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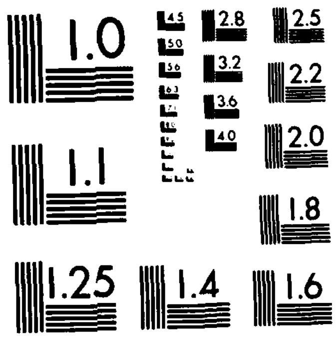

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Starting with the Schwarzschild solution to Einstein's field equations, the equations for relativistic corrections to satellite tracking are derived. Included are corrections for range from satellite to earth station, range from satellite to satellite, and Doppler range difference from satellite to earth station. In addition, the classical and relativistic Doppler effects are derived in the appendixes.

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FOREWORD

This report presents a mathematical derivation of the relativistic effects in satellite tracking. Although the results are *not* new, the presentation of the derivations in this systematic, self-contained way can help the user easily make modifications to suit his requirements.

This report was reviewed by Mr. Robert W. Hill, Head of the Space Flight Sciences Branch, Space and Surface Systems Division.

Released by:

OLIVER F. BRAXTON, Head
Strategic Systems Department

CONTENTS

INTRODUCTION

Since the advent of the NAVSTAR GPS (Global Positioning System) era, it has become more important to correct for relativistic effects in satellite orbit determination. This is primarily so because range to the GPS satellites is now being determined by timing the propagation of a pulse from the satellites to the receivers. Prior to GPS, Doppler data were used to determine the orbits from range difference or range rate.

When using Doppler data, the relativistic effects can be absorbed into a frequency bias that is formally eliminated. No explicit mathematical equation needs to be incorporated into the computer program. This assumes, of course, that the periodic relativistic effect due to orbital eccentricity is negligible.

When using range data, however, the situation becomes more complicated. A pulse emitted from the satellite carries information telling the proper time of the transmitter (satellite) clock at the time of emission, τ_{T_e}. When the pulse is received, the proper time of reception is noted by the receiver (station) clock, τ_{R_r}. The relativistically uncorrected range then is

$$\rho_u = c(\tau_{R_r} - \tau_{T_e})$$

where c is the vacuum speed of light. We say this range is relativistically uncorrected because two clocks at different gravitational potentials do not run at the same rate: for two clocks at rest, the one in the stronger gravitational field runs slower.

Because the clocks run at different rates, even if they are perfect clocks and are perfectly synchronized initially, they will grow more and more out of synchronization and $\tau_{T_e} \neq \tau_{R_e}$ where τ_{R_e} is the proper time of emission as read on the receiver clock. The relativistically corrected range then is

$$\rho = c(\tau_{R_r} - \tau_{R_e}).$$

Although the equations for the relativistic corrections are well known and have been published in several papers,[1,2] the purpose of this paper is to derive them in a systematic way. The only part that is not derived is the Schwarzschild solution to Einstein's field equations. All other derivations are included in this paper in addition to the Doppler effect.

PRELIMINARY DISCUSSION

From the field equations of general relativity, the squared line element in a centrally symmetric gravitational field is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + c^2\left(1 - \frac{2GM}{c^2 r}\right) dt^2 \tag{1}$$

where G is the gravitational constant, M the mass of the central body, and c the velocity of light in vacuum. This exact solution to the field equations was obtained by Schwarzchild.[3,4]

To first order in $1/c^2$, Equation 1 can be written as

$$ds^2 \approx -\left(1 + \frac{2GM}{c^2 r}\right) dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + c^2\left(1 - \frac{2GM}{c^2 r}\right) dt^2$$

$$= c^2\, dt^2 \left[-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 - \frac{r^2}{c^2}\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 - \frac{2GM}{c^4 r}\left(\frac{dr}{dt}\right)^2 + \left(1 - \frac{2GM}{c^2 r}\right)\right]. \tag{2}$$

The first three terms in the bracket are $-\frac{v^2}{c^2}$ expressed in spherical coordinates and since $\frac{v^2}{c^2} \gg \frac{2GM}{c^4 r}\left(\frac{dr}{dt}\right)^2$, the fourth term can be neglected and Equation 2 can be written as

$$ds = c\,dt\left[1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right]^{1/2} . \tag{3}$$

The Newtonian potential for a spherically symmetric field due to a sphere or point mass is $\Phi(r) = -\frac{\mu}{r}$ ($\mu = GM$) so in terms of the Newtonian potential, Equation 3 can be written

$$ds = c\,dt\left[1 + \frac{2\Phi(r)}{c^2} - \frac{v^2}{c^2}\right]^{1/2} . \tag{4}$$

Proper time $d\tau$ is the time interval measured by an observer at rest in any system and is defined by

$$d\tau = \frac{ds}{c} .$$

An observer at rest in a system that is moving with a velocity, v, in a potential $\Phi(r)$ then will measure a proper time interval given by Equation 4

$$d\tau = \left[1 + \frac{2\Phi(r)}{c^2} - \frac{v^2}{c^2}\right]^{1/2} dt. \tag{5}$$

The quantity dt is called the "coordinate time interval." For two observers with different velocities and at different potentials, $d\tau_1 \neq d\tau_2$ whereas $dt_1 = dt_2$; i.e., $dt_1 = dt_2$ as objects observed, not as observers.

