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A TRANSFORMATION METHOD FOR THE EVALUATION OF THE RESPONSE  
MOMENTS OF ELASTIC STRUCTURES WITH RANDOM PROPERTIES

BY

RAYMOND H. BENNETT

B.S. Civil Engineering

University of Virginia, 1978

M.E. Civil Engineering

University of Virginia, 1979

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DISSERTATION

Submitted in Partial Fulfillment of the  
Requirements for the Degree of

Doctor of Philosophy in Engineering

The University of New Mexico  
Albuquerque, New Mexico

April 1985

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| REPORT DOCUMENTATION PAGE                                                                                                                                                                 |                                      | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                    |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------|----------------------------------------------------------------|
| 1. REPORT NUMBER<br>AFIT/CI/NR 85-46D                                                                                                                                                     | 2. GOVT ACCESSION NO.<br>AD-A156 882 | 3. RECIPIENT'S CATALOG NUMBER                                  |
| 4. TITLE (and Subtitle)<br>A Transformation Method For The Evaluation Of The<br>Response Moments Of Elastic Structures With<br>Random Properties                                          |                                      | 5. TYPE OF REPORT & PERIOD COVERED<br>THESIS/DISSERTATION      |
| 7. AUTHOR(s)<br>Raymond H. Bennett                                                                                                                                                        |                                      | 6. PERFORMING ORG. REPORT NUMBER                               |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>AFIT STUDENT AT: The University of New Mexico                                                                                              |                                      | 8. CONTRACT OR GRANT NUMBER(s)                                 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>AFIT/NR<br>WPAFB OH 45433                                                                                                                      |                                      | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)                                                                                                               |                                      | 12. REPORT DATE<br>April 1985                                  |
|                                                                                                                                                                                           |                                      | 13. NUMBER OF PAGES<br>150                                     |
|                                                                                                                                                                                           |                                      | 15. SECURITY CLASS. (of this report)<br>UNCLASS                |
|                                                                                                                                                                                           |                                      | 15a. DECLASSIFICATION/DOWNGRADING<br>SCHEDULE                  |
| 16. DISTRIBUTION STATEMENT (of this Report)<br>APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED                                                                                        |                                      |                                                                |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)<br>1                                                                                           |                                      |                                                                |
| 18. SUPPLEMENTARY NOTES<br>APPROVED FOR PUBLIC RELEASE: IAW AFR 190-1X<br>14 May 85<br>LYNN E. WOLAVER<br>Dean for Research and<br>Professional Development<br>AFIT, Wright-Patterson AFB |                                      |                                                                |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)                                                                                                        |                                      |                                                                |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>ATTACHED                                                                                             |                                      |                                                                |

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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In this investigation a new method for computing the first and second moments of the displacement response of linear elastic structures with random stiffness and damping is developed.

The method accurately computes the mean and variance of the displacement of a dynamically loaded structure as functions of time. The probability that the displacement will be greater than a specified value is also computed.

The results of the transformation method are compared to exact solutions for a number of cases.

The influence of the randomness in the underlying parameters on the randomness of the response is discussed.

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## CHAPTER I

### INTRODUCTION

#### 1.1 OBJECTIVES AND SCOPE

The objective of this dissertation is to develop a method for determining the first and second moments of the dynamic response of structural systems with random parameters.

The randomness in the response of structural systems is due to two primary factors. The first of these is the randomness in the loading which has been discussed extensively in the past. The other is the randomness associated with the structure and has usually been ignored in computing the moments of the response.

The dissertation will develop a new method for including the structural uncertainties directly in the computation of the response moments.

The systems under study are limited to those with linear, elastic properties. The random structural properties considered are the stiffness and damping. The response parameters of interest are the deflections at key points on the structure. The desired results are the mean and covariances of the deflections as functions of time.

The method developed will not require knowledge of the distribution of the structural parameter other than their means and variances. Similarly the distribution of the response will not be computed. For the computation of first passage probabilities the response will be assumed Gaussian.

#### 1.2 MOTIVATION

The randomness in structural parameters, such as stiffness, strength and damping, has long been realized by designers. For typical

structures which are designed to resist normal fluctuations in environmental loads, such uncertainties may be safely accounted for by using conservative values for the stiffness or strength rather than the mean values. However, for structures which must resist extreme loads such as earthquakes, accidental blast, or weapons effects, such a conservative approach is often economically impractical. A preferable approach establishes a probability of failure that is deemed acceptable, and the structure is designed to insure such a probability is not exceeded.

For the engineer to design a structure with a specific probability of failure three things are required. The first of these is a probabilistic description of the loading; secondly, the probabilistic characterization of the structure is required. Third, and the most difficult, is a method for computing the moments of the response and, hence, the probability of failure.

The probability of failure is the probability that a response parameter will exceed an allowable value which may or may not be a random variable. When the allowable value is deterministic the moments of the response plus the assumption or computation of its distribution function will be sufficient to compute the probability of failure. When the allowable value is a random variable then the joint distribution of the allowable value and the response values is required.

Typical response parameters of interest in computing structural failure include strains, stresses, displacements, velocities, accelerations, number of stress reversals and ductility ratios. Strains, stresses and ductility ratios are usually of interest when examining the possibility of overload resulting in breaking of a member. Stress

reversals may be of interest in examining fatigue failure. Displacement, velocity and acceleration are of interest when examining stability, service or shock isolation type failures. For example when the deflection of a shock isolated piece of equipment exceeds available rattle space the system may be assumed to have failed. Similarly a tall building may suffer a service failure if the displacement of the upper floors in high winds results in the occupants feeling uncomfortable. This type of failure may occur when the stresses and strains are still well below a dangerous level.

While much work has been done in the area of load modeling for earthquakes, wind and blast loads, and in the calculation of the response of deterministic systems to random loads, little has been done in establishing techniques for computing the response moments in the presence of randomness in the structural parameters.

Recent studies described in the literature review which follows have concentrated on quantifying the structural uncertainties, but not on their effects on response uncertainties.

### 1.3 LITERATURE REVIEW

This review deals with studies on the theory and calculation of random dynamic structural response. Books of general interest will be discussed first followed by papers dealing with the quantification of structural uncertainties. Papers involving the computation of response characteristics in the presence of structural uncertainties will then be discussed. Finally, some papers dealing with the first passage and peak values of structural response will be reviewed.

The theory of random vibrations of linear elastic structures with deterministic mass, damping and stiffness properties is well developed.

The equation of motion for such systems when they are mass excited is,

$$[M]\{\ddot{X}(t)\} + [C]\{\dot{X}(t)\} + [K]\{X(t)\} = \{P(t)\} \quad (1-1)$$

where

$[M]$  = System Mass Matrix

$[C]$  = System Damping Matrix

$[K]$  = System Stiffness Matrix

$\{X\}$  = System Displacement Vector

$\{P(t)\}$  = System Forcing Vector

When  $\{P(t)\}$  is deterministic  $\{X(t)\}$  will be as well.

Lin (1), Crandall (2), and Papoulis (3) all address the stationary and transient response of such linear systems for both stationary and nonstationary loadings. Single-degree-of-freedom, (SDOF), systems may be solved using the system's impulse response function. Multiple-degree-of-freedom, (MDOF), systems are analyzed via their normal modes. Expressions for the autocorrelation functions and first passage probabilities of the response are developed as well as the expressions for the mean responses. In a more recent work Vanmarke (4) addresses the extension of this work to include first crossing probabilities for both stationary and nonstationary envelopes about the process. Elishakoff (5) provides an excellent text for quantifying the response statistics for both static and dynamic loads as well as a discussion of the Monte Carlo method of analysis. Soong (6) expresses the solution of differential equations with random coefficients as a function of the coefficients. The mean of the solution at any time may then be obtained by

integrating the product of the solution and the joint probability density function of the coefficients over the range of the coefficients. The variance may be obtained in a similar manner. However, the resulting integrals may have to be evaluated numerically at each time step.

Several studies have addressed the quantification of structural uncertainties. MacGregor, Mirza and Ellingwood (7) compiled information on the means and variances for concrete strength, steel yield strength, error in steel placement and total resistance. Ellingwood (8) has also examined the statistical properties of reinforced concrete beam columns. These studies were based on tests of existing structures and laboratory specimens.

An analytical study was performed by Grant, Mirza and MacGregor (9) to examine the strength of concrete columns based on the underlying random variables. The formula for the strength of the columns was evaluated repeatedly with a Monte Carlo procedure to determine the mean and variance. Ramsey, Mirza and MacGregor (10) applied a similar technique to the computation of deflections of concrete beams. Mlakar (11) used a Taylor series expansion about the mean to evaluate the mean and variance of the resistance of a beam column. Thielen (12) used a similar technique to examine the behavior of concrete beams near the limit state.

The consideration of structural uncertainties in response analysis is fairly recent. The building codes have recognized strength variation through the specification of strength reduction factors and working stresses well below the average values. The field of reliability and

safety analysis was among the first to handle the uncertainties explicitly.

Studies within this field have concentrated on the proper design loads and strengths to use for structures in routine circumstances. Ang and Cornell (13) investigated the influence random properties such as yield, bond and shear strength have on structural reliability and proper design levels. Ellingwood and Ang (14) proposed procedures for risk based design as did Ravionda, Lind and Siu (15). Moses (16) examined the reliability of structural systems.

There have been few studies concerned with the calculation of the moments of the dynamic response of a system with random stiffness and damping. Collins and Thompson (17) addressed the problem of random eigenvalues and eigenvectors for a system such as Equation (1-1) with a random stiffness matrix. Hasselman and Hart (18) developed expressions for the modal response of MDOF systems based on random eigenvectors.

In both of these papers the expressions for the eigenvalue means were taken to be the same values that would result from the deterministic expression and the variances were evaluated by Taylor series expansions about the means. Hisada and Nakgiri (19) used a Taylor series expansion about the means to quantify the effects of node location uncertainty on the stiffness matrix and resulting stresses and strains for the static case. Handa and Anderson (20) developed an approximation for solving the static problem by separating the random variables into a deterministic variable equal to the mean and a random variable with mean zero but a variance equal to the variance of the

original random variable. However, the cross-correlation between the stiffness and deflection is ignored. This results in the mean response being equal to the response of the system when the system parameters are treated as being deterministic and equal to their means. This is known to be incorrect.

There are many papers related to first passage probabilities and peak values for stochastic processes. The topic is also well covered in many of the basic texts listed earlier.

In a classic paper Rice (21) developed the expressions for first passage times of a white noise process. Karlin and Taylor (22) developed expressions for first passage probabilities of a random process represented by Brownian motion with a drift in the mean. Corotis, Vanmarke and Cornell (23) developed expressions for the first passage of nonstationary random processes. Crandall, Chandiramani and Cook (24) used a numerical simulation technique to examine first passage probabilities of the response of a SDOF system. The system was linear, elastic and the system properties were deterministic. The loading was repeatedly applied in a Monte Carlo style and the statistics of the response including the expected time to first passage beyond a barrier computed. A number of types of barriers were investigated as were the zero and stationary start conditions. These authors also developed a model where the sample space is divided into cells and the probability mass is repeatedly redistributed in accordance with theoretically derived transition probabilities. Krenk, Madsen and Madsen (25) developed a method for modifying stationary crossing frequencies for use in transient response envelopes. Ang (26) developed a method for computing the response

moments of a linear system, and then developed first passage probabilities based on these. Simple nonlinear elastic structures were examined.

## CHAPTER 2

### RESPONSE OF SDOF SYSTEMS WITH RANDOM STIFFNESS AND DAMPING

#### 2.1 INTRODUCTION

In this chapter a new method is developed for analyzing linear, elastic, SDOF systems with random stiffness and damping. The system may be excited by either a deterministic signal or random process. In either case the response of the system will be a stochastic process. Such processes may frequently be satisfactorily characterized by their first two moments, the mean and variance. For the processes considered here these will vary as functions of time.

The method developed involves the replacement of the random variables appearing in the system's equation of motion with two new variables. The first of the new variables is deterministic and equal to the mean of the original random variable. The second variable is random with mean zero and has a variance equal to that of the original random variable.

This substitution has been used by Handa and Anderson (20) for analyzing the static loading of structures but this is its first application to the dynamic response of structures. Handa and Anderson also neglect the correlation between the randomness in the system parameters and the randomness in the response introducing an error in the mean which they acknowledge. The correlation is included here and hence the mean generated is the true mean.

The results of the method are the mean and variance of the system's displacement and velocity as functions of time. The correlation between the displacement and velocity may be readily computed as well.

The details of the transformation of the random variables appearing in the equation of motion are explained in Section 2.2. The resulting transformed equation of motion is solved using forward differences, the details of this operation are explained in Section 2.3. A computer code was developed to perform the computations and its basic numerical algorithm is provided in Section 2.4. The effects of some higher order terms which have been neglected in the computation of the variance are examined in Section 2.5. Several numerical examples verifying the method are provided in Section 2.6. The method will be referred to as the transformation method.

The following assumptions are made concerning the system's parameters and response characteristics,

- a) The mass of the system is deterministic.
- b) The system response is twice mean square differentiable with respect to time.
- c) The system's damping and stiffness are random variables whose values are not functions of time. The two may be correlated or uncorrelated.
- d) When the forcing function is stochastic it has an auto-correlation function with one factor which is a delta function.
- e) The forcing function is independent of the system parameters.

## 2.2 TRANSFORMATION OF THE EQUATIONS OF MOTION

The equation of motion of a mass excited SDOF system is,

$$M\ddot{X} + C\dot{X} + KX = F(t) \quad (2-1)$$

where,

$M$  = System Mass

$C$  = Damping Resistance

$K$  = Stiffness

$X(t)$  = System Displacement

$F(t)$  = Forcing Function

When  $K$  and  $C$  are deterministic Equation (2-1) may be solved exactly for either deterministic or stochastic forcing functions. When  $K$  and  $C$  are random variables the response,  $X(t)$ , is a stochastic process, the mean and variance of which may be evaluated by the method of Soong (6). However, this method will in general require numerical integration at each time of interest.

To develop expressions for the mean and mean square values of  $X(t)$  a substitution is made in Equation (2-1) for the random variables. The substitution replaces the original variables with the sum of two other variables, one of which is deterministic and equal to the mean of the original variable; the second variable is random with mean zero and variance equal to the variances of the original variable. The substitution is made for  $K$ ,  $C$ ,  $X(t)$ ,  $\dot{X}(t)$ ,  $\ddot{X}(t)$  and  $F(t)$ .

The deterministic portion of the response  $X(t)$  is represented by  $\mu(t)$  and the random portion by  $\delta(t)$ . Hence the substitution for  $X(t)$  is,

$$X(t) = \mu(t) + \delta(t) \quad (2-2)$$

Taking the expected value of Equation (2-2) yields,

$$E(X(t)) = E(\mu(t)) + E(\delta(t)) \quad (2-3a)$$

$$= \mu(t) + 0 \quad (2-3b)$$

To develop an expression for the mean square value of the response one squares Equation (2-2) which yields,

$$X(t)^2 = \mu(t)^2 + 2\mu(t)\delta(t) + \delta(t)^2 \quad (2-4)$$

Taking the expected value of Equation (2-4) yields an expression for the mean square response of  $X(t)$

$$E\{X(t)^2\} = \mu(t)^2 + 2\mu(t) \cdot 0 + E\{\delta(t)^2\} \quad (2-5a)$$

$$E\{X(t)^2\} = \mu(t)^2 + E\{\sigma(t)^2\} \quad (2-5b)$$

The variance of  $X(t)$  is given by,

$$\text{Var}\{X(t)\} = E\{X(t)^2\} - E\{X(t)\}^2 \quad (2-6)$$

Substituting the expressions in Equation (2-5b) for  $E\{X(t)^2\}$  and Equation (2-3b) for  $E\{X(t)\}$  yields,

$$\text{Var}\{X(t)\} = E\{\delta(t)^2\} \quad (2-7)$$

Similar substitutions may be developed for the derivatives of  $X(t)$  by differentiating Equation (2-2). This yields for the velocity,

$$\dot{X}(t) = \dot{\mu}(t) + \dot{\delta}(t) \quad (2-8)$$

And for the acceleration,

$$\ddot{X}(t) = \ddot{\mu}(t) + \ddot{\delta}(t) \quad (2-9)$$

Taking the expected value of Equation (2-8) yields an expression for the mean of the velocity,

$$E\{\dot{X}(t)\} = E\{\dot{\mu}(t)\} + E\{\dot{\delta}(t)\} \quad (2-10)$$

Since  $\mu(t)$  is deterministic its derivatives are as well, hence,

$$E\{\dot{\mu}(t)\} = \dot{\mu}(t) \quad (2-11)$$

Since  $X(t)$  and thus  $\delta(t)$  have been assumed mean square differentiable the expectation and differentiation operations may be interchanged. And

since  $E(\delta(t))$  is zero for all times the expected value of the change of  $\delta(t)$  with time must also be zero, yielding,

$$E(\dot{\delta}(t)) = 0 \quad . \quad (2-12)$$

When the relation in Equations (2-11) and (2-12) are substituted into Equation (2-10) the resulting expression for the mean of the velocity is,

$$E(\dot{X}(t)) = \dot{\mu}(t) \quad . \quad (2-13)$$

Squaring Equation (2-8) and taking the expected value provides the expression for the mean square value of the velocity,

$$E(\dot{X}(t)^2) = \dot{\mu}(t)^2 + E(\dot{\delta}(t)^2) \quad . \quad (2-14)$$

Similar expressions for the mean and mean square values of the acceleration may be developed from Equation (2-9). The expected value of the acceleration is,

$$E(\ddot{X}(t)) = \ddot{\mu}(t) \quad . \quad (2-15)$$

The mean square value of the acceleration is given by,

$$E(\ddot{X}(t)^2) = \ddot{\mu}(t)^2 + E(\ddot{\delta}(t)^2) \quad . \quad (2-16)$$

Representing the stiffness  $k$  as the sum of  $K_D$  and  $K_R$  with  $K_D$  deterministic and  $K_R$  a random variable with mean zero yields,

$$K = K_D + K_R \quad . \quad (2-17)$$

Taking the expected value of Equation (2-17) yields,

$$E(K) = K_D + E(K_R) \quad . \quad (2-18)$$

And since  $K_R$  is mean zero,

$$E(K) = K_D \quad . \quad (2-19)$$

Squaring Equation (2-17) to develop an expression for the square of  $K$  yields,

$$K^2 = K_D^2 + 2K_R K_D + K_P^2 \quad . \quad (2-20)$$

Taking the expected value of Equation (2-20) yields

$$E(K^2) = K_D^2 + E(K_R^2) \quad . \quad (2-21)$$

Upon rearranging one has the expression for the variance of  $K$ .

$$\text{Var}(K) = E(K^2) - E(K)^2 \quad (2-22a)$$

$$= E(K_R^2) \quad . \quad (2-22b)$$

For the damping,  $C$ , let  $C_D$  and  $C_R$  be the deterministic and random portions, respectively. Then following the development used for the stiffness, the expressions for  $C$ , the mean of  $C$ , and the variance of  $C$  are,

$$C = C_D + C_R \quad (2-23)$$

$$E(C) = C_D \quad (2-24)$$

$$\text{Var}(C) = E(C_R^2) \quad . \quad (2-25)$$

The substitution for the forcing function,  $F(t)$ , consists of the deterministic portion  $P(t)$  and a random portion  $q(t)$ . Making this substitution and following the development above results in the following

expressions for the forcing function,  $F(t)$ , the mean forcing function,  $P(t)$ , and the variance of the forcing function,

$$F(t) = P(t) + q(t) \quad (2-26)$$

$$E(F(t)) = P(t) \quad (2-27)$$

$$\text{Var}(F(t)) = E(q^2(t)) \quad (2-28)$$

Substituting the results of Equations (2-2), (2-8), (2-9), (2-17), (2-23), and (2-26) into the equation of motion, Equation (2-1), results in,

$$M(\ddot{\mu} + \ddot{\delta}) + (C_D + C_R)(\dot{\mu} + \dot{\delta}) + (K_D + K_R)(\mu + \delta) = P(t) + q(t) \quad (2-29)$$

When the system parameters and response are considered correlated the expected value of Equation (2-29) is,

$$M\ddot{\mu} + C_D\dot{\mu} + K_D\mu + E(K_R\delta) + E(C_R\dot{\delta}) = P(t) \quad (2-30)$$

A solution for  $\mu(t)$ , the mean response of the system, may be sought based on this expression.

To develop an expression for  $\delta(t)$  one may subtract Equation (2-30) from Equation (2-29) yielding the result,

$$M\ddot{\delta} + C_D\dot{\delta} + K_D\delta + CRT + KRT = q(t) - K_R\mu - C_R\dot{\mu} \quad (2-31a)$$

where

$$CRT = C_R\dot{\delta} - E(C_R\dot{\delta}) \quad (2-31b)$$

$$KRT = K_R\delta - E(K_R\delta) \quad (2-31c)$$

The term CRT represents the fluctuation of the random portion of the damping force about its mean. The fluctuation of the random portion of the stiffness force is represented by KRT.

When the random portions of the restoring forces are considered small in comparison to the deterministic portions,  $C_D \dot{\mu}$  and  $K_D \mu$ , the fluctuations may be considered negligible and set to zero. This assumption is made in the development of the technique. The implications of the assumption are examined in Section 2.5 and the examples in Section 2.6.

