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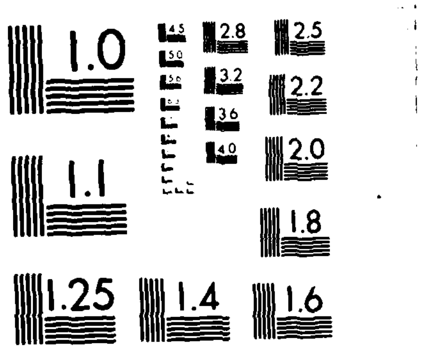
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Variability of Measures of Weapons Effectiveness

Volume III: Application to Fragment Sensitive Targets in the Presence of Delivery Error

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The problem of computing the uncertainty associated with the probability of kill P_{kf} due to fragmentation in the presence of aiming error and in the absence of blast is considered. It is assumed that the damage due to fragmentation can be approximated by the two-parameter Carleton damage function. The aiming error in the delivery of the weapon is assumed to be Gaussian and independent in each of the range and deflection directions.

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Application to Fragment Sensitive Targets in the Presence of Delivery Error

PREFACE

This report describes work done in the summer of 1983 by Dr. B. D. Sivazlian, Department of Industrial and Systems Engineering, the University of Florida, Gainesville, Florida 32611 under Contract No. F08635-83-C-0202, with the Air Force Armament Laboratory (AFATL), Armament Division, Eglin Air Force Base, Florida 32542. The program manager was Mr Daniel A. McInnis, (DLYW).

The work was initiated under a 1982 USAF-SCEEE Summer Faculty Research program sponsored by the Air Force Office of Scientific Research conducted by the Southeastern Center for Electrical Engineering Education under Contract No. F49620-82-C-0035.

This work addresses itself to the problem of computing the uncertainty associated with the probability of kill P_{kf} due to fragmentation in the presence of aiming error and in the absence of blast. Let P_k be the probability of kill due to fragmentation in the absence of aiming error. Assume that points on the ground surface are referenced relative to a system of Cartesian coordinates where the x-axis is pointed in the direction of range and the y-axis is pointed in the direction of deflection. The origin O is arbitrarily selected. For a point target located at (x,y) assuming that the weapon bursts at a point (u,v), the probability of kill due to fragmentation is given by the two-parameter Carleton damage function

$$P_k(x-u,\ y-v) = \exp\left[-\left(\frac{x-u}{R_x}\right)^2 - \left(\frac{y-v}{R_y}\right)^2\right] .$$

The aiming errors in the direction of range and deflection are assumed to be unbiased centered at (x,y) independent of each other having a Guassian distribution with respective standard deviation σ_x and σ_y.

It is shown that P_{kf} can be expressed as a mathematical function of the four parameters R_x, R_y, σ_x and σ_y. Moreover, under the assumption that R_x, R_y, σ_x and σ_y are not known with certainty but are estimates, explicit expressions are obtained for $E[P_{kf}]$ and $Var[P_{kf}]$.

The author has benefited from helpful discussions with several people. Particular thanks are due to Mr Jerry P. Bass, Mr Daniel A. McInnis and Mr Charles A. Reynolds, all from DLYW, who have read the report and who have contributed to it through helpful comments.

The report is the third of a series of four reports dealing with the uncertainty associated with various weapon effectiveness indices, and details methodologies and techniques used in computing such uncertainties in the presence of error in the input parameters.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Milton D. Kingcaid

MILTON D. KINGCAID, Colonel, USAF
Chief, Analysis and Strategic Defense Division

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SECTION I

INTRODUCTION

This report considers the problem of estimating the probability of kill due to fragmentation, P_{kf}, in the presence of aiming error. Prior to solving the estimation problem, it is necessary to derive a mathematical expression for P_{kf} in order to apply the usual statistical techniques to arrive at confidence intervals for P_{kf}. In Section II, the derivation of P_{kf} is endeavored based on several explicitly stated assumptions. In Section III, the estimation of $E[P_{kf}]$ and $Var[P_{kf}]$ is carried out when using the subjective estimation procedure. In Section IV the estimation technique using Taylor's series is applied to obtain $E[P_{kf}]$ and $Var[P_{kf}]$. Finally, Section V provides some conclusive remarks.

A comparison of Section III with Section IV will show that the subjective estimation procedure involves complex mathematical expressions which result in cumbersome calculations. Furthermore, the procedure assumes that the estimates are independent random variables. This assumption is not necessary when using the Taylor's series estimation technique. In addition, the latter allows one to segregate the contribution of each variance component to the total $Var[P_{kf}]$.

SECTION II

MATHEMATICAL MODEL FOR P_{kf}

1. Background

In this section, a mathematical model for the probability of kill due to fragmentation, P_{kf}, in the presence of aiming error is developed. The basic situation that one is facing consists of the following.

A weapon whose main effect is kill due to fragmentation is delivered from air to a target point located on the ground surface. The target's position is stationary. The weapon may not directly hit the target due to the presence of aiming errors. These errors are assumed to be unbiased; that is, centered at the location of the target. The target may or may not be killed by the effect of fragmentation; thus, the target is killed with a given probability level. The probability of kill due to fragmentation is related to the distance between weapon and target by a well-defined mathematical function. In addition, the aiming error is not known precisely but is expressed by a probability density function which provides a mathematical formula for computing the probability that the weapon will impact in an interval du dv close to a point (u,v) on the ground surface.

The technique that will be used to compute the expression for the probability of kill due to fragmentation, P_{kf}, in the presence of aiming error is based on the laws of conditional probability. Ultimately, P_{kf} is not going to depend on the position of the target if the aiming error is unbiased, and if weapon delivery can theoretically result in a point of impact which can be anywhere on the ground surface. On the other hand, P_{kf} will depend on:

a. the parameters specifying the functional form relating probability of kill to distance;

b. the statistical parameters of aiming error distribution.

2. Assumptions

The following assumptions pertaining to this situation will be made:

a. Each of the target and weapon is idealized as a point, and the weapon is aimed at the target.

b. The direction of the weapon delivery range and deflection are, respectively, parallel to the (x-y) coordinate system on the ground plane. Since the coordinate system can be arbitrarily selected, there is no loss in generality in making this specific assumption. The position of the target has coordinates (x,y).

c. The aiming error (distance) in each of the x and y directions are independently and normally distributed with respective means x and y and standard deviation σ_x and σ_y. Let (du, dv) be the infinitesimal rectangle close to the point (u,v) at which the weapon impacts, and define the random variables U and V which measure, respectively, the distances between the target point and the weapon impact point along the abcissa and the ordinate. Then, the probability that the weapon will impact in the rectangle du dv is

$$f_{U,V}(u-x,\ v-y)\ du\ dv = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{(u-x)^2}{2\sigma_x^2}\right] \cdot \exp\left[-\frac{(v-y)^2}{2\sigma_y^2}\right] du\ dv\ . \tag{1}$$

d. The probability of kill due to fragmentation at a point (x,y) given that the weapon impacts at (u,v) is given by the Carleton damage function

$$P_k(x-u,\ y-v) = \exp\left[-\left(\frac{x-u}{R_x}\right)^2 - \left(\frac{y-v}{R_y}\right)^2\right]. \tag{2}$$

This is sometimes known as the elliptical damage function involving the two parameters R_x and R_y. These parameters are often identified as the weapon radii: R_x is known as the range weapon radius, and R_y is known as the deflection weapon radius.

e. Blast effect is neglected. This implies that the target is not blast sensitive. Or, if weapon blast exists, its effect on the target is negligible, hence not resulting in a kill.

f. In general, the point in space from which weapon is delivered is nonstationary, the weapon is subject to ballistic errors, etc. It shall be assumed that all these factors combine into a single source of error which is incorporated in the aiming error.

g. The fragmentation does not contribute to the aiming error.