If we imagine all points in space containing clocks which by some mechanism are kept in sychronization, the clocks will always be in agreement and show the coordinate time. This coordinate time will be the same as the proper time for an observer at rest at infinity since from Equation 5 as $v \rightarrow 0$ and $\Phi(r) \rightarrow 0$, $d\tau \rightarrow dt$.

The distinction between proper time and coordinate time is very important. To paraphrase Eddington,[4] proper time is the time of consciousness and coordinate time is the time in physical and astronomical reckoning.

RELATIVISTIC CORRECTION TO RANGE

SATELLITE TO EARTH STATION

We can write Equation 5 for a transmitter (any satellite, subscript T) and for a receiver (earth fixed station or any other satellite, subscript R) and we have

$$d\tau_T = \left[1 + \frac{2\Phi(r_T)}{c^2} - \frac{v_T^2}{c^2}\right]^{1/2} dt \tag{6}$$

$$d\tau_R = \left[1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}\right]^{1/2} dt. \tag{7}$$

Note that there is no subscript on the coordinate time interval, dt, as discussed in the previous section. The reason for no subscripts is that we are assuming that the transmission time has already been taken into account. We are only comparing clock rates.

When Equation 6 is divided by Equation 7

$$\frac{d\tau_T}{d\tau_R} = \left[1 + \frac{2\Phi(r_T)}{c^2} - \frac{v_T^2}{c^2}\right]^{1/2} \left[1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}\right]^{-1/2}$$

$$= \left[1 + \frac{1}{c^2}\left(\Phi(r_T) - \frac{v_T^2}{2}\right) - \cdots\right] \left[1 - \frac{1}{c^2}\left(\Phi(r_R) - \frac{v_R^2}{2}\right) + \cdots\right]$$

which gives to order $1/c^2$

$$\frac{d\tau_T}{d\tau_R} = 1 + \frac{1}{c^2}\left(\Phi(r_T) - \frac{v_T^2}{2}\right) - \frac{1}{c^2}\left(\Phi(r_R) - \frac{v_R^2}{2}\right). \tag{8}$$

Potential energy is

$$V = m\Phi(r_T)$$

and the nonrelativistic kinetic energy (valid to our order of approximation) is

$$K = \frac{1}{2} m v_T^2 .$$

Also, from classical mechanics the satellite's total energy is

$$E = K + V = - \frac{\mu m}{2a_T}$$

where $\mu = GM$, m is the satellite mass, and a_T is the semimajor axis. From above then

$$K = E - V = - \frac{\mu m}{2a_T} - V$$

and the first bracket in Equation 8 becomes

$$\frac{1}{c^2}\left[\Phi(r_T) - \frac{v_T^2}{2}\right] = \frac{1}{mc^2}\left[V - K\right] = \frac{1}{mc^2}\left[2V + \frac{\mu m}{2a_T}\right] = - \frac{2\mu}{c^2}\left[\frac{1}{r_T} - \frac{1}{4a_T}\right]$$

so

$$\frac{d\tau_T}{d\tau_R} = 1 + \underline{\frac{1}{c^2}\left[\frac{\mu}{r_R} + \frac{v_R^2}{2}\right]} - \frac{2\mu}{c^2}\left[\frac{1}{r_T} - \frac{1}{4a_T}\right]. \tag{9}$$

If the receiver is a ground fixed station, then the underlined term is constant for a given station and can be handled separately. Nevertheless, we will carry the term along.

For an elliptical orbit

$$r_T = a_T \left(1 - e_T \cos E_T(\tau_R)\right) \tag{10}$$

where $E_T(\tau_R)$ is the eccentric anomaly, e_T is the eccentricity of the satellite orbit, and a_T is the semi-major axis. E_T is written as a function of τ_R since the ground station clock is used as a standard.

When we solve Equation 10 for $\frac{1}{r_T}$ and substitute into Equation 9

$$\frac{d\tau_T}{d\tau_R} = 1 + \frac{1}{c^2}\left[\frac{\mu}{r_R} + \frac{v_R^2}{2}\right] - \frac{\mu}{2a_T c^2}\left[\frac{3 + e_T \cos E_T(\tau_R)}{1 - e_T \cos E_T(\tau_r)}\right] \tag{11}$$

but since

$$\frac{3 + e \cos E}{1 - e \cos E} = 3 + \frac{4\, e \cos E}{1 - e \cos E}$$

Equation 11 can be rewritten as

$$\frac{d\tau_T}{d\tau_R} = 1 + \frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right] - \frac{2\mu e_T}{a_T c^2}\left[\frac{\cos E_T(\tau_R)}{1 - e_T \cos E_T(\tau_R)}\right]. \tag{12}$$

The problem now is to find how far out of synchronization the clocks become after a long time interval (of the order of seconds, minutes, or even hours). To do this, we integrate Equation 12

$$\tau_T = \tau_R + \frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right]\tau_R - \frac{2\mu e_T}{a_T c^2}\int \frac{\cos E_T(\tau_R)\, d\tau_R}{1 - e_T \cos E_T(\tau_R)} + K \tag{13}$$

where K is a constant of integration.