When the CRT and KRT terms are neglected Equation (2-31a) becomes,

$$M\ddot{\delta} + C_D \dot{\delta} + K_D \delta = q(t) - K_R \mu - C_R \dot{\mu} \quad (2-32)$$

The response characteristic of interest in this analysis is  $E(\delta(t)^2)$ . Furthermore, the equations for  $\mu(t)$  and  $\delta(t)$  are coupled in that  $\mu(t)$  appears in the expression for  $\delta(t)$  and vice versa. The separate the equations and permit the computation of  $\mu(t)$  and  $E(\delta(t)^2)$  the forward difference technique will be applied as explained in the following section.

### 2.3 APPLICATION OF FINITE DIFFERENCES

Many structural mechanics problems involve differential equations. They arise in both static and dynamic analysis and may be solved directly for a relatively small number of cases. For those cases which may not be solved directly, various numerical integration schemes have been developed. A common method of performing the integration is the application of finite differences. This technique may be applied to a differential equation provided the function and its derivatives as they appear in the equation are continuous.

Consider the definition of a derivative:

$$\frac{df(y)}{dy} = \lim_{\Delta y \rightarrow 0} \frac{f(y+\Delta y) - f(y)}{\Delta y} . \quad (2-33)$$

When  $\Delta y$  approaches a small, fixed value,  $s$ , rather than zero an approximation to the derivative of  $f(y)$  is:

$$\frac{df(y)}{dy} \approx \frac{f(y+s) - f(y)}{s} . \quad (2-34)$$

This expression is the basis for the forward difference technique. If one considers a time,  $\tau$ , to equal  $n\Delta t$ , where  $n$  is a positive integer and  $\Delta t$  is a small, positive time constant, the expressions for  $f(\tau)$  and its derivatives are,

$$f(\tau) = f_n \quad (2-35)$$

$$\dot{f}(\tau) \approx \frac{f_{n+1} - f_n}{\Delta t} \quad (2-36)$$

$$\ddot{f}(\tau) \approx \frac{f_{n+2} - 2f_{n+1} + f_n}{\Delta t^2} . \quad (2-37)$$

For further information on the finite difference approach and its applications Wang (31) provides a good introductory discussion. Bathe (32) addresses numerical integration schemes in considerable detail as well.

To develop the relations required to evaluate  $\mu(t)$ , Equations (2-35), (2-36), and (2-37) are applied to Equation (2-30) in place of  $\mu(t)$  and its derivatives resulting in,

$$\begin{aligned} \frac{M}{\Delta t^2} (\mu_{n+2} - 2\mu_{n+1} + \mu_n) + \frac{C_D}{\Delta t} (\mu_{n+1} - \mu_n) + K_D (\mu_n) = \\ = P_n - E(K_R \delta)_n - E(C_R \dot{\delta})_n \quad . \end{aligned} \quad (2-38)$$

Combining terms and solving for  $\mu_{n+2}$  yields,

$$\mu_{n+2} = \frac{1}{\bar{M}} (P_n - E(K_R \delta)_n - E(C_R \dot{\delta})_n + A_1 \mu_{n+1} + A_2 \mu_n) \quad . \quad (2-39a)$$

where

$$\bar{M} = M/\Delta t^2 \quad (2-39b)$$

$$\bar{C}_D = C_D/\Delta t \quad (2-39c)$$

$$A_1 = 2\bar{M} - \bar{C}_D \quad (2-39d)$$

$$A_2 = \bar{C}_D - \bar{M} - K_D \quad (2-39e)$$

Equation (2-39a) provides an expression by which  $\mu(t)$  may be calculated in a step-by-step manner, based on information calculated at previous time steps, provided that information can be developed. The main difficulty lies in the calculation of  $E(K_R \delta)_n$  and  $E(C_R \dot{\delta})_n$ . These are the terms which have been ignored or assumed zero in many previous studies. Their inclusion here represents an improvement over previous approximations. The means for calculating these terms will be discussed shortly.

Applying the finite difference relations of Equations (2-35), (2-36), and (2-37) to  $\delta(t)$  and its derivatives in Equation (2-32) yields,

$$\overline{M}(\delta_{n+2} - 2\delta_{n+1} + \delta_n) + \overline{C}_D(\delta_{n+1} - \delta_n) + K_D(\delta_n) = q_n - K_R \dot{\mu}_n - C_R \dot{\mu}_n \quad (2-40)$$

Rearranging and solving for  $\delta_{n+2}$ ,

$$\delta_{n+2} = \frac{1}{\overline{M}} (q_n - K_R \dot{\mu}_n - C_R \dot{\mu}_n + A_1 \delta_{n+1} + A_2 \delta_n) \quad (2-41)$$

Since an expression for  $\delta(t)^2$  is sought, Equation (2-41) is squared resulting in,

$$\begin{aligned} \delta_{n+2}^2 = & \frac{1}{\overline{M}^2} (q_n^2 - 2q_n K_R \dot{\mu}_n - 2q_n C_R \dot{\mu}_n + 2q_n A_1 \delta_{n+1} \\ & + 2q_n A_2 \delta_n + K_R^2 \dot{\mu}_n^2 + 2K_R C_R \dot{\mu}_n^2 \\ & - 2\mu_n K_R A_1 \delta_{n+1} - 2\mu_n K_R A_2 \delta_n + C_R^2 \dot{\mu}_n^2 \\ & - 2C_R \dot{\mu}_n A_1 \delta_{n+1} - 2C_R \dot{\mu}_n A_2 \delta_n \\ & + A_1^2 \delta_{n+1}^2 + 2A_1 A_2 \delta_{n+1} \delta_n + A_2^2 \delta_n^2) \end{aligned} \quad (2-42)$$

When expected values are taken in the above expression some terms on the right side will be zero. Due to the assumption outlined at the end of Section 2.1, terms involving a product of  $K_R$  and  $q_n$  or  $C_R$  and  $q_n$  will be zero, for the randomness in the loading is independent of the

system parameters. Hence, when the expected value of Equation (2-42) is taken the result is,

$$\begin{aligned}
 E(\delta_{n+2}^2) = & \frac{1}{M} \left[ \text{Var}(F_n) + \mu_n^2 \text{Var}(K) + \dot{\mu}_n^2 \text{Var}C \right. \\
 & + 2\mu_n \dot{\mu}_n \text{Cov}(K,C) - 2\mu_n A_1 E(K_R \delta_{n+1}) \\
 & - 2\mu_n A_2 E(K_R \delta_n) - 2\dot{\mu}_n A_1 E(C_R \delta_{n+1}) \\
 & - 2\dot{\mu}_n A_2 E(C_R \delta_n) + A_1^2 E(\delta_{n+1}^2) \\
 & + A_2^2 E(\delta_n^2) + 2A_1 A_2 E(\delta_{n+1} \delta_n) + 2A_1 E(q_n \delta_{n+1}) \\
 & \left. + 2A_2 E(q_n \delta_n) \right] \quad (2-43)
 \end{aligned}$$

Note in Equation (2-43) that all terms which involve the randomness of the system parameters are multiplied by the mean response or its derivatives. Hence, when the system has a zero mean response the variance and mean square values are functions of the randomness of the excitation only. This is due to the assumptions regarding the terms designated CRT and KRT in Equations (2-31a), (2-31b), and (2-31c). These terms are discussed further in Section 2.5.

The terms  $\text{Var}(K)$ ,  $\text{Var}(C)$ ,  $\text{Cov}(K,C)$  and  $E(q_n^2)$  must be specified prior to commencing the solution. The terms involving  $E(\delta_{n+1}^2)$  and  $E(\delta_n^2)$  on the right hand side are the results of Equation (2-43) at

previous time steps. The formulation of the remaining terms,  $E(C_R \delta_n)$ ,  $E(K_R \delta_n)$ ,  $E(\delta_{n+1} \delta_n)$ , and  $E(q_n \delta_n)$  proceeds as follows. Consider Equation (2-41) with the index shifted by one. This provides an expression for  $\delta_{n+1}$ .

$$\delta_{n+1} = \frac{1}{M} (q_{n-1} - K_R \mu_{n-1} - C_R \dot{\mu}_{n-1} + A_1 \delta_n + A_2 \delta_{n-1}) \quad (2-44)$$

Multiplying this expression by  $K_R$  yields

$$K_R \delta_{n+1} = \frac{1}{M} (K_R q_{n-1} - K_R^2 \mu_{n-1} - K_R C_R \dot{\mu}_{n-1} + A_1 K_R \delta_n + A_2 K_R \delta_{n-1}) \quad (2-45)$$

Taking the expected value of both sides results in,

$$E(K_R \delta_{n+1}) = \frac{1}{M} (-\text{Var}(K) \mu_{n-1} - \dot{\mu}_{n-1} \text{Cov}(K, C) + A_1 E(K_R \delta_n) + A_2 E(K_R \delta_{n-1})) \quad (2-46)$$

This expression may then be readily evaluated in a step-by-step manner from the zero start condition. The expression for  $E(C_R \delta_{n+1})$  is developed similarly and results in,

$$E(C_R \delta_{n+1}) = \frac{1}{M} (-\mu_{n-1} \text{Cov}(K, C) - \dot{\mu}_{n-1} \text{Var}(C) + A_1 E(C_R \delta_n) + A_2 E(C_R \delta_{n-1})) \quad (2-47)$$

The expression for  $E(\delta_{n+1} \delta_n)$  is developed in a similar manner. It is,

$$E(\delta_{n+1} \delta_n) = \frac{1}{M} (-\mu_{n-1} E(K_R \delta_n) - \dot{\mu}_{n-1} E(C_R \delta_n) + A_1 E(\delta_n^2) + A_2 E(\delta_n \delta_{n-1})) \quad (2-48)$$

The expression for  $E(q_n \delta_{n+1})$  is

$$\begin{aligned}
 E(q_n \delta_{n+1}) = & \frac{1}{M} (E(q_n q_{n-1}) - E(q_n K_R) \mu_{n-1} \\
 & - E(q_n C_R) \dot{\mu}_{n-1} + A_1 E(q_n \delta_n) \\
 & + A_2 E(q_n \delta_{n-1}))
 \end{aligned} \tag{2-49}$$

However, since the loading is independent of the system parameters and uncorrelated with itself for any time shift, all terms in the above expression are zero. Hence,

$$E(q_n \delta_{n+1}) = E(q_n \delta_n) = 0 \tag{2-50}$$

Since these terms are zero the randomness in the response due to the loading and that due to the system parameters are independent. Thus they may be computed separately and combined merely by adding the two independent variances.

The expressions for the variance of the velocity may be developed from Equations (2-48), (2-43) and the finite different approximation to the velocity.

$$\dot{\delta}_n \approx \frac{\delta_{n+1} - \delta_n}{\Delta t} \tag{2-37}$$

Squaring and taking the expected value yields,

$$E(\dot{\delta}_n^2) = \frac{E(\delta_{n+1}^2) - 2E(\delta_{n+1} \delta_n) + E(\delta_n^2)}{\Delta t^2} \tag{2-51}$$

Hence the method of transforming the random variables by a shift about the mean provides expressions which enable the analyst to evaluate the means, variances and covariance of displacement and velocity. Note these expressions are independent of the distribution of the system parameters and the loading function. This is due to the linearization of the solution to the equation of motion at each time step.

An algorithm for computing the mean and variance of the displacement response of SDOF systems is outlined in the following section.

#### 2.4 THE COMPUTATION ALGORITHM

The following algorithm was developed to compute the mean and variance of the displacement response of an SDOF system. The system stiffness and damping are considered random. They are described in terms of their means and variances, which are input at the beginning of the calculation as is their correlation coefficient. The randomness in the loading is described in terms of its mean and variance which are defined at the beginning of the calculation as well. These may vary with time provided the function describing the variation is defined.

The algorithm is based on a zero start condition and proceeds as follows,

1. Read input parameters;  
 $K$ ,  $\text{Var}(K)$ ,  $C$ ,  $\text{Var}(C)$ ,  $\text{Cov}(K,C)$   
 $M$ ,  $\Delta t$ , and problem duration.
2. Initialize all variables for zero start condition.
3. Set,  $NT$ , the total number of time steps equal to the problem duration/ $\Delta t$ .

4. Do for each step  $N=1, NT$ 
  - a) Compute the mean and variance of the loading function based on predetermined functions.
  - b) Compute  $E(\delta_{n+1} \delta_n)$  from Equation (2-48).
  - c) Compute  $\mu_{n+2}$  from Equation (2-39a).
  - d) Compute  $E(\delta_{n+2}^2)$  from Equation (2-43).
  - e) Compute  $E(K_R \delta_{n+2})$  from Equation (2-46).
  - f) compute  $E(C_R \delta_{n+2})$  from Equation (2-47).
  - g) Write desired results.
  - h) Time = Time +  $\Delta t$ .
  - i) Go to step 4a until  
Time = Problem duration.

This computational algorithm is implemented in the computer program RSDOF, a listing of which is included in Appendix A. Numerical examples in Section 2.6 use the code to verify the adequacy of the method as well as to investigate the influence the random system parameters have on the response moments.

## 2.5 THE EFFECT OF HIGHER ORDER TERMS

This section discusses the effects of the assumptions made in Section 2.2 regarding the fluctuations in the random portion of the system's stiffness and damping restoring forces.

Equation (2-31a) may be rewritten as,

$$M\ddot{\delta} + C_D\dot{\delta} + K_D\delta = q(t) - K_R\mu - C_R\dot{\mu} - CRT - KRT \quad (2-52)$$

where

$$CRT = C_R\dot{\delta} - E(C_R\dot{\delta}) \quad (2-31b)$$

$$KRT = K_R\delta - E(K_R\delta) \quad (2-31c)$$

In Section 2.2 KRT and CRT were assumed to be zero. Hence, they do not appear in the development of the finite difference expressions for  $\delta_{n+2}$ , Equation (2-41) and  $E(\delta_{n+2}^2)$ , Equation (2-43) in Section 2.3. In the following development no assumptions regarding the values of CRT and KRT are made and they are included in the development of new expressions for  $\delta_{n+2}$  and  $E(\delta_{n+2}^2)$ .

When the difference expressions are substituted for  $\delta(t)$  and its derivatives on the left hand side of Equation (2-52) the result is:

$$\begin{aligned} \overline{M}(\delta_{n+2} - 2\delta_{n+1} + \delta_n) + \overline{C}_D(\delta_{n+1} - \delta_n) + K_D\delta_n \\ = q_n - K_R\mu_n - C_R\dot{\mu}_n - CRT - KRT \end{aligned} \quad (2-53)$$

Solving Equation (2-53) for  $\delta_{n+2}$  yields,

$$\delta_{n+2} = \frac{1}{\overline{M}} (q_n - K_R\mu_n - C_R\dot{\mu}_n - CRT - KRT + A_1\delta_{n+1} + A_2\delta_n) \quad (2-54)$$

Note the difference expressions could have been applied to the  $\delta(t)$  terms in CRT and KRT as well, but this would only complicate the algebra without changing the results.

To develop an expression for the variance of the response one first squares Equation (2-54) resulting in,

$$\begin{aligned}
 \delta_{n+2}^2 = & \frac{1}{M^2} (q_n^2 - 2q_n K_R \mu_n - 2q_n C_R \dot{\mu}_n + 2q_n A_1 \delta_{n+1} \\
 & + 2q_n A_2 \delta_n + K_R^2 \mu_n^2 + 2K_R C_R \mu_n \dot{\mu}_n \\
 & - 2A_1 K_R \delta_{n+1} - 2A_2 K_R \delta_n \mu_n + C_R^2 \dot{\mu}_n^2 - 2A_1 C_R \delta_{n+1} \dot{\mu}_n \\
 & - 2A_2 C_R \delta_n \dot{\mu}_n + A_1^2 \delta_{n+1}^2 + 2A_1 A_2 \delta_{n+1} \delta_n + A_2^2 \delta_n^2 \\
 & + \{C_R T^2 + 2C_R T K_R T + K_R T^2 - 2q_n C_R T \\
 & + 2\mu_n K_R C_R T + 2\dot{\mu}_n C_R C_R T - 2A_1 \delta_{n+1} C_R T \\
 & - 2A_2 \delta_n C_R T - 2q_n K_R T + 2K_R \mu_n K_R T \\
 & + 2\dot{\mu}_n C_R K_R T - 2A_1 \delta_{n+1} K_R T - 2A_2 \delta_n K_R T\}) \quad (2-55)
 \end{aligned}$$

Equation (2-55) is identical to Equation (2-42) developed in Section 2.3 with the exception of the terms involving CRT and KRT contained within the brackets, { }, above. Hence when one takes the expected value of Equation (2-55) to determine the variance of the response the result is Equation (2-43) plus the expected value of the terms within the brackets. Thus the expected value of the terms within the brackets provides an estimate of the error in the variance of the response due to assuming CRT and KRT are zero. Designating this term REM one finds,

$$\begin{aligned}
\text{REM}_n &= \frac{1}{M^2} E \{ \text{CRT}_n^2 + 2\text{CRT}_n \text{KRT}_n + \text{KRT}_n^2 \\
&\quad + 2(\text{K}_R \mu_n + \text{C}_R \dot{\mu}_n - q_n - A_{1n+1}^\delta - A_{2n}^\delta) \text{CRT}_n \\
&\quad + 2(\text{K}_R \mu_n + \text{C}_R \dot{\mu}_n - q_n - A_{1n+1}^\delta - A_{2n}^\delta) \text{KRT}_n \} \quad (2-56)
\end{aligned}$$

Evaluating the right hand side of Equation (2-56) term by term yields,

$$\begin{aligned}
E(\text{CRT}^2) &= E[(\text{C}_R \dot{\delta} - E(\text{C}_R \dot{\delta}))^2] \\
&= \text{Var}(\text{C}_R \dot{\delta})
\end{aligned}$$

Likewise,

$$E(\text{KRT}^2) = \text{Var}(\text{K}_R \delta) \quad (2-58)$$

And,

$$\begin{aligned}
E(\text{KRTCRT}) &= E[(\text{C}_R \dot{\delta} - E(\text{C}_R \dot{\delta}))(\text{K}_R \delta - E(\text{K}_R \delta))] \\
&= \text{Cov}(\text{K}_R \delta, \text{C}_R \dot{\delta}) \quad (2-59)
\end{aligned}$$

For the remaining terms on the right hand side of Equation (2-56) involving CRT one gets,

$$\begin{aligned}
E((\text{K}_R \mu_n + \text{C}_R \dot{\mu}_n - q_n - A_{1n+1}^\delta - A_{2n}^\delta) \text{CRT}_n) &= \\
&= E((\text{K}_R \mu_n + \text{C}_R \dot{\mu}_n - q_n - A_{1n+1}^\delta - A_{2n}^\delta) (\text{C}_R \dot{\delta})_n) \\
&= E((\text{K}_R \mu_n + \text{C}_R \dot{\mu}_n - q_n - A_{1n+1}^\delta - A_{2n}^\delta) E(\text{C}_R \dot{\delta})_n) \quad (2-60)
\end{aligned}$$

Since  $E(K_R)$ ,  $E(C_R)$ ,  $E(\delta_{n+1})$ , and  $E(\delta_n)$  are all zero, and the  $E(C_R \delta)_n$  is deterministic the result is,

$$\begin{aligned} E((K_R \mu_n + C_R \dot{\mu}_n - q_n - A_1 \delta_{n+1} - A_2 \delta_n) CRT_n) &= \\ &= \mu_n E(K_R C_R \dot{\delta}_n) + \dot{\mu}_n E(C_R^2 \dot{\delta}_n) - E(q_n C_R \dot{\delta}_n) \\ &\quad - A_1 E(\delta_{n+1} \dot{\delta}_n C_R) - A_2 E(\delta_n \dot{\delta}_n C_R) \end{aligned} \quad (2-61)$$

The corresponding expression for the terms involving KRT in Equation (2-56) is,

$$\begin{aligned} E((K_R \mu_n + C_R \dot{\mu}_n - q_n - A_1 \delta_{n+1} - A_2 \delta_n) KRT_n) &= \\ &= \mu_n E(K_R^2 \delta_n) + \dot{\mu}_n E(K_R C_R \delta_n) - E(q_n K_R \delta_n) \\ &\quad - A_1 E(\delta_{n+1} \delta_n K_R) - A_2 E(\delta_n \delta_n K_R) \end{aligned} \quad (2-62)$$

Combining Equations (2-57), (2-58), (2-59), (2-61) and (2-62) the resulting expression for the remainder terms, REM is,

$$\begin{aligned} REM &= \frac{1}{M^2} (\text{Var}(K_R \delta)_n + \text{Var}(C_R \dot{\delta})_n + 2\text{Cov}(K_R \delta, C_R \dot{\delta})_n) \\ &\quad + 2\{\mu_n [E(K_R C_R \delta_n) + E(K_R^2 \delta_n)] + \dot{\mu}_n [E(K_R C_R \delta_n) + E(C_R^2 \dot{\delta}_n)] \\ &\quad - E(q_n C_R \dot{\delta}_n) - E(q_n K_R \delta_n) - A_1 [E(\delta_{n+1} \dot{\delta}_n C_R) \\ &\quad + E(\delta_{n+1} \delta_n K_R)] - A_2 [E(\delta_n^2 K_R) + E(\delta_n \dot{\delta}_n C_R)]\} \end{aligned} \quad (2-63)$$

This expression may be further reduced since  $q_n$  is independent of the system parameters  $C_R$  and  $K_R$  and is independent of  $\delta_{n+1}$  and  $\delta_n$  as well.

