \rightarrow T_g - \ell$, and,

2.2.3.2.1 If $L_A < R$, move to step 2.2.3 a).

2.2.3.2.2 If $L_A = R$, move to step 2.2.3 b).

2.2.3.2.3 If $L_A > R$, set $L_A \rightarrow L_A - 1$, and move to step 2.

2.2.3.3 If $x_T = 1$ and $L_A = R$, set $T_g \rightarrow T_g - \ell$, and move to step 2.2.2a).

2.2.3.4 If $x_T = 1$ and $L_A \neq R$, set $T_g \rightarrow T_g + 1$, set $T \rightarrow T + 1$, and,

2.2.3.4.1 If $x_T = c$, move to step 2.2.3.

2.2.3.4.2 If $x_T = 0$ or $x_T = 1$, set $T_g \rightarrow T_g - \ell$, set $L_A = 1$, and move to step 2.

In figures 1 and 2 we present the flow chart of the algorithm. We observe that the storage requirements are reasonable, and that only seven parameters are maintained and updated. Among them, the parameters R and L_A are integers, and as we will discuss in the next section, they correspond to numbers of slots.

4. Qualitative Properties

In the description of the algorithm, class 1 reflects the operations of packets in classes A and B, while class 2 reflects the operations of packets that are placed in class C. The algorithm basically selects arrival intervals for transmission. Let us call examined, arrival intervals that have been resolved by the algorithm. The parameters used by the algorithm are then interpreted as follows:

- R: An upper bound to the number of consecutive empty slots allowed, during the resolution of some initial collision, where $R \geq 1$. By design, no more than R such slots are allowed. Thus, when packets observe R+1 consecutive empty slots, they know that there is no collision resolution in process (step 1.3.4).
- T: The time elapsed from the arrival instant of the packet, to the current time.
- L_A: The number of slots containing packets from class A, from the arrival instant of the packet to the current time. If the slot within which the packet arrived is a slot as above, then it is included in the number L_A.
- T_g: The time elapsed from the arrival instant of the packet to the current time, minus the examined interval after the above arrival instant.
- ℓ: The total length of the arrival interval that is transmitted in the current slot.
- Δ: An initial arrival interval, that represents a design parameter.
- T_1: The time length between the arrival instant of the packet and the ending point of the most recent arrival interval currently chosen for transmission (Figure 3). All the packets in the arrival interval that corresponds to the length T_1 belong to class C.

From the operation of the algorithm we conclude that if $x_T = 1$ and $x_{T+1} = 0$ or 1, then either an existing collision is resolved at T + 1, or no such collision is in process. Also, if a packet arrives within the time interval $[T-1,T)$, and $x_{T-1} = 0$ or 1, then the packet moves to class A. If, instead, $x_{T-1} = c$, then the packet moves to class B. The throughput of the algorithm can be as close to 0.487 as desired, if the design parameter R increases. As R increases, and for low Poisson rates, the expected per packet delay induced by the algorithm increases as well. Thus, the selection of the R value is based on the desired tradeoff between throughput and expected delay. In the presence of feedback errors, as in [3] and [4], the throughput of the algorithm deteriorates gracefully, and no deadlocks occur (in contrast to the algorithm in [1]). That should be clear from the operational characteristics of the algorithm. We point out that the events represented by steps 2.2.3.2.3, 2.2.3.3, and 2.2.3.4.2 for $x_T = 0$, in the description of the algorithm, can only occur in the presence of feedback errors. Also, for R = 1, binary (collision versus noncollision) feedback suffices, with minor modifications in the algorithmic rules.

5. System Stability

Let us now consider the evolution of the algorithm, as seen by an outside observer. Let T measure time in slot units, and let the algorithmic operation start at T = 0. At T = R + 1 there will then be, R slots containing packets from class A, and one slot containing packets from class C. Let us define the variables, T_n, d_n, $L_n(A)$, and I_n, as follows:

$T_o = R + 1$

T_n ; $n \geq 1$: The first time after T_{n-1}, such that there are no slots containing packets from class B.

D_n ; $n \geq 0$: The total length of arrival intervals containing packets from class C, at time T_n; where $d_0 = 1$.

$L_n(A)$; $n \geq 0$: The number of slots containing packets from class A, at time T_n.

$$I_n = \begin{cases} 1 \; ; \text{ if } x_{T_{n-1}} = 1 \\ 0 \; ; \text{ if } x_{T_{n-1}} = 0 \end{cases}$$

The triple, $(D_n, L_n(A), I_n)$, describes the state of the system at time T_n, as induced by the operation of the algorithm. The sequence, $\{S_n\} = \{(D_n, L_n(A), I_n)\}$, is a Markov chain. That is, given S_n, the statistics of the states S_{n+k} ; $k \geq 1$ are fully determined, and they are independent of the states S_{n-k}; $k \geq 1$. The above is easily concluded from the operation of the algorithm, which also gives:

$$1 \leq L_n(A) \leq R + 1 \; ; \; \forall n \geq 0 \tag{1}$$

$$d_n \geq 0 \; ; \; \forall \, n \geq 0 \tag{2}$$

$$D_n = k - \Delta \sum_{i=0}^{M} \frac{\ell_i}{2^i} \; ; \; k \, , \, M \in N \, , \, \ell_i = 0 \text{ or } 1, \; M < \infty \tag{3}$$

; where N denotes the set of natural numbers.

From expressions (2) and (3), we conclude that the values of D_n are denumerable. In combination with expression (1), we then conclude that the state space of the Markov chain, $\{S_n\}$, is denumerable. It can be easily seen that the state, $S_0 = (1, R, 0)$, is accessible by any state in $\{S_n\}$; thus, the state space of the Markov chain, $\{S_n\}$, has at most one minimal closed subset. Let us denote by $\mathcal{D}$, the set of state values, $(d, \ell(A), I)$, that are accessible from the state, $S_0 = (1, R, 0)$. That is, given $(d, \ell(A), I)$ in $\mathcal{D}$, there exists some n, such that,

$$\Pr\{S_n = (d, \ell(A), I) \,|\, S_0 = (1, R, 0)\} > 0$$

The Markov chain, $\{S_n\}$, is then irreducible on $\mathcal{D}$. As it can be easily shown, $\{S_n\}$ is also aperiodic on $\mathcal{D}$. Let us now define a set, $\{H_n\}$, of random variables, such that,

$$H_n = T_{n+1} - T_n \tag{4}$$

Given n, given a state value, $(d_n, \ell_n(A), I_n)$, such that, $d_n \geq \Delta$, an arrival interval of length Δ is then chosen by the algorithm for transmission. The statistics of the random variable H_n in (4) are then similar to the statistics of the number of slots needed, for the resolution of an arrival interval of length Δ by the algorithm in [1]. In appendix B, it is shown that the following holds.

$$E\{H_n \,|\, d_n \geq \Delta\} = E\{H_n \,|\, d_n = \Delta\} < \infty \; ; \; \forall \, \Delta < \infty \tag{5}$$

Given n, let now the state value, $(d_n, \ell_n(A), I_n)$, be such that, $d_n < \Delta$. Then, no packet knows the value d_n, and each packet in class C selects the algorithmic parameter ℓ, equal to Δ. In appendix A, we show that the following relatioship holds, however.

$$E\{H_n|d_n<\Delta\} \leq E\{H_n|d_n=\Delta\} \tag{6}$$

Considering Markov chains with stationary transition probabilities, we now express two helpful theorems and a corollary. The proofs of the theorems are included in appendix A. The corollary is a generalization of theorem 9.1a in [8], while theorem 2 is a consequence of theorem 9.1b in the same reference.