Kepler's equation is

$$n(\tau_R - \tau_{R_O}) = E_T(\tau_R) - e_T \sin E_T(\tau_R)$$

where

$$n = \mu^{1/2} / a_T{}^{3/2}$$

so

$$d\tau_R = \frac{a_T{}^{3/2}}{\mu^{1/2}} \left(1 - e_T \cos E_T(\tau_R)\right) dE_T. \tag{14}$$

If we substitute Equation 14 into Equation 13 and carry out the integration, we get

$$\tau_T = \tau_R + \frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right]\tau_R - \frac{2\mu^{1/2}\, a_T{}^{1/2} e_T}{c^2} \sin E_T(\tau_R) + K. \qquad (15)$$

If desired, we can evaluate K in terms of the time when the clocks were synchronized. If $\tau_T = \tau_R = \tau_0$, then from Equation 15

$$K = \frac{2\mu^{1/2}\, a_T{}^{1/2} e_T}{c^2} \sin E_T(\tau_0) - \frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right]\tau_0. \qquad (16)$$

For the time of emission of a pulse, Equation 15 gives

$$\tau_{T_e} = \tau_{R_e} + \frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right]\tau_{Re} - \frac{2\mu^{1/2}\, a_T{}^{1/2} e_T}{c^2} \sin E_T(\tau_{Re}) + K \qquad (17)$$

where τ_{Te} is the time of emission by the satellite clock and τ_{Re} is the time of emission by the station clock. If we let τ_{Rr} be the reception time by the station clock, the uncorrected range is obtained by using the satellite clock reading τ_{Te} for the emission time

$$\rho_u = c(\tau_{R_r} - \tau_{T_e})$$

and the corrected range is

$$\rho = c(\tau_{R_r} - \tau_{R_e})$$

where both times are read on the station clock.

From Equation 17, the corrected range then is

$$\rho = \rho_u + \frac{1}{c}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right]\tau_{R_e} - \frac{2\mu^{1/2}\, a_T{}^{1/2}\, e_T}{c} \sin E_T(\tau_{R_e}) + cK \qquad (18)$$

where cK is a range bias due to relativistic effects.

If the user has an orbit determination program that uses range difference data, he can write Equation 18 in the form*

$$\Delta\rho = \Delta\rho_u + \underline{\frac{1}{c}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right]} \Delta\tau_{R_r} - \frac{2\mu^{1/2}\, a_T^{\;1/2} e_T}{c}\left[\sin\left(E_T + \Delta E_T\right) - \sin E_T\right] \qquad (19)$$

where E_T is the eccentric anomaly at the time of emission of the first pulse according to the station clock.

Also, if the user has already integrated an orbit such that he has satellite position $\underline{r}$ and velocity $\underline{v}$ at various times. he can avoid the calculation of E_T in Equation 18 and write the periodic relativistic correction as

$$\frac{2\mu^{1/2}\, a_T^{\;1/2} e_T}{c} \sin E_T = \frac{2}{c}\, \underline{r} \cdot \underline{v}. \qquad (20)$$

This relation was called to the author's attention by John T. Carr of the Space and Surface Systems Division, Strategic Systems Department, NSWC, Dahlgren, Va. The derivation of Equation 20 is given in Appendix A.

Note that the underscored term in all equations throughout is constant and can be handled in various ways. For example, since the underscored quantity in Equation 17 causes the GPS clock to run fast compared with a station clock, prior to launch the satellite clock is purposely set low to 10.22999999545 Mhz or 4.45 x 10^{-10} low in frequency relative to a nominal 10.23 Mhz.[5] This in effect removes the underscored term and leaves only the periodic relativistic term.

SATELLITE TO SATELLITE

For the case of satellite-to-satellite position determination, e.g., a host vehicle using GPS satellites to navigate via GPSPAC, we write Equation 15 for each satellite using subscript G for GPS and H for the host vehicle. We still use R for the ground station whose clock serves as a standard.

From Equation 15 then

$$\tau_{G_e} = \tau_{R_e} + \underline{\frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_G} + \frac{v_R^2}{2}\right]} \tau_{R_e} - \frac{2\mu^{1/2}\, a_G^{\;1/2}\, e_G}{c^2} \sin E_G(\tau_{R_e}) + K_G \qquad (21)$$

$$\tau_{H_r} = \tau_{R_r} + \underline{\frac{1}{c^2}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_H} + \frac{v_R^2}{2}\right]} \tau_{R_r} - \frac{2\mu^{1/2}\, a_H^{\;1/2}\, e_H}{c^2} \sin E_H(\tau_{R_r}) + K_H \qquad (22)$$

where the subscripts e and r mean emission and reception, respectively, with K_G and K_H corresponding to K of Equation 15.