Hence,

$$E(q_n C_R \dot{\delta}_n) = E(q_n) E(C_R \dot{\delta}_n) \quad (2-64)$$

And,

$$E(q_n K_R \delta_n) = E(q_n) E(K_R \delta_n) \quad (2-65)$$

However, since  $q_n$  is mean zero,

$$E(q_n C_R \dot{\delta}_n) = 0 \quad (2-66)$$

And,

$$E(q_n K_R \delta_n) = 0 \quad (2-67)$$

Thus Equation (2-63) is reduced to,

$$\begin{aligned} \text{REM} = \frac{1}{M} & \left( \text{Var}(K_R \delta)_n + \text{Var}(C_R \dot{\delta})_n + 2\text{Cov}(K_R \delta, C_R \dot{\delta})_n \right. \\ & + 2\{\mu_n [E(K_R C_R \dot{\delta}_n) + E(K_R^2 \delta_n)] \\ & + \dot{\mu}_n [E(K_R C_R \delta_n) + E(C_R^2 \dot{\delta}_n)] \\ & - A_1 [E(\delta_{n+1} \dot{\delta}_n C_R) + E(\delta_{n+1} \delta_n K_R)] \\ & \left. - A_2 [E(\delta_n^2 K_R) + E(\delta_n \dot{\delta}_n C_R)] \right\} \quad (2-68) \end{aligned}$$

Examination of the terms on the right hand side of Equation (2-68) reveals that these terms provide a means for the uncertainties in the system parameters entering the computation of the response's variance when the response is mean zero.

When the variation of the response is small compared to its mean then one would expect that the value of  $\text{Var}(K_R \delta)$  would be small compared to that of  $E(K_R^2) \mu^2$ . Similarly one would expect that the  $\text{Var}(C_R \delta)$  would be small compared to  $E(C_R^2) \mu^2$  and  $\text{Cov}(K_R \delta, C_R \dot{\delta})_n$  would be small in comparison to  $E(K_R C_R) \dot{\mu}_n \mu_n$ . Hence one would expect that these terms have little influence on the response. Similar arguments may be made for discounting the rest of the terms in Equation (2-68) as they all have corresponding terms which one would expect to be much larger in Equation (2-43).

Several of the following examples verify this assumption by comparing the results of the computer code RSDOF to analytical methods which do not ignore the terms included in Equation (2-68).

## 2.6 NUMERICAL EXAMPLES

This section includes examples of the transformation technique developed earlier in this chapter applied to the solution of specific problems. The following problems are covered.

- 1) Response of deterministic random systems to a white noise loading.
- 2) Response of a random system to a deterministic constant loading.
- 3) Response of a random system to a deterministic blast loading.

The first two problems were used as a method of checking the results of the technique. These examples were also used to quantify the effects the random system parameters have on the response. The third demonstrates the use of the technique in a practical application.

### EXAMPLE 1 - White Noise Excitation of An SDOF System

Consider the system diagramed in Figure 2-1. All system parameters are unitless. When  $F(t)$  is white noise with spectral density  $\phi$  the mean square response of the system with deterministic parameters, is given by Lin (1) as,

$$E(X(t)^2) = \frac{\pi\phi}{2\epsilon\omega_0^3 m^2} \left\{ 1 - \frac{\exp(-2\epsilon\omega_0 t)}{\omega_D^2} [\omega_0^2 + 2(\epsilon\omega_0 \sin\omega_D t)^2 + \epsilon\omega_0 \sin 2\omega_D t] \right\} \quad (2-69)$$

where 
$$\epsilon = \frac{C}{2\sqrt{KM}} \quad (2-70)$$

$$\omega_0 = \sqrt{\frac{K}{M}} \quad (2-71)$$

$$\omega_D = \omega_0 \sqrt{1-\epsilon^2} \quad (2-72)$$

As the response reaches its stationary value Equation (2-69) reduces to,

$$E_s(X(t)^2) = \frac{\pi\phi}{2\epsilon\omega_0^3 m^2} \quad (2-73)$$

To model the response of the system one first has to develop a suitable model for the white noise excitation. A white noise random process with spectral density  $\phi$  has an autocorrelation function,  $R_{FF}(\tau)$  of,

$$R_{FF}(\tau) = 2\pi\phi\delta(\tau) \quad (2-74)$$

where  $\delta(\tau)$  is the dirac delta function.

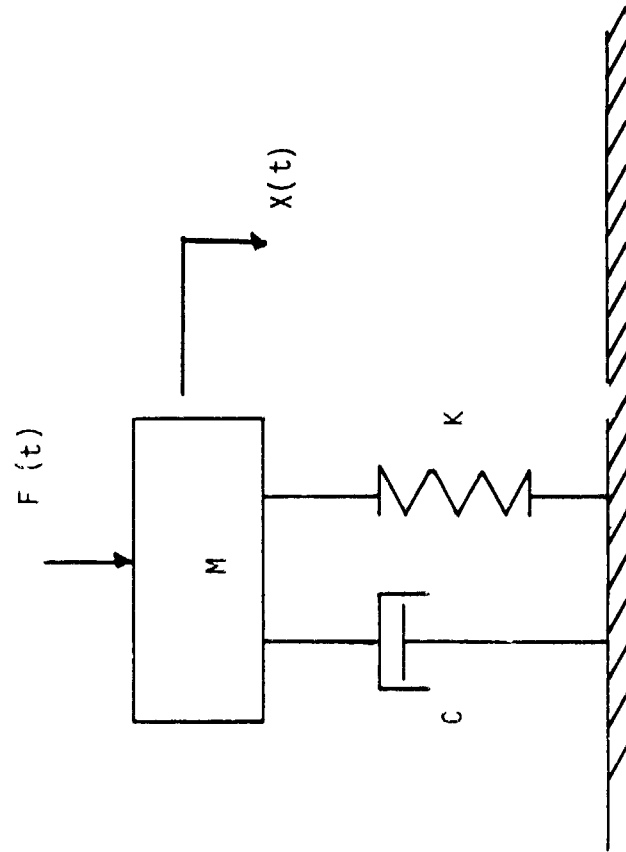


Figure 2-1 Mass Excited SDOF System

When one wishes to excite a structure using a white noise input and perform response computations in a discrete time framework he can model the input as a frequency of independent, identically distributed random variables applied to the structure at  $\Delta t$  time intervals. This is an IID (independent identically distributed) random process (sometimes referred to as a band-limited white noise random process). Let  $\{F_j = F(j\Delta t), j = 0, 1, 2, \dots\}$  be an IID random process. The random process has mean zero and autocorrelation function,

$$R_{FF}((j-k)\sigma t) = \begin{cases} \sigma^2 & j = k \\ 0 & j \neq k \end{cases} \quad (2-75)$$

$$= \sigma^2 \delta_{j-k} \quad (2-76)$$

where  $\delta_{j-k}$  is the Kronecker delta (equal to zero when the subscript is nonzero, and equal to one when the subscript is zero). The spectral density of this random process is,

$$S_{FF}(\omega) = \begin{cases} \frac{\Delta t \sigma^2}{2\pi} & -\frac{\pi}{\Delta t} \leq \omega \leq \frac{\pi}{\Delta t} \\ 0 & \text{elsewhere} \end{cases} \quad (2-77)$$

In view of this, the IID random process is equivalent to the white noise random process if  $\sigma^2$  is chosen as follows,

$$\sigma^2 = \frac{2\pi}{\Delta t} \phi \quad (2-78)$$

This definition for  $\sigma^2$  insures equivalence between the white noise and IID random process in terms of their spectral densities. However, equivalence in terms of the structural response these inputs excite is only guaranteed for certain values of  $\Delta t$ .

Specifically, the cutoff frequency of the IID random process,  $\pi/\Delta t$ , must be much greater than the natural frequency of the structure being excited in order for the IID random process to appear similar to the white noise input, as far as structural response is concerned. This establishes the requirement,

$$\Delta t \ll \frac{\omega_n}{\pi} \quad (2-79)$$

where  $\omega_n$  is the natural frequency of the structure being excited.

To address the issue of how small a time step,  $\Delta t$ , is required to insure an adequate approximation of the structural response characteristics a series of calculations with decreasing values of  $\Delta t$  were performed. The ratio of the natural period to the time step is termed  $N$ . The computations were run for the case of  $C = 20$ , ( $\epsilon = 0.2$ ), until the response became stationary. With  $\sigma^2 = 100/\Delta t$ ,  $K = 500$ , and  $M = 5$ , the exact solution for the mean square value of the response,  $E_S(X(t)^2)$ , as defined in Equation (2-73) is 0.005. The values for  $E_S(X(t)^2)$  computed for successive values of  $N$  are plotted vs  $\ln(N)$  in Figure 2-2.

As  $\Delta t$  becomes small (that is as  $N$  becomes large), the values obtained from the computations very closely approximate the exact solution, being in error by less than 1% for values of  $N$  greater than 1500.

To examine the variation of the error with time the results of computations for different values of the damping parameter  $C$ , were compared to the exact solution as given by Equation (2-69). The values of  $C$  chosen for the comparison correspond to values of  $\epsilon$ , the critical damping ratio, of 0.05, 0.10, and 0.2. The computed values of  $E(X(t)^2)$  are

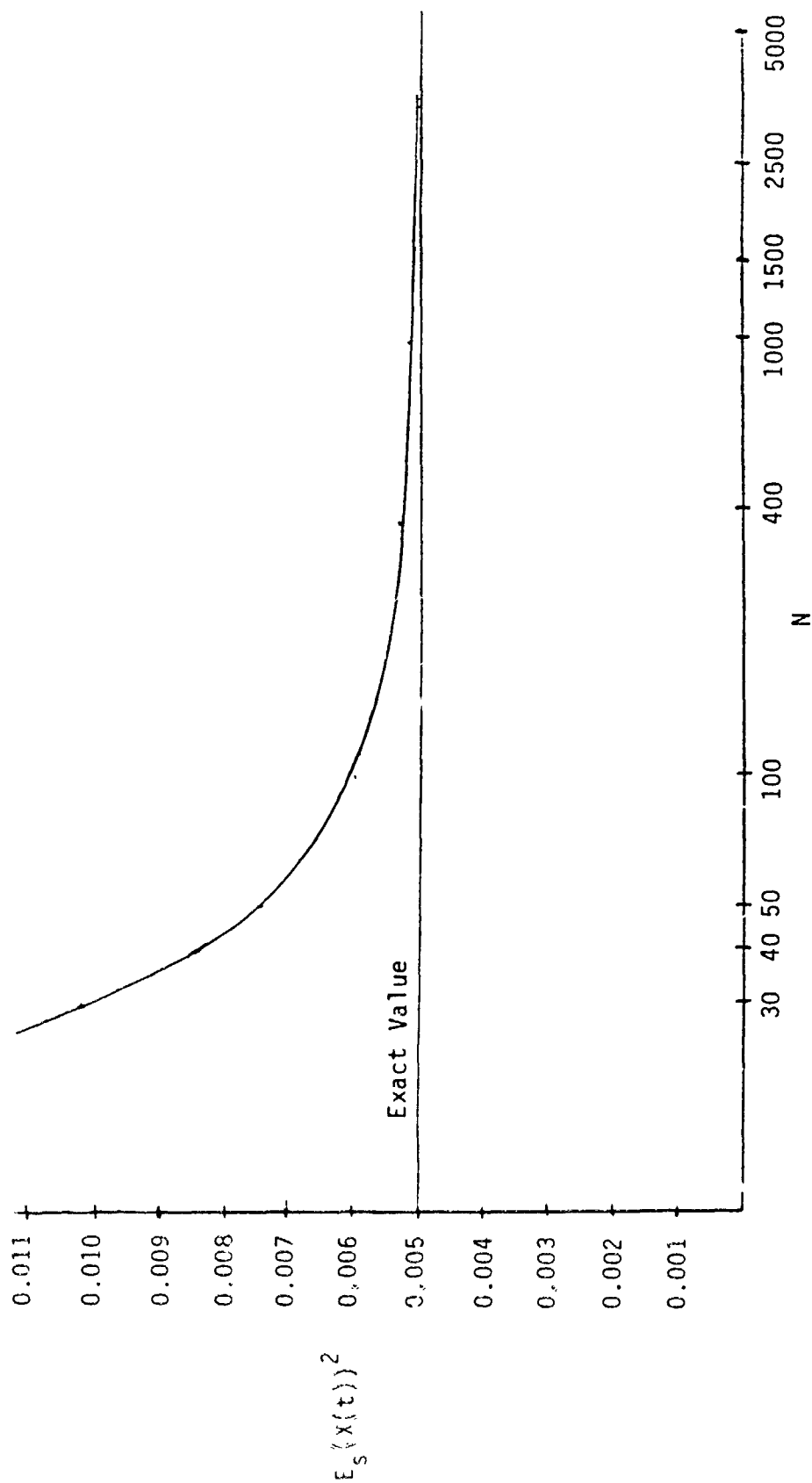


Figure 2-2 Values of  $E_s(X(t))^2$  for Various Values of  $N$ , where  $N = \text{Period}/\Delta t$

plotted versus time in Figure 2-3. The relative error,  $E$ , was computed at each time step by,

$$E = \frac{E(X(t)^2)_{\text{comp}} - E(X(t)^2)_{\text{exac}}}{E(X(t)^2)_{\text{exac}}} \times 100 \quad (2-80)$$

where  $E(X(t)^2)_{\text{comp}}$  is the value of the mean square response as computed and  $E(X(t)^2)_{\text{exac}}$  is the value for the mean square response given by Equation (2-69). The error was found to be monotonically increasing with time for all three cases, reaching its maximum value when the system reached the point of stationary response.

The values for  $E_{\text{max}}$ , the maximum error for the three cases are listed in Table (2-1). The maximum error was found to increase with decreasing values of  $\epsilon$ . However, since the mean square response goes to infinity as  $\epsilon$  goes to zero this is not surprising. A smaller value of  $\Delta t$  would result in smaller errors for the very lightly damped system.

If one now considers the system diagramed in Figure 2-1 to have both random stiffness  $K$  and damping  $C$  the values obtained from Equation (2-42) for the mean square response will be the same as for when the system is deterministic. This is due to the assumption made in Section 2.1 regarding the variations of the random components of the restoring forces. Specifically it was assumed that,

$$E(C_R \dot{\delta}) = C_R \dot{\delta}$$

and

$$E(K_R \delta) = K_R \delta$$

(2-81)

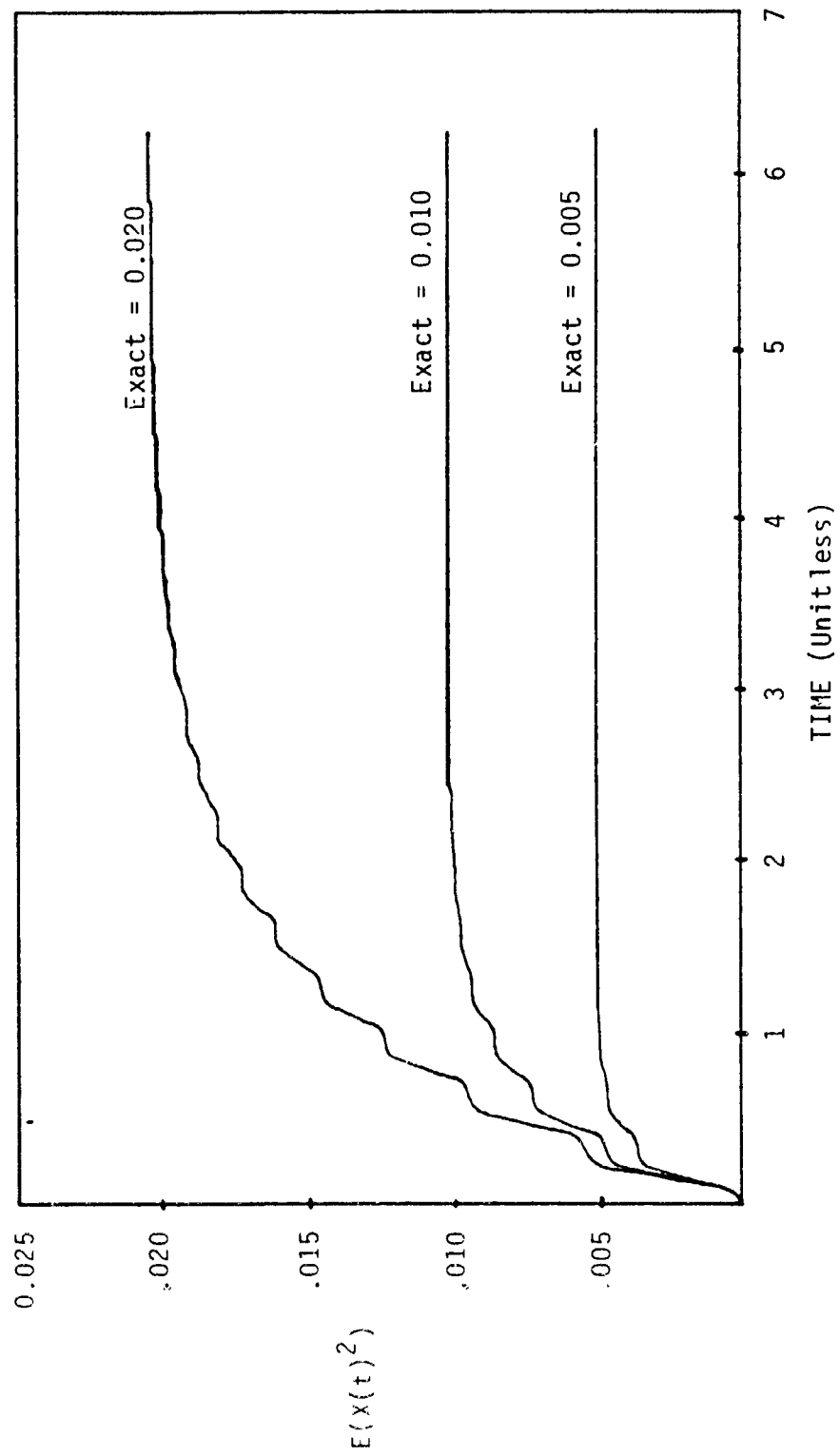


Figure 2-3 Mean Square Response for White Noise Loading

TABLE 2-1  
MAX ERROR IN THE COMPUTED VALUE OF  
 $E(X(t)^2)$  FOR VARIOUS VALUES OF  $\epsilon$

| $\epsilon$ | $E(X(t)^2)_{\text{comp}}$ | Max error |
|------------|---------------------------|-----------|
| 0.05       | 0.020472                  | 2.55%     |
| 0.10       | 0.010128                  | 1.28%     |
| 0.20       | 0.005032                  | 0.64%     |

N = 2500 for all cases.

These assumptions have no effect on the mean response of the system but do result in the contribution of the randomness in the stiffness and damping to the randomness of the response being multiplied by a factor containing the mean response. Hence when the mean response is zero the effects of the randomness in the system parameters is lost.

To answer the question as to how significant this error is one may write the expression for the mean square response of the stationary system as a function of  $K$  and  $C$ . Rearranging Equation (2-73) results in,

$$E_s(X(t)^2) = \frac{\pi\phi}{2KC} \quad (2-82)$$

For purposes of illustration one may consider  $K$  random and  $C$  deterministic, resulting in Equation (2-82) being a function of the random variable  $K$ . Taking the expected value of Equation (2-82) results in,

$$E(E_s(X(t)^2)) = \frac{\pi\phi}{2C} E\left(\frac{1}{K}\right) \quad (2-83)$$

And since the expectation of an expected value is the expected value the left hand side may be rewritten and Equation (2-83) becomes,

$$E_s(X(t)^2) = \frac{\pi\phi}{2C} E\left(\frac{1}{K}\right) \quad (2-84)$$

The  $E\left(\frac{1}{K}\right)$  depends on the distribution of  $K$ . When the mean stiffness  $K_D$ , is equal to 500 and the coefficient of variation  $\delta_K$  is 0.15,

and  $K$  is uniformly distributed with upper bound  $K_U$  and lower bound  $K_L$ ,  $E(\frac{1}{K})$  is given by,

$$E(\frac{1}{K}) = \int_{K_L}^{K_U} \frac{1}{K} \frac{1}{\text{SPAN}} dK \quad (2-85)$$

where  $\text{SPAN} = K_U - K_L$  and is given by

$$\text{Var } K = \frac{1}{K} (\text{SPAN})^2 \quad (2-86)$$

And since,

$$\left. \begin{aligned} \text{Var } K &= (0.15)^2 K_D^2 \\ \text{Var } K &= 5625 \end{aligned} \right\} \quad (2-87)$$

The span of the distribution is,

$$\text{SPAN} = 259.8 \quad (2-88)$$

Yielding values of  $K_U$  and  $K_L$  the upper and lower limits of the distribution of,

$$\left. \begin{aligned} K_U &= 629.9 \\ K_L &= 370.1 \end{aligned} \right\} \quad (2-89)$$

The resulting value for  $E(\frac{1}{K})$  is thus,

$$E(\frac{1}{K}) = 0.002046 = \frac{1}{488.7} \quad (2-90)$$

this differs from  $\frac{1}{K_D}$  by only 2.35%. Thus the true value for the station any mean square response of a white noise excited system with

random stiffness is underpredicted by only about 2.35% for the case illustrated. A suitable rule of thumb would be

$$E_s(X(t)^2) = \frac{\phi\pi}{2C K_D} (1 + \delta_K^2) \quad (2-91)$$

where  $\delta_K$  is the coefficient of variation for  $K$  and  $K_D$  is the mean value. This example provides considerable reassurance that the assumptions made in Section 2.1 will produce only a small error.