Theorem 1

Let $\{F_n\}$ be a Markov chain with denumerable state space F. Let g be a nonnegative scalar real functional defined on F, such that, $g(s)<\infty$; $\forall$ $s\epsilon F$. Let there exist constants $\varepsilon>0$ and $0\leq M<\infty$, and a set $A\subset F$, $A\neq F$, such that,

$$\text{i)}\ 0 \leq \sup_{s\epsilon A} [E\{g(F_1)|F_0=s\} - g(s)] = M$$

$$\text{ii)}\ E\{g(F_1)|F_0=s\} - g(s) \leq -\varepsilon \ ;\ \forall\ s\epsilon A^c$$

Then,

$$\overline{\lim_{n\to\infty}} \Pr\{F_n\epsilon A|F_0=t\} \geq \frac{\varepsilon}{M+\varepsilon} \ ;\ \forall\ t\epsilon F$$

From theorem 1, we can directly express the following corollary.

Corollary 1

Let $\{F_n\}$ be an irreducible Markov chain with denumerable space F. let g be a functional as in theorem 1, and let conditions i) and ii) in the latter theorem be satisfied. Let in addition the set A in theorem 1 be such that, if $\{F_n\}$ is nonpositive reccurent, then,

$$\text{i)}\ \overline{\lim_{n\to\infty}}\ \Pr\{F_n \in A|F_0 = i\} < \frac{\varepsilon}{\varepsilon+M} \ ;\ \text{for some } i \text{ in } F.$$

; where ε, and M are as in theorem 1.

Then, the chain $\{F_n\}$ is positive recurrent .

Theorem 2

Let $\{F_n\}$ be an irreducible Markov chain with denumerable state space F. Let g be a nonnegative scalar real functional defined on F, such that, $g(s) < \infty$; $\forall\, s \in F$. Let there exist $\beta > 0$, and a set $A \subset F : A \neq F$, such that:

i) $E\{g(F_1)|F_0=s\} - g(s) \geq 0$; $\forall\, s \in A^c$

ii) $E\{|g(F_1)-g(F_0)||F_0=s\} \leq \beta$; $\forall\, s \in F$

iii) $g(s) > \sup\limits_{t \in A} g(t)$; $\forall\, s \in A^c$

Then, the chain $\{F_n\}$ is nonpositive recurrent.

Let us now consider the Markov chain $\{S_n\}$ induced by the algorithm, and let us consider the following functional,

$$g((d, \ell(A), I)) \triangleq d + \ell(A) \tag{7}$$

Let us define, $s_n = (d_n, \ell_n(A), I_n)$ and $\Delta_n = \min(d_n, \Delta)$. An arrival interval of length Δ_n is then chosen by the algorithm for transmission, at time T_n. At time T_{n+1}, the algorithm examines a subset, δ_n of Δ_n for transmission, and at the same time the remaining packets, and those that arrived in the interval $[T_n, T_{n+1})$ are either in calss C or in class A. Therefore, denoting by h_n the value of the variable H_n in (4), and considering (7), we have,

$$g(s_{n+1}) = g(s_n) - \delta_n + h_n \tag{8}$$

Let us define the following set of states:

$$A = \{s \in \mathcal{D} : s = (d, \ell(A), I) : d < \Delta\} \tag{9}$$

We can then express the following lemma, whose proof if in appendix B.

Lemma 1

There exist constants, $\varepsilon > 0$, $0 < M < \infty$, such that condition i) in corollary 1 is satisfied, for A as in (9), and for F_n ; $n \geq 0$ substituted by S_n ; $n \geq 0$.

Using the quantitites, Δ_n and H_n, defined earlier, using the functional $g(\cdot)$ in (7) and the expression in (8), using the set A in (9) and the result in lemma 1, using theorems 1 and 2, we can now express the main theorem of this section, whose proof is in appendix B.

Theorem 3

The Markov chain $\{S_n\}$ is positive recurrent, if and only if:

$$E\{\delta_0 | S_0 = (\Delta, \ell(A), I)\} > E\{H_0 | S_0 = (\Delta, \ell(A), I)\}$$

The inequality in theorem 3 provides the necessary and sufficient condition for the stability of the Markov chain $\{S_n\}$. We will now show that the satisfaction of this inequality also guarantees the existence of a steady-state distribution, for the per packet delay. Let W_n be the random variable that denotes the delay of the nth successfully transmitted packet. We can then express the following theorem, whose proof is in appendix B.

Theorem 4

If, $E\{\delta_0 | S_0 = (\Delta, \ell(A), I)\} > E\{H_0 | S_0 = (\Delta, \ell(A), I)\}$

Then, there exists some proper random variable, W_0^* (that is $\Pr\{W_0^* < \infty\} = 1$), such that,

$$\Pr\{W_n \leq b | S_0 = t\} \underset{n\to\infty}{\to} \Pr\{W_0^* \leq b\} \; ; \; \forall\, b \in R, \; \forall\, t \in \mathcal{D}$$

Due to theorems 3 and 4, we conclude that the inequality in theorem 3 basically expresses the condition for stability of the LSTFA; it thus provides the algorithmic throughput, for every given system parameter value, R. Given the system parameters, Δ and R, the throughput, $\lambda^*(\Delta, R)$, of the LSTFA is the maximum Poisson intensity of the input traffic, that maintains the condition for positive recurrence in theorem 3. Given the system parameter R, the throughput, $\lambda^*(R)$, of the LSTFA is then defined as follows.

$$\lambda^*(R) = \sup_{\Delta} \lambda^*(\Delta, R) = \lambda^*(\Delta^*, R) \tag{10}$$

For various R choices, we computed the throughput, $\lambda^*(R)$, in (10), using the recursions in appendix B, in conjuction with tight lower and upper bounds on the quantitites in theorem 3. In table 1, we list the values $\lambda^*(R) = \lambda^*(\Delta^*, R)$ and Δ^*, for various R choices. The throughput, $\lambda^*(R)$, approaches the value 0.48711 in [1], as R increases. We point out that for R = 1, the algorithm can be modified to operate with binary (collision versus noncollision) feedback, maintaining the throughput 0.4493, versus the throughput 0.429 in Capetanakis' algorithm, and the throughput 0.42 in [4].

6. Conclusions

In this paper, we presented a synchronous limited sensing random access algorithm, for the Poisson user model, and for ternary feedback. The operational characteristics of the algorithm are controlled by a system parameter, R, that takes positive integer values. The throughput of the algorithm approaches the value 0.48711, as the value of the parameter R increases. As the latter value increases, however, the expected per packet delay at relatively low Poisson intensities, and the sensitivity to feedback errors increase as well. This property should be qualitatively clear, from the description of the algorithm. The choice of the system parameter, R, is thus based on a tradeoff, between throughput and expected delays at low Poisson rates as well as error sensitivity. For relatively small values of R, the algorithmic throughput is close to the limit 0.487, while the delays at low Poisson intensities are simultaneously acceptable, and while at the same time the algorithmic throughput deteriorates gracefully in the presence of feedback errors (in contrast to the algorithm in [1], that then reaches deadlocks). For R = 1, the algorithm basically operates with binary (collision versus noncollision) feedback.

We studied the stability properties of the algorithm analytically. We used simulations, however, to initially derive quantitative results on the induced delays, and on the behavior of the algorithm in the presence of feedback errors. In this paper, we do not include the latter results, for two reasons: First, because the delays, as functions of the system parameter R, behave as explained in the above paragraph, and because the response to feedback errors is qualitatively as that of the algorithm in [4]. Second, because we prefer analytical methods, and we are presently in the process of developing the appropriate analytical tools, for studying the delay characteristics of the algorithm.