3. The Mathematical Model

The probability of kill at (x,y) due to fragmentation is:

$$P_{kf} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\text{Probability of kill at } (x,y) \mid \text{weapon impacts at } (u,v)]$$
$$[\text{Probability that the weapon impacts between } (u,v) \text{ and } (u+du,\ v+du)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_k(x-u,\ y-v) \cdot f_{U,V}(u-x,\ v-y)\, du\, dv$$

$$P_{kf} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x-u}{R_x}\right)^2 - \left(\frac{y-v}{R_y}\right)^2\right] \cdot$$
$$\frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{(u-x)^2}{2\sigma_x^2}\right] \cdot \exp\left[-\frac{(v-y)^2}{2\sigma_y^2}\right] du\, dv \ .$$

Making the change in variables $w = u-x$ and $z = v-y$ yields

$$P_{kf} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{w^2}{R_x^2} + \frac{z^2}{R_y^2}\right) \cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\left(\frac{w^2}{2\sigma_x^2} + \frac{z^2}{2\sigma_y^2}\right)\right] dw\, dz \ .$$

This integral may be expressed as the product of two single integrals as follows:

$$P_{kf} = \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{R_x^2} + \frac{1}{2\sigma_x^2}\right) w^2\right] dw$$

$$\cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{R_y^2} + \frac{1}{2\sigma_y^2}\right) z^2\right] dz \ . \tag{3}$$

The first integral is computed:

$$I_x = \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{R_x^2} + \frac{1}{2\sigma_x^2}\right) w^2\right] dw \ . \tag{4}$$

Let $\qquad \dfrac{\theta}{\sqrt{2}} = \sqrt{\dfrac{1}{R_x^2} + \dfrac{1}{2\sigma_x^2}}\ w \ .$

Then $\qquad I_x = \dfrac{1}{\sqrt{2\pi}\,\sigma_x} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{\theta^2}{2}} \cdot \dfrac{1}{\sqrt{2}\sqrt{\dfrac{1}{R_x^2} + \dfrac{1}{2\sigma_x^2}}}\, d\theta$

Since $\qquad \displaystyle\int_{-\infty}^{\infty} e^{-\frac{\theta^2}{2}} d\theta = \sqrt{2\pi}$

It follows that

$$I_x = \frac{1}{\sigma_x \sqrt{2}\sqrt{\dfrac{1}{R_x^2} + \dfrac{1}{2\sigma_x^2}}}$$

$$= \frac{1}{\sqrt{\dfrac{2\sigma_x^2}{R_x^2} + 1}}$$

$$= \frac{R_x}{\sqrt{R_x^2 + 2\sigma_x^2}} \ . \tag{5}$$

Similarly, one obtains for the second integral in (3)

$$I_y = \frac{1}{\sqrt{2\pi}\,\sigma_y} \int_{-\infty}^{\infty} \exp[-(\frac{1}{R_y^2} + \frac{1}{2\sigma_y^2})\, z^2]\, dz$$

$$= \frac{R_x}{\sqrt{R_y^2 + 2\sigma_y^2}} \,. \qquad (6)$$

Substituting (5) and (6) in (3) yields

$$P_{kf} = I_x\, I_y = \frac{R_x}{\sqrt{R_x^2 + 2\sigma_x^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2\sigma_y^2}} \,. \qquad (7)$$

SECTION III

ESTIMATION OF $E[P_{kf}]$ AND $Var[P_{kf}]$ USING THE SUBJECTIVE ESTIMATION PROCEDURE

1. Background

Relation (7) shows that the mathematical expression for P_{kf} is a function of the four parameters R_x, R_y, σ_x, and σ_y. In the subjective estimation procedure, it is assumed that the uncertainty level of each of the four input parameters is provided as a subjective information. A lower and upper bound value for each parameter is obtained. These may be determined, for example, through a subjective procedure in which individuals are requested to provide a lower and upper bound values on R_x, R_y, σ_x, and σ_y based on their judgement and their experience. The value of a particular parameter is assumed to take on equally likely values between its two extreme points. This is equivalent to assuming that each parameter is a random variable uniformly distributed over its range of values. Further, the parameters are assumed to be mutually independent random variables. With this statistical information, the evaluation of $E[P_{kf}]$ and $Var[P_{kf}]$ are reduced to the computation of a set of definite integrals.

We now assume that the four parameters R_x, R_y, σ_x, and σ_y are estimated independently and uniformly distributed over the following ranges:

$$R_{x_1} < R_x < R_{x_2} \quad ; \quad R_{y_1} < R_y < R_{y_2}$$

$$\sigma_{x_1} < \sigma_x < \sigma_{x_2} \quad ; \quad \sigma_{y_1} < \sigma_y < \sigma_{y_2} .$$

2. Estimation of $E[P_{kf}]$

From relation (7), it is clear from the stated assumptions that I_x and I_y are independently distributed. Hence,

$$E[P_{kf}] = E[I_x]\,E[I_y]\,. \tag{8}$$

Thus, the problem reduces to finding $E[I_x]$ and $E[I_y]$. To simplify notations, let

$$I = \frac{R}{\sqrt{R^2+2\sigma^2}} \tag{9}$$

where R and σ are independently and uniformly distributed over the respective ranges $R_1 < R < R_2$ and $\sigma_1 < \sigma < \sigma_2$. If $E[I]$ is calculated, then $E[I_x]$ and $E[I_y]$ can be immediately determined. Now

$$E[I] = \frac{1}{(R_2-R_1)(\sigma_2-\sigma_1)} \int_{\sigma_1}^{\sigma_2} \int_{R_1}^{R_2} \frac{R}{\sqrt{R^2+2\sigma^2}}\, dR\, d\sigma\,. \tag{10}$$

Now

$$\int_{R_1}^{R_2} \frac{R}{\sqrt{R^2+2\sigma^2}}\, dR = \int_{R_1}^{R_2} d(\sqrt{R^2+2\sigma^2})$$

$$= \sqrt{R_2^2+2\sigma^2} - \sqrt{R_1^2+2\sigma^2}\,. \tag{11}$$

Substituting (11) in (10) results in

$$E[I] = \frac{1}{(R_2-R_1)(\sigma_2-\sigma_1)} \int_{\sigma_1}^{\sigma_2} \left(\sqrt{R_2^2+2\sigma^2} - \sqrt{R_1^2+2\sigma^2}\right) d\sigma$$

$$= \frac{\sqrt{2}}{(R_2-R_1)(\sigma_2-\sigma_1)} \int_{\sigma_1}^{\sigma_2} \left[\sqrt{\left(\frac{R_2}{\sqrt{2}}\right)^2+\sigma^2} - \sqrt{\left(\frac{R_1}{\sqrt{2}}\right)^2+\sigma^2}\right] d\sigma$$

$$= \frac{\sqrt{2}}{(R_2-R_1)(\sigma_2-\sigma_1)} \Big\{\frac{1}{2}\Big[\sigma\sqrt{\sigma^2+\left(\frac{R_2}{\sqrt{2}}\right)^2}$$

$$+ \frac{R_2^2}{2} \ln\left(\sigma + \sqrt{\sigma^2+\frac{R_2^2}{2}}\right)\Big]_{\sigma_1}^{\sigma_2}$$

$$- \frac{1}{2} \lfloor \sigma \sqrt{\sigma^2 + \left(\frac{R_1}{\sqrt{2}}\right)^2} + \frac{R_1^2}{2} \ln \left(\sigma + \sqrt{\sigma^2 + \frac{R_1^2}{2}}\right) \rfloor_{\sigma_1}^{\sigma_2} \}$$

$$E[I] = \frac{1}{\sqrt{2}(R_2-R_1)(\sigma_2-\sigma_1)} \{ \lfloor \sigma_2 \sqrt{\sigma_2^2 + \frac{R_2^2}{2}} - \sigma_1 \sqrt{\sigma_1^2 + \frac{R_2^2}{2}}$$

$$+ \frac{R_2^2}{2} \ln \frac{\sigma_2 + \sqrt{\sigma_2^2 + \frac{R_2^2}{2}}}{\sigma_1 + \sqrt{\sigma_1^2 + \frac{R_2^2}{2}}} \rfloor - \lfloor \sigma_2 \sqrt{\sigma_2^2 + \frac{R_1^2}{2}}$$

$$- \sigma_1 \sqrt{\sigma_1^2 + \frac{R_1^2}{2}} + \frac{R_1^2}{2} \ln \frac{\sigma_2 + \sqrt{\sigma_2^2 + \frac{R_1^2}{2}}}{\sigma_1 + \sqrt{\sigma_1^2 + \frac{R_1^2}{2}}} \rfloor \} .$$