**EXAMPLE 2** - Comparison of the Results of the Transformation Method to the Exact Solution for the Response of an SDOF System with Random Parameters

The objective of this example is the verification of the transformation method by comparing its results with those obtained by numerically integrating an exact solution for the mean and variance of the system response.

The system under consideration has the following unitless mean values,

$$K_D = 500$$

$$\epsilon = 0.1$$

$$M = 5$$

and zero initial conditions.

The loading function used for the comparison was deterministic. The load was instantaneously applied at time 0 and remained in place indefinitely. The loading function is diagramed in Figure 2-4.

The system response to such a loading is,

$$x(t) = \frac{F}{K} \left( 1 - \exp(-\epsilon \omega_0 t) \left( \frac{\epsilon \omega_D}{\omega_0} \sin \omega_D t + \cos \omega_D t \right) \right) \quad (2-92)$$

where

$$\omega_0 = \sqrt{\frac{K}{M}} \quad (2-93)$$

$$\omega_D = \omega_0 \sqrt{1 - \epsilon^2} \quad (2-94)$$

The first comparison consists of evaluating the response of the SDOF system using the code RSDOF with the randomness in the system parameters set to zero. The results of this computation were plotted with values obtained from Equation (2-92) in Figure 2-5. As can be seen in the figure the plots are indistinguishable.

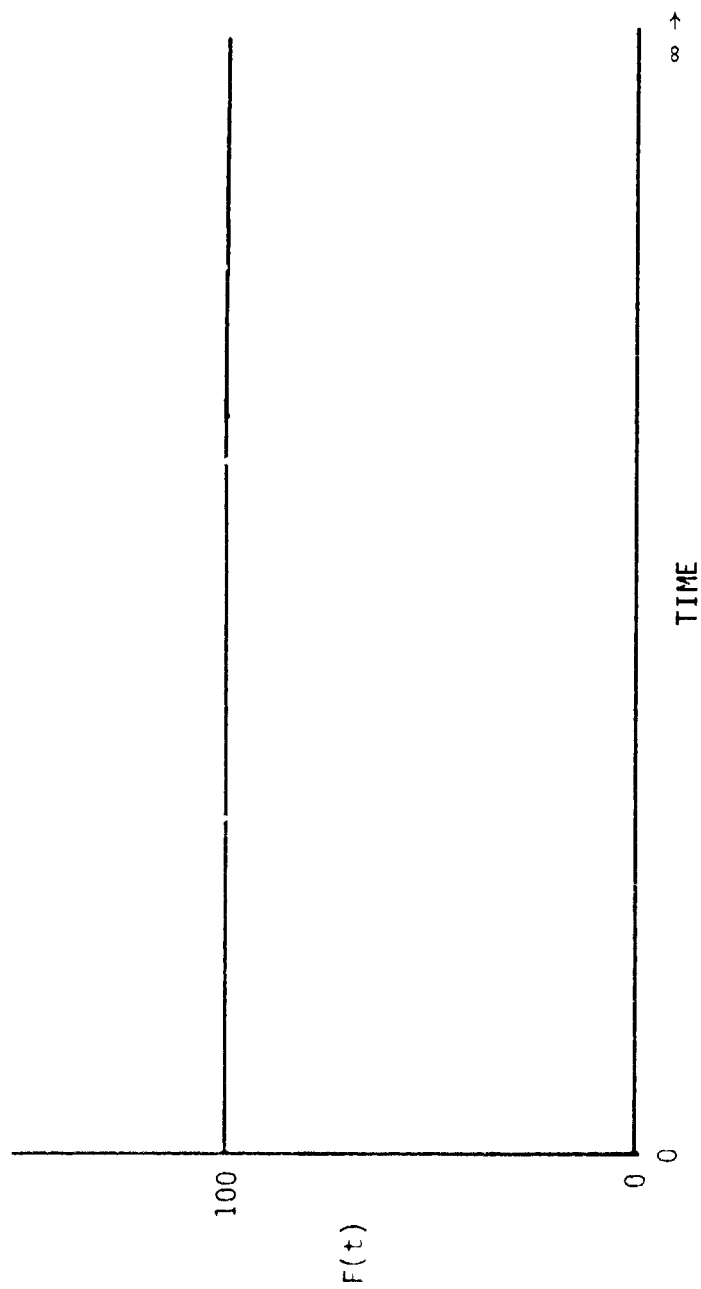


Figure 2-4 Dimensionless Load vs Time for Example 2

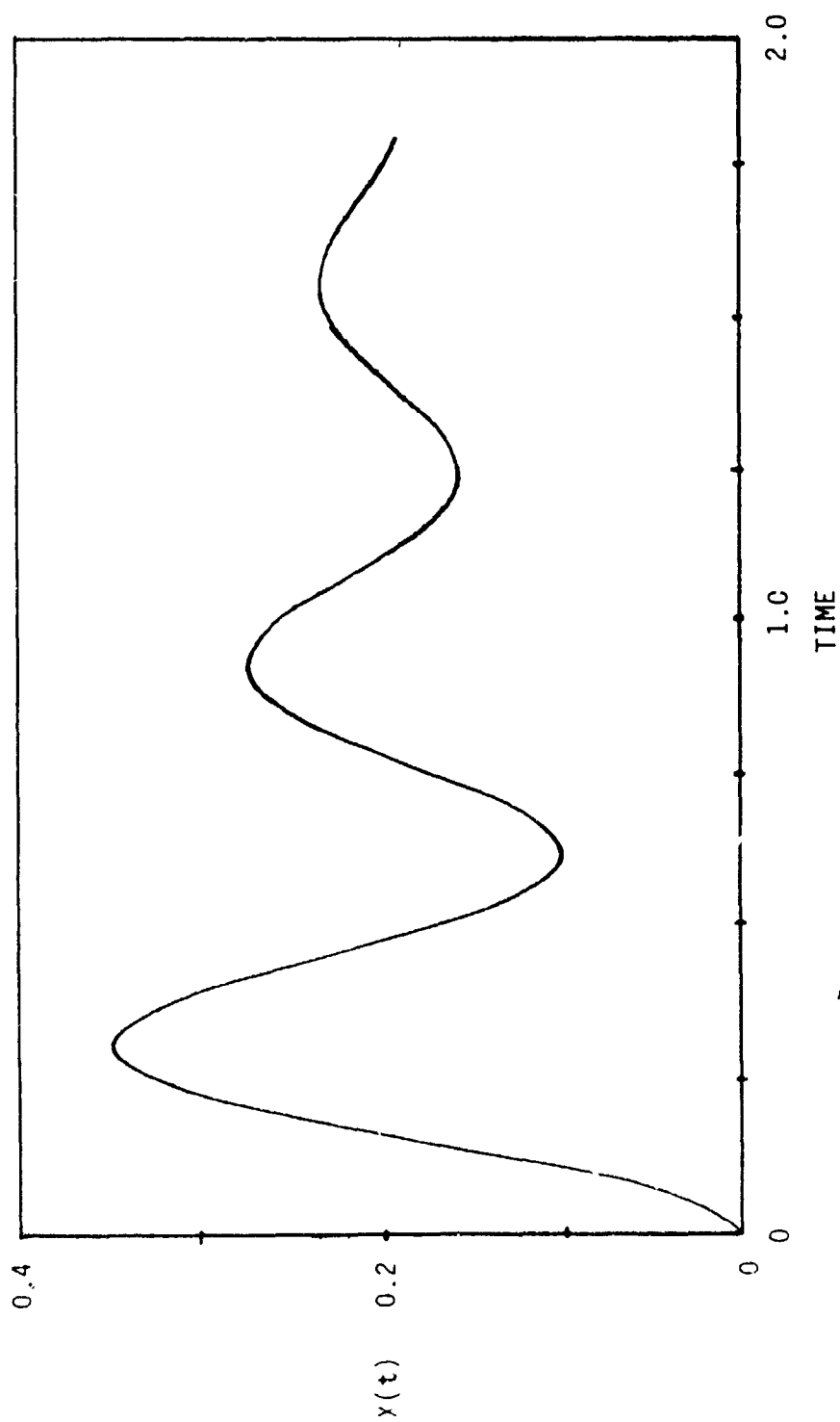


Figure 2-5 Exact and Numerical Solutions for the Deterministic Case, Example 2

For the second comparison the stiffness was treated as a random variable with a coefficient of variation of 0.1. To generate the exact solution for the mean and variance of  $X(t)$  Equation (2-92) was treated as a function of  $K$  at any time  $t$ . Thus,

$$X(t) = G(K)$$

$$\text{And} \quad E(X(t)) = E(G(K)) = \int_{-\infty}^{\infty} G(K) P_K dK \quad (2-95)$$

$$\text{And} \quad E(X^2(t)) = E(G^2(K)) = \int_{-\infty}^{\infty} G^2(K) P_K dK \quad (2-96)$$

$$\text{Var}(X(t)) = E(G^2(K)) - E(G(K))^2 \quad (2-97)$$

where  $P_K$  is the probability density function for  $K$ .

One notes that the exact values for  $E(X(t))$  and  $\text{Var}(X(t))$  are functions of the probability distribution function of  $K$ . This information is not included in the equations of the transformation method since the method replaces the derivatives with linear approximations. The means and variances of the resulting linear equations are independent of the distributions.

The integrations of Equations (2-95) and (2-96) were performed for two different distributions of the random variable  $K$ . The first of these was the uniform distribution. The probability density function for  $K$  is then,

$$\left. \begin{aligned} P_K &= \frac{1}{K_U - K_L} & K_L \leq K \leq K_U \\ &= 0 & \text{elsewhere} \end{aligned} \right\} \quad (2-98)$$

where  $K_U$  is the upper limit of integration and  $K_L$  the lower. Values of  $K_K$  and  $K_U$  were 413.4 and 586.6 respectively. These yield a mean value of  $K$  of 500 and a coefficient of variation of .1.

The numerical integration was performed using Newton-Quotes algorithm available via the QUANC8 subroutine available on the University of New Mexico's VSPC computer system.

The resulting means and standard deviations for approximately 2.5 cycles of response are plotted in Figures 2-6 and 2-7 respectively. The means are indistinguishable. The transformation method appears to seriously undershoot the troughs of the standard deviation curve. However, since the response at or near the peaks is of primary concern to the engineer this is not considered a serious shortcoming.

For a second comparison the stiffness  $K$  was assumed to have a truncated normal distribution. The probability density function for  $K$  is then,

$$p_K = \frac{1.00067}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2\sigma^2} (K - \mu_K)^2 \right] \quad (2-99)$$

$$\text{for } K_L < K < K_U$$

$$= 0 \quad \text{elsewhere}$$

The means are once again indistinguishable, and are plotted in Figure 2-8. The standard deviations are plotted in Figure 2-9. Once again the transformation method undershoots the troughs of the standard deviation vs time curve.

Based on the above information the transformation method provides an adequate approximation for the mean and standard deviation, or variance, of an SDOF system with random parameters.

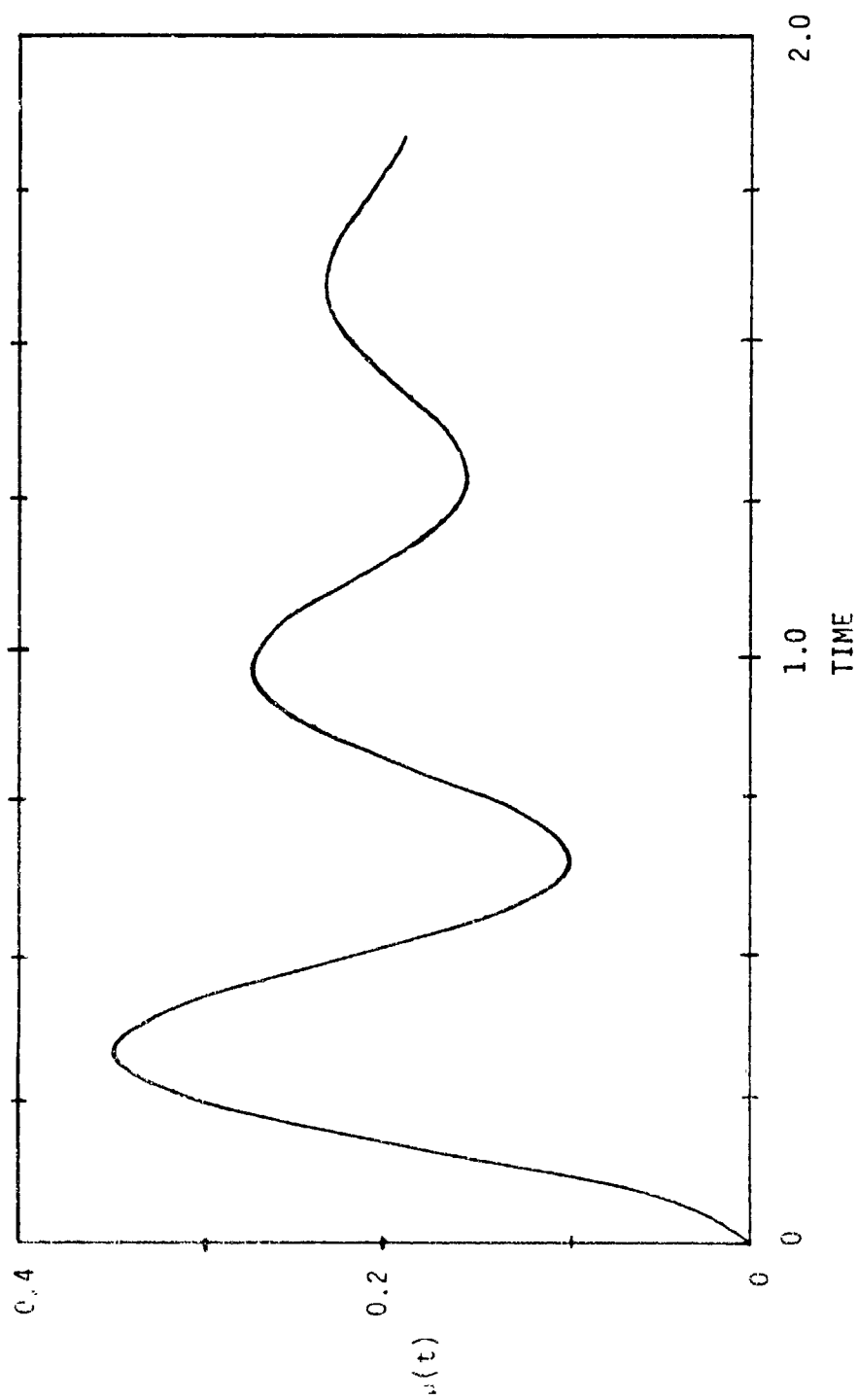


Figure 2-6 Transformation Approximation and Exact Solutions for  $\mu(t)$  Uniformly Distributed Stiffness

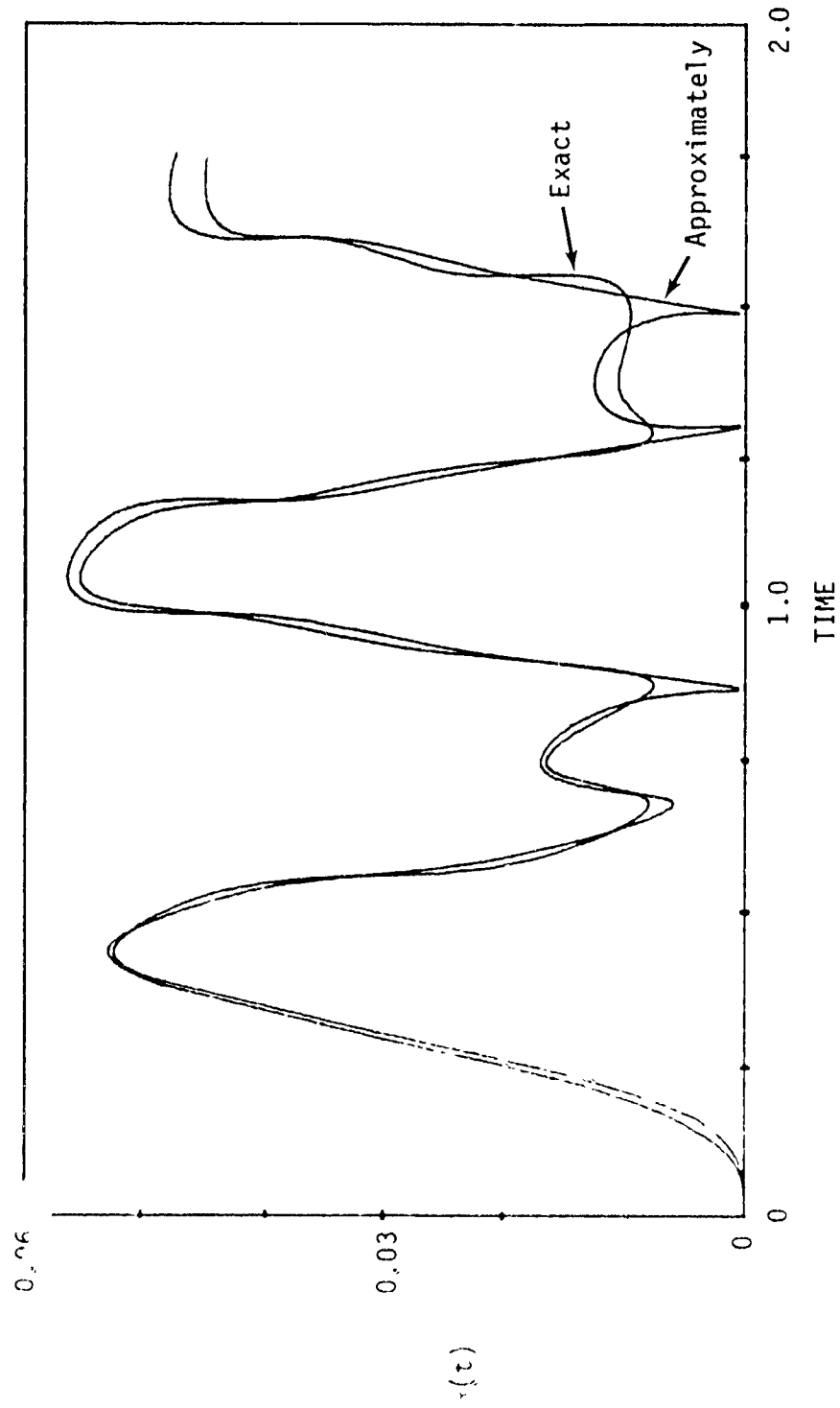


Figure 2-7 Exact and Transformation Approximation for  $\sigma(t)$ ,  
Uniformly Distributed Stiffness

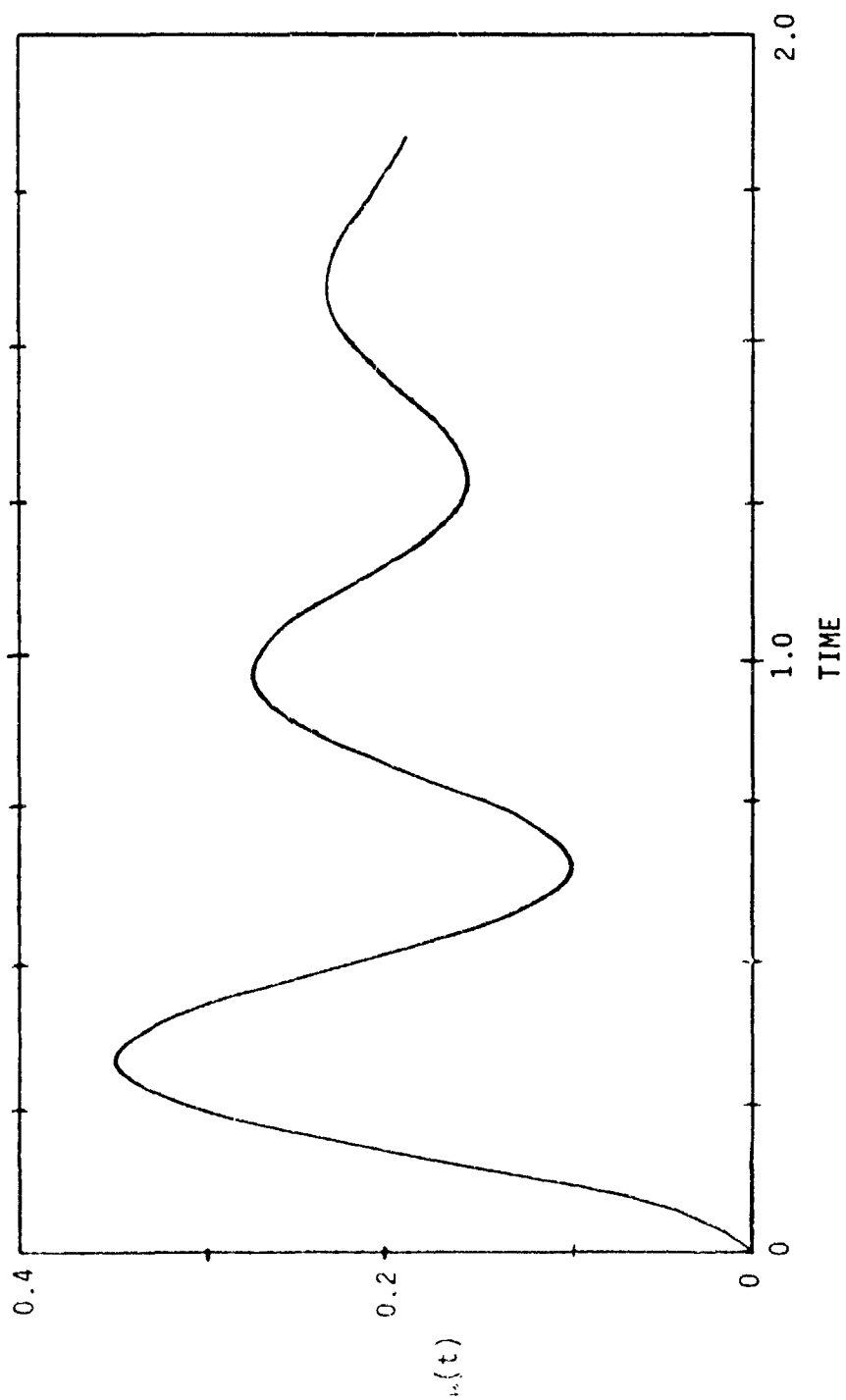


Figure 2-8 Exact and Transformation Approximation for  $u(t)$ , Normally Distributed Stiffness