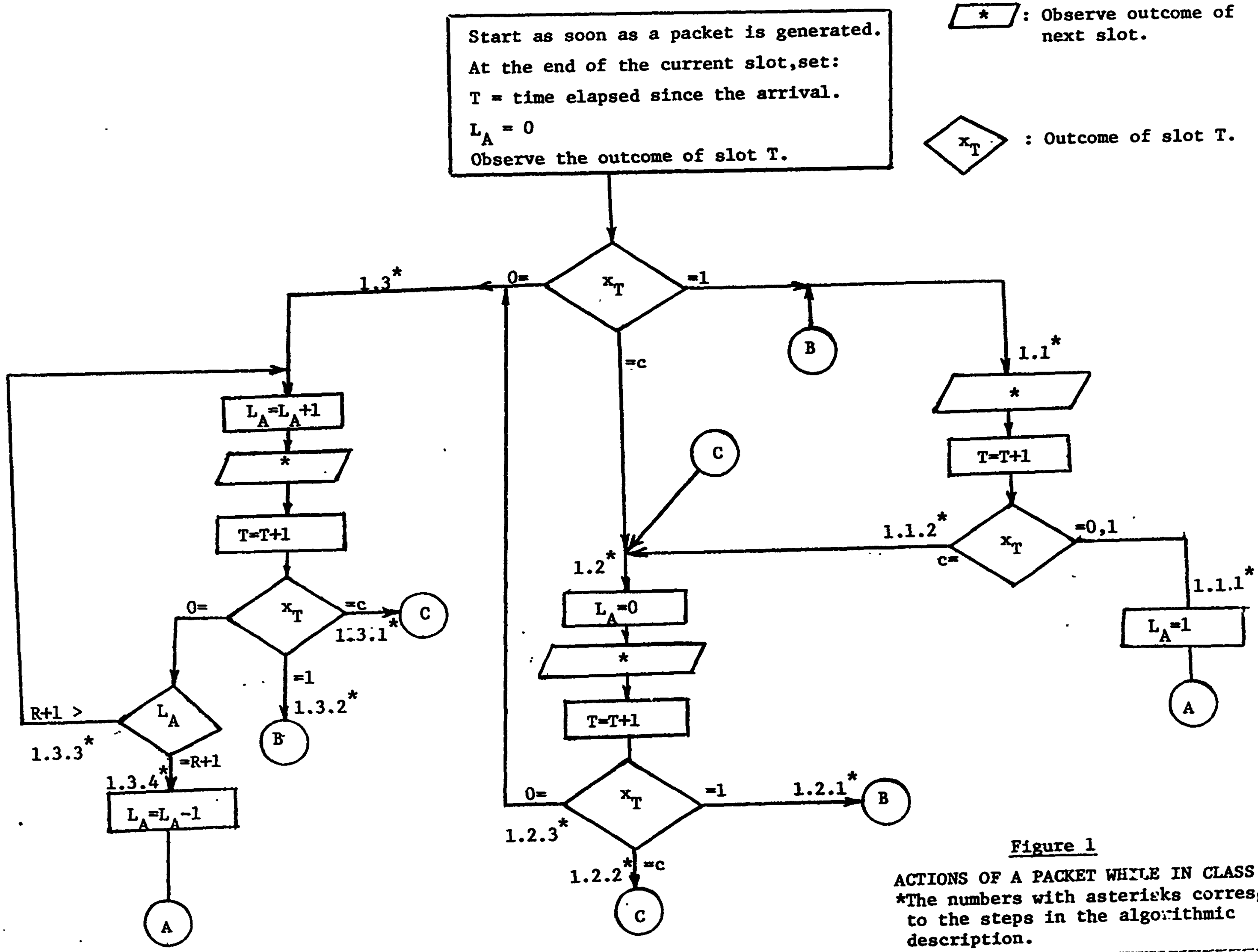


Figure 1

ACTIONS OF A PACKET WHILE IN CLASS 1
*The numbers with asterisks correspond to the steps in the algorithmic description.

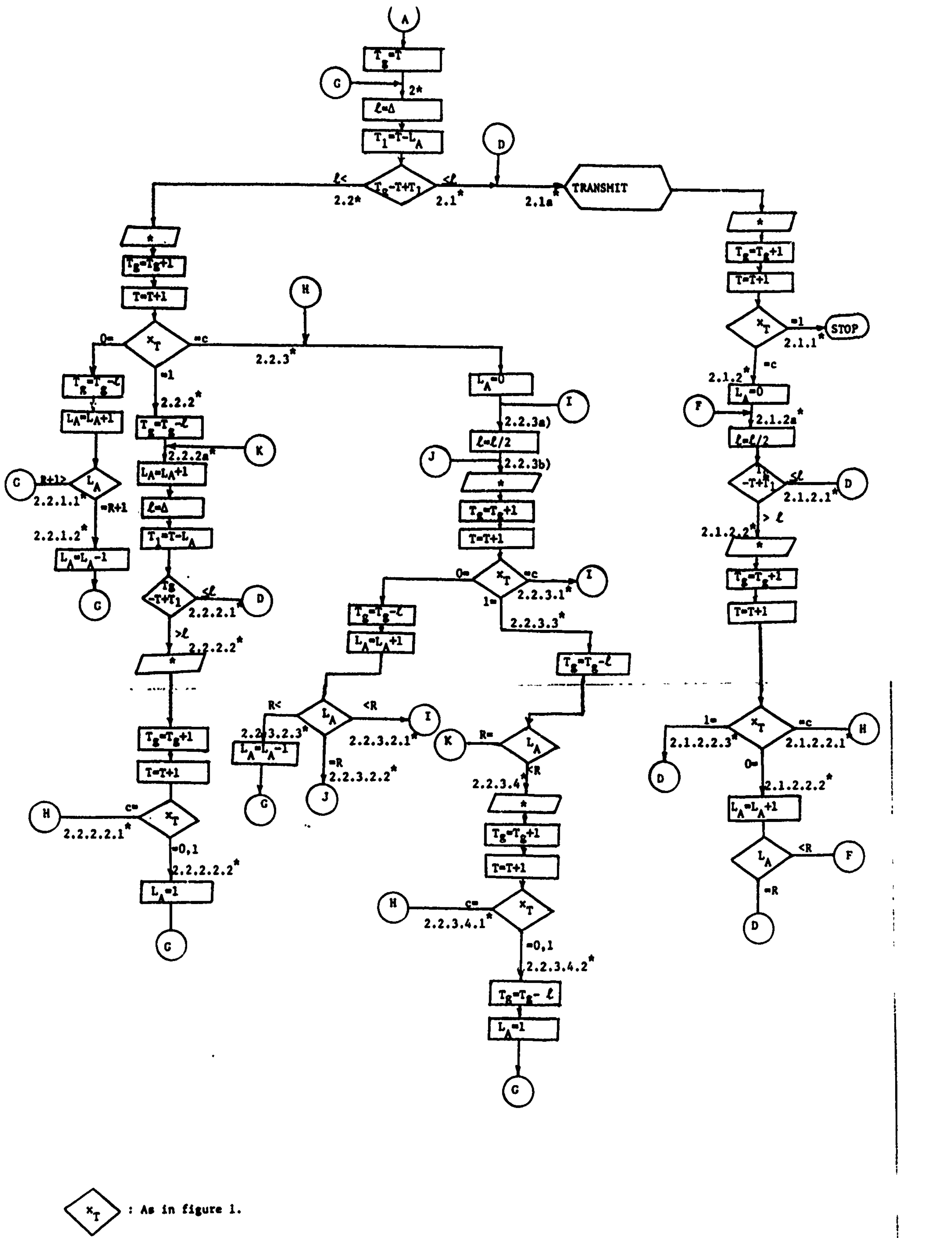
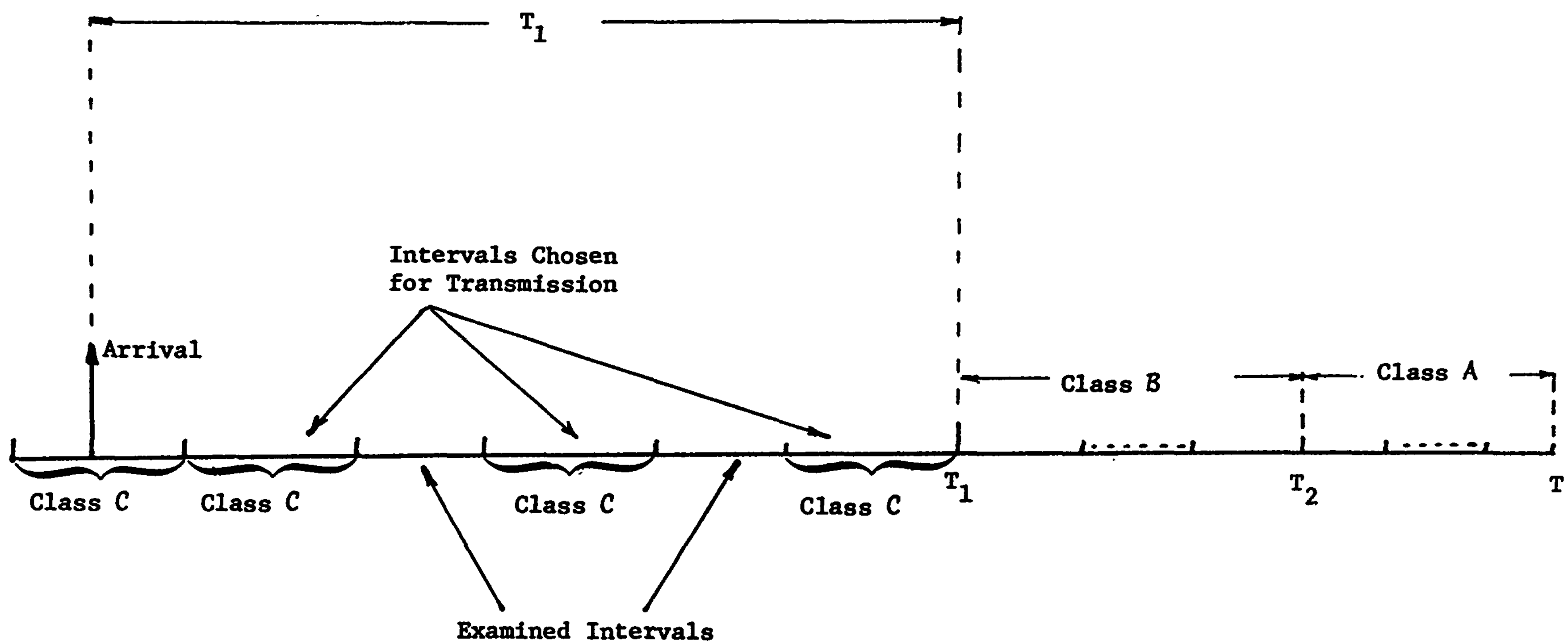