After a slight rearrangement one obtains

$$E[I] = \frac{1}{\sqrt{2}\,(R_2-R_1)(\sigma_2-\sigma_1)} \lfloor \sigma_2^2 \left(\sqrt{1 + \frac{R_2^2}{2\sigma_2^2}} - \sqrt{1 + \frac{R_1^2}{2\sigma_2^2}}\right)$$

$$- \sigma_1^2 \left(\sqrt{1 + \frac{R_2^2}{2\sigma_1^2}} - \sqrt{1 + \frac{R_1^2}{2\sigma_1^2}}\right)$$

$$+ \frac{R_2^2}{2} \ln \frac{\sigma_2}{\sigma_1} \left(\frac{1 + \sqrt{1 + \frac{R_2^2}{2\sigma_2^2}}}{1 + \sqrt{1 + \frac{R_2^2}{2\sigma_1^2}}} \right)$$

$$- \frac{R_1^2}{2} \ln \frac{\sigma_2}{\sigma_1} \left(\frac{1 + \sqrt{1 + \frac{R_1^2}{2\sigma_2^2}}}{1 + \sqrt{1 + \frac{R_1^2}{2\sigma_1^2}}} \right)] .$$

From this relation $E[I_x]$ and $E[I_y]$ can be obtained

$$E[I_x] = \frac{1}{\sqrt{2}\,(R_{x_2}-R_{x_1})(\sigma_{x_2}-\sigma_{x_1})} [\sigma_{x_2}^2 \left(\sqrt{1 + \frac{R_{x_2}^2}{2\sigma_{x_2}^2}} - \sqrt{1 + \frac{R_{x_1}^2}{2\sigma_{x_2}^2}} \right)$$

$$- \sigma_{x_1}^2 \left(\sqrt{1 + \frac{R_{x_2}^2}{2\sigma_{x_1}^2}} - \sqrt{1 + \frac{R_{x_1}^2}{2\sigma_{x_1}^2}} \right)$$

$$+ \frac{R_{x_2}^2}{2} \ln \frac{\sigma_{x_2}}{\sigma_{x_1}} \left(\frac{1 + \sqrt{1 + \frac{R_{x_2}^2}{2\sigma_{x_2}^2}}}{1 + \sqrt{1 + \frac{R_{x_2}^2}{2\sigma_{x_1}^2}}} \right)$$

$$- \frac{R_{x_1}^2}{2} \ln \frac{\sigma_{x_2}}{\sigma_{x_1}} \left(\frac{1 + \sqrt{1 + \frac{R_{x_1}^2}{2\sigma_{x_2}^2}}}{1 + \sqrt{1 + \frac{R_{x_1}^2}{2\sigma_{x_1}^2}}} \right)] . \qquad (12)$$

Let

$$X_{ij} = \sqrt{1 + \frac{R_{x_i}^2}{2\sigma_{x_j}^2}} \qquad i,j = 1,2 \; . \tag{13}$$

Using (13) in (12) one obtains

$$E[I_x] = \frac{1}{\sqrt{2}\,(R_{x_2}-R_{x_1})(\sigma_{x_2}-\sigma_{x_1})} \Big[\sigma_{x_2}^2 \,(X_{22}-X_{12}) - \sigma_{x_1}^2 \,(X_{21} - X_{11})$$

$$+ \frac{R_{x_2}^2}{2} \ln \frac{\sigma_{x_2}}{\sigma_{x_1}} \frac{(1+X_{22})}{(1+X_{21})} - \frac{R_{x_1}^2}{2} \ln \frac{\sigma_{x_2}}{\sigma_{x_1}} \frac{(1+X_{12})}{(1+X_{11})}\Big] \; . \tag{14}$$

Similarly,

$$E[I_y] = \frac{1}{\sqrt{2}\,(R_{y_2}-R_{y_1})(\sigma_{y_2}-\sigma_{y_1})} \Big[\sigma_{y_2}^2 \Big(\sqrt{1 + \frac{R_{y_2}^2}{2\sigma_{x_2}^2}} - \sqrt{1 + \frac{R_{y_1}^2}{2\sigma_{y_2}^2}}\Big)$$

$$- \sigma_{y_1}^2 \Big(\sqrt{1 + \frac{R_{y_2}^2}{2\sigma_{y_1}^2}} - \sqrt{1 + \frac{R_{y_1}^2}{2\sigma_{y_1}^2}}\Big)$$

$$+ \frac{R_{y_2}^2}{2} \ln \frac{\sigma_{y_2}}{\sigma_{y_1}} \left(\frac{1 + \sqrt{1 + \frac{R_{y_2}^2}{2\sigma_{y_2}^2}}}{1 + \sqrt{1 + \frac{R_{y_2}^2}{2\sigma_{y_1}^2}}} \right)$$

$$- \frac{R_{y_1}^2}{2} \ln \frac{\sigma_{y_2}}{\sigma_{y_1}} \left(\frac{1 + \sqrt{1 + \frac{R_{y_1}^2}{2\sigma_{y_2}^2}}}{1 + \sqrt{1 + \frac{R_{y_1}^2}{2\sigma_{y_1}^2}}} \right) \Big] \ . \tag{15}$$

Let

$$Y_{ij} = \sqrt{1 + \frac{R_{y_i}^2}{2\sigma_{y_j}^2}} \qquad i,j = 1,2 \ . \tag{16}$$

Using (16) in (15) one obtains

$$E[I_y] = \frac{1}{\sqrt{2}\,(R_{y_2} - R_{y_1})(\sigma_{y_2} - \sigma_{y_1})} \Big[\sigma_{y_2}^2 \left(Y_{22} - Y_{12}\right) - \sigma_{y_1}^2 \left(Y_{21} - Y_{11}\right)$$

$$+ \frac{R_{y_2}^2}{2} \ln \frac{\sigma_{y_2}}{\sigma_{y_1}} \frac{(1+Y_{22})}{(1+Y_{21})} - \frac{R_{y_1}^2}{2} \ln \frac{\sigma_{y_2}}{\sigma_{y_1}} \frac{(1+Y_{12})}{(1+Y_{11})} \Big] \ . \tag{17}$$

3. Estimation for $Var[P_{kf}]$

From relation (7) one obtains

$$Var[P_{kf}] = E[I_x^2] \cdot E[I_y^2] - \{E[I_x] \cdot E[I_y]\}^2 \ .$$

Since $E[I_x]$ and $E[I_y]$ have been obtained, it suffices to find $E[I_x^2]$ and $E[I_y^2]$. Expression (9) is referred to and one has

$$I^2 = \frac{R^2}{R^2 + 2\sigma^2} \ . \tag{18}$$

Thus:

$$E[I^2] = \frac{1}{(R_2-R_1)(\sigma_2-\sigma_1)} \int_{R_1}^{R_2} \int_{\sigma_1}^{\sigma_2} \frac{R^2}{R^2+2\sigma^2} \, d\sigma dR \ . \tag{19}$$

Now

$$\int_{\sigma_1}^{\sigma_2} \frac{R^2}{R^2+2\sigma^2} \, d\sigma = \frac{R^2}{2} \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\sigma^2 + \left(\frac{R}{\sqrt{2}}\right)^2}$$

$$= \frac{R^2}{2} \frac{1}{\frac{R}{\sqrt{2}}} \tan^{-1} \frac{\sigma}{\frac{R}{\sqrt{2}}} \Bigg|_{\sigma_1}^{\sigma_2}$$

$$= \frac{R}{\sqrt{2}} \tan^{-1} \frac{\sigma_2 \sqrt{2}}{R} - \frac{R}{\sqrt{2}} \tan^{-1} \frac{\sigma_1 \sqrt{2}}{R}$$

$$= \frac{R}{\sqrt{2}} \cot^{-1} \frac{R}{\sigma_2 \sqrt{2}} - \frac{R}{\sqrt{2}} \cot^{-1} \frac{R}{\sigma_1 \sqrt{2}} \ . \tag{20}$$