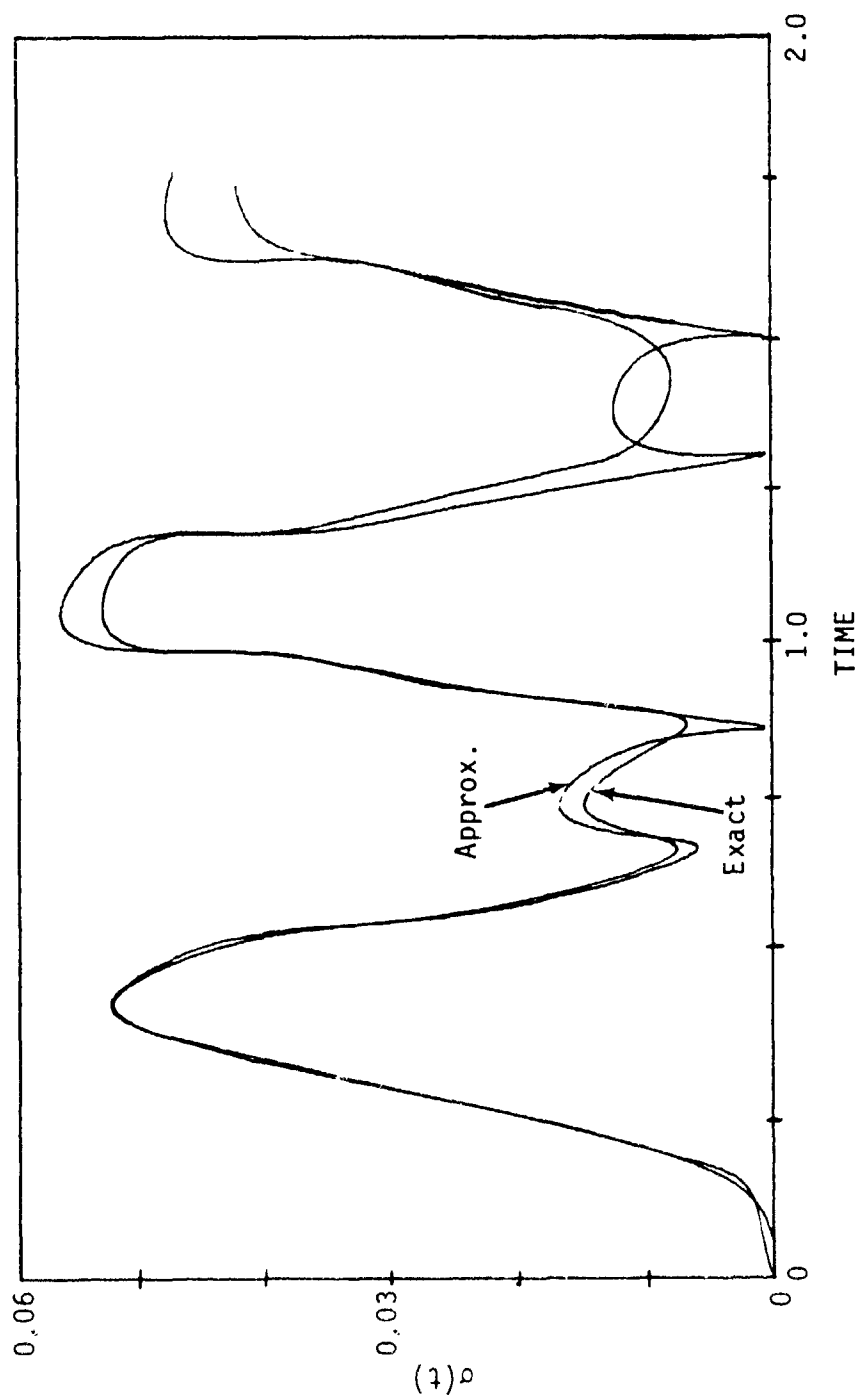


Figure 2-9 Exact and Transformation Approximation for  $\sigma(t)$ , Normally Distributed Stiffness

EXAMPLE 3 - Evaluation of the Influence of Random Stiffness and Damping on the Response Statistics for an SDOF System

This example consists of a set of computations of the mean and standard deviation for an SDOF system. The system's mean stiffness and mass were held constant. The loading function was the same as in Example 2 and is shown in Figure 2-4. The mean damping ratio, the standard deviations for the stiffness and damping, and the correlation coefficient between the stiffness and damping were varied in individual cases. The values used for these parameters in each case are shown in Table 2-2.

The results of cases 1 and 2 and the deterministic case were used to examine the influence of the randomness in the stiffness on the randomness of the response. The resulting values for the mean,  $\mu(t)$ , and the standard deviation,  $\sigma(t)$ , for each case are plotted in Figures 2-10, and 2-11. The deterministic case was discussed in Example 2 and its mean response is plotted in Figure 2-5. One notes that the value for the peak response increases as the standard deviation of the stiffness increases. Furthermore the standard deviation curve lags behind the response curve though they are of similar shape. The trend of increasing peak response is evident by comparing the values for the peak response obtained for cases 1 and 2 and the deterministic value. However, the variation between these cases is smaller than it would be for the static case, hence the variation in the stiffness has a smaller effect on the peak dynamic response than it does on the static. Furthermore the values obtained from the code are at preselected times based on the period of the response as determined by its mean parameters, when the

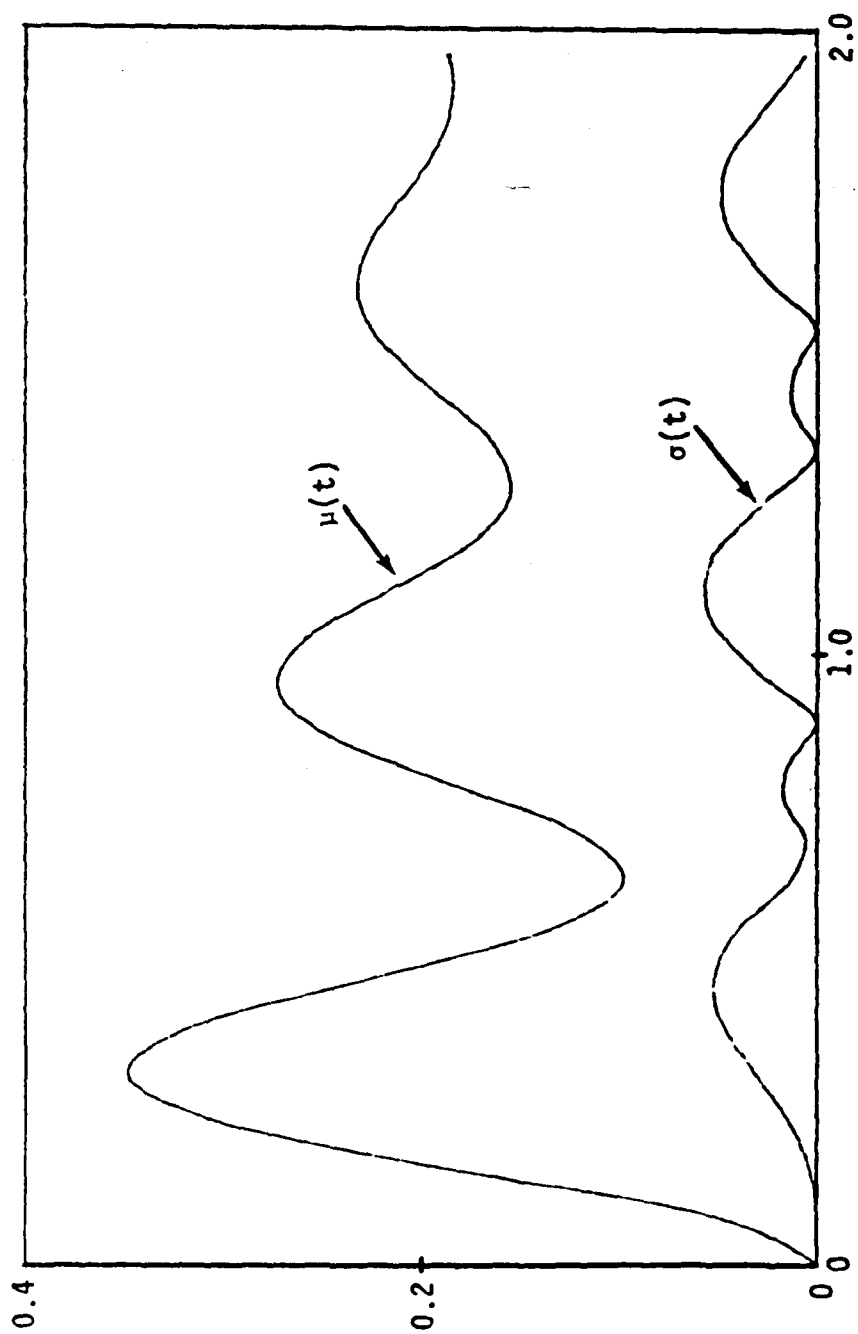


Figure 2-10  $\mu(t)$  and  $\sigma(t)$  Case 1, Example 2

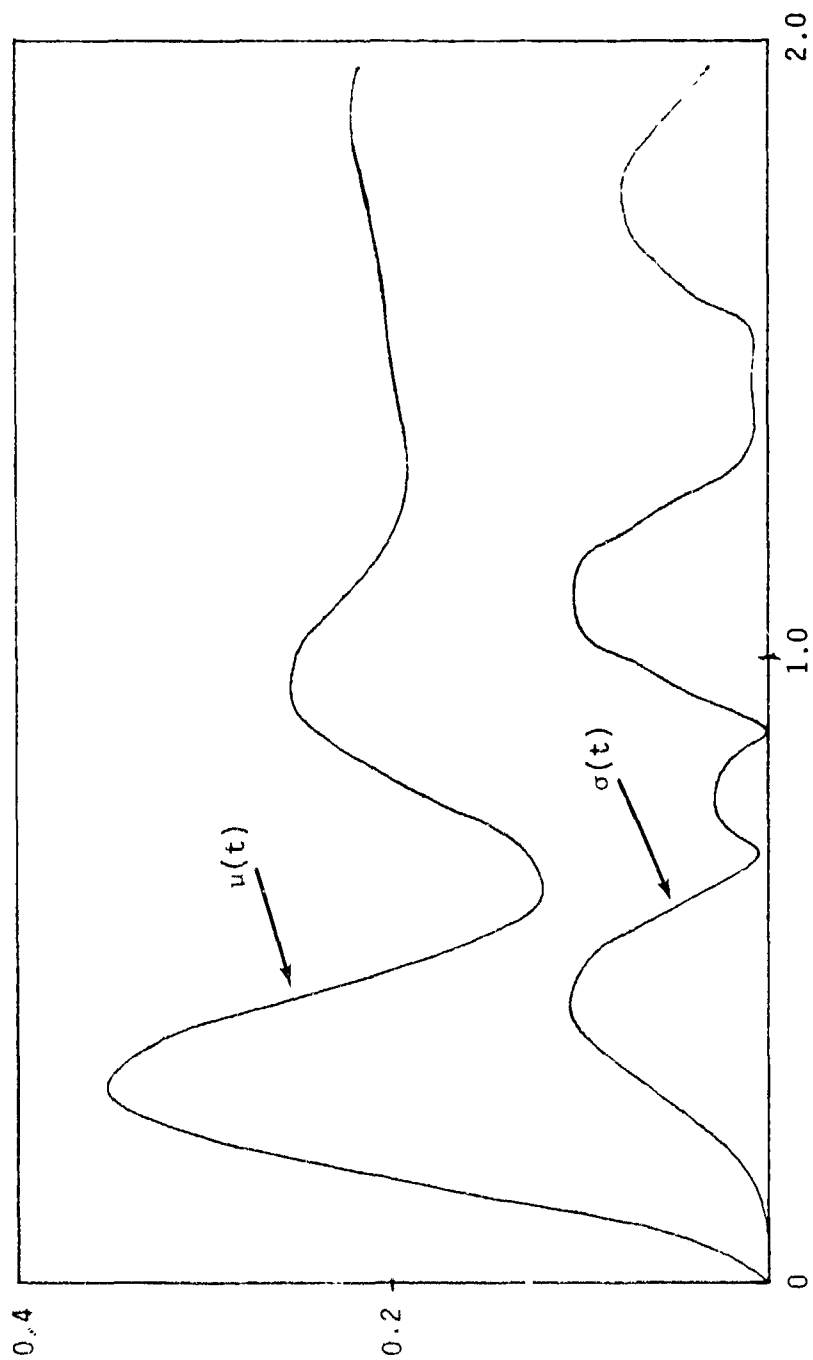


Figure 2-11  $u(t)$  and  $\sigma(t)$  Case 2, Example 2

TABLE 2-2  
PARAMETERS USED IN CASE STUDIES

| Case | $\epsilon$ | $\sigma_{\epsilon}$ | $\sigma_K$ | $P_{KC}$ |
|------|------------|---------------------|------------|----------|
| 1    | 0.1        | 0                   | 50         | 0        |
| 2    | 0.1        | 0                   | 100        | 0        |
| 3    | 0.5        | 0                   | 100        | 0        |
| 4    | 0.1        | 0.01                | 0          | 0        |
| 5    | 0.1        | 0.02                | 0          | 0        |
| 6    | 0.5        | 0.02                | 0          | 0        |
| 7    | 0.1        | 0.01                | 50         | 0        |
| 8    | 0.1        | 0.01                | 50         | 1.0      |
| 9    | 0.1        | 0.01                | 50         | -1.0     |

system has random parameters there can be a slight phase shift resulting in the peak values printed being off by one or two time steps from the true peaks.

The peak response for each of the cases,  $\sigma_k = 0, 50, 100$  are listed in Table 2-3.

The results of cases 4, 5, and 6 allow one to examine the influence of the randomness in the damping on the randomness of the response. The means and standard deviations for these cases are plotted in Figures 2-12, 2-13, and 2-14 respectively.

By comparing the results of cases 1 and 4 one notes that the values obtained for  $\alpha_X(t)$  resulting from a random stiffness are an order of magnitude larger than those obtained for random damping. Cases 2 and 5 show the same effect for larger variations in the stiffness and damping. However, one must note that these results are for a system with damping which is only 10% of the critical value. When the damping is increased to 50% of critical as in cases 3 and 6, plotted in Figures 2-15 and 2-14 respectively, the resulting values for  $\alpha_X(t)$  resulting from randomness in the stiffness and damping are still not of the same magnitude.

From these results one concludes that for lightly damped systems which do not have a zero mean, the effects on response of randomness in the damping are minimal compared to those resulting from a randomness in the stiffness. For the case of white noise response the effects of randomness in  $K$  and randomness in  $C$  will be the same when the coefficients of variation of  $K$  and  $C$  are equal.