Figure 2

ACTIONS OF A PACKET WHILE IN CLASS 2

* The numbers with asterisks correspond to the steps in the description of the algorithm.



T_1
Intervals Chosen for Transmission
Arrival
Class B
Class A
Class C
Class C
Class C
Class C
T_1
T_2
T
Examined Intervals

Figure 3

R	$\lambda^*(R)$	$\Delta_1^* = \Delta^*\lambda^*(R)$	Δ^*
1	.4493	1.16	2.58
2	.4793	1.24	2.60
3	.48529	1.262	2.60
4	.4866	1.264	2.60
5	.487	1.266	2.60
6	.48709	1.266	2.60
7	.48711	1.268	2.60

Table 1

Throughput

Appendix A

Proof of expression (6)

Let $d_n = d < \Delta$. Let us then assume that there are q packets contained in d, and there are m packets contained in Δ-d, where $m \geq 0$ and $q \geq 0$. Let us assume that the whole interval Δ is chosen for transmission, and let us then denote by L(d, m,q) the number of slots needed to resolve the initial collision. From the operation of the algorithm, we then conclude,

$$L(d,m,q) \geq L(d,0,q), \text{ a.e.}$$

We note that L(d,0,q) is identical to the number of slots needed to resolve a collision, when $d_n = d$ and there are q packets contained in d; thus,

$$E\{L(d,0,q)\} = E\{H_n | d_n = d, \text{ q packets in d}\} \; ; \; d < \Delta \tag{A.1}$$

Also,

$$E\{H_n | d_n = \Delta\} = \sum_{k=0}^{\infty} E\{H_n | d_n = \Delta, \text{ k packets in } \Delta\} \Pr\{\text{k packets in } \Delta\}$$

$$= \sum_{k=0}^{\infty} \sum_{q=0}^{k} E\{H_n | d_n = \Delta, \text{ k packets in } \Delta, \text{ q packets in d}\}.$$

$$\cdot \Pr\{\text{q packets in d} | \text{k packets in } \Delta, \; d_n = \Delta\}.$$

$$\cdot \Pr\{\text{k packets in } \Delta\}$$

$$= \sum_{k=0}^{\infty} \sum_{q=0}^{k} E\{L(d,k-q,q)\} \Pr\{\text{q packets in d} | \text{k packets in } \Delta, \; d_n = \Delta\}$$

$$\cdot \Pr\{\text{k packets in } \Delta\}$$

$$= \sum_{q=0}^{\infty} \sum_{k=q}^{\infty} E\{L(d,k-q,q)\} \Pr\{\text{q packets in d} | \text{k packets in } \Delta\}.$$

$$\cdot \Pr\{\text{k packets in } \Delta\} \geq$$

$$\geq \sum_{q=0}^{\infty} E\{L(q,0,q)\} \sum_{k=q}^{\infty} Pr\{q \text{ packets in } d | k \text{ packets in } \Delta\} Pr\{k \text{ packets in } \Delta\} =$$

$$(\text{due to A.1}) = \sum_{q=0}^{\infty} E\{H_n | d_n = d, \ q \text{ packets in } d\} \ . \ Pr\{q \text{ packets in } d\} =$$

$$= E\{H_n | d_n = d < \Delta\}$$

<u>Proof of Theorem 1</u>

Let us define,

$$P_{ts}^{n} = Pr\{F_n = s | F_0 = t\} \tag{A.2}$$

From conditions i) and ii) in the theorem, we easily find by induction,

$$0 \leq E\{g(F_n) | F_0 = s\} \leq g(s) + n\, M \overset{\Delta}{=} B_n(s) < \infty; \ \forall\, n < \infty; \ \forall\, s \in F \tag{A.3}$$

Let us define,

$$\Theta_n(s) \overset{\Delta}{=} E\{g(F_{n+1}) - g(F_n) | F_n = s\} \tag{A.4}$$

We will prove that,

$$\overline{\lim_{n\to\infty}} \sum_{s \in F} \Theta_n(s)\, P_{ts}^{n} \geq 0 \ ; \ \forall\, t \in F \tag{A.5}$$

Let us temporarily assume that (A.5) is false. Then, there exist some state $t \in F$, a $\delta > 0$, and some $N_\delta < \infty$, such that,

$$\sum_{s \in F} \Theta_n(s)\, P_{ts}^{n} < -\delta \ ; \ \forall\, n \geq N_\delta \tag{A.6}$$