*In going from Equation 18 to Equation 19, we have used $\tau_{R_e} = \tau_{R_r} - \frac{\rho}{c}$, $\Delta\tau_{R_e} = \Delta\tau_{R_r} - \frac{\Delta\rho}{c}$ and dropped terms of order $1/c^2$.

When we use

$$\frac{1}{1+\frac{1}{c^2}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_G}+\frac{v_R^2}{2}\right]} \approx 1-\frac{1}{c^2}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_G}+\frac{v_R^2}{2}\right]$$

and

$$\frac{1}{1+\frac{1}{c^2}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_H}+\frac{v_R^2}{2}\right]} \approx 1-\frac{1}{c^2}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_H}+\frac{v_R^2}{2}\right]$$

Equations 21 and 22 can be inverted to $O\left(\frac{1}{c^2}\right)$ to give

$$\tau_{R_e} = \tau_{G_e} - \underline{\frac{1}{c^2}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_G}+\frac{v_R^2}{2}\right]}\tau_{G_e} + \frac{2\mu^{1/2}\, a_G^{\;1/2} e_G}{c^2}\sin E_G(\tau_{R_e}) - K_G \qquad (21')$$

$$\tau_{R_r} = \tau_{H_r} - \underline{\frac{1}{c^2}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_H}+\frac{v_R^2}{2}\right]}\tau_{H_r} + \frac{2\mu^{1/2}\, a_H^{\;1/2} e_H}{c^2}\sin E_H(\tau_{R_r}) - K_H . \qquad (22')$$

The uncorrected range is

$$\rho_u = c(\tau_{H_r} - \tau_{G_e})$$

and the corrected range is

$$\rho = c(\tau_{R_r} - \tau_{R_e})$$

where τ_{R_e} is the time the pulse is emitted by the GPS satellite and τ_{R_r} is the time of reception by the host vehicle, each measured on the earth station clock.

If Equation 21′ is subtracted from Equation 22′ and multiplied by c

$$\rho = \rho_u + \underline{\frac{1}{c}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_G}+\frac{v_R^2}{2}\right]}\tau_{G_e} - \underline{\frac{1}{c}\left[\frac{\mu}{r_R}-\frac{3\mu}{2a_H}+\frac{v_R^2}{2}\right]}\tau_{H_r}$$

$$+\frac{2\mu^{1/2}}{c}\left[a_H^{\;1/2}\, e_H \sin E_H(\tau_{H_r}) - a_G^{\;1/2}\, e_G \sin E_G(\tau_{G_e})\right] - c\,(K_H - K_G). \qquad (23)$$

In Equation 23, the station clock has been eliminated by using $\tau_{R_r} = \tau_{H_r} + O(\frac{1}{c^2})$ and $\tau_{R_e} = \tau_{G_e} + O(\frac{1}{c^2})$ from Equations 21 and 22.

Equation 23 was developed keeping both of the underscored terms in Equations 21 and 22. If the GPS clock frequency was set low before launch as discussed at the end of the previous section, then in effect the first underscored term in Equation 23 is removed and we have

$$\rho = \rho_u - \underline{\frac{1}{c}\left[\frac{\mu}{r_R} - \frac{3\mu}{2a_H} + \frac{v_R^2}{2}\right]} \tau_{H_r} + \frac{2\mu^{1/2}}{c}\left[a_H{}^{1/2} e_H \sin E_H(\tau_{H_r}) - a_G{}^{1/2} e_G \sin E_G(\tau_{G_e})\right] - c(K_H - K_G{}') \tag{24}$$

leaving one cumulative time effect. Also $K_G{}' \neq K_G$ due to vanishing of the underscored term in Equation 16.

RELATIVISTIC CORRECTION TO DOPPLER DETERMINATION OF RANGE DIFFERENCE

In Appendix C, Equation C-6 gives the general relativistic Doppler effect to be

$$f_R = \frac{\left[1 + \frac{2\Phi(r_T)}{c^2} - \frac{v_T^2}{c^2}\right]^{1/2}}{\left[1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}\right]^{1/2}} \frac{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}\right)}{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}\right)} f_T \tag{25}$$

and Equation C-5 gives

$$\frac{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}\right)}{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}\right)} = 1 - \frac{\dot{\rho}}{c} \tag{26}$$

where subscript T refers to transmitter (any satellite), subscript R refers to receiver (earth fixed station), and $\dot{\rho} = \frac{d\rho}{d\tau_R}$.