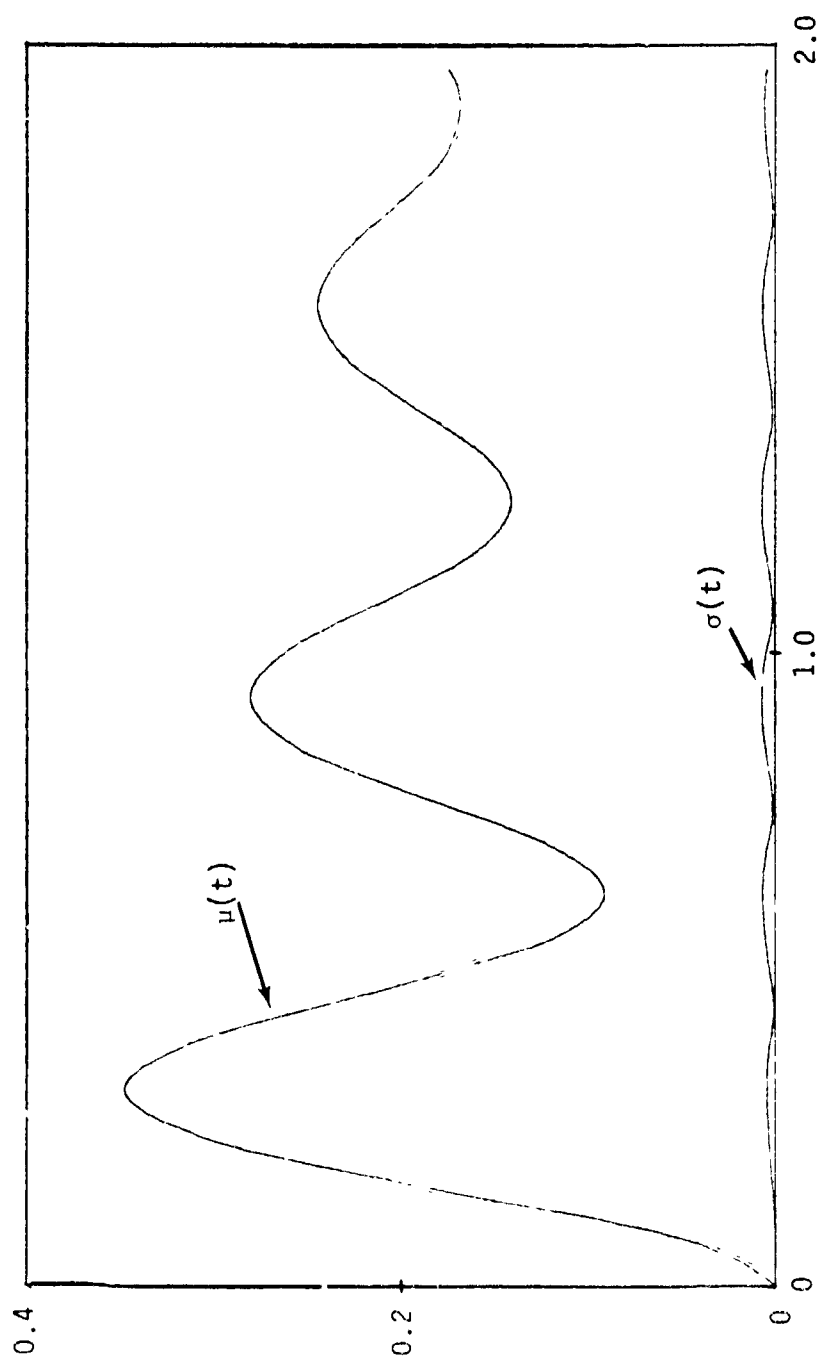


Figure 2-12  $\mu(t)$  and  $\sigma(t)$  Case 4, Example 2

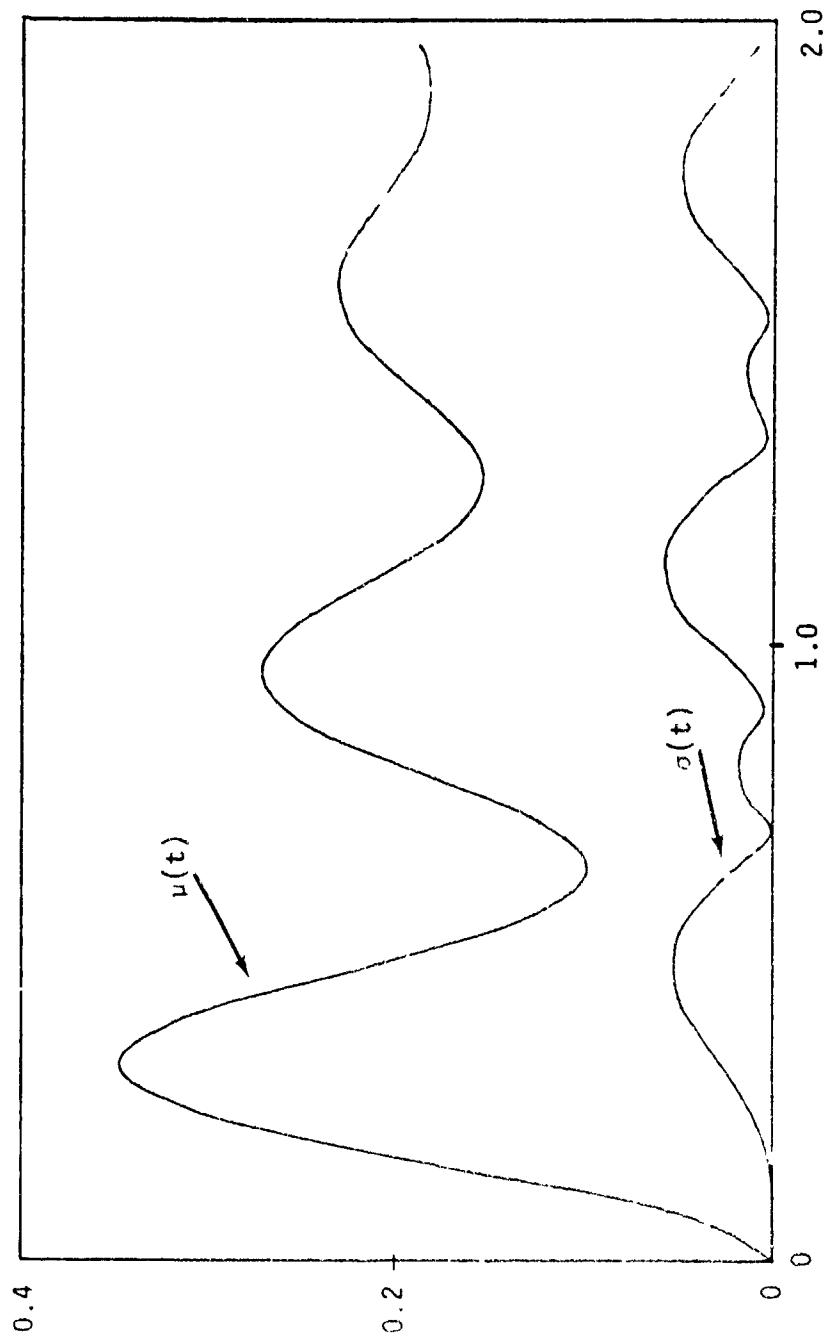


Figure 2-13  $\mu(t)$  and  $\sigma(t)$  Case 5, Example 2

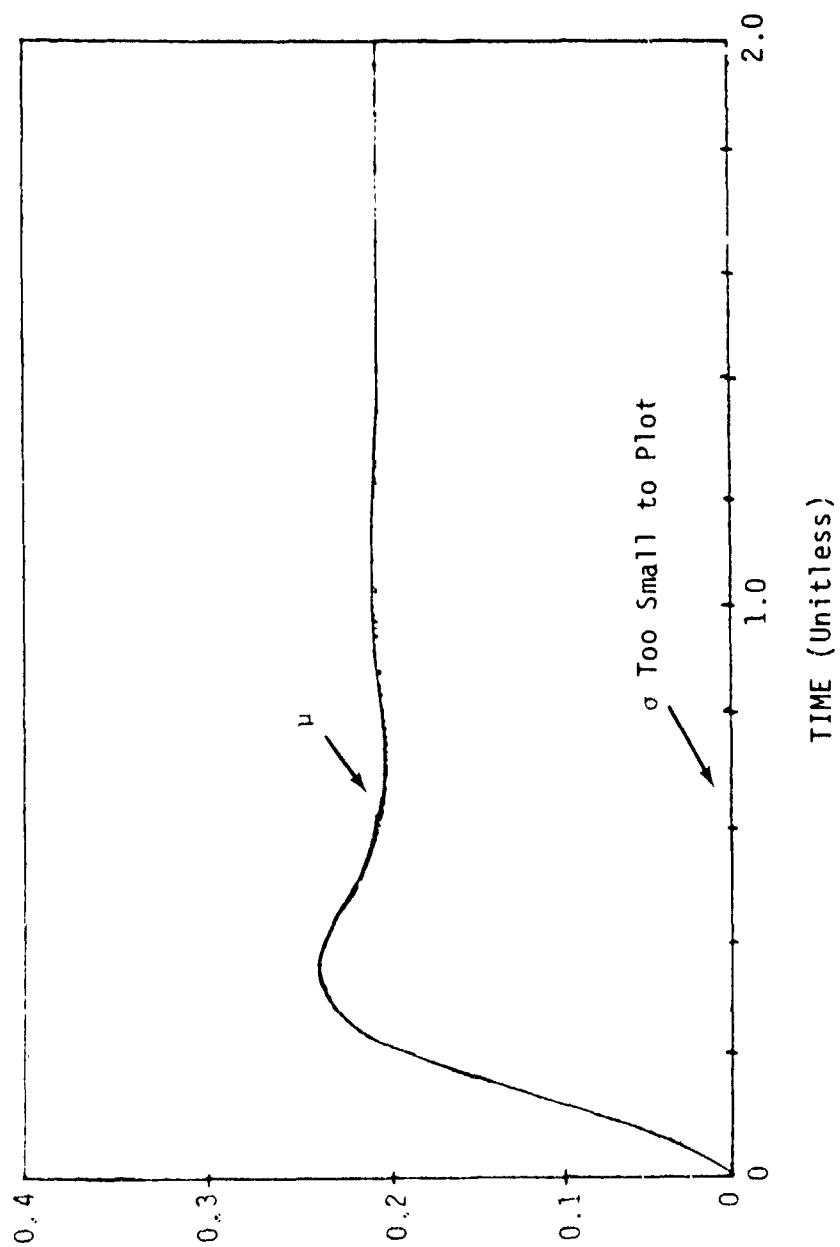


Figure 2-14  $\mu$  and  $\sigma$  for Case 6, Example 3

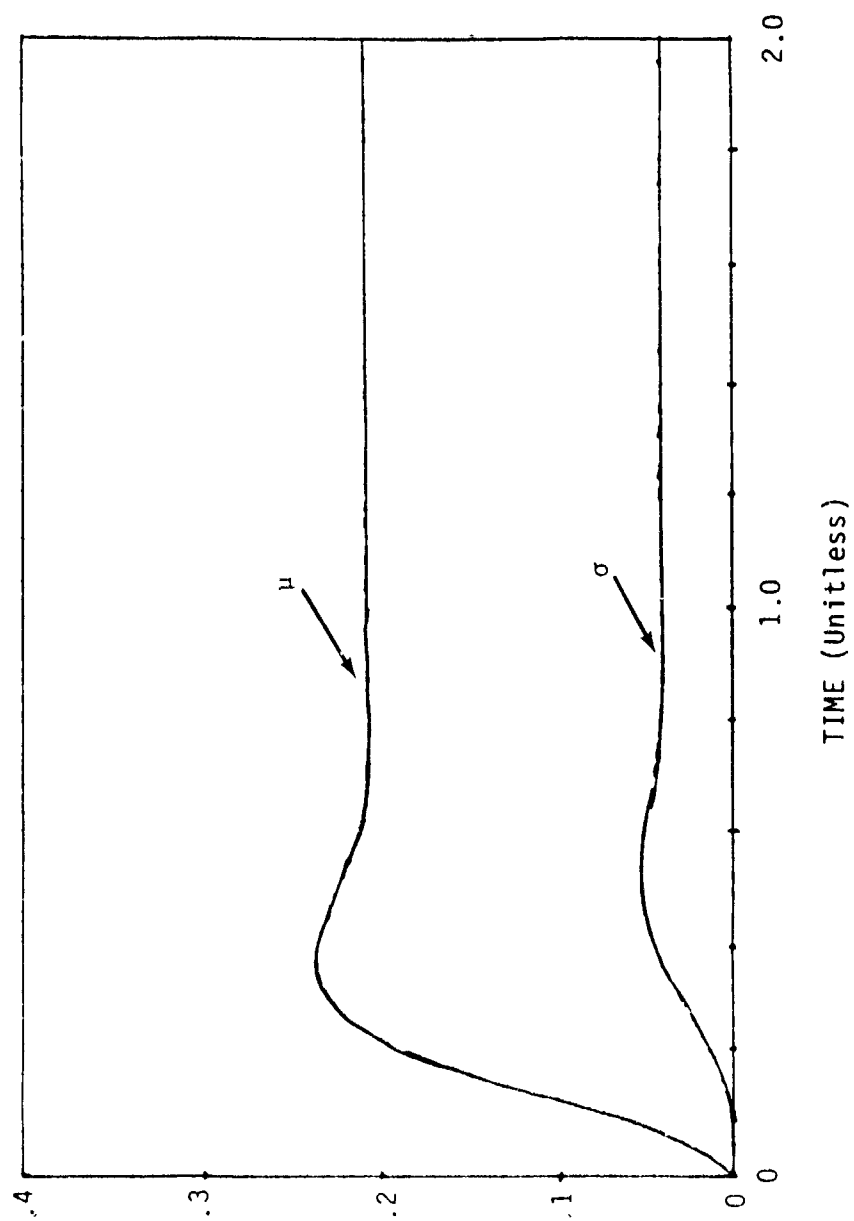


Figure 2-15  $\mu$  and  $\sigma$  for Case 3, Example 3

TABLE 2-3  
STANDARD DEVIATION OF K AND  
CORRESPONDING PEAK RESPONSE VALUES

| $\sigma_K$ | Peak Response |
|------------|---------------|
| 0          | 0.3473        |
| 50         | 0.3485        |
| 100        | 0.3521        |

Furthermore while the randomness in the stiffness effects the response for its entire duration, the effects of the random damping dissipate as the system converges to its static deflection which is independent of the damping. However, when the loading is transitory the effects of the random stiffness will also dissipate as the system returns to its initial state.

The results of cases 7, 8, and 9 allow one to assess the impact of correlation between the stiffness and damping. In these cases the correlation coefficient between the stiffness and damping,  $P_{KD}$ , was given the values of 0, 1, and -1, respectively.

The difference between these cases were too small to be effectively plotted, however, case 8,  $P_{KC} = 1$ , produced the largest values of both  $\mu_X(t)$  and  $\sigma_X(t)$ . Case 9,  $P_{KC} = -1$ , produced the smallest values of  $\mu(t)$  and  $\sigma_X(t)$ .

Based on these results one concludes that  $\sigma_X(t)$  at time of peak response increases with increasing correlation between the stiffness and damping. The values for the mean and standard deviation at the time of peak response are provided in Table 2-4 as an illustration.

TABLE 2-4  
MEAN AND STANDARD DEVIATION OF  
 $X(t)$  AT TIME OF PEAK RESPONSE

| $P_{KC}$ | $\mu(t)$ | $\sigma_k(t)$ |
|----------|----------|---------------|
| -1       | 0.3479   | 0.276         |
| 0        | 0.3486   | 0.326         |
| 1        | 0.3493   | 0.370         |

#### EXAMPLE 4 - Blast Loading of a Simply Supported Beam

This example illustrates the application of the methodology developed in the previous sections to a problem of practical interest.

The structure considered is a simply supported reinforced concrete beam. The loading is a simplified approximation to a blast loading. The structure and its loading time history are shown in Figure 2-16.

A deterministic solution using standard analysis procedures is provided in Biggs (33). The peak deflection resulting from the deterministic and stochastic solutions are compared.

When modeling such a structure with an SDOF system the stiffness is,

$$K_D = \frac{384EI}{5L^3} \quad (2-100)$$

for the present case this yields a value of  $K_D$  of  $2.08 \times 10^6$  lb/in. The stiffness is assumed to have a coefficient of variation of 0.15. This results in the variance of  $K, E(K_K^2)$ , being  $9.734 \times 10^{10}$  lb<sup>2</sup>/in<sup>2</sup>.

The damping was assumed deterministic and equal to 8% of critical.

The results of the stochastic computation, the mean of the midspan deflection and its standard deviation are plotted in Figure 2-17. The maximum value obtained for the mean deflection was 0.2450". Biggs estimated an upper bound of 0.2726" for the undamped structure.

The key question of interest for the engineer is, "what is the probability of the peak response exceeding a desired level,  $b$ ?". This question is addressed in Example 2 of Chapter 4 where the passage probabilities for this problem are computed.

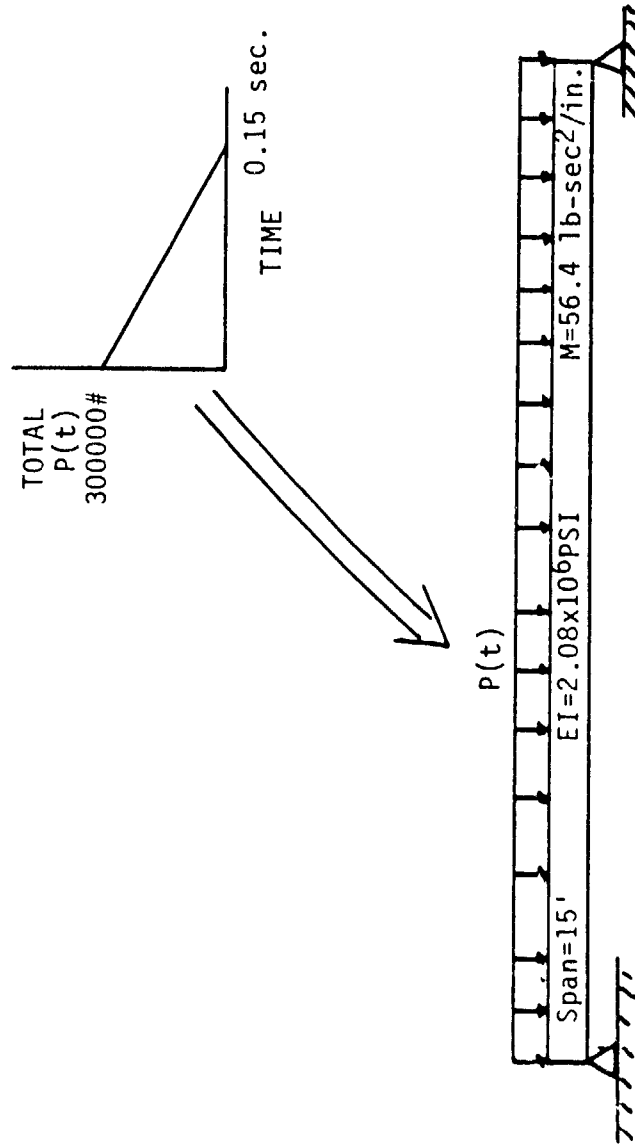


Figure 2-16 Simply Supported, Blast Excited Beam of Example 4

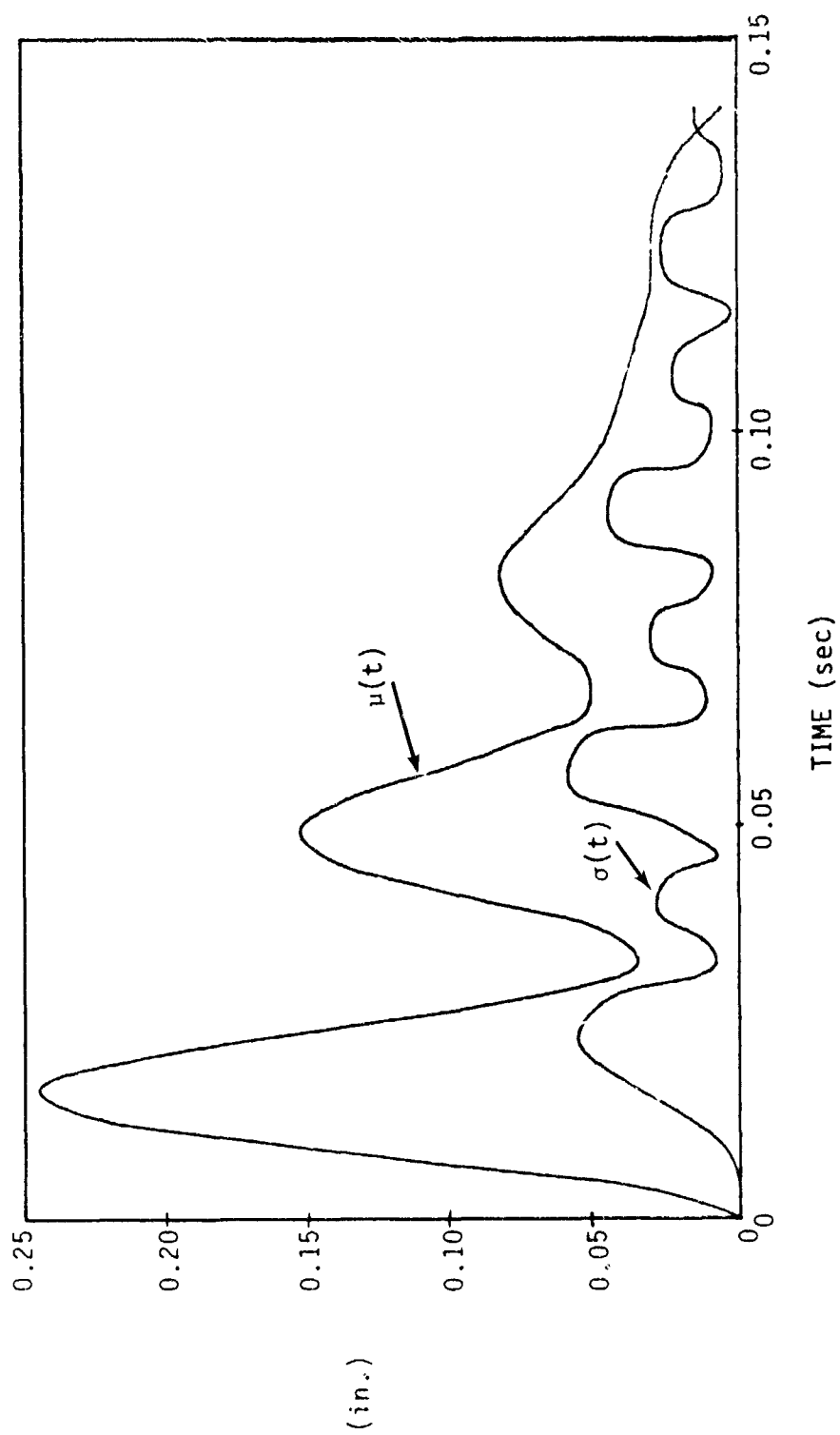


Figure 2-17  $\mu(t)$  and  $\sigma(t)$  of Midspan Deflection, Example 4

## CHAPTER 3

### RESPONSE OF MDOF SYSTEMS WITH RANDOM STIFFNESS AND DAMPING

#### 3.1 INTRODUCTION

In this chapter the transformation method developed in Chapter 2 is expanded to MDOF systems. This introduces a number of complications involving the correlation between the random variables describing the system and the response parameters. Expressions for the general case where the correlation between system parameters may be user specified and for a simpler case of individual terms of the system matrices being uncorrelated are developed. The latter case is used in the computer code developed simply to reduce computation times.

The assumptions regarding the stochastic nature of the displacement response,  $X(t)$ , and of the loading,  $F(t)$ , for the SDOF system of Chapter 2 to apply to  $\{X(t)\}$ , the vector displacement response, and  $\{F(t)\}$ , the forcing function vector as well.

The desired results of the computation are the mean response vector,  $\{\mu(t)\}$ , and the Covariance Matrix,  $\text{Cov}(\{X(t)\}, \{X(t)\}^T)$  as functions of time.

No attempt is made to model the distribution of the response vector or to determine the correlation between values at different times.

#### 3.2 TRANSFORMATION OF THE EQUATIONS OF MOTION

The equation of motion for a mass excited MDOF system is,

$$[M]\{\ddot{X}(t)\} + [C]\{\dot{X}(t)\} + [K]\{X(t)\} = \{F(t)\} \quad (3-1)$$

where

$$\begin{aligned}
 [M] &= \text{System Mass Matrix} \\
 [C] &= \text{System Damping Matrix} \\
 [K] &= \text{System Stiffness Matrix} \\
 \{X(t)\} &= \text{System Displacement Vector} \\
 \{F(t)\} &= \text{Forcing Function Vector}
 \end{aligned}$$

When  $[K]$  and  $[C]$  are deterministic the response of the system may be obtained by several methods. For a deterministic loading  $\{F(t)\}$ , Equation (3-1) may be solved by a variety of numerical schemes including modal decomposition, explicit integration, and implicit integration.

When  $\{F(t)\}$  is a stochastic process modal decomposition is the method most frequently used. However when  $[K]$  and  $[C]$  are matrices of random variables the solution of Equation (3-1) is not readily available.

The transformation method developed in Chapter 2 provides a new method for addressing this problem.