Expression (A.6) implies that,

$$E\{g(F_{n+1}) | F_0 = t\} < E\{g(F_n) | F_0 = t\} - \delta \ ; \ \forall\, n \geq N_\delta$$

and thus,

$$E\{g(F_{N_\delta+k})|F_0 = t\} < E\{g(N_\delta)|F_0 = t\} - k\,\delta \; ; \; \forall\, k \geq 1 \tag{A.7}$$

But due to (A.3) we have, $E\{g(N_\delta)|F_0 = t\} < \infty$; thus, (A.7) gives, $\lim\limits_{k\to\infty} E\{g(F_{N_\delta+k})|F_0 = t\} = -\infty$, which contradicts the left part of (A.3). Thus, expression (A.5) holds. Using (A.5) and the conditions in the theorem, we can now write,

$$\sum_{s\in F}\Theta_n(s)\; P^n_{ts} = \sum_{s\in A^c} \Theta_n(s)\; P^n_{ts} + \sum_{s\in A} \Theta_n(s)\; P^n_{ts} \leq$$

$$\leq -\varepsilon\; \Pr\{F_n \in A^c|F_0 = t\} + M\; \Pr\{F_n \in A|F_0 = t\} =$$

$$= -\varepsilon + (\varepsilon+M)\; \Pr\{F_n \in A|F_0 = t\} \rightarrow$$

$$\rightarrow 0 \leq \overline{\lim_{n\to\infty}} \sum_{s\in F} \Theta_n(s)\; P^n_{ts} \leq -\varepsilon + (\varepsilon+M)\; \overline{\lim_{n\to\infty}}\; \Pr\{F_n \in A|F_0 = t\}$$

$$\rightarrow 0 < \frac{\varepsilon}{M+\varepsilon} \leq \overline{\lim_{n\to\infty}}\; \Pr\{F_n \in A|F_0 = t\}$$

<u>Proof of Theorem 2</u>

We will prove the theorem in a number of steps. We first state and prove theorem A below.

<u>Theorem A</u>

Let $\{F_n\}$ be an irreducible and positive recurrent Markov chain, with stationary transition probabilities, $\{\pi_s\}$, and with denumerable state space F. Let g be a positive scalar real functional defined on F, such that, $g(s) < \infty$; $\forall\, s \in F$. If there exists, $0 < \beta < \infty$, such that:

$$E\{|g(F_1) - g(F_0)|\,|F_0 = s\} \leq \beta;\; \forall\, s \in F \tag{A.8}$$

Then,

$$\sum_{s\in F} E\{g(F_1) - g(F_0)|F_0 = s\}\; \pi_s = 0 \tag{A.9}$$

Proof

Let there exist some state t in F, such that, $\Pr\{S_0 = t\} = 1$, and let us define,

$$\Theta_n(s) \triangleq E\{g(F_{n+1}) - g(F_n) | F_n = s\} \tag{A.10}$$

Let T_ℓ ; $\ell \geq 1$ be the time of the ℓth visit to state t. Then, since $\{F_n\}$ is positive recurrent, we have,

$$E\{T_1\} < \infty \tag{A.11}$$

and due to (A.8),

$$|\Theta_n(s)| \leq \beta \ ; \ \forall \ s \in F \tag{A.12}$$

Thus,

$$\left| E\left\{\sum_{n=0}^{T_1-1} \Theta_n(F_n)\right\}\right| \leq \beta \ E\{T_1\} < \infty \tag{A.13}$$

From (A.13), and from theorems 2 and 4 in [9], we conclude,

$$\frac{E\left\{\sum_{n=0}^{T_1-1} \Theta_n(F_n)\right\}}{E\{T_1\}} = \sum_{s \in F} \Theta_0(s) \ \pi_s \tag{A.14}$$

Let us define,

$$Y_k = \sum_{n=1}^{k} [E\{g(F_n) | F_{n-1}\} - g(F_n)] \ ; \ k \geq 1 \tag{A.15}$$

$$Z_k = \sum_{n=0}^{k-1} \Theta_n(F_n) \ ; \ k \geq 1 \tag{A.16}$$

Then,

$$Z_k = g(F_k) - g(t) + Y_k \tag{A.17}$$

But, as in the proof of theorem 1, we have, $E\{|g(F_n)| \,|\, F_0 = t\} < \infty$; $\forall$ n, and $E\{g(F_n)|F_{n-1}, F_{n-2},\ldots,F_0\} = E\{g(F_n)|F_{n-1}\}$, due to the Markovian assumption. Thus, the process $\{Y_k\}$ in (A.15) is a martingale with respect to the Markov chain $\{F_n\}$ (p. 240, ex. b, in [10]). Also,

$$\begin{aligned} |Y_{k+1} - Y_k| &= \left|E\{g(F_{k+1})|F_k\} - g(F_{k+1})\right| = \\ &= |E\{g(F_{k+1}) - g(F_k)|F_k\} - [g(F_{k+1}) - g(F_k)]| \leq \\ &\leq \beta + |g(F_{k+1}) - g(F_k)| \end{aligned}$$

$$\begin{aligned} E\{|Y_{k+1} - Y_k| \,|\, F_k, F_{k-1},\ldots,F_0\} &\leq \beta + E\{|g(F_{k+1}) - g(F_k)| \,|\, F_k,\ldots,F_0\} = \\ &= \beta + E\{|g(F_{k+1}) - g(F_k)| \,|\, F_k\} \leq 2\beta \;;\; \forall\, k \geq 1 \end{aligned} \tag{A.18}$$

Now, T_1 is a Markov time with respect to $\{F_n\}$. Thus, in conjuction with (A.11) and (A.18), we conclude (corol. 3.1, p. 260 in [10]),

$$E\{Y_{T_1}\} = E\{Y_1\} = E\left\{E\{g(F_1)|F_0\} - E\{g(F_1)\}\right\} = 0$$

And,

$$E\{Z_{T_1}\} = E\{g(F_{T_1}) - g(t) + Y_{T_1}\} = g(t) - g(t) + E\{Y_{T_1}\} = 0 \tag{A.19}$$

From (A.14) and (A.19), we conclude, $\sum_{s\in F} \theta_0(s)\pi_s = 0$, that proves the theorem.

We now prove the following lemma.

<u>Lemma A</u>

Let $\{F_n\}$ be an irreducible Markov chain, with denumerable state space F. Let g be a nonnegative scalar real functional defined on F. Let there exist some $0 < \beta < \infty$, and an element τ in F; $\tau \neq F$, such that,

a) $E\{g(F_1)|F_0=s\} - g(s) \geq 0$; $\forall\ s \in F : s \neq \tau$

b) $E\{|g(F_1)-g(F_0)|\,|F_0=s\} \leq \beta$; $\forall\ s \in F$

c) $g(s) > g(\tau)$; $\forall\ s \in F : s \neq \tau$

Then, the chain $\{F_n\}$ is nonpositive recurrent.

Proof

Let us assume that $\{F_n\}$ is positive recurrent. Due to theorem A, in conjuction with property b), we then conclude,

$$\sum_{s\in F} E\{g(f_1)-g(F_0)|F_0=s\}\ \pi_s = 0 \tag{A.20}$$

; where, $\{\pi_s\}$; $\pi_s > 0$, $\forall\ s\in F$, are the stationary probabilities of $\{F_n\}$.