The ratio of radicals is the same as in the Relativistic Correction to Range section, so by the same procedure used to obtain Equation 12, Equations 25 and 26 above become

$$f_R = \left[1 + \underline{\frac{1}{c^2}\left(\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right)} - \frac{2\mu e_T}{a_T c^2}\left(\frac{\cos E_T(\tau_R)}{1 - e_T \cos E_T(\tau_R)}\right)\right]\left(1 - \frac{1}{c}\frac{d\rho}{d\tau_R}\right) f_T. \tag{27}$$

If we multiply the terms in Equation 27 together and neglect terms of higher order than $1/c^2$, this becomes

$$f_R = f_T + \underline{\frac{f_T}{c^2}\left(\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right)} - \frac{2f_T \mu e_T}{a_T c^2}\left(\frac{\cos E_T(\tau_R)}{1 - e_T \cos E_T(\tau_R)}\right) - \frac{f_T}{c}\frac{d\rho}{d\tau_R}. \tag{28}$$

The underscored term in Equation 28 can be seen to be a constant frequency bias that could be solved in the computer orbit determination program. In fact, this procedure has been followed over the years in Doppler tracking of satellites; i.e., this relativistic frequency bias has been absorbed into any other frequency biases caused by other effects. By handling the constant relativistic effect in this manner, the only remaining relativistic effect is the periodic term caused by the nonzero eccentricity.

In the following, we will carry the constant term along but continue to underscore it for identification.

In tracking the satellite, we have a station oscillator of frequency, f_S, with the frequency chosen so that $f_S > f_{R\,max}$. Then, if we mix f_S with the incoming signal f_R, the beat frequency is

$$\frac{dN}{d\tau_R} = f_S - f_R$$

so when Equation 28 is subtracted from f_S

$$\frac{dN}{d\tau_R} = \Delta f - \underline{\frac{f}{c^2}\left(\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right)} + \frac{2f_T \mu e_T}{a_T c^2}\left(\frac{\cos E_T(\tau_R)}{1 - e_T \cos E_T(\tau_R)}\right) + \frac{f_T}{c}\frac{d\rho}{d\tau_R} \tag{29}$$

where $\Delta f = f_S - f_T$.

If we integrate both sides of Equation 29 from τ_R to $\tau_R + \Delta\tau_R$ and let N be the number of Doppler counts taken over the time interval $\Delta\tau_R$, we get

$$N = \Delta f \Delta\tau_R - \underline{\frac{f_T}{c^2}\left(\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right)\Delta\tau_R} + \frac{2f_T \mu e_T}{a_T c^2} \int_{\tau_R}^{\tau_R + \Delta\tau_R} \frac{\cos E_T(\tau_R)}{1 - e_T \cos E_T(\tau_R)}\, d\tau_R + \frac{f_T}{c}\, \Delta\rho. \quad (30)$$

As in Equation 13, the integral above is

$$\frac{2f_T \mu^{1/2} a_T^{1/2} e_T}{c^2}\, [\sin (E_T + \Delta E_T) - \sin E_T].$$

so if Equation 30 is multiplied by $\frac{c}{f_T}$ and solved for $\Delta\rho$

$$\Delta\rho = \frac{c}{f_T}\left[N - \Delta f\, \Delta\tau_R\right] + \underline{\frac{1}{c}\left(\frac{\mu}{r_R} - \frac{3\mu}{2a_T} + \frac{v_R^2}{2}\right)\Delta\tau_R} - \frac{2\mu^{1/2} a_T^{1/2} e_T}{c}\left[\sin (E_T + \Delta E_T) - \sin E_T\right]. \quad (31)$$

In Equation 31, the first term is the usual nonrelativistic Doppler result corresponding to $\Delta\rho_u$ (uncorrected range difference), the second term is the constant portion of the relativistic effect, and the third term is the periodic relativistic effect. Thus, Equation 31 above for Doppler data corresponds to Equation 19, which was obtained by differencing two range measurements.

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APPENDIX A

AN ALTERNATE FORM FOR THE PERIODIC RELATIVISTIC EFFECT

From Equation 18, the periodic relativistic effect (PRE) is

$$\text{PRE} = \frac{2\mu^{1/2}a^{1/2}e}{c} \sin E. \tag{A-1}$$

The elliptic orbit equation is

$$r = a\,(1 - e \cos E) \tag{A-2}$$

and Kepler's equation is

$$\frac{\mu^{1/2}}{a^{3/2}}(t - t_O) = E - e \sin E. \tag{A-3}$$

Differentiating Equations A-2 and A-3 with respect to time

$$\dot{r} = ae \sin E\,\dot{E} \tag{A-4}$$

$$\frac{\mu^{1/2}}{a^{3/2}} = (1 - e \cos E)\,\dot{E}. \tag{A-5}$$

When Equation A-5 is solved for $\dot{E}$, substituted into Equation A-4, and Equation A-2 is used to eliminate the quantity $(1 - e \cos E)$, one obtains

$$r\dot{r} = \mu^{1/2}a^{1/2}e \sin E. \tag{A-6}$$

Since $r^2 = \underline{r} \cdot \underline{r}$, then $r\,\dot{r} = \underline{r} \cdot \dot{\underline{r}} = \underline{r} \cdot \underline{v}$, so

$$\text{PRE} = \frac{2\mu^{1/2}a^{1/2}e}{c} \sin E = \frac{2}{c}\underline{r} \cdot \underline{v} \tag{A-7}$$