Substituting (20) in (19) yields

$$E[I^2] = \frac{1}{(R_2-R_1)(\sigma_2-\sigma_1)} \int_{R_1}^{R_2} \left[\frac{R}{\sqrt{2}} \cot^{-1} \frac{R}{\sigma_2\sqrt{2}} - \frac{R}{\sqrt{2}} \cot^{-1} \frac{R}{\sigma_1\sqrt{2}}\right] dR. \tag{21}$$

Making use of the following result

$$\int x \, (\cot^{-1} ax) \, dx = \frac{1+a^2x^2}{2a^2} \cot^{-1} ax + \frac{x}{2a} + \text{const},$$

expression (21) becomes

$$E[I^2] = \frac{1}{(R_2-R_1)(\sigma_2-\sigma_1)\sqrt{2}} \left[\frac{1+\frac{1}{2\sigma_2^2}R^2}{2\,\frac{1}{2\sigma_2^2}} \cot^{-1} \frac{R}{\sigma_2\sqrt{2}} \right.$$

$$\left. + \frac{R}{2\,\frac{1}{\sigma_2\sqrt{2}}} - \frac{1+\frac{1}{2\sigma_1^2}R^2}{2\,\frac{1}{2\sigma_1^2}} \cot^{-1} \frac{R}{\sigma_1\sqrt{2}} - \frac{R}{2\,\frac{1}{\sigma_1\sqrt{2}}} \Bigg|_{R_1}^{R_2} \right]$$

$$= \frac{\sqrt{2}}{(R_1-R_1)(\sigma_2-\sigma_1)} \left[\frac{R^2+2\sigma_2^2}{4} \tan^{-1} \frac{\sigma_2\sqrt{2}}{R} + \frac{\sigma_2 R}{2\sqrt{2}} \right.$$

$$\left. - \frac{R^2+2\sigma_1^2}{4} \tan^{-1} \frac{\sigma_1\sqrt{2}}{R} - \frac{\sigma_1 R}{2\sqrt{2}} \Bigg|_{R_1}^{R_2} \right] .$$

This expression finally reduces to

$$E[I^2] = \frac{\sqrt{2}}{(R_2-R_1)(\sigma_2-\sigma_1)} \left[\frac{R_2^2+2\sigma_2^2}{4} \tan^{-1} \frac{\sigma_2\sqrt{2}}{R_2} - \frac{R_2^2+2\sigma_1^2}{4} \tan^{-1} \frac{\sigma_1\sqrt{2}}{R_2} \right.$$

$$\left. - \frac{R_1^2+2\sigma_2^2}{4} \tan^{-1} \frac{\sigma_2\sqrt{2}}{R_1} + \frac{R_1^2+2\sigma_1^2}{4} \tan^{-1} \frac{\sigma_1\sqrt{2}}{R_1} \right] + \frac{1}{2} . \tag{22}$$

The expressions for $E[I_x^2]$ and $E[I_y^2]$ are immediately obtained as

$$E[I_x^2] = \frac{1}{2} + \frac{\sqrt{2}}{(R_{x_2}-R_{x_1})(\sigma_{x_2}-\sigma_{x_1})} \left[\frac{R_{x_2}^2+2\sigma_{x_2}^2}{4} \tan^{-1} \frac{\sigma_{x_2}\sqrt{2}}{R_{x_2}} \right.$$

$$- \frac{R_{x_2}^2+2\sigma_{x_1}^2}{4} \tan^{-1} \frac{\sigma_{x_1}\sqrt{2}}{R_{x_2}} - \frac{R_{x_1}^2+2\sigma_{x_2}^2}{4} \tan^{-1} \frac{\sigma_{x_2}\sqrt{2}}{R_{x_1}}$$

$$\left. + \frac{R_{x_1}^2+2\sigma_{x_1}^2}{4} \tan^{-1} \frac{\sigma_{x_1}\sqrt{2}}{R_{x_1}} \right] . \tag{23}$$

$$E[I_y^2] = \frac{1}{2} + \frac{\sqrt{2}}{(R_{y_2}-R_{y_1})(\sigma_{y_2}-\sigma_{y_1})} \left[\frac{R_{y_2}^2+2\sigma_{y_2}^2}{4} \tan^{-1} \frac{\sigma_{y_2}\sqrt{2}}{R_{y_2}}\right.$$

$$- \frac{R_{y_2}^2+2\sigma_{y_1}^2}{4} \tan^{-1} \frac{\sigma_{y_1}\sqrt{2}}{R_{y_2}} - \frac{R_{y_1}^2+2\sigma_{y_2}^2}{4} \tan^{-1} \frac{\sigma_{y_2}\sqrt{2}}{R_{y_1}}$$

$$\left. + \frac{R_{y_1}^2+2\sigma_{y_1}^2}{4} \tan^{-1} \frac{\sigma_{y_1}\sqrt{2}}{R_{y_1}}\right] . \tag{24}$$

4. Example

A weapon whose main effect is kill due to fragmentation is aimed at a particular target from a flying aircraft. Following its release, it is estimated that the weapon impact angle will be 60°. The standard deviation of the aiming error in range is 150 ± 50 ft (to be interpreted as equally likely between 100 and 200 ft). The standard deviation of the aiming error in deflection is 80 ± 10 ft. For the given impact angle and weapon/target situation, it is estimated that the range weapon radius and the deflection weapon radius are, respectively, 85 ± 5 ft and 170 ± 8 ft. It is required to determine the probability of kill P_{kf} of the weapon and to provide a two-standard deviation confidence interval for P_{kf}. Using the formally introduced notations one has

$$R_{x_1} = 80 \text{ ft}, \; R_{x_2} = 90 \text{ ft}, \; R_{y_1} = 162 \text{ ft}, \; R_{y_2} = 178 \text{ ft},$$

$$\sigma_{x_1} = 100 \text{ ft}, \; \sigma_{x_2} = 200 \text{ ft}, \; \sigma_{y_1} = 70 \text{ ft, and } \sigma_{y_2} = 90 \text{ ft}.$$

From equation (13) the values of X_{ij} are first derived

$$X_{11} = \sqrt{1 + \frac{R_{x_1}^2}{2\sigma_{x_1}^2}} = \sqrt{1 + \frac{(80)^2}{2(100)^2}} = 1.148,912,5$$

$$X_{12} = \sqrt{1 + \frac{R_{x_1}^2}{2\sigma_{x_2}^2}} = \sqrt{1 + \frac{(80)^2}{2(200)^2}} = 1.039,230,5$$

$$X_{21} = \sqrt{1 + \frac{R_{x_2}^2}{2\sigma_{x_1}^2}} = \sqrt{1 + \frac{(90)^2}{2(100)^2}} = 1.185,326,9$$

$$X_{22} = \sqrt{1 + \frac{R_{x_2}^2}{2\sigma_{x_2}^2}} = \sqrt{1 + \frac{(90)^2}{2(200)^2}} = 1.049,404,6.$$

From equation (16) the values of Y_{ij} are next derived

$$Y_{11} = \sqrt{1 + \frac{R_{y_1}^2}{2\sigma_{y_1}^2}} = \sqrt{1 + \frac{(162)^2}{2(70)^2}} = 1.917,800,6$$

$$Y_{12} = \sqrt{1 + \frac{R_{y_1}^2}{2\sigma_{y_2}^2}} = \sqrt{1 + \frac{(162)^2}{2(90)^2}} = 1.618,641,4$$

$$Y_{21} = \sqrt{1 + \frac{R_{y_2}^2}{2\sigma_{y_1}^2}} = \sqrt{1 + \frac{(178)^2}{2(70)^2}} = 2.057,440,5$$

$$Y_{22} = \sqrt{1 + \frac{R_{y_2}^2}{2\sigma_{y_2}^2}} = \sqrt{1 + \frac{(178)^2}{2(90)^2}} = 1.719,244,7.$$

