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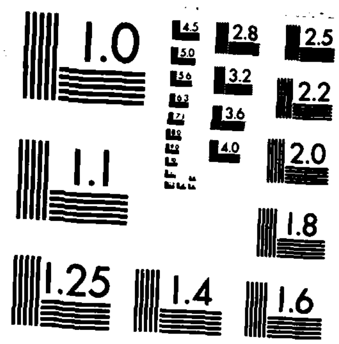

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ON THE TURNING LOZANGE OF CONSTANT SIDE WHOSE VERTICES ALTERNATE BETWEEN TWO FIXED CIRCLES OF A RING

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ABSTRACT

This note was written to call attention to the remarkable paper [1] by W. L. Black, H. C. Howland and B. Howland. Their result recalls the classical theorems of Poucelet and Steiner from the golden age of Geometry. [1] starts with two circles Γ and $\tilde{\Gamma}$ in R^3 satisfying the qualitative condition (A) and assumes that there is a closed equilateral polygon

$$\Pi_{2n} = P_1 P_2 \cdots P_{2n} \quad (n > 2) \quad ,$$

having the sides $P_k P_{k+1} = d$ $(k = 1,2,\ldots,n;\ P_{2n+1} + P_1)$ such that all vertices P_{2k-1} are on Γ, and P_{2k} are on $\tilde{\Gamma}$ $(k = 1,\ldots,n)$. It is shown that Π_{2n} can be turned around with constant d, so as to return to its initial position. This note, independent of [1], settles the case when the circles Γ and $\tilde{\Gamma}$ are coplanar and $n = 2$, when Π_4 becomes a lozange. It is shown that $d^2 = a^2 + b^2 - c^2$, where a and b are the radii of Γ and $\tilde{\Gamma}$, respectively, and c is the distance between the centers of Γ and $\tilde{\Gamma}$.

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cont
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SIGNIFICANCE AND EXPLANATION

on a theorem about zig-zags between two circles,

This note is to call attention to the ~~remarkable~~ paper [1] by W. L. Black, H. C. Howland and B. Howland, and may be read independently of [1]. It explains the construction ~~(Fig. 1)~~ of a linkage composed of four equal bars $P_1P_2 = P_2P_3 = P_3P_4 = P_4P_1 = d$, joined at the points P_k, with the following property: With P_1 and P_3 moving in the circular groove Λ, and P_3, P_4 moving in the circular groove Λ', it is shown that the lozange $P_1P_2P_3P_4$ can be turned around. → cont keywords include: see page -A-

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ON THE TURNING LOZANGE OF CONSTANT SIDE WHOSE VERTICES ALTERNATE BETWEEN TWO FIXED CIRCLES OF A RING

I. J. Schoenberg

I am writing this note to call attention to the remarkable paper [1] by W. L. Black, H. C. Howland and B. Howland. Their result which recalls the two classical theorems of V. Poucelet and J. Steiner (see e.g. [2, Chapter 14]), may be described as follows.

Let Γ and $\tilde{\Gamma}$ be two circles in the 3-dimensional space R^3 with the property

(A) There exists a number d such that each point of either circle is at the distance d from exactly two points on the other circle

Should the circles $\Gamma, \tilde{\Gamma}$ be coplanar, then (A) will be satisfied if we make the assumption

(B) The smaller circle encloses the center of the larger circle.

Furthermore, the authors assume that there is a d with the following property: There is an equilateral closed polygon

$$\Pi_{2n} = P_1 P_2 \ldots P_{2n} \quad (n \geq 2) \quad ,$$

having all its sides $= d$, such that

$$P_{2k-1} \in \Gamma \text{ , } P_{2k} \in \tilde{\Gamma} \quad (k = 1,2,\ldots,n) \quad . \tag{1}$$

Then the polygon Π_{2n} can be turned around, with all its sides $= d$, so that the zig-zag property (1) holds, and returned to its initial position.

The authors do not describe a way of determining the (constant) size d of the side of Π_{2n}.