But,

$$\sum_{s\in F} E\{g(F_1)-g(F_0)|F_0=s\}\ \pi_s = \pi_\tau\ E\{g(F_1)-g(\tau)|F_0=\tau\} + $$

$$+ \sum_{\substack{s\in F\\ s\neq\tau}} E\{g(F_1)-g(F_0)|F_0=s\} \tag{A.21}$$

And, for $\{P_{\ell s}\}$: $\pi_s = \sum_{\ell\in F} P_{\ell s}\ \pi_\ell$,

$$E\{g(F_1)-g(\tau)|F_0=\tau\} = \sum_{\substack{\ell\in F\\ \ell\neq\tau}} [g(\ell)-g(\tau)]P_{\tau\ell} \tag{A.22}$$

Since the chain $\{F_n\}$ is irreducible, there exists some $\ell:\ell\in F$, $\ell\neq\tau$, such that, $P_{\tau\ell} > 0$. Due to that, in conjuction with condition c), we obtain,

$$E\{g(F_1)-g(\tau)|F_0=\tau\} > 0, \text{ where } \pi_\tau > 0 \tag{A.23}$$

Due to condition a), we have,

$$\sum_{\substack{s\in F\\ s\neq \tau}} E\{g(F_1)-g(F_o)|F_o=s\}\ \pi_s \geq 0 \tag{A.24}$$

From (A.21), (A.23), and (A.24), we conclude,

$$\sum_{s\in F} E\{g(F_1)-g(F_o)|F_o=s\}\ \pi_s > 0 \tag{A.25}$$

(A.25) contradicts (A.20); thus, the chain $\{F_n\}$ is nonpositive recurrent.

Let us now refer to the statements in theorem 2. Since the Markov chain $\{F_n\}$ is irreducible, there exists some state τ in the set A, and some state τ_1 in the set A^c, such that,

$$P_{\tau\tau_1} > 0 \tag{A.26}$$

Let $\{F'_n\}$ be some Markov chain with state space, $F' = A^c \cup \tau$, and with transition probabilities, $\{P'_{ts}\}$, given as follows:

$$P'_{ts} = \begin{cases} P_{ts} & ;\ s \in A^c \\ \sum_{\ell\in A} P_{t\ell} ; & s = \tau \end{cases} \tag{A.27}$$

; where $\{P_{ts}\}$ denote the transition probabilities of the chain $\{F_n\}$.

Let us now define,

$$\begin{aligned} \rho_{ts} &\triangleq \min\{n \geq 1 : F_n=s|F_o=t\} \\ \hat{\rho}_{tA} &\triangleq \min\{n \geq 1 : F_n\epsilon A|F_o=t\} \\ \rho'_{ts} &\triangleq \min\{n \geq 1 : F'_n=s|F'_o=t\} \end{aligned} \tag{A.28}$$

Clearly $\rho_{\tau A} \leq \rho_{\tau\tau}$, a.e. Also,

$$\Pr\{\rho'_{t\tau}=1\} = P'_{t\tau} = \sum_{\ell\varepsilon A} P_{t\ell} \qquad \text{(A.29)}$$

$$\Pr\{\rho'_{t\tau}=n\} = \sum_{s\varepsilon A^c} P_{ts}\Pr\{\rho'_{s\tau}=n-1\} \; ; \; n \geq 2$$

But the recursions in (A.29) are also satisfied by the probabilities $\Pr\{\rho_{tA}=n\}$; $n\geq 1$. Thus, $\rho'_{\tau\tau} \leq \rho_{\tau\tau}$ stochastically, and to prove theorem 2, it suffices to show that,

$$E\{\rho'_{\tau\tau}\} = \infty \qquad \text{(A.30)}$$

Indeed, if (A.30) is true, then also $E\{\rho_{\tau\tau}\} = \infty$, and $\{F_n\}$ is then nonpositive recurrent. Let us now define the following functional, g', on F':

$$g'(s) = \begin{cases} g(s) & ; s \varepsilon A^c \\ \sup\limits_{t\varepsilon A} g(t) & ; s = \tau \end{cases} \qquad \text{(A.31)}$$

We will prove that the functional in (A.31) satisfies the following properties.

d) $E\{g'(F'_1)|F'_o=s\} - g'(s) \geq 0$; $\forall\, s \varepsilon A^c$

e) $E\{|g'(F'_1)-g'(F'_o)|\,\big|F'_o=s\} \leq \beta$; $\forall\, s \varepsilon F$

f) $g'(s) > g'(\tau)$; $\forall\, s \varepsilon A^c$

Property f) evolves from condition iii) in the theorem. Property d) holds, because due to condition i) in the theorem, we have,

$$\forall\, s \varepsilon A^c;\; E\{g'(F'_1)|F'_o=s\} - g'(s) = \sum_{\ell\varepsilon A^c} g'(\ell)\, P'_{s\ell} + g'(\tau)\, P'_{s\tau} - g'(s) =$$

$$= \sum_{\ell\varepsilon A^c} g(\ell)\, P_{\ell s} + \left\{\sup_{\ell\varepsilon A} g(\ell)\right\} \sum_{\ell\varepsilon A} P_{s\ell} - g(s) \geq \sum_{\ell\varepsilon F} g(\ell)\, P_{s\ell} - g(s) =$$

$$= E\{g(F_1)|F_o=s\} - g(s) \geq 0$$

To prove property e), let us first consider the following derivations:

For $s \in A^c$; $E\{|g'(F_1')-g'(F_o)|\,\Big|\,F_o'=s\} = \sum_{\ell\in A^c} |g'(\ell)-g'(s)|P'_{s\ell}+|g'(\tau)-g'(s)|P'_{s\tau} =$

$$= \sum_{\ell\in A^c} |g(\ell)-g(s)|P_{s\ell} + |g'(\tau)-g(s)|\sum_{\ell\in A} P_{s\ell} \le$$

$$\le \sum_{\ell\in F} |g(\ell)-g(s)|P_{s\ell} = E\{|g(F_1)-g(F_o)|\,|F_o=s\} \le \beta \text{ ; since}$$

$$g(s) > g'(\tau) \ge g(t) \ge 0 \text{ ; } \forall\, t \in A,\ \forall\, s \in A^c, \text{ and thus,}$$

$$|g'(\tau)-g(s)| \le |g(\ell)-g(s)| \text{ ; } \forall\, \ell \in A,\ \forall\, s \in A^c. \tag{A.32}$$

Also,

$$E\{|g'(F_1')-g'(F_o')|\,\Big|\,F_o'=\tau\} = \sum_{\ell\in A^c} |g(\ell)-g'(\tau)|\ P_{\tau\ell} \le \sum_{\ell\in A^c} |g(\ell)-g(\tau)|P_{\tau\ell} \le$$

$$< \sum_{\ell\in F} |g(\ell)-g(\tau)|\ P_{\tau\ell} = E\{|g(F_1)-g(F_o)|\,\Big|\,F_o= \tau\} \le \beta \tag{A.33}$$

Expressions (A.32) and (A.33) prove condition e). For the Markov chain $\{F_n'\}$, the state τ is accessible by any state in $A^c \subset F'$. Let C denote the set of all states in F' that are accessible by the state τ, where $C \subset F'$. Then C is an essential class; that is all states in C communicate. The set C is thus closed. Let us now denote by $\{F_n''\}$, the restriction of the Markov chain $\{F_n'\}$ on C. Then, properties d), e), and f) hold trivially for $\{F_n''\}$, where $\tau\notin C$ due to (A.26). Since the chain $\{F_n''\}$ is irreducible, it is also nonpositive recurrent, by lemma A. Thus, (A.30) holds, and the proof of theorem 2 is now complete.