APPENDIX B

THE CLASSICAL DOPPLER EFFECT

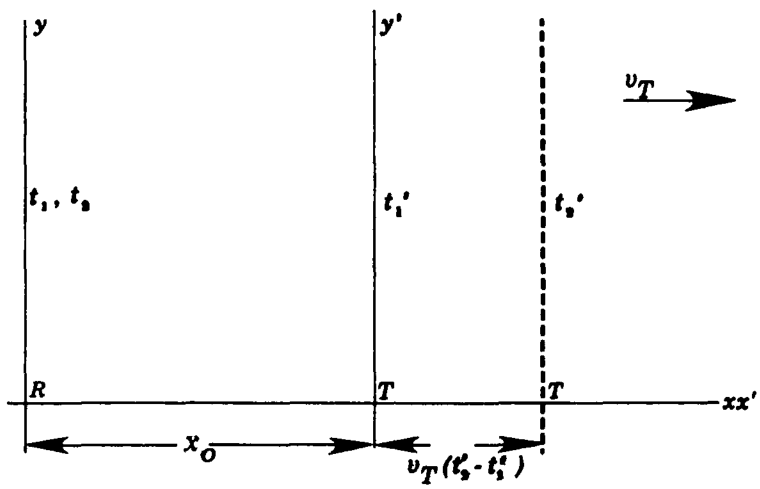

FIGURE B-1. RECEIVER AT REST, TRANSMITTER MOVING AWAY

When T is at position x_O, we imagine an event occurring at its origin (e.g., a light flash) at time t_1' by T's clock. Since the light travels at velocity c, when R sees the event its clock reads

$$t_1 = t_1' + \frac{x_O}{c} \tag{B-1}$$

a later time because of the transmission time x_O/c (Figure B-1).

At a later time, t_2', another event takes place; R sees this event when its clock reads

$$t_2 = t_2' + \frac{x_O}{c} + \frac{v_T}{c}(t_2' - t_1'). \tag{B-2}$$

The additional distance of travel is $v_T\ (t_2' - t_1')$, not $v_T\ (t_2 - t_1)$ since the former is the true distance that T has traveled in the actual time interval between the events, while $t_2 - t_1$ is the apparent time interval according to R.

When Equation B-1 is subtracted from Equation B-2

$$t_2 - t_1 = t_2' - t_1' + \frac{v_T}{c}(t_2' - t_1')$$

or let $\Delta t_R = t_2 - t_1$, $\Delta t_T = t_2' - t_1'$ and we have

$$\Delta t_R = \Delta t_T \left(1 + \frac{v_T}{c}\right). \quad \text{(B-3)}$$

This is the difference in time intervals because of the motion changing the transmission times. The clocks are assumed to be identical and running at the same rates; i.e., no relativistic effects.

Equation B-3 is (classically) true for any pair of events; they do not have to be periodic. If the events are periodic (such as the vibrations of an electron producing a light wave), then $f_R = \frac{1}{\Delta t_R}$, $f_T = \frac{1}{\Delta t_T}$ so using Equation B-3

$$f_R = \frac{f_T}{1 + \frac{v_T}{c}}. \quad \text{(B-4)}$$

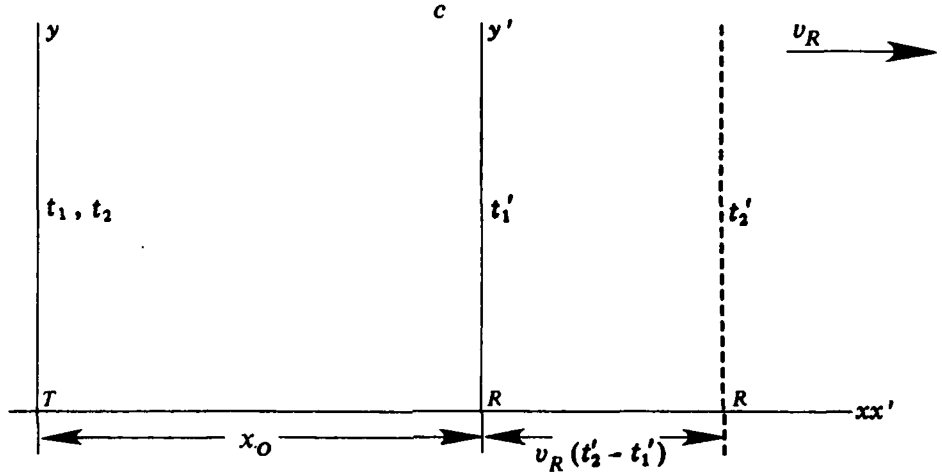

FIGURE B-2. TRANSMITTER AT REST, RECEIVER MOVING AWAY

In the same manner as before (Figure B-2)

$$t_1' = t_1 + \frac{x_O}{c} \quad \text{(B-5)}$$

and

$$t_2' = t_2 + \frac{x_O}{c} + \frac{v_R}{c}(t_2' - t_1'). \quad \text{(B-6)}$$

Note that one might be tempted to write for the additional transmission distance in Equation B-6 $v_R\ (t_2 - t_1)$ since $t_2 - t_1$ is the true time interval between the events. R, however, which is in motion, has been traveling for a time $t_2' - t_1'$ during the interval in which it has observed both events. The additional distance is then $v_R\ (t_2' - t_1')$.