Following the development of Section 2.2 results in,

$$\{X(t)\} = \{\mu(t)\} + \{\delta(t)\} \quad (3-2)$$

Taking the expected value of Equation (3-2) yields,

$$E\{X(t)\} = E\{\mu(t)\} + E\{\delta(t)\} \quad (3-3a)$$

$$= \{\mu(t)\} \quad (3-3b)$$

To develop an expression for the mean square value of the response one multiplies Equation (3-2) by its transpose yielding,

$$\begin{aligned}
 \{X(t)\}\{X(t)\}^T &= \{\mu(t)\}\{\mu(t)\}^T + \{\mu(t)\}\{\delta(t)\}^T \\
 &\quad + \{\delta(t)\}\{\mu(t)\}^T + \{\delta(t)\}\{\delta(t)\}^T
 \end{aligned} \quad (3-4)$$

Taking the expected value of Equation (3-4) yields,

$$E[\{X(t)\}\{X(t)\}^T] = E[\{\dot{\mu}(t)\}\{\dot{\mu}(t)\}^T] + E[\{\delta(t)\}\{\delta(t)\}^T] \quad (3-5)$$

The covariance matrix of the response is given by,

$$\text{Cov}[\{X(t)\}, \{X(t)\}^T] = E[\{X(t)\}\{X(t)\}^T] - E\{X(t)\} \cdot E\{X(t)\}^T \quad (3-6a)$$

$$\text{Cov}[\{X(t)\}, \{X(t)\}^T] = E[\{\delta(t)\}\{\delta(t)\}^T] \quad (3-6b)$$

One notes that the expressions in Equation (3-6b) are nxn matrices for a system with n degrees of freedom. The diagonal terms in the matrices are the variances of the displacement responses at the degrees of freedom, and the off diagonal terms are the covariances between displacement responses at the degrees of freedom.

A similar substitution results in the expression for the mean and covariance terms for the velocity vector,  $\{\dot{X}(t)\}$ , which are,

$$\{\dot{X}(t)\} = \{\dot{\mu}(t)\} + \{\dot{\delta}(t)\} \quad (3-7)$$

$$E\{\dot{X}(t)\} = \{\dot{\mu}(t)\} \quad (3-8)$$

$$\text{Cov}(\{\dot{X}(t)\}, \{\dot{X}(t)\}^T) = E[\{\dot{\delta}(t)\}\{\dot{\delta}(t)\}^T] \quad (3-9)$$

The expressions for the mean and covariance of the acceleration vector,  $\{\ddot{X}(t)\}$  are similarly derived and are,

$$\{\ddot{X}(t)\} = \{\ddot{\mu}(t)\} + \{\ddot{\delta}(t)\} \quad (3-10)$$

$$E\{\ddot{X}(t)\} = \{\ddot{\mu}(t)\} \quad (3-11)$$

$$\text{Cov}(\{\ddot{X}(t)\}, \{\ddot{X}(t)\}^T) = E[\{\ddot{\delta}(t)\}\{\ddot{\delta}(t)\}^T] \quad (3-12)$$

Representing the stiffness matrix,  $[K]$ , as the sum of  $[K_D]$  and  $[K_R]$  with  $[K_D]$  deterministic and  $[K_R]$  a matrix of random variables with mean zero yields,

$$[K] = [K_D] + [K_R] \quad (3-13)$$

Taking the expected value of Equation (3-13) yields

$$E[K] = E[K_D] + E[K_R] \quad (3-14)$$

And since  $[K_R]$  is mean zero,

$$E[K] = [K_D] \quad (3-15)$$

To develop covariance terms one must consider individual elements in the matrix  $[K]$ . Let  $(K)_{ij}$  denote the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $[K]$ . Then,

$$(K)_{ij} = (K_D)_{ij} + (K_R)_{ij} \quad (3-16)$$

The product between this and the arbitrary element  $(K)_{lm}$  is,

$$\begin{aligned} (K)_{ij}(K)_{lm} &= (K_D)_{ij}(K_D)_{lm} + (K_D)_{ij}(K_R)_{lm} + (K_R)_{ij}(K_D)_{lm} \\ &\quad + (K_R)_{ij}(K_R)_{lm} \end{aligned} \quad (3-17)$$

Taking the expected value of Equation (3-17) yields,

$$E[(K)_{ij}(K)_{lm}] = (K_D)_{ij}(K_D)_{lm} + E[(K_R)_{ij}(K_R)_{lm}] \quad (3-18)$$

because the random components are mean zero. This quantity is the correlation between the random variables  $(K)_{ij}$  and  $(K)_{lm}$ . The covariance between random variable pairs is its correlation minus the product of means. In the present case this is,

$$\text{Cov}[(K)_{ij}, (K)_{lm}] = E[(K_R)_{ij}(K_R)_{lm}] \quad (3-19)$$

Similar expressions are developed for the damping matrix,  $[C]$ , with  $[C_D]$  and  $[C_R]$  representing the deterministic and random, mean zero terms respectively,

$$[C] = [C_D] + [C_R] \quad (3-20)$$

The resulting expression for the mean damping matrix is,

$$E[C] = [C_D] \quad (3-21)$$

The covariance between terms of  $[C]$  is,

$$\text{Cov}[(C)_{ij}, (C)_{lm}] = E[(C_R)_{ij}(C_R)_{lm}] \quad (3-22)$$

The substitution for the vector of forcing functions,  $\{F(t)\}$ , consists of the sum of deterministic mean portion  $\{P(t)\}$  and the random portion  $\{q(t)\}$ . Making this substitution and following the development above results in the following expressions for  $\{F(t)\}$ , its mean and covariance.

$$\{F(t)\} = \{P(t)\} + \{q(t)\} \quad (3-23)$$

$$E\{F(t)\} = \{P(t)\} \quad (3-24)$$

$$\text{Cov}(\{F(t)\}\{F(t)\}^T) = E(\{q(t)\}\{q(t)\}^T) \quad (3-25)$$

Substituting the results of Equations (3-2), (3-7), (3-10), (3-13), (3-20), and (3-23) into Equation (3-1) yields,

$$[M]\{\ddot{\mu} + \dot{\delta}\} + [C_D][C_R]\{\dot{\mu} + \dot{\delta}\} + [K_D][K_R]\{\mu + \delta\} = \\ = \{P(t)\} + \{q(t)\} \quad (3-26)$$

When the system parameters and response are considered correlated the expected value of Equation (3-26) is,

$$[M]\{\ddot{\mu}\} + [C_D]\{\dot{\mu}\} + [K_C]\{\mu\} + E([K_R]\{\delta\}) + E([C_R]\{\dot{\delta}\}) \\ = \{P(t)\} \quad (3-27)$$

A solution for  $\{\mu(t)\}$ , the mean response vector may be sought based on this expression.

To develop an expression for  $\{\delta(t)\}$  one may subtract Equation (3-27) from Equation (3-26) resulting in,

$$[M]\{\dot{\delta}\} + [C_D]\{\dot{\delta}\} + [K_D]\{\delta\} = \{CRT\} + \{KRT\} = \\ = \{q(t)\} - [K_R]\{\mu\} - [C_R]\{\dot{\mu}\} \quad (3-28a)$$

where  $\{CRT\} = [C_R]\{\dot{\delta}\} - E([C_R]\{\dot{\delta}\}) \quad (3-28b)$

$$\{KRT\} = [K_R]\{\delta\} - E([K_R]\{\delta\}) \quad (3-28c)$$

The terms in Equations (3-28b) and (3-28c) may be considered very small and may be neglected under the arguments presented in Sections 2.2, and 2.5, resulting in,

$$[M]\{\ddot{\delta}\} + [C_D]\{\dot{\delta}\} + [K_D]\{\delta\} = \{q(t)\} - [K_R]\{\mu\} - [C_R]\{\dot{\mu}\} \quad (3-29)$$

The forward difference method is used to solve Equation (3-27) for the mean response of the system. The expression for the variance of the response is developed by applying the forward difference equations to Equation (3-29), yielding an expression for  $\{\delta(t)\}$  in terms of the underlying random variables, and  $\{\delta(t)\}$  at earlier times. This expression is then multiplied by its transpose, and the expected value of the result is equal to the variance at time  $t$ .

The details of this development are provided in Section 3.4 after some comments on the development of the system matrices in Section 3.3.

### 3.3 FORMULATION OF THE SYSTEM MATRICES

The finite element method was used to develop the system's stiffness, damping and mass matrices. The structure of interest is represented as the combination of a number of plane frame elements. Each element has the six degrees of freedom shown in Figure 3-1.

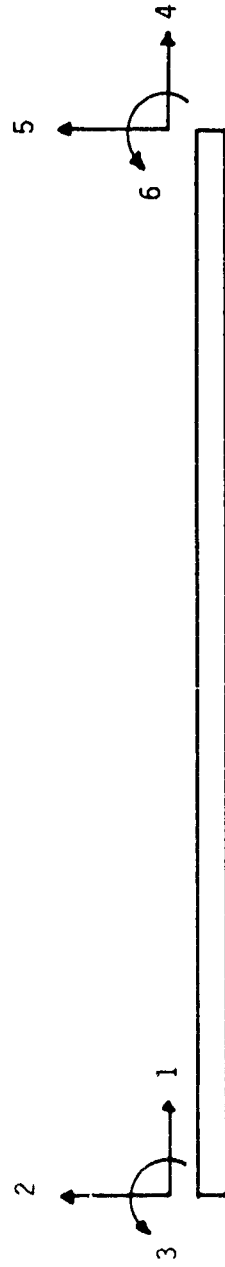


Figure 3-1 Plane Frame Element Degrees of Freedom

The stiffness matrix,  $[K]$ , for such an element is,

$$[k] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \quad (3-30)$$

where  $AE$  = Extensional Rigidity of the Element

$EI$  = Flexural Rigidity of the Element

$L$  = Length of the Element

When  $EI$  and  $AE$  are random variables each term of  $[k]$  consists of a random component with mean zero and a deterministic component equal to the values obtained from Equation (3-30) evaluated at the mean values of  $EI$  and  $AE$ . Thus,

$$[k] = [k_R] + [k_D] \quad (3-31)$$

Taking the expected value of Equation (3-30) yields,

$$E[k] = E[k_R] + E[k_D] \quad (3-32)$$

And since  $[k_R]$  is mean zero and  $[k_D]$  is deterministic,

$$E[k] = [k_D] \quad (3-33)$$

When individual realizations of EI and AE appearing in Equation (3-30) are assumed uncorrelated then the elements of  $[k_R]$  are independent of one another. This assumption is made for ease of computation only. In fact the correlation functions between the terms of the system stiffness matrix will seldom be known in practice. The method developed in the following sections may be applied to cases where the terms in the stiffness matrix are correlated, provided the correlation functions are determined and stored for use at each time step. This becomes apparent in the formulation of the finite difference expressions and is discussed in detail in Section 3.4.

The system stiffness matrix,  $[K]$ , is formed by summing the contributions of each element for each degree of freedom. When the orientation of the system degrees of freedom differ by an angle  $\gamma$  from the element degrees of freedom the matrix defined by Equation (3-30) must be rotated to the proper coordinates. This rotation is accomplished by pre- and post-multiplying  $[k]$  by a transformation matrix  $[T]$ . When one considers an element oriented as in Figure 3-2 the stiffness matrix expresses the relations between forces and displacements in the X and Y direction.

The transformation matrix  $[T]$ , used to develop the rotated stiffness matrix,  $[k']$ , which expresses the relations between forces and displacements in the X' and Y' directions is,

$$[T] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 & 0 & 0 & 0 \\ -\sin\gamma & \cos\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma & \sin\gamma & 0 \\ 0 & 0 & 0 & -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-34)$$

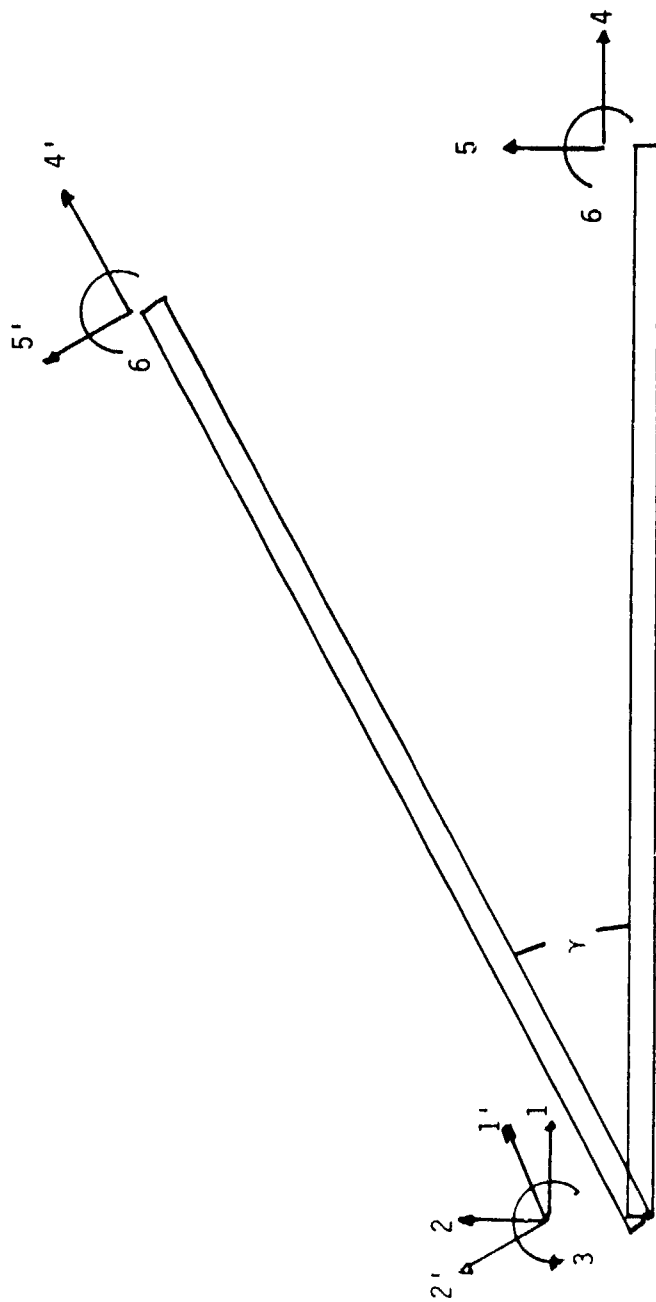


Figure 3-2 Rotation of Plane Frame Element Through an Angle  $\gamma$

And the rotated stiffness matrix is,

$$[k'] = [T]^T [k] [T] \quad (3-35)$$

Since  $[k]$  has a deterministic and random components  $[k']$  will as well.

Thus  $[k']$  may be expressed as,

$$[k'] = [k_R'] + [k_D'] \quad (3-36)$$

with the random component  $[k_R']$  given by,

$$[k_R'] = [T]^T [k_R] [T] \quad (3-37)$$

and the deterministic component  $[k_D']$  given by

$$[k_D'] = [T]^T [k_D] [T] \quad (3-38)$$

The expected value of  $[k']$  is

$$E[k'] = [k_D'] = [T]^T [k_D] [T] \quad (3-39)$$

The covariance of  $[k']$  is given by,

$$\text{Cov}[k'] = E[k_R'] [k_R']^T = E[k_R'] [k_R']^T \quad (3-40)$$

Replacing  $[k_R']$  with the Equation (3-27) yields

$$E[k_R'] [k_R']^T = E[T]^T [k_R] [T] [T]^T [k_R] [T] \quad (3-41)$$

However,  $[T]$  is orthogonal to  $[T]^T$  so  $[T][T]^T = [I]$  resulting in,

$$E[k_R'] [k_R']^T = E[T]^T [k_R] [k_R]^T [T] \quad (3-42)$$

And since  $[t]$  is deterministic

$$E[k_R'] [k_R']^T = [T]^T E([k_R] [k_R]^T) [T] \quad (3-43)$$

Resulting in

$$\text{Cov}[k'] [k']^T = [T]^T (\text{Cov}[k] [k]^T) [T] \quad (3-44)$$

However due to the manner in which  $[K_R]$  appears in the expressions developed in the following section the properties of interest are the variance of each term of  $[K]$  and hence each term of  $[k']$ . The  $ij^{\text{th}}$  component of  $[k']$  may be written using indicial notation as

$$k'_{ij} = T_{il}^T j_{lp} T_{pj} \quad (3-45)$$

Hence the  $\text{Var}(k_{ij})$  may be expressed as

$$\text{Var}(k'_{ij}) = (T_{il}^T)^2 \text{Var}(k_{lp}) (T_{pj})^2 \quad (3-46)$$

when the terms of  $k$  are independent as assumed earlier. When this is not the case the values of the covariances of  $(k_{ij} k_{lp})$  must be known. When the correlation between AE and EI is known these may be readily computed from Equation (3-30). However, there will be  $6^4$  of them. The final assembly of the global matrices would result in  $n^4$  terms of the form  $\text{Cov}(K_{ij} K_{lp})$ .

The system stiffness matrix is assembled by summing the components of the element stiffness matrices at each degree of freedom. Thus  $[K]$  may be written as,

$$[K] = \sum_{i=1}^{\ell} [k']_i \quad (3-47)$$

where  $[k']_i$  is the stiffness matrix for the  $i^{\text{th}}$  element, and  $\ell$  is the total number of elements.

Similar expressions for the random,  $[K_R]$  and deterministic,  $[K_D]$ , components of  $[K]$  are,

$$[K_R] = \sum_{i=1}^{\ell} [k_R']_i \quad (3-48)$$

And

$$[K_D] = \sum_{i=1}^{\ell} [k_D']_i \quad (3-49)$$

The variance of each term  $K_{ij}$  is the sum of the variances of  $k'_{mp}$  for each element where element degrees of freedoms  $m$  and  $p$  correspond to system degrees of freedom  $i$  and  $j$  respectively.

The mass matrix for a frame element may take one of several forms. The one chosen for use here is presented by Cook (33). The matrix is diagonal with no zero entries which makes it trivial to invert as well as offering several other advantages in forming the finite difference expressions. The terms of the mass matrix  $m$  are given by,

$$(m) = \frac{M}{420} \begin{pmatrix} 210 & 210 & L^2 & 210 & 210 & L^2 \end{pmatrix} \quad (3-50)$$

where  $M$  is the total mass of the element. When required  $(M)$  may be rotated as was  $[k]$  in Equation (3-35). This results in,

$$(m') = [T]^T (m) [T] \quad (3-51)$$

The system mass matrix,  $(M)$  will also be diagonal and is formed by summing the element matrices at appropriate degrees of freedom. Thus  $(M)$  may be expressed as

$$(M) = \sum_{i=1}^{\ell} (m')_i \quad (3-52)$$

Where  $(m)_i$  is the mass matrix of the  $i^{\text{th}}$  element and  $\ell$  is the total number of elements.

The system damping matrix is a linear combination of  $[K]$  and  $[M]$ . This formulation is called Rayleigh damping and results in,

$$[C] = \alpha[K] + \beta(M) \quad (3-53)$$

where  $\alpha$  and  $\beta$  are deterministic constants. Substituting Equations (3-20) and (3-13) for  $[C]$  and  $[K]$  above yields.

$$[C]_D + [C]_R = \alpha[K]_D + \alpha[K]_R + \beta(M) \quad (3-54)$$

Taking the expected value of Equation (3-54) yields

$$[C]_D = \alpha[K]_D + \beta(M) \quad (3-55)$$

and hence the random component of  $[C]$  is

$$[C]_R = \alpha[K]_R \quad (3-56)$$

Alpha and Beta may be chosen such that the damping of the system is a predetermined portion of critical at two separate frequencies. Considering the frequencies  $\omega_1$  and  $\omega_2$  with corresponding percentages of critical damping  $\epsilon_1$  and  $\epsilon_2$ ,  $\beta$  and  $\alpha$  are given by,

$$\alpha = \frac{2(\omega_2 \epsilon_2 - \omega_1 \epsilon_1)}{\omega_2^2 - \omega_1^2} \quad (3-57)$$

$$\beta = 2\omega_1 \epsilon_1 - \alpha \omega_1^2 \quad (3-58)$$

Once the system matrices,  $[K_D]$ ,  $[K_R]$ ,  $[C_D]$ ,  $[C_R]$ , and  $(M)$  have been obtained the forward difference method is used to compute the mean  $\{\mu(t)\}$ , and covariance  $\{X(t)\}\{X(t)\}^T$  of the response. The details of this computation are explained in the following section.