Referring to (14), $E[I_x]$ is computed using the numerical values of X_{ij} (i,j = 1,2).

$$E[I_x] = \frac{1}{\sqrt{2}\,(90-80)(200-100)} \lfloor (200)^2 (1.049,404,6 - 1.039,230,5)$$

$$- (100)^2 (1.185,326,9 - 1.148,912,5)$$

$$+ \frac{(90)^2}{2} \ln \frac{200}{100} \frac{1 + 1.049,404,6}{1 + 1.185,326,9}$$

$$+ \frac{(80)^2}{2} \ln \frac{200}{100} \frac{1+1.039,230,5}{1+1.148,912,5} \rfloor$$

$$E[I_x] = \frac{1}{\sqrt{2}\,(10)(100)} [(40,000)(.010,174,1)-(10,000)(.036,414,4)$$

$$+ \left(\frac{8100}{2}\right)(.628,931,1) - \left(\frac{6,400}{2}\right)(.640,757,8)]$$

$$= .381,530,8. \tag{25}$$

Referring to (17), $E[I_y]$ is computed using the numerical values of Y_{ij} (i,j = 1,2).

$$E[I_y] = \frac{1}{\sqrt{2}\,(178-162)(90-70)} \lfloor (90)^2 (1.719,244,7 - 1.618,641,4)$$

$$- (70)^2 (2.057,440,5 - 1.978,006,0)$$

$$+ \frac{(178)^2}{2} \ln \frac{90}{70} \quad \frac{1 + 1.719,244,7}{1 + 2.057,440,5}$$

$$- \frac{(162)^2}{2} \ln \frac{90}{70} \quad \frac{1 + 1.618,641,4}{1 + 1.917,800,6} \rfloor .$$

$$E[I_y] = \frac{1}{\sqrt{2}\,(16)\,(20)} \lfloor (8100)\,(.100,603,3) - (4,900)\,(.139,639,9)$$

$$+ \left(\frac{31,684}{2}\right)(.134,090,5) - \left(\frac{26,244}{2}\right)(.143,139,9) \rfloor$$

$$= .832,242,5. \tag{26}$$

Using (23) $E[I_x^2]$ is next computed

$$E[I_x^2] = \frac{1}{2} + \frac{\sqrt{2}}{(90-80)(200-100)} \left[\frac{(90)^2 + 2(200)^2}{4} \tan^{-1} \frac{200\sqrt{2}}{90}\right.$$

$$- \frac{(90)^2 + 2(100)^2}{4} \tan^{-1} \frac{100\sqrt{2}}{90} - \frac{(80)^2 + 2(200)^2}{4} \tan^{-1} \frac{200\sqrt{2}}{80}$$

$$\left. + \frac{(80)^2 + 2(100)^2}{4} \tan^{-1} \frac{100\sqrt{2}}{80}\right]$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{(10)(100)} [(22{,}025)(1.262{,}728{,}8) - (7{,}025)\ (1.004{,}044{,}0)$$

$$- (21{,}600)\ (1.295{,}153{,}5) + (6{,}600)\ (1.055{,}990{,}3)]$$

$$= .149{,}859{,}2. \qquad (27)$$

Finally, using (24), $E[I_y^2]$ is computed

$$E[I_y^2] = \frac{1}{2} + \frac{\sqrt{2}}{(178-162)\ (90-70)} \left[\frac{(178)^2 + 2(90)^2}{4} \tan^{-1} \frac{90\sqrt{2}}{178}\right.$$

$$- \frac{(178)^2 + 2(70)^2}{4} \tan^{-1} \frac{70\sqrt{2}}{178} - \frac{(162)^2 + 2(90)^2}{4} \tan^{-1} \frac{90\sqrt{2}}{162}$$

$$\left. + \frac{(162)^2 + 2(70)^2}{4} \tan^{-1} \frac{70\sqrt{2}}{162}\right]$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{(16)\ (20)} [(11{,}971)(.620{,}756{,}6) - (10{,}371)(.507{,}553{,}7)$$

$$- (10{,}611)(.665{,}944{,}4) + (9{,}011)(.548{,}526{,}7)]$$

$$= .693{,}018{,}1. \qquad (28)$$

$E[P_{kf}]$ and $Var[P_{kf}]$ may now be computed. Now using (24) and (25) one obtains

$$E[P_{kf}] = E[I_x]\ E[I_y]$$

$$= (.381,530,8)\ (.832,242,5)$$

$$= .317,526,147 \tag{29}$$

It immediately follows that

$$\{E[P_{kf}]\}^2 = (.317,526,147)^2$$

$$= .100,822,854. \tag{30}$$

The expression for the variance was obtained as

$$Var[P_{kf}] = E[I_x^2]\ E[I_y^2] - \{E[P_{kf}]\}^2 .$$

Using (27), (28) and (30) one obtains

$$Var[P_{kf}] = (.149,859,2)(.693,018,1) - (.100,822,854)$$

$$= .003,032,284.$$

The standard deviation of P_{kf} is

$$\sigma_{P_{kf}} = \sqrt{.003,032,284}$$

$$= .055. \tag{31}$$

The two-standard deviation confidence interval on P_{kf} is immediately obtained from (29) and (31); thus

$$\hat{P}_{kf} = .318 \pm .110 \tag{32}$$

SECTION IV

ESTIMATION OF $E[P_{kf}]$ and $Var[P_{kf}]$ USING THE TAYLOR'S SERIES ESTIMATION PROCEDURE

1. Background

Recalling that P_{kf} is a function of the four input parameters R_x, R_y, σ_x and σ_y, one may write

$$P_{kf} = P_{kf}(R_x,R_y,\sigma_x,\sigma_y) = \frac{R_x}{\sqrt{R_x^2 + 2\sigma_x^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2\sigma_y^2}} . \tag{33}$$

Let $\bar{R}_x$, $\bar{R}_y$, $\bar{\sigma}_x$ and $\bar{\sigma}_y$ refer, respectively, to the mean of R_x, R_y, σ_x, and σ_y. Expanding P_{kf} about the point $(\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y)$ one obtains up to the first order terms:

$$\begin{aligned} P_{kf}(R_x,R_y,\sigma_x,\sigma_y) &= P_{kf}(\bar{R}_x,\bar{R}_y,\bar{\sigma}_x,\bar{\sigma}_y) \\ &+ (R_x - \bar{R}_x) \left.\frac{\partial P_{kf}}{\partial R_x}\right|_{\bar{R}_x,\bar{R}_y,\bar{\sigma}_x,\bar{\sigma}_y} + (R_y-\bar{R}_y) \left.\frac{\partial P_{kf}}{\partial R_y}\right|_{\bar{R}_x,\bar{R}_y,\bar{\sigma}_x,\bar{\sigma}_y} \\ &+ (\sigma_x-\bar{\sigma}_x) \left.\frac{\partial P_{kf}}{\partial \sigma_x}\right|_{\bar{R}_x,\bar{R}_y,\bar{\sigma}_x,\bar{\sigma}_y} + (\sigma_y-\bar{\sigma}_y) \left.\frac{\partial P_{kf}}{\partial \sigma_y}\right|_{\bar{R}_x,\bar{R}_y,\bar{\sigma}_x,\bar{\sigma}_y} . \end{aligned} \tag{34}$$

Note that the partial derivatives of P_{kf} with respect to the four variables R_x, R_y, σ_x, and σ_y are to be evaluated at the mean values of the variables.