If Equation B-5 is subtracted from Equation B-6

$$t_2' - t_1' = t_2 - t_1 + \frac{v_R}{c}(t_2' - t_1'),$$

and put $\Delta t_R = t_2' - t_1'$, $\Delta t_T = t_2 - t_1$, then

$$\Delta t_R = \Delta t_T + \frac{v}{c} \Delta t_R$$

or

$$\Delta t_R = \frac{\Delta t_T}{1 - \frac{v_R}{c}} \quad . \qquad \text{(B-7)}$$

Again, assuming perfect clocks, this is the difference in time intervals because of the transmission time.

If the events are periodic, $f_R = \frac{1}{\Delta t_R}$, $f_T = \frac{1}{\Delta t_T}$ so from Equation B-7

$$f_R = f_T \left(1 - \frac{v_R}{c}\right). \qquad \text{(B-8)}$$

To summarize to this point

$$f_R = \frac{f_T}{1 + \frac{v_T}{c}} \quad \text{for } R \text{ at rest, } T \text{ moving away,} \qquad \text{(B-4)}$$

$$f_R = f_T \left(1 - \frac{v_R}{c}\right) \quad \text{for } T \text{ at rest, } R \text{ moving away.} \qquad \text{(B-8)}$$

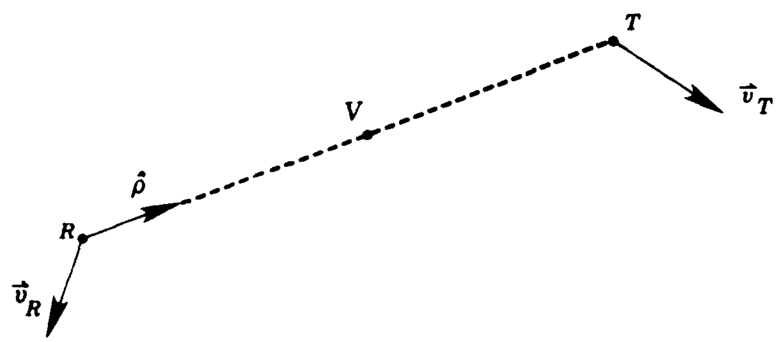

FIGURE B-3. TRANSMITTER AND RECEIVER MOVING IN ARBITRARY DIRECTIONS

To be more general, both Equation B-4 and Equation B-8 assumed R and T were separating, so in those equations $-v_R \rightarrow \hat{\rho} \cdot \vec{v}_T$, $v_T \rightarrow \hat{\rho} \cdot \vec{v}_T$ where $\hat{\rho}$ is a unit vector directed from R toward T (Figure B-3). So Equation B-4 and Equation B-8 can be written

$$f_R = \frac{f_T}{1 + \frac{\hat{\rho} \cdot \vec{v}_T}{c}} \quad R \text{ at rest} \qquad \text{(B-9)}$$

$$f_R = f_T \left(1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}\right) \quad T \text{ at rest.} \qquad \text{(B-10)}$$

To put both R and T in motion, we imagine a "virtual" observer V at rest in inertial space on the line joining R and T. Then, since the observer is at rest, the frequency it receives from T according to Equation B-9 is

$$f_{R_V} = \frac{f_T}{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}\right)} \tag{B-11}$$

and the frequency received by R transmitted by V is by Equation B-10

$$f_R = f_{T_V} \left(1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}\right). \tag{B-12}$$

Since V is an empty point in space, the signal simply passes through so

$$f_{T_V} = f_{R_V} . \tag{B-13}$$

When we substitute Equation B-11 into Equation B-12 using Equation B-13, we have

$$f_R = \frac{1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}}{1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}} \quad f_T \tag{B-14}$$

which is the classical Doppler equation when both R and T are in motion.

In Doppler tracking of satellites, we write the Doppler equation as

$$f_R = \left(1 - \frac{\dot{\rho}}{c}\right) f_T \tag{B-15}$$

where $\dot{\rho}$ is the time rate of change of the range between the station and satellite. At first glance, it might appear that in Equation B-14 we are neglecting the receiver (station) velocity and writing $\hat{\rho} \cdot \vec{v}_T = \dot{\rho}$ so that to first order Equation B-14 is

$$f_R = \frac{1}{1 + \frac{\dot{\rho}}{c}} \quad f_T \approx \left(1 - \frac{\dot{\rho}}{c}\right) f_T .$$

This is not the case!