The present note should be regarded as an appendix to [1] that solves the case when $n = 2$ and the circles are coplanar. It can be read independently. We prove

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__Theorem 1__. __Let__ 0 __and__ $\tilde{0}$ __be the centers of the coplanar circles__ Γ __and__ $\tilde{\Gamma}$, __respectively, and let__

(2) $$a = \text{radius of } \Gamma, \quad b = \text{radius of } \tilde{\Gamma}, \quad c = 0\tilde{0} \quad .$$

__We assume that the assumption__ (B) __is satisfied. Then there is a__ 1-__parameter family of lozanges__ Π_4, __of constant sides__ $= d$, __having the zig property__ (1) __for__ $n = 2$, __with__ d __given by the equation__

(3) $$d^2 = a^2 + b^2 - c^2 \quad .$$

Proof.

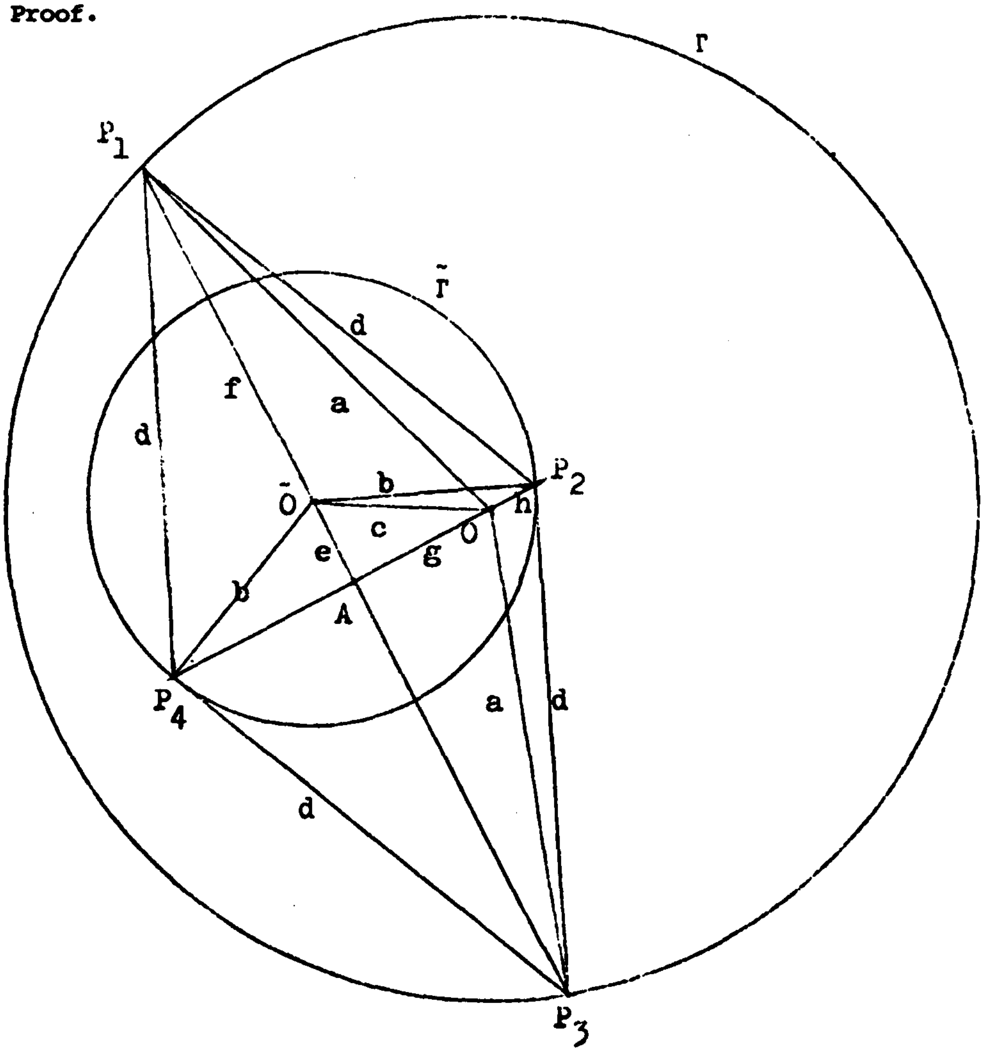

Figure 1

I. Let us first show that there is a 1-parameter family of lozanges $\Pi_4 = P_1P_2P_3P_4$ each of which enjoys the zig-zag property (1) for $n = 2$. Afterwards, we will show that the side d of Π_4 satisfies the equation (3) so that d has the same value for all members of the family. Referring to Fig. 1 let $\Pi_4 = P_1P_2P_3P_4$ be a lozange such that P_1 and P_3 are on Γ, and P_2, P_4 on $\tilde{\Gamma}$. Evidently $P_1P_2P_3$ is isosceles and so is $P_1\ 0\ P_3$. It follows that 0 is on the diagonal P_2P_4. Likewise $P_4P_1P_2$ and $P_4\ \tilde{0}\ P_2$ are isosceles triangle, and therefore $\tilde{0}$ is on the diagonal P_1P_3. Moreover P_1P_3 and P_2P_4 are evidently perpendicular to each other.

This makes the construction of $\Pi_4 = P_1P_2P_3P_4$ evident: Having picked P_1 arbitrarily on Γ, we draw the line joining P_1 to $\tilde{0}$ and intersecting Γ again in P_3. Next we drop the perpendicular from 0 onto $P_1\ \tilde{0}\ P_3$. With $A = P_1P_3 \cap P_2P_4$ we evidently have $P_1A = AP_3$ and $P_2A = AP_4$, so that $\Pi_4 = P_1P_2P_3P_4$ is a lozange having the side $d = P_1P_2$.