Appendix B

Proof of Lemma 1

Let us define the following set,

$$B = \{s \in \mathcal{D} : s=(d,\ell(A),\ I) : d \text{ integer},\ 0 \leq d \leq R+1\} \tag{B.1}$$

Then,

$$\Pr\{S_{n+1} \in B|S_o=i\} = \sum_{s\in\mathcal{D}} \Pr\{S_{n+1} \in B|S_n=s\} \Pr\{S_n=s|S_o=i\}$$

$$\geq \sum_{s\in A} \Pr\{S_{n+1} \in B|S_n=s\} \Pr\{S_n=s|S_o=i\} \tag{B.2}$$

We now observe that if $s_n = (d,\ \ell(A),\ I)$, $d < \Delta$, and there are no packets in Δ, then $S_{n+1} \in B$. Thus,

$$\Pr\{S_{n+1} \in B|S_n=(d,\ \ell(A),\ I),\ d < \Delta\} \geq$$

$$\geq \Pr\{0 \text{ packets in } d|S_n=(d,\ \ell(A),\ I),\ d < \Delta\} =$$

$$= e^{-\lambda d} > e^{-\lambda\Delta} \tag{B.3}$$

From (B.2) and (B.3) we conclude,

$$\Pr\{S_{n+1} \in B\ |S_o=i\} \geq e^{-\lambda\Delta} \sum_{s\in A} \Pr\{S_n=s|S_o=i\} = e^{-\lambda\Delta} \Pr\{S_n \in A|S_o=i\} \tag{B.4}$$

But the set B is finite; thus, if $\{S_n\}$ is nonpositive recurrent, then,

$$\Pr\{S_{n+1} \in B|S_o=i\} \underset{n\to\infty}{\to} 0\ ;\ \forall\ i \in \mathcal{D} \tag{B.5}$$

From (B.4) and (B.5) we then conclude that if $\{S_n\}$ is nonpositive recurrent, then,

$$\Pr\{S_{n+1} \in A|S_o=i\} \underset{n\to\infty}{\to} 0\ ;\ \forall\ i \in \mathcal{D} \tag{B.6}$$

The proof of the lemma is now complete.

Proof of Theorem 3

Considering expressions (8), (5), and (6), we obtain,

$$E\{|g(S_1)-g(S_o)|\,|S_o=s\} = E\{|-\delta_o + H_o|\,|S_o=s\} \le$$

$$\le E\{|\delta_o|\,|S_o=s\} + E\{|H_o|\,|S_o=s\} \le$$

$$\le \Delta + E\{H_o|d_o = \Delta\} \overset{\Delta}{=} M < \infty \ ; \ \forall\, s \in \mathcal{D} \tag{B.7}$$

; since $0 < \delta_o \le \Delta$, a.e.

Due to (B.7), we easily conclude,

$$E\{g(S_1)|S_o=s\} - g(s) \le M < \infty \ ; \ \forall\, s \in \mathcal{D} \tag{B.8}$$

Part 1: Let us now assume that the following inequality holds,

$$E\{\delta_o|\ S_o=(\Delta,\ell(A),I)\} > E\{H_o|S_o=(\Delta,\ell(A),I)\} \tag{B.9}$$

Then, due to (8), and for A as in (9), we have,

$$E\{g(S_1)|S_o=s\} - g(s) = - E\{\delta_o|S_o=(\Delta,\ell(A),\ I)\} +$$

$$+ E\{H_o|S_o=(\Delta,\ell(A),I)\} \overset{\Delta}{=} -\varepsilon < 0 \ ; \ \forall\, s \in A^c \tag{B.10}$$

But the conditions (B.8) and (B.10) are as those in theorem 1. Then, in conjuction with lemma 1 and corollary 1 we condluce that if (B.9) holds, then the Markov chain $\{S_n\}$ is positive recurrent.

Part 2: Let us now assume that the following inequality holds,

$$E\{\delta_o|S_o=(\Delta,\ell(A),I)\} \le E\{H_o|S_o=(\Delta,\ell(A),I)\} \tag{B.11}$$

Consider the set,

$$B^c = \{s\epsilon\mathcal{D}:\ g(s) > \Delta+R+1\} \tag{B.12}$$

Then, we necessarily have, $s=(d,\ell(A),I);\ d>\Delta\ ;\forall\ s\epsilon B^c$, and due to (8),

$$E\{g(S_1)|S_0=s\} - g(s) = -E\{\delta_0|S_0=(\Delta,\ell(A),I)\} + E\{H_0|S_0=(\Delta,\ell(A),I)\} \geq 0\ ;\ \forall\ s\epsilon B^c \tag{B.13}$$

Also,

$$g(t) \leq \Delta+R+1\ ;\ \forall\ t\epsilon B \rightarrow \sup_{t\epsilon B} g(t) \leq g(s)\ ;\ \forall\ s\epsilon B^c \tag{B.14}$$

Conditions (B.7), (B.13), and (B.14) are as the conditions in theorem 2, with set A in the latter theorem being substituted by the set B, where B^c is as in (B.12). Thus, if (B.11) is satisfied, then the Markov chain $\{S_n\}$ is nonpositive reccurent. Parts 1 and 2 above complete the proof of theorem 3.

<u>Proof of Theorem 4</u>

If the condition in the theorem holds, and due to theorem 3, the Markov chain $\{S_n\}$ is positive reccurent. Let us now define,

$$\begin{aligned} &S_0 = (0,\ R+1,\ 1) \triangleq t_0 \\ &\phi_0 \triangleq 0,\ \rho_n \triangleq \min\{k\ :\ S_{\phi_n+k} = t_0\}\ ,\ \phi_{n+1} = \phi_n+\rho_n \\ &\tau_n \triangleq T_{\phi_n}\ ,\ \tau_{1,n} \triangleq \tau_n - (R+1) \end{aligned} \tag{B.15}$$

The instants, $\{\tau_n\}$, are thus such that, there are no slots containing packets from classes $\mathcal{B}$ and $\mathcal{C}$, there are R+1 slots containing packets from class $\mathcal{A}$, and the last slot just before each of those instants contains a single successfully transmitted packet. The interval, $[\tau_{1,n}, \tau_n)$, contains all the R+1 slots, which at time τ_n contain packets from class $\mathcal{A}$. Since the Markov chain, $\{S_n\}$, is positive reccurent, the random variables, $\{\rho_n\}$, are i.i.d., with finite expected value. Let us define,

$$\Theta_n \triangleq \tau_{n+1} - \tau_n = \sum_{i=0}^{\rho_n-1} H_{\phi_n+i} = \tau_{1,n+1} - \tau_{1,n} \tag{B.16}$$

;where H_n is given by expression (4).

Then, the random variables, $\{\Theta_n\}$, are i.i.d., and,

$$\sum_{i=0}^{\rho_0-1}(H_i - \delta_i) = \sum_{i=0}^{\rho_0-1} [\, g(S_{i+1})-g(S_i)] = g(S_{\rho_0}) - g(t_0) = 0$$

$$\rightarrow 0 \le \sum_{i=0}^{\rho_0-1} H_i = \sum_{i=0}^{\rho_0-1} \delta_i \le \rho_0\Delta \rightarrow 0 \le \Theta_0 \le \rho_0\Delta \rightarrow$$

$$\rightarrow 0 \le E\{\Theta_0\} \le \Delta\, E\{\rho_0\} \tag{B.17}$$

Let us now denote by M_n, the number of packets that arrive in $[\tau_{1,n+1}, \tau_{1,n})$. Then, the random variables, $\{M_n\}$, are i.i.d., and as in [12] it can be shown that,

$$E\{M_0\} = \lambda\, E\{\Theta_0\} < \infty \tag{B.18}$$

Since all the M_n packets are transmitted in $[\tau_{1,n}, \tau_{1,n+1})$, the process, $\{W_n\}$, is regenarative, with respect to $\{M_n\}$. Moreover, it is easily concluded that, $\Pr\{M_0=1\} > 0$; thus, the distribution of M_0 is aperiodic. Taking into account expression (5), we thus conclude that the assertion in theorem 4 holds (see [13], th. 2). Also, since $\{S_n\}$ is positive reccurent, the variable, $\rho_0 = \min\{k: S_{\phi_n+k} = t_0 | S_0=t\}$, is a proper random variable; thus, $\{M_n\}$, is then a delayed renewal process, and the theorem also holds for $S_0=t\neq t_0$.