To derive Equation B-15, we write the range vector as

$$\vec{\rho} = \vec{r}_T(t_e) - \vec{r}_R\ (t_r) \tag{B-16}$$

where t_e = time emitted and t_r = time received. The time emitted is obtained from

$$t_e = t_r - \frac{\rho}{c} . \tag{B-17}$$

From Equation B-16

$$\dot{\vec{\rho}} = \frac{d\vec{r}_T(t_e)}{dt_r} - \frac{d\vec{r}_R\ (t_r)}{dt_r}$$

$$\dot{\vec{\rho}} = \frac{d\vec{r}_T}{dt_e}(t_e)\frac{dt_e}{dt_r} - \frac{dr_R}{dt_r}(t_r) \,. \tag{B-18}$$

From Equation B-17 $\frac{dt_e}{dt_r} = 1 - \frac{\dot{\rho}}{c}$ so (B-18) is

$$\dot{\vec{\rho}} = \vec{v}_T \left(1 - \frac{\dot{\rho}}{c}\right) - \vec{v}_R \,. \tag{B-19}$$

Define a unit vector $\hat{\rho}$ directed from the station toward the satellite. Then if we take the dot product of $\hat{\rho}$ with $\dot{\vec{\rho}}$, we obtain from Equation B-19

$$\dot{\rho} = \hat{\rho} \cdot \dot{\vec{\rho}}$$

$$\dot{\rho} = \hat{\rho} \cdot \vec{v}_T - \hat{\rho} \cdot \vec{v}_T \frac{\dot{\rho}}{c} - \hat{\rho} \cdot \vec{v}_R \tag{B-20}$$

From Equation B-20

$$\dot{\rho} = \frac{\hat{\rho} \cdot \vec{v}_T - \hat{\rho} \cdot \vec{v}_R}{1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}}$$

so

$$1 - \frac{\dot{\rho}}{c} = \frac{1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}}{1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}} \,. \tag{B-21}$$

When Equation B-21 is substituted into Equation B-14, the Doppler equation is put in the form of Equation B-15. It is clear then that Equation B-15 is an exact expression of the classical Doppler equation. There is no dropping of $\dot{\rho}^2/c^2$ and higher-order terms, and the station is not considered at rest.

APPENDIX C

THE RELATIVISTIC DOPPLER EFFECT

The proper time in any system is given by Equation 5. We can then write for transmitter and receiver

$$d\tau_T = dt_T \sqrt{1 + \frac{2\Phi(r_T)}{c^2} - \frac{v_T^2}{c^2}} \qquad \text{(C-1)}$$

$$d\tau_R = dt_R \sqrt{1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}} \quad . \qquad \text{(C-2)}$$

Note that here, unlike Equations 6 and 7, we have put subscripts on the coordinate times. The reason is that in the earlier section of the report our goal was to compare clock rates assuming that the transmission time was taken into account.

The Doppler effect, however, arises because of the transmission time, as was shown in Appendix B. Toward our goal of deriving the relativistic Doppler effect then, t_T in Equation C-1 is the coordinate time when the satellite emitted the signal, and t_R in Equation C-2 is the coordinate time when it is received by the station.

The proper frequency is $f_R = \frac{1}{d\tau_R}$, $f_T = \frac{1}{d\tau_T}$ so if we divide Equation C-1 by Equation C-2

$$\frac{f_R}{f_T} = \frac{\sqrt{1 + \frac{2\Phi(r_T)}{c^2} - \frac{v_T^2}{c^2}}}{\sqrt{1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}}} \frac{dt_T}{dt_R} \; . \qquad \text{(C-3)}$$

In order to obtain an expression for $\frac{dt_T}{dt_R}$, we write from Equation C-2

$$d\tau_{Re} = dt_T \sqrt{1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}}$$

$$d\tau_{Rr} = dt_R \sqrt{1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}}$$

so

$$\frac{dt_T}{dt_R} = \frac{d\tau_{Re}}{d\tau_{Rr}}.$$

Since the corrected range is

$$\rho = c\,(\tau_{R_r} - \tau_{R_e})$$

then

$$\tau_{R_e} = \tau_{R_r} - \frac{\rho}{c}$$

so

$$\frac{dt_T}{dt_R} = \frac{d\tau_{R_e}}{d\tau_{R_r}} = 1 - \frac{\dot{\rho}}{c}. \qquad \text{(C-4)}$$

From Appendix B, Equation B-21, we see that Equation C-4 can be written as

$$\frac{dt_T}{dt_R} = 1 - \frac{\dot{\rho}}{c} = \frac{1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}}{1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}} \qquad \text{(C-5)}$$

so Equation C-3 becomes

$$\frac{f_R}{f_T} = \frac{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_R}{c}\right)\sqrt{1 + \frac{2\Phi(r_T)}{c^2} - \frac{v_T^2}{c^2}}}{\left(1 + \hat{\rho} \cdot \frac{\vec{v}_T}{c}\right)\sqrt{1 + \frac{2\Phi(r_R)}{c^2} - \frac{v_R^2}{c^2}}}. \qquad \text{(C-6)}$$

Equation C-6 is the general relativistic Doppler effect. If R and T are in free space far from any source of gravitation, $\Phi(r_T) = \Phi(r_R) = o$ and Equation C-6 reduces to the special relativistic Doppler effect. If also $\frac{v_T^2}{c^2} \ll 1$ and $\frac{v_R^2}{c^2} \ll 1$, then Equation C-6 reduces to the classical Doppler effect.

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