2. Estimation of $E[P_{kf}]$

Taking expectations on both sides of (34) yields as a first approximation

$$E[P_{kf}(R_x,R_y,\sigma_x,\sigma_y)] = P_{kf}(\bar{R}_x,\bar{R}_y,\bar{\sigma}_x,\bar{\sigma}_y).$$

Thus, at this level of approximation the expected value of P_{kf} is directly obtained by setting in (33) the values of R_x, R_y, σ_x, and σ_y equal to their expected value. More explicitly one can write using (33)

$$E[P_{kf}] = E[P_{kf}(R_x, R_y, \sigma_x, \sigma_y)]$$

$$= \frac{\overline{R}_x}{\sqrt{\overline{R}_x^2 + 2\overline{\sigma}_x^2}} \cdot \frac{\overline{R}_y}{\sqrt{\overline{R}_y^2 + 2\overline{\sigma}_y^2}} . \qquad (35)$$

We now refer to the definitions of I_x and I_y as given in expressions (6) and (7). It is then clear that

$$E[I_x] = \frac{\overline{R}_x}{\sqrt{\overline{R}_x^2 + 2\overline{\sigma}_x^2}}$$

and

$$E[I_y] = \frac{\overline{R}_y}{\sqrt{\overline{R}_y^2 + 2\overline{\sigma}_y^2}}$$

and $$E[P_{kf}] = E[I_x] \cdot E[I_y] .$$

It should be noted that an improved approximation in the value of $E[P_{kf}]$ can be obtained by incorporating additional terms in the Taylor's series expansion given by (34).

3. <u>Estimation of $Var[P_{kf}]$</u>

First, expression (34) is written as

$$P_{kf}(R_x, R_y, \sigma_x, \sigma_y) - P_{kf}(\overline{R}_x, \overline{R}_y, \overline{\sigma}_x, \overline{\sigma}_y)$$

$$= (R_x - \bar{R}_x) \left.\frac{\partial P_{kf}}{\partial R_x}\right|_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y} + (R_y - \bar{R}_y) \left.\frac{\partial P_{kf}}{\partial R_y}\right|_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ (\sigma_x - \bar{\sigma}_x) \left.\frac{\partial P_{kf}}{\partial \sigma_x}\right|_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y} + (\sigma_y - \bar{\sigma}_y) \left.\frac{\partial P_{kf}}{\partial \sigma_y}\right|_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y} . \tag{36}$$

Squaring and taking expectations on both sides of (36) yields

$$\mathrm{Var}[P_{kf}(R_x, R_y, \sigma_x, \sigma_y)] = \mathrm{Var}[R_x] \left(\frac{\partial P_{kf}}{\partial R_x}\right)^2_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ \mathrm{Var}[R_y] \left(\frac{\partial P_{kf}}{\partial R_y}\right)^2_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y} + \mathrm{Var}[\sigma_x] \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ \mathrm{Var}[\sigma_y] \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y} + 2\,\mathrm{Cov}[R_x, R_y] \left[\left(\frac{\partial P_{kf}}{\partial R_x}\right) \cdot \left(\frac{\partial P_{kf}}{\partial R_y}\right)\right]_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ 2\,\mathrm{Cov}[R_x, \sigma_x] \left[\left(\frac{\partial P_{kf}}{\partial R_x}\right) \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)\right]_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ 2\,\mathrm{Cov}[R_x, \sigma_y] \left[\left(\frac{\partial P_{kf}}{\partial R_x}\right) \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)\right]_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ 2\,\mathrm{Cov}[R_y, \sigma_x] \left[\left(\frac{\partial P_{kf}}{\partial R_y}\right) \left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)\right]_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ 2\,\mathrm{Cov}[R_y, \sigma_y] \left[\left(\frac{\partial P_{kf}}{\partial R_y}\right) \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)\right]_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y}$$

$$+ 2\,\mathrm{Cov}[\sigma_x, \sigma_y] \left[\left(\frac{\partial P_{kf}}{\partial \sigma_x}\right) \left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)\right]_{\bar{R}_x, \bar{R}_y, \bar{\sigma}_x, \bar{\sigma}_y} . \tag{37}$$

In the special case when R_x, R_y, σ_x and σ_y are independently distributed, expression (37) becomes

$$\mathrm{Var}[P_{kf}(R_x,R_y,\sigma_x,\sigma_y)] = \mathrm{Var}[R_x]\left(\frac{\partial P_{kf}}{\partial R_x}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}_x,\overline{\sigma}_y}$$

$$+ \mathrm{Var}[R_y]\left(\frac{\partial P_{kf}}{\partial R_y}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}_x,\overline{\sigma}_y} + \mathrm{Var}[\sigma_x]\left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}_x,\overline{\sigma}_y}$$

$$+ \mathrm{Var}[\sigma_y]\left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}_x,\overline{\sigma}_y} . \tag{38}$$

We now obtain the expressions for the partial derivatives

$$\frac{\partial P_{kf}}{\partial R_x} = \frac{R_y}{\sqrt{R_y^2 + 2\sigma_y^2}} \cdot \frac{\sqrt{R_x^2 + 2\sigma_x^2} - R_x \cdot \frac{1}{2} \cdot 2R_x \left(R_x^2 + 2\sigma_x^2\right)^{-\frac{1}{2}}}{(R_x^2 + 2\sigma_x^2)}$$

$$= \frac{2\sigma_x^2 R_y}{\sqrt{R_y^2 + 2\sigma_y^2}\,(R_x^2 + 2\sigma_x^2)^{\frac{3}{2}}} . \tag{39}$$

Hence,

$$\left(\frac{\partial P_{kf}}{\partial R_x}\right)^2 = \frac{4\sigma_x^4 R_y^2}{(R_y^2 + 2\sigma_y^2)(R_x^2+2\sigma_x^2)^3} . \tag{40}$$

Similarly,

$$\frac{\partial P_{kf}}{\partial R_y} = \frac{2\sigma_y^2 R_x}{\sqrt{R_x^2 + 2\sigma_x^2}\,(R_y^2 + 2\sigma_y^2)^{\frac{3}{2}}} . \tag{41}$$

Hence,

$$\left(\frac{\partial P_{kf}}{\partial R_y}\right)^2 = \frac{4\sigma_y^4 R_x^2}{(R_x^2 + 2\sigma_x^2)\,(R_y^2 + 2\sigma_y^2)^3} . \tag{42}$$

The evaluation of $\frac{\partial P_{kf}}{\partial \sigma_x}$ and $\frac{\partial P_{kf}}{\partial \sigma_y}$ proceeds similarly

$$\frac{\partial P_{kf}}{\partial \sigma_x} = (R_x)\ (-\tfrac{1}{2})\ (4\sigma_x)\ (R_x^2 + 2\sigma_x^2)^{-\frac{3}{2}}\ \frac{R_y}{\sqrt{R_y^2 + 2\sigma_y^2}}$$

$$= -\frac{2\sigma_x R_x}{(R_x^2 + 2\sigma_x^2)^{\frac{3}{2}}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2\sigma_y^2}} \ . \qquad (43)$$

Hence,

$$\left(\frac{\partial P_{kf}}{\partial \sigma_x}\right)^2 = \frac{4\sigma_x^2\ R_x^2}{(R_x^2 + 2\sigma_x^2)^3} \cdot \frac{R_y^2}{(R_y^2 + 2\sigma_y^2)} \ . \qquad (44)$$

Similarly,

$$\frac{\partial P_{kf}}{\partial \sigma_y} = -\frac{2\sigma_y R_y}{(R_y^2 + 2\sigma_y^2)^{\frac{3}{2}}} \cdot \frac{R_x}{\sqrt{R_x^2 + 2\sigma_x^2}} \qquad (45)$$

$$\left(\frac{\partial P_{kf}}{\partial \sigma_y}\right)^2 = \frac{4\sigma_y^2\ R_y^2}{(R_y^2 + 2\sigma_y^2)^3} \cdot \frac{R_x^2}{R_x^2 + 2\sigma_x^2} \ . \qquad (46)$$