We recall that P_1 was an arbitrary point of Γ; the constancy of d will follow as soon as we establish the equation (3), for it shows that d does not depend on the choice of P_1 on Γ.

II. Let us prove the equation (3). To simplify our notations, and referring to Fig. 1, we write

$$e = A\tilde{0},\ f = \tilde{0}P_1,\ g = A0,\ h = 0P_2\quad .$$

The Pythagorean theorem gives the following equations

$$d^2 = (P_1P_2)^2 = (e+f)^2 + (g+h)^2 = e^2 + f^2 + g^2 + h^2 + 2(ef+gh)\ ,$$

$$a^2 = (e+f)^2 + g^2 = e^2 + g^2 + f^2 + 2ef\ ,$$

$$b^2 = (g+h)^2 + e^2 = g^2 + h^2 + e^2 + 2gh\ ,$$

$$c^2 = e^2 + g^2\quad .$$

These show that $d^2 + c^2 = a^2 + b^2$ and the equation (3) is established. The constancy of d immediately implies the turning property of the lozange Π_4.

Remarks. 1. On Fig. 1 we see that if $O = \tilde{O}$, hence $c = 0$ then $a = AP_1$ and $b = AP_2$, and therefore the equation (3) may be regarded as a generalization of the Pythagorean theorem

$$(P_1P_2)^2 = (AP_1)^2 + (AP_2)^2 \quad .$$

2. Our result would make an attractive linkage that would be fun to handle: The four equal side of Π_4 should be linked at the vertices, while the two pairs of opposite vertices P_1, P_3 and P_2, P_4 should fit into the circular grooves Γ and $\tilde{\Gamma}$, respectively. From the general result of [1] we may similarly make a linkage from the closed equilateral polygon Π_{2n}, with P_{2k-1} fitting into the circular groove Γ, and P_{2k} into $\tilde{\Gamma}$. Clearly $n = 3$, or $n = 4$, are the most likely choices. For $n > 2$ the side d must be well approximated graphically, by trial and error, after the grooves Γ and $\tilde{\Gamma}$ have been chosen.

References

1. W. L. Black, H. C. Howland and B. Howland, A theorem about zig-zags between two circles, American Mathematical Monthly, 81 (1974), 754-757.

2. I. J. Schoenberg, Mathematical Time Exposures, Mathematical Association of America, 1982.

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ABSTRACT (continued)

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