Recursions

Given Δ and R, let us define the sequences $\{L_k\}$ and $\{Q_k\}$, and the quantities p_i^k, as follows,

$$L_k \triangleq E\{H_0 | D_0=\Delta,\ k \text{ packets in } \Delta\}$$

$$Q_k \triangleq E\{\delta_0 | D_0=\Delta,\ k \text{ packets in } \Delta\}$$

$$p_i^k \triangleq \binom{k}{i} 2^{-k} \tag{B.19}$$

; where H_0 and D_0 are respectively defined by (4) and (3), and where δ_0 is as in (8).

From the operation of the algorithm, we then easily derive the following recursive expressions.

$$L_0 = L_1 = 1$$

(B.20)

$$\underset{;k\geq 2}{L_k} = \begin{cases} 1+\ell+1+L_{k-1} & ;\ \text{w.p.} \left(p_o^k\right)^\ell p_1^k \ ;\ 0 \leq \ell \leq R-1 \\ 1+\ell+L_i & ;\ \text{w.p.} \left(p_o^k\right)^\ell p_i^k \ ;\ 0 \leq \ell \leq R-1\ ,\ k \geq i \geq 2 \\ 1+R+L_k & ;\ \text{w.p.} \left(p_o^k\right)^R \end{cases}$$

$$Q_o = Q_1 = \Delta$$

(B.21)

$$\underset{;k\geq 2}{Q_k} = \begin{cases} \Delta \sum_{j=1}^{\ell} 2^{-j} + \Delta\, 2^{-\ell-1} + 2^{-\ell-1} Q_{k-1} & ;\ \text{w.p.} \ \left(p_o^k\right)^\ell p_1^k\ ,\ 0 \leq \ell \leq R-1 \\ \Delta \sum_{j=1}^{\ell} 2^{-j} + 2^{-\ell-1} Q_k & ;\ \text{w.p.} \left(p_o^k\right)^\ell \ p_i^k\ ,\ 0 \leq \ell \leq R-1,\ k \geq i \geq 2 \\ \Delta \sum_{j=1}^{R} 2^{-j} + 2^{-R} Q_k & ;\ \text{w.p.} \ \left(p_o^k\right)^R \end{cases}$$

Defining $\sum_{i=k}^{n} x_i = 0$; if $n < k$, and respectively from (B.20), and (B.21), we conclude,

$$\left\{1 - \left(p_o^k\right)^R - [1-p_o^k]^{-1}[1-(p_o^k)^R]\ p_o^k\right\} L_k =$$

$$= 1 + [1-p_o^k]^{-1}[1-(p_o^k)^R] \left\{ p_o^k + p_1^k + p_1^k\, L_{k-1} + \sum_{i=2}^{k-1} p_i^k\, L_i \right\} \ ;\ k \geq 2$$

$$L_o = L_1 = 1 \qquad \text{(B.22)}$$

$$\left\{1 - 2^{-R}(p_o^k)^R - 2[2-p_o^k]^{-1}[1-2^{-R}(p_o^k)^R]p_o^k\right\} Q_k =$$

$$= [2-p_o^k]^{-1}[1-2^{-R}(p_o^k)^R] \left\{ \Delta p_o^k + \Delta p_1^k + p_1^k\, Q_{k-1} + \sum_{i=2}^{k-1} p_i^k\, Q_i \right\} \ ;\ k \geq 2$$

(B.23)

$$Q_o = Q_1 = \Delta$$

Note that for $R = \infty$, the sequences, $\{L_k\}$ and $\{Q_k\}$, are exactly as in Gallager's algorithm [1], [11].

If we put $Q_k = \Delta Q_k'$, then we can easily see that Q_k' satisfies the following recrusions:

$$\left\{1 - 2^{-R}(p_o^k)^R - 2[2-p_o^k]^{-1}[1-2^{-R}(p_o^k)^R]p_o^k\right\} Q_k' =$$

$$= [2-p_o^k]^{-1}[1-2^{-R}(p_o^k)^R]\left\{p_o^k + p_1^k + p_1^k Q'_{k-1} + \sum_{i=2}^{k-1} p_i^k Q_i'\right\} ; k \geq 2 \tag{B.24}$$

$$Q_o' = 1, \; Q_1' = 1$$

From (B.22), we can easily conclude by induction that,

$$0 \leq L_k \leq 3k; \; \forall \; k \geq 1 \; , \; \forall \; R \geq 1, \; R \epsilon N^+ \tag{B.25}$$

Therefore,

$$0 \leq E\{H_0/D_0=\Delta\} = \sum_{k=0}^{\infty} L_k Pr(k \text{ packets in } \Delta) \leq 3\lambda \; \Delta+1 < \infty; \forall 0 \leq \Delta < \infty, \; \forall \; R \epsilon N^+ \tag{B.26}$$

This proves the inequality in (5).

Formula (B.25) permits us to develop tight lower and upper bounds on $E\{H_0/D_0=\Delta\}$ as follows: From (B.22) we can directly compute a finite number, M, of terms of L_k. Then, by using (B.25) we have,

$$H_\ell(\lambda\Delta) = \sum_{k=0}^{M} L_k \; e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!} \leq E\{H_0/D_0=\Delta\} \leq \sum_{k=0}^{M} L_k \; e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!} +$$

$$+ 3\lambda\Delta - 3 \sum_{k=0}^{M} k \; e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!} = H_u(\lambda\Delta) \tag{B.27}$$

Similarly, since $0 \leq Q_k' \leq 1$, we have,

$$\delta_\ell(\lambda\Delta) = \lambda\Delta \sum_{k=0}^{M} Q'_k\, e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!} \leq \lambda\, E\{\delta_0/D_0=\Delta\} \leq$$

$$\leq \lambda\Delta \left[\sum_{k=0}^{M} Q'_k\, e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!} + 1 - \sum_{k=0}^{M} e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!}\right] = \delta_u(\lambda\Delta) \qquad \text{(B.28)}$$

Now, the condition, $E\{\delta_0/D_0=\Delta\} > E\{H_0/D_0=\Delta\}$, is equivalent to, $f(\lambda\Delta) = \frac{\lambda E\{\delta_0/D_0=\Delta\}}{E\{H_0/D_0=\Delta\}} > \lambda$ and therefore, the throughput of the algorithm is

$$\lambda^*(R) = \sup_{\Delta_1>0} f(\Delta_1) \text{ , where } \Delta_1 = \lambda\Delta$$

But

$$\frac{\delta_\ell(\Delta_1)}{H_u(\Delta_1)} \leq f(\Delta_1) \leq \frac{\delta_u(\Delta_1)}{H_\ell(\Delta_1)} \rightarrow \sup_{\Delta_1>0} \frac{\delta_\ell(\Delta_1)}{H_u(\Delta_1)} \leq \lambda^*(R) \leq \sup_{\Delta_1>0} \frac{\delta_u(\Delta_1)}{H_\ell(\Delta_1)} \qquad \text{(B.29)}$$

Based on (B.29), we give in table 1, the values of $\lambda^*(R)$, $\Delta_1^* = \Delta^*\lambda^*(R)$, Δ^*, for different Rs. In the computation we took $M = 25$. The values of $\lambda^*(R)$ in the table are correct, up to the digit referred to in this table.

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