### 3.4 APPLICATION OF FINITE DIFFERENCES

The forward difference technique is used to express the equations of motion generated in Section 3.2. The resulting expressions are thus evaluated in a step-by-step manner to yield values for the mean and variance of the response over times

Let  $t_n = n\Delta t$ ,  $r = 0, 1, \dots$ ,  $\Delta t$  a "small value", be a discretized time variable. The vector forms of the difference expressions developed in Section 2.2 are,

$$\{f(t_n)\} = \{f_n\} \quad (3-59)$$

$$\{\dot{f}(t_n)\} = \frac{1}{\Delta t} (\{f_{n+1}\} - \{f_n\}) \quad (3-60)$$

$$\{\ddot{f}(t_n)\} = \frac{1}{\Delta t^2} (\{f_{n+1}\} - 2\{f_{n+1}\} + \{f_n\}) \quad (3-61)$$

To develop the relations required to evaluate  $\{u(t_n)\}$  and its derivatives in Equation (3-27). This yields,

$$\begin{aligned} (\overline{M}) (\{u_{n+2}\} - 2\{u_{n+1}\} + \{u_n\}) + [C_D] (\{u_{n+1}\} - \{u_n\}) \\ + [K_D]\{u_n\} = \{P_n\} - E([K_R]\{\delta_n\}) - E([C_R]\{\dot{\delta}_n\}) \end{aligned} \quad (3-62a)$$

where

$$(\overline{M}) = \frac{1}{\Delta t^2} (M) \quad (3-62b)$$

$$[C_D] = \frac{1}{\Delta t} [C_D] \quad (3-62c)$$

If the relationship between  $[K_R]$  and  $[C_R]$  as defined in Equation (3-56) is now imposed, Equation (3-62a) reduces to,

$$\begin{aligned} (\overline{M}) (\{u_{n+2} - 2u_{n+1} + u_n\}) + [\overline{C}_D] (\{u_{n+1} - u_n\}) + [K_D]\{u_n\} = \\ = \{P_n\} - E([K_R]\{\eta_n\}) \end{aligned} \quad (3-63a)$$

where

$$\{\eta_n\} = \{\delta_n\} + \alpha\{\dot{\delta}_n\} \quad (3-63b)$$

Solving Equation (3-63a) for  $\{u_{n+2}\}$  yields,

$$\{u_{n+1}\} = (\overline{M})^{-1} (\{P_n\} - E([K_R]\{\eta_n\}) + [A_1]\{u_{n+1}\} + [A_2]\{u_n\}) \quad (3-64a)$$

where

$$[A_1] = 2 (\overline{M}) - [\overline{C}_D] \quad (3-64b)$$

$$[A_2] = [C_D] - (\overline{M}) - [K_D] \quad (3-64c)$$

Replacing  $[C_D]$  with Equation (3-55) yields,

$$[A_1] = (2 - \Delta t \beta) (\overline{M}) - \left( \frac{\alpha}{\Delta t} \right) [K_D] \quad (3-65a)$$

$$[A_2] = (\beta \Delta t - 1) (\overline{M}) + \left( \frac{-\alpha}{\Delta t} \right) [K_D] \quad (3-65b)$$

Equation (3-64a) provides an expression by which  $\{u_{n+2}\}$  may be evaluated in a step-by-step manner similar to that used for the SDOF system discussed in Chapter 2. The values for  $\{u_{n+1}\}$  and  $\{u_n\}$  are available from previous time steps.

The expected value of  $[K_R]\{u_n\}$  may be developed from the expressions for  $\{\delta_n\}$  which now follows.

The process  $\{\delta(t)\}$  is implicitly defined by Equation (3-29). Applying the forward difference relations to  $\{\delta(t)\}$  and its derivatives results in,

$$\begin{aligned} (\overline{M}) (\{\delta_{n+2}\} - 2\{\delta_{n+1}\} + \{\delta_n\}) + [\overline{C}_D](\{\delta_{n+1}\} - \{\delta_n\}) + [K_D]\{\delta_n\} \\ = \{q_n\} - [K_R]\{Z_n\} \end{aligned} \quad (3-66a)$$

where 
$$\{Z_n\} = \{u_n\} + \alpha \{\dot{u}_n\} \quad (3-66b)$$

Solving Equation (3-65a) for  $\{\delta_{n+2}\}$  yields,

$$\{\delta_{n+2}\} = (\overline{M})^{-1} (\{q_n\} - [K_R]\{Z_n\} + [A_1]\{\delta_{n+1}\} + [A_2]\{\delta_n\}) \quad (3-67)$$

To evaluate the expected value of the product  $(K_R)_{ij}(\delta_{n+2})_k$  one writes the expression for  $(\delta_{n+2})_k$  by evaluating Equation (3-67) for its  $k_{th}$  element and multiplying by  $(K_R)_{ij}$  and taking the expected value. The equation for the  $k_{th}$  element of  $\{\delta_{n+2}\}$  is,

$$(\delta_{n+2})_k = (\bar{M})_{kk}^{-1} ((q_n)_k - (K_R)_{k\ell}(Z_n)_\ell + (A_1)_{kr}(\delta_{n+1})_r + (A_2)_{kp}(\delta_n)_p) \quad (3-68)$$

Premultiplying both sides by  $(K_R)_{ij}$  yields,

$$(K_R)_{ij}(\delta_{n+2})_k = (\bar{M})_{kk}^{-1} ((K_R)_{ij}(q_n)_k - (K_R)_{ij}(K_R)_{k\ell}(Z_n)_\ell + (K_R)_{ij}(A_1)_{kr}(\delta_{n+1})_r + (K_R)_{ij}(A_2)_{kp}(\delta_n)_p) \quad (3-69)$$

Taking the expected value of Equation (3-68) yields,

$$E((K_R)_{ij}(\delta_{n+2})_k) = (\bar{M})_{kk}^{-1} (E((K_R)_{ij}(q_n)_k) - E((K_R)_{ij}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{kr}E((K_R)_{ij}(\delta_{n+1})_r) + (A_2)_{kp}E((K_R)_{ij}(\delta_n)_p)) \quad (3-70)$$

And since  $[K_R]$  is independent of  $\{q_n\}$  Equation (3-70) reduces to,

$$E((K_R)_{ij}(\delta_{n+2})_k) = (\bar{M})_{kk}^{-1} (-E((K_R)_{ij}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{kr}E((K_R)_{ij}(\delta_{n+1})_r) + (A_2)_{kp}E((K_R)_{ij}(\delta_n)_p)) \quad (3-71)$$

Thus, the  $n^3$  terms of the  $E((K_R)_{ij}(\delta_{n+2})_k)$  may be evaluated in a step-by-step manner. However, the evaluation will require  $n^2$  calculations for each of the  $n^3$  terms. To reduce the number of computations a simplifying assumption may be imposed. The assumption is that the deflection at degree-of-freedom, (DOF),  $k$  is independent of the entries of the random stiffness matrix at dof's other than  $k$ .

This may be written

$$E((K_R)_{ij}(\delta_{n+2})_k) = 0 \quad \text{when } k \neq i \quad (3-72)$$

$$\text{or } k \neq j$$

When  $k = i$  the  $E((K_R)_{ij}(\delta_{n+2})_k)$  is equal to  $E((K_R)_{kj}(\delta_{n+2})_k)$  and is,

$$E((K_R)_{kj}(\delta_{n+2})_k) = (\bar{M})_{kk}^{-1} [-E((K_R)_{kj}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{kr}E((K_R)_{kj}(\delta_{n+1})_r) + (A_2)_{kp}E((K_R)_{kj}(\delta_n)_p)] \quad (3-73)$$

(no sum on  $k$  or  $j$ )

Due to the independence of the elements of  $[K_R]$  the expected value of  $(K_R)_{kj}(K_R)_{k\ell}$  is zero unless  $\ell = j$  then

$$E((K_R)_{kj}(K_R)_{j\ell}) = E(K_R)_{kj}^2 \quad (3-74)$$

Furthermore, by taking advantage of the assumption expressed in Equation (3-71) for the terms  $(K_R)_{ki}(\delta_{n+1})_r$  and  $(K_R)_{kj}(\delta_n)_p$ , Equation (3-73) may be rewritten as,

$$E((K_R)_{kj}(\delta_{n+2})_k) = (\bar{M})_{kk}^{-1} [-E(K_R)_{kj}^2(Z_n)_j + (A_1)_{kk}E((K_R)_{kj}(\delta_{n+1})_k) + (A_1)_{kj}E((K_R)_{kj}(\delta_{n+1})_j) + (A_2)_{kk}E((K_R)_{kj}(\delta_n)_k) + (A_2)_{kj}E((K_R)_{kj}(\delta_n)_j)] \quad (3-75)$$

(no sum on  $k$  or  $j$ )

When we consider the case of  $k=j$  Equation (3-71) may be written,

$$E((K_R)_{ij}(\delta_n)_j) = (\overline{M})_{jj}^{-1} [-E((K_R)_{ij}(K_R)_{j\ell})(Z_n)_\ell + (A_1)_{jr}E((K_R)_{ij}(\delta_{n+1})_r) + (A_2)_{jp}E((K_R)_{ij}(\delta_n)_p)] \quad (3-76)$$

Once again taking advantage of the independence of the elements of  $[K_R]$  and the assumption expressed in Equation (3-72), Equation (3-76) is reduced to

$$E((K_R)_{ij}(\delta_{n+2})_j) = (\overline{M})_{ij}^{-1} [-E((K_R)_{ij}(K_R)_{k\ell})(Z_n)_\ell + (A_1)_{ij}E((K_R)_{ij}(\delta_{n+1})_j) + (A_1)_{ji}E((K_R)_{ij}(\delta_{n+1})_i) + (A_2)_{jj}E((K_R)_{ij}(\delta_n)_i)] \quad (3-77)$$

no sum on  $j$  or  $i$

However, the expressions in Equations (3-75) and (3-77) allow the  $2n^2$  terms of  $(K_R)_{ij}(\delta_n)_k$  for  $k=j$  and  $k=i$  to be evaluated at each point in time. When  $i=j$ , Equation (3-71) with the assumption of Equation (3-71) yields,

$$E((K_R)_{ij}(\delta_{n+2})_i) = (\overline{M})_{ii}^{-1} [-E(K_R)_{ii}^2(Z_n)_i + (A_1)_{ii}E((K_R)_{ii}(\delta_{n+1})_i) + (A_2)_{ii}E((K_R)_{ii}(\delta_{n+1})_i)] \quad (3-78)$$

However, if one considers a slightly more restrictive assumption than that of Equation (3-72) the calculations may be reduced still further.

Consider the product  $(K_R)_{ij}(\delta_n)_j$ ; this is the component of the random portion of the static restoring force at DOF  $i$  due to a deflection

at DOFj. If one considers the product  $(K_R)_{ij}(\delta_n)_i$  there is no such clear physical meaning. One may therefore say that the terms involving  $(K_R)_{ij}(\delta_n)_j$  will be representative of the uncertainties in actual forces and are the terms of primary interest. Since the terms involving  $(K_R)_{ij}(\delta_n)_i$  have no such clear physical meaning it is assumed they will not have a significant effect on the solution of the problem. The assumption is made for computational convenience. The more liberal assumption of  $E[(K_R)_{ij}(\delta_n)_i]$  and  $E[(K_R)_{ij}(\delta_n)_j]$  are both not equal zero would be more accurate provided computation and storage capabilities exist to execute it. The means for doing so are provided by Equations (3-75) and (3-77). The full set of  $n^3$  terms,  $E[(K_R)_{ij}(\delta_n)_k]$  may also be assembled at each time step by Equation (3-71), provided once again that sufficient storage and computation capabilities are available.

Evaluating Equation (3-71) under the constraint,

$$E((K_R)_{ij}(\delta_n)_j) = 0 \quad k \neq j \quad (3-79)$$

yields,

$$\begin{aligned} E[(K_R)_{ij}(\delta_{n+2})_j] &= (\overline{M})_{jj} [-E(K_R)_{ij}(K_R)_{j\ell}](Z_n)_\ell \\ &+ (A_1)_{jr} E[(K_R)_{ij}(\delta_{n+1})_r] + (A_2)_{jp} E[(K_R)_{ij}(\delta_n)_p] \end{aligned} \quad (3-80)$$

Applying Equation (3-79) to the  $K_R \delta$  terms on the right hand side of Equation (3-80) yields,

$$\begin{aligned} E((K_R)_{ij}(\delta_{n+2})_j) &= (\overline{M})_{jj}^{-1} [-E((K_{R1j})(K_{RJ\ell}))](Z_n)_\ell \\ &+ (A_1)_{jj} E((K_R)_{1j}(\delta_{n+1})_j) + (A_2)_{jj} E((K_R)_{1j}(\delta_n)_j) \end{aligned} \quad (3-81)$$

and due to the independence of the terms of  $(K_R)$  when  $j \neq i$  this yields,

$$E((K_R)_{ij}(\delta_{n+2})_j) = (\overline{M})_{jj}^{-1} [(A_1)_{jj}E((K_R)_{ij}(\delta_{n+1})_j) + (A_1)_{jj}E((K_R)_{ij}(\delta_n)_j)] \quad (3-82)$$

And so when  $E(K_R)_{ij}(\delta_n)_j = 0$ , i.e. the case of zero start, Equation (3-82) results in,

$$E((K_R)_{ij}(\delta_{n+2})_j) = 0 \quad \text{all } n. \quad (3-83)$$

then  $i=j$  Equation (3-81) becomes,

$$E((K_R)_{ii}(\delta_{n+2})_i) = (\overline{M})_{ii}^{-1} [-E(K_{Rii})^2(Z_n)_i + (A_1)_{ii}E(K_{Rii})(\delta_{n+1})_i] + (A_2)_{ii}E((K_R)_{ii}(\delta_{n+1})_i) \quad (3-84)$$

Hence through the assumption of independent elements in the stiffness matrix and the constraint of Equation (3-79) the expression of Equation (3-71) involving  $n^3$  terms requiring a total of  $n^5$  ( $n^2$  per term), computations per time step is reduced to  $n$  terms involving a total of  $n$  computations per time step.

Equation (3-84) now allows one to evaluate Equation (3-64a) for the mean response since  $E[K_R]\{\zeta_n\}$  is a vector whose  $i^{\text{th}}$  term is given by

$$(E[K_R]\{\eta_n\})_i = \frac{\alpha E(K_R)_{ii}(\delta_{n+1})_i}{\Delta t} + E(K_R)_{ii}(\delta_n)_i(1 - \frac{\alpha}{\Delta t}) \quad (3-85)$$

To develop an expression for the variance of the response one first multiplies Equation (3-67), the expression for  $\{\delta_{n+2}\}$  by its transpose and then takes the expected value of the resulting expression. This results in,

$$\begin{aligned}
 E(\{\delta_{n+2}\}\{\delta_{n+2}\}^T) &= (\bar{M})^{-1} (E\{q_n\})\{q_n\}^T - [E_1]^T - [E_1] + [E_2] \\
 &+ [E_2]^T + [E_3] + [E_3]^T + E[K_R]\{Z_n\}\{Z_n\}^T[K_R]^T \\
 &- [E_4] - [E_4]^T - [E_5] - [E_5]^T + E[A_1]\{\delta_{n+1}\}\{\delta_{n+1}\}^T[A_1]^T + [E_6]^T + [E_6] \\
 &+ E[A_2]\{\delta_n\}\{\delta_n\}^T[A_2] (\bar{M})^{-1}
 \end{aligned} \tag{3-86a}$$

$$\text{where} \quad [E_1] = E(\{q_n\}\{Z_n\}^T[K_R]^T) \tag{3-86b}$$

$$[E_2] = E(\{q_n\}\{\delta_n\}^T[A_2]^T) \tag{3-86c}$$

$$[E_3] = E(\{q_n\}\{\delta_{n+1}\}^T[A_1]^T) \tag{3-86d}$$

$$[E_4] = E([K_R]\{Z_n\}\{\delta_{n+1}\}^T[A_1]^T) \tag{3-86e}$$

$$[E_5] = E([K_R]\{Z_n\}\{\delta_n\}^T[A_2]^T) \tag{3-86f}$$

$$[E_6] = E([A_1]\{\delta_{n+1}\}\{\delta_n\}^T[A_2]^T) \tag{3-86g}$$

Since the loading and stiffness are independent  $[E_1]$  may be written,

$$[E_1] = E\{q_n\}\{Z_n\}^T E[K_R]^T \quad (3-87)$$

And since  $\{q_n\}$  and  $[K_R]$  are mean zero,

$$[E_1] = [0] \quad (3-88)$$

Upon examination of Equation (3-67) one notes that  $\{\delta\}_n$  is independent of  $\{q_{n-1}\}$  and  $\{q_n\}$ . Hence  $[E_2]$  and  $[E_3]$  may be written as

$$[E_2] = E\{q_n\}E\{\delta_n\}^T [A_2]^T \quad (3-89)$$

$$[E_3] = E\{q_n\}E\{\delta_{n+1}\}^T [A_1]^T \quad (3-90)$$

And since  $\{q_n\}$ ,  $\{\delta_n\}$  and  $\{\delta_{n+1}\}^T$  are all mean zero,

$$[E_2] = [0] \quad (3-91)$$

and

$$[E_3] = [0] \quad (3-92)$$

To evaluate  $[E_4]$  one may write out the expression for its in  $ij^{th}$  term,

$$(E_4)_{ij} = E((K_R)_{il}(Z_n)_l(\delta_{n+1})_p(A_1^T)_{pj}) \quad (3-93)$$

However, from Equation (3-84) this reduces to,

$$(E_4)_{ij} = E((K_R)_{ii}(\delta_{n+1}))Z_{ni}A_{1ij} \quad (3-94)$$

Similarly,

$$(E_5)_{ij} = E((K_R)_{ii}(\delta_{n+1}))Z_{ni}A_2^T A_{1ij} \quad (3-95)$$

The matrix  $[E_6]$  may be written as

$$[E_6] = [A_1]E(\{\delta_{n+1}\}\{\delta_n\}^T)[A_2]^T \quad (3-96)$$

Note that  $\{\delta_{n+1}\}$  may be written

$$\{\delta_{n+1}\} = (\overline{M})^{-1}(\{q_{n-1}\} - [K_R]\{Z_{n-1}\} + [A_1]\{\delta_n\} + [A_2]\{\delta_{n-1}\}) \quad (3-97)$$

Postmultiplying the above by  $\{\delta_n\}^T$  yields,

$$\begin{aligned} \{\delta_{n+1}\}\{\delta_n\}^T &= (\overline{M})^{-1}(\{q_{n-1}\}\{\delta_n\}^T - [K_R]\{Z_{n-1}\}\{\delta_n\}^T + [A_1]\{\delta_n\}\{\delta_n\}^T \\ &\quad + [A_2]\{\delta_{n-1}\}\{\delta_n\}^T) \end{aligned} \quad (3-98)$$

Taking the expected value yields,

$$\begin{aligned} E\{\delta_{n+1}\}\{\delta_n\}^T &= (\overline{M})^{-1}[0 - E[K_R]\{Z_{n-1}\}\{\delta_n\}^T + [A_1]E\{\delta_n\}\{\delta_n\}^T \\ &\quad + A_2(E\{\delta_n\}\{\delta_{n-1}\}^T)^T] \end{aligned} \quad (3-99)$$

The expression  $(E\{\delta_n\}\{\delta_{n-1}\}^T)^T$  is merely the transpose of the left hand side evaluated at the previous time step. The  $E\{[K_R]\{Z_{n-1}\}\{\delta_n\}^T\}$  is a diagonal matrix whose entries are,

$$(E[K_R]\{Z_{n-1}\}\{\delta_n\}^T)_{ii} = (Z_{n-1})_i E((K_R)(\delta_n)_i) \quad (3-100)$$

The expression  $E[K_R]\{Z_n\}\{Z_n\}^T[K_R]^T$  may be written in tensor notation as

$$E((K_R)_{i\ell}(Z_n)_\ell(Z_n)_p(K_R^T)_{pj}) = (Z_n)_\ell(Z_n)_p E(K_{Ri\ell}K_{Rpj}^T) \quad (3-101)$$

And since  $E(K_{Ri\ell}K_{Rpj}^T) = 0$  unless  $j=i$ ,  $p=\ell$ , Equation (3-101) reduces to a diagonal matrix with entries given by

$$E[K_R]\{Z_n\}\{Z_n\}^T[K_R]^T = E(K_{Rij})^2 Z_{nj}^2$$

Equation (3-86c) may now be evaluated at each time step yielding an expression for the  $\text{Cov}(\{X(t)\}\{X(t)\}^T)$  at each time step.

The numerical algorithm for performing the computation is presented in Section 3.5. The computation provides the expected value of the displacement response at each degree of freedom and the covariance between the responses at various degrees of freedom.

### 3.5 THE MULTIPLE DEGREE OF FREEDOM COMPUTATIONAL ALGORITHM

The following algorithm is used to evaluate the mean response,  $\{\mu(t)\}$ , and the covariance of  $\{X(t)\}$  using the forward difference equations developed in Section 3.4. The algorithm is implemented in the code RMDOF. A listing of the code is provided in Appendix B.

The algorithm is based on a zero start condition and proceeds as follows,

1. Read the problem description parameters, the number of nodes, the number of elements and the number of restraints.
2. Read the nodal locations.
3. For each element, read the nodes to which it is connected, the material properties, and their uncertainties. Formulate the element stiffness matrix and compute the variance of each term in the stiffness matrix. Sum these into the global matrices. Formulate the element mass matrix and sum the results into the global mass matrix.
4. Read the damping parameters,  $\omega_1, \epsilon_1, \omega_2, \epsilon_2$  and calculate  $\alpha$  and  $\beta$  from Equations (3-57) and (3-58).
5. Formulate the problem solution matrices  $[A_1]$  and  $[A_2]$  defined in Equations (3-65a) and (3-65b).
6. Impose the constraint conditions.
7. For each time step do the following:
  - a) Update time and the index on the arrays  $\{\mu_{n+1}\}$ ,  
 $\{\mu_n\}$ ,  $E[\{\delta_{n+1}\}\{\delta_{n+1}\}^T]$ ,  $E[\{\delta_n\}\{\delta_n\}^T]$ ,  
 $E[\{\delta_{n+1}\}, \{\delta_n\}^T]$ ,  $E[[K_R]\{\delta_n\}]$ ,  
 $E[[K_R]\{\delta_{n+1}\}]$ .
  - b) Form  $E[[K_R]\{\delta_{n+2}\}]$  from Equation (3-85).
  - c) Compute  $E[\{\delta_{n+1}\}\{\delta_n\}^T]$  from Equation (3-98).

- d) Compute  $\{u_{n+2}\}$  from Equation (3-64a).
- e) Compute  $E[\{\delta_{n+2}\}\{\delta_{n+2}\}^T]$  from Equation (3-86a).
- f) Write appropriate results.

8. Repeat 7 until the problem is completed.

### 3.6