Expressions (39) through (45) can then be substituted either in (37) or (38) to obtain $Var[P_{kf}(R_x,R_y,\sigma_x,\sigma_y)]$. Writing $Var[P_{kf}]$ for $Var[P_{kf}(R_x,R_y,\sigma_x,\sigma_y)]$, one obtains, for example, from (38)

$$Var[P_{kf}] = \frac{4\bar{\sigma}_x^4\ \bar{R}_y^2}{(\bar{R}_y^2+2\bar{\sigma}_y^2)(\bar{R}_x^2+2\bar{\sigma}_x^2)^3}\ Var[R_x]$$

$$+ \frac{4\bar{\sigma}_y^4\ \bar{R}_x^2}{(\bar{R}_x^2+2\bar{\sigma}_x^2)(\bar{R}_y^2+2\bar{\sigma}_y^2)^3}\ Var[R_y]$$

$$+ \frac{4\bar{\sigma}_x^2 \bar{R}_x^2 \bar{R}_y^2}{(\bar{R}_y^2+2\bar{\sigma}_y^2)(\bar{R}_x^2+2\bar{\sigma}_x^2)^3} \mathrm{Var}[\sigma_x]$$

$$+ \frac{4\bar{\sigma}_y^2 \bar{R}_x^2 \bar{R}_y^2}{(\bar{R}_y^2+2\bar{\sigma}_y^2)^3 (\bar{R}_x^2+2\bar{\sigma}_x^2)} \mathrm{Var}[\sigma_y] \qquad (47)$$

or

$$\mathrm{Var}[P_{kf}] = \frac{4\bar{\sigma}_x^2 \bar{R}_y^2}{(\bar{R}_x^2+2\bar{\sigma}_x^2)^3 (\bar{R}_y^2+2\bar{\sigma}_y^2)} \left(\bar{\sigma}_x^2 \mathrm{Var}[R_x] + \bar{R}_x^2 \mathrm{Var}[\sigma_x]\right)$$

$$+ \frac{4\bar{\sigma}_y^2 \bar{R}_x^2}{(\bar{R}_x^2+2\bar{\sigma}_x^2)(\bar{R}_y^2+2\bar{\sigma}_y^2)^3} \left(\bar{\sigma}_y^2 \mathrm{Var}[R_y] + \bar{R}_y^2 \mathrm{Var}[\sigma_y]\right) . \qquad (48)$$

4. Example

Referring to the example cited in Section III-4, the following values are computed for the necessary data to compute $E[P_{kf}]$ and $\mathrm{Var}[P_{kf}]$

$$\bar{R}_x = 85 \text{ ft}; \qquad \mathrm{Var}[R_x] = \frac{(90-80)^2}{12} = \frac{100}{12} \text{ ft}^2$$

$$\bar{R}_y = 170 \text{ ft}; \qquad \mathrm{Var}[R_y] = \frac{(178-162)^2}{12} = \frac{256}{12} \text{ ft}^2$$

$$\bar{\sigma}_x = 150 \text{ ft}; \qquad \mathrm{Var}[\sigma_x] = \frac{(200-100)^2}{12} = \frac{10{,}000}{12} \text{ ft}^2$$

$$\bar{\sigma}_y = 80 \text{ ft}; \qquad \mathrm{Var}[\sigma_y] = \frac{(90-70)^2}{12} = \frac{400}{12} \text{ ft}^2$$

To compute $E[P_{kf}]$, refer to expression (35) to obtain

$$E[P_{kf}] = \frac{\bar{R}_x}{\sqrt{\bar{R}_x^2 + 2\bar{\sigma}_x^2}} \cdot \frac{\bar{R}_y}{\sqrt{\bar{R}_y^2 + 2\bar{\sigma}_y^2}}$$

$$= \frac{85}{\sqrt{(85)^2 + (2)(150)^2}} \cdot \frac{170}{\sqrt{(170)^2 + (2)(80)^2}}$$

$$E[P_{kf}] = \left(\frac{85}{228.52}\right)\left(\frac{170}{204.20}\right)$$

$$= (.372)\ (.832)$$

$$= .310. \qquad (49)$$

To compute $Var[P_{kf}]$ one can refer either to expression (47) or expression (48). Expression (47) is used since the contribution of each variance component can be identified.

a. <u>Contribution of $Var[R_x]$</u>

$$\frac{4\bar{\sigma}_x^4\ \bar{R}_y^2\ Var[R_x]}{(\bar{R}_x^2 + 2\bar{\sigma}_x^2)^3\ (\bar{R}_y^2 + 2\bar{\sigma}_y^2)} = \frac{(4)(150)^4\ (170)^2\ \left(\frac{100}{12}\right)}{[(85)^2 + 2(150)^2]^3\ [(170)^2 + 2(80)^2]}$$

$$= .000,082,1.$$

b. <u>Contribution of $Var[R_y]$</u>

$$\frac{4\bar{\sigma}_y^4\ \bar{R}_x^2\ Var[R_y]}{(\bar{R}_x^2 + 2\bar{\sigma}_x^2)(\bar{R}_y^2 + 2\bar{\sigma}_y^2)^3} = \frac{(4)\ (80)^4\ (85)^2\ \left(\frac{256}{12}\right)}{[(85)^2 + 2(150)^2][(170)^2 + 2(80)^2]^3}$$

$$= .000,006,7.$$

c. <u>Contribution of $Var[\sigma_x]$</u>

$$\frac{4\bar{\sigma}_x^2\ \bar{R}_y^2\ \bar{R}_x^2\ Var[\sigma_x]}{(\bar{R}_x^2 + 2\bar{\sigma}_x^2)^3\ (\bar{R}_y^2 + 2\bar{\sigma}_y^2)} = \frac{(4)\ (150)^2\ (170)^2\ (85)^2\ \left(\frac{10,000}{12}\right)}{[(85)^2 + 2(150)^2]^3\ [(170)^2 + 2(80)^2]}$$

$$= .002,636,5.$$

d. Contribution of $Var[\sigma_y]$

$$\frac{4\bar{\sigma}_y^2\ \bar{R}_x^2\ \bar{R}_y^2\ Var[\sigma_y]}{(\bar{R}_x^2 + 2\bar{\sigma}_x^2)(\bar{R}_y^2 + 2\bar{\sigma}_y^2)^3} = \frac{(4)\ (80)^2\ (85)^2\ (170)^2\ (\frac{400}{12})}{[(85)^2 + 2(150)^2][(170)^2 + 2(80)^2]^3}$$

$$= .000,047,1.$$

The variance of P_{kf} is thus given by

$$Var[P_{kf}] = .000,082,1 + .000,006,7$$
$$+ .002,636,5 + .000,047,1 = .002,772,4.$$

The standard deviation of P_{kf} is

$$\sigma_{P_{kf}} = \sqrt{.002,772,4} = .053 \tag{50}$$

The two-standard deviation confidence interval on P_{kf} is obtained from (48) and (49) and is

$$P_{kf} = .310 \pm .106. \tag{51}$$

The results are summarized as follows

Component	Variance	Percentage
Range weapon radius R_x	.000,082,1	2.96
Deflection weapon radius R_y	.000,006,7	.24
St. Dev. of aiming error in range σ_x	.002,636,5	95.10
St. Dev. of aiming error in deflection σ_y	.000,047,1	1.70
Total	.002,772,4	100.00

Comparing the results of (32) obtained through the subjective estimation procedure which the results of (50) obtained using the Taylor's series estimation procedure, it becomes clear that the two methods agree very closely and that the estimates are robust.

5. Further Considerations

The previous model could be adapted to account for the following three special situations:

1. $R_x = R_y = R;\ \sigma_x \neq \sigma_y$
2. $R_x \neq R_y;\quad \sigma_x = \sigma_y = \sigma$
3. $R_x = R_y = R;\ \sigma_x = \sigma_y = \sigma$.

The most interesting of these cases is the second one as it corresponds to guided weapons in which the aiming error is the same in all directions. It can be shown then that given that the weapon is aimed in the neighborhood of an origin point, the probability that the weapon will impact between radius r and r + dr (of concentric circles centered at the origin) is given by the Rayleigh density function

$$h(r)\ dr = \frac{1}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right] r\ dr\ .$$

The probability of kill due to fragmentation can be found directly from (7) by setting $\sigma_x = \sigma_y = \sigma$. Thus,

$$P_{kf}(R_x, R_y, \sigma) = \frac{R_x}{\sqrt{R_x^2 + 2\sigma^2}} \cdot \frac{R_y}{\sqrt{R_y^2 + 2\sigma^2}}\ . \tag{52}$$

Using the technique based on the Taylor's series expansion one finds

$$E[P_{kf}(R_x,R_y,\sigma)] = \frac{\overline{R}_x}{\sqrt{\overline{R}_x^2 + 2\overline{\sigma}^2}} \cdot \frac{\overline{R}_y}{\sqrt{\overline{R}_y^2 + 2\overline{\sigma}^2}} \tag{53}$$

$$\begin{aligned} Var[P_{kf}(R_x,R_y,\sigma)] &= Var[R_x] \left(\frac{\partial P_{kf}}{\partial R_x}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}} \\ &+ Var[R_y] \left(\frac{\partial P_{kf}}{\partial R_y}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}} \\ &+ Var[\sigma] \left(\frac{\partial P_{kf}}{\partial \sigma}\right)^2_{\overline{R}_x,\overline{R}_y,\overline{\sigma}} . \end{aligned} \tag{54}$$

Carrying on the differentiation process, one obtains similar to (39) and (41):

$$\frac{\partial P_{kf}}{\partial R_x} = \frac{2\sigma^2 R_y}{\sqrt{R_y^2 + 2\sigma^2}\,(R_x^2 + 2\sigma^2)^{\frac{3}{2}}} \tag{55}$$

$$\frac{\partial P_{kf}}{\partial R_y} = \frac{2\sigma^2 R_x}{\sqrt{R_x^2 + 2\sigma^2}\,(R_y^2 + 2\sigma^2)^{\frac{3}{2}}} . \tag{56}$$

To obtain $\partial P_{kf}/\partial\sigma$ refer to (52) and differentiate partially with respect to σ to yield

$$\begin{aligned} \frac{\partial P_{kf}}{\partial \sigma} = R_x\, R_y\, [&- \tfrac{1}{2}\,(R_x^2 + 2\sigma^2)^{-\frac{3}{2}}\,(4\sigma)(R_y^2 + 2\sigma^2)^{-\frac{1}{2}} \\ &+ (R_x^2 + 2\sigma^2)^{-\frac{1}{2}}\,(-\tfrac{1}{2})\,(R_y^2 + 2\sigma^2)^{-\frac{3}{2}}\,(4\sigma)] \end{aligned}$$

$$= -2\sigma\, R_x\, R_y\, (R_x^2 + 2\sigma^2)^{-\frac{1}{2}}\, (R_y^2 + 2\sigma^2)^{-\frac{1}{2}}$$

$$[(R_x^2 + 2\sigma^2)^{-1} + (R_y^2 + 2\sigma^2)^{-1}]$$

$$= \frac{-2\sigma(R_x R_y)\,(R_x^2 + R_y^2 + 4\sigma^2)}{(R_x^2 + 2\sigma^2)^{\frac{3}{2}}\,(R_y^2 + 2\sigma^2)^{\frac{3}{2}}} \,. \qquad (57)$$

Substituting (55), (56) and (57) in (54) yields

$$\mathrm{Var}[P_{kf}] = \frac{4\bar{\sigma}^4\, \bar{R}_y^2}{(\bar{R}_y^2 + 2\bar{\sigma}^2)(\bar{R}_x^2 + 2\bar{\sigma}^2)^3}\, \mathrm{Var}[R_x]$$

$$+ \frac{4\bar{\sigma}^4\, \bar{R}_x^2}{(\bar{R}_x^2 + 2\bar{\sigma}^2)(\bar{R}_y^2 + 2\bar{\sigma}^2)^3}\, \mathrm{Var}[R_y]$$

$$+ \frac{4\bar{\sigma}^2\, \bar{R}_x^2\, \bar{R}_y^2\, (\bar{R}_x^2 + \bar{R}_y^2 + 4\bar{\sigma}^2)^2}{(\bar{R}_x^2 + 2\bar{\sigma}^2)^3\, (\bar{R}_y^2 + 2\bar{\sigma}^2)^3}\, \mathrm{Var}[\sigma] \,. \qquad (58)$$

Example

In example III-4 assume the weapon to be guided and let the parameters of interest be: $R_x = 85 \pm 5$ ft, $R_y = 170 \pm 8$ ft and $\sigma = 30 \pm 5$ ft. It is required to determine P_{kf}.

Now $\quad \bar{R}_x = 85$ ft, $\quad \mathrm{Var}[R_x] = \dfrac{(90-80)^2}{12} = \dfrac{100}{12}$ ft^2

$\bar{R}_y = 170$ ft, $\quad \mathrm{Var}[R_y] = \dfrac{(178-162)^2}{12} = \dfrac{256}{12}$ ft^2

$\bar{\sigma} = 30$ ft, $\quad \mathrm{Var}[\sigma] = \dfrac{(35-25)^2}{12} = \dfrac{100}{12}$ ft^2

Using (53) yields

$$E[P_{kf}] = \frac{85}{\sqrt{(85)^2 + 2(30)^2}} \cdot \frac{170}{\sqrt{(170)^2 + 2(30)^2}}$$

$$= (.895)(.970) = .868 \; . \tag{59}$$

Expression (58) is now used to determine the contribution of the various variance components to the total variance which is $Var[P_{kf}]$.

a. <u>Contribution of $Var[R_x]$</u>

$$\frac{4\bar{\sigma}^4\, \bar{R}_y^2\, Var[R_x]}{(\bar{R}_x^2 + 2\bar{\sigma}^2)^3\, (\bar{R}_y^2 + 2\bar{\sigma}^2)} = \frac{(4)\;(30)^4\;(170)^2\;(\frac{100}{12})}{[(85)^2 + (2)(30)^2]^3\, [(170)^2 + (2)(30)^2]}$$

$$= .000,034,6.$$

b. <u>Contribution of $Var[R_y]$</u>

$$\frac{4\bar{\sigma}^4\, \bar{R}_x^2\, Var[R_y]}{(\bar{R}_x^2 + 2\bar{\sigma}^2)(\bar{R}_y^2 + 2\bar{\sigma}^2)^3} = \frac{(4)(30)^4\;(85)^2\;(\frac{256}{12})}{[(85)^2 + (2)(30)^2][(170)^2 + (2)(30)^2)^3}$$

$$= .000,001,9.$$

c. <u>Contribution of $Var[\sigma]$</u>

$$\frac{4\bar{\sigma}^2\, \bar{R}_x^2\, \bar{R}_y^2\, (\bar{R}_x^2 + \bar{R}_y^2 + 4\bar{\sigma}^2)^2\; Var[\sigma]}{(\bar{R}_x^2 + 2\bar{\sigma}^2)^3\; (\bar{R}_y^2 + 2\bar{\sigma}^2)^3}$$

$$= \frac{(4)(30)^2\;(85)^2\;(170)^2\;[(85)^2 + (170)^2 + (4)(30)^2]^2\;(\frac{100}{12})}{[(85)^2 + (2)(30)^2]^3\;[(170)^2 + (2)(30)^2]^3}$$

$$= .000,464,8.$$

The variance of P_{kf} is thus given by

$$\mathrm{Var}[P_{kf}] = .000{,}034{,}6 + .000{,}001{,}9 + .000{,}464{,}8$$

$$= .000{,}501{,}3.$$

The standard deviation of P_{kf} is

$$\sigma_{P_{kf}} = \sqrt{.000{,}501{,}3} = .022. \tag{60}$$

The two-standard deviation confidence interval on P_{kf} is obtained from (59) and (60) and is

$$P_{kf} = .868 \pm .044.$$

The variance results are summarized as follows:

Component	Variance	Percentage
Range weapon radius R_x	.000,034,6	6.90
Deflection weapon radius R_y	.000,001,9	.38
St. Dev. of aiming error σ	.000,464,8	92.72
Total	.000,501,3	100.00

SECTION V

CONCLUSIONS

In this report, two methods are developed to measure the variability of the probability of kill P_{kf} due to fragmentation of weapons delivered from air. It is assumed that delivery error is present in killing the target, both weapon and target being idealized as points. A two-parameter Carleton damage function is used to describe mathematically P_{kf} as a function of the distance between the location of the target and the location of weapon explosion.

It is shown that it is possible to provide two-standard deviation confidence intervals on P_{kf} and that the estimates are robust, yielding approximately the same result for the two methods.

It is suggested that future work investigates the case of the more general three-parameter Carleton damage function using the same methodology to compute confidence intervals on P_{kf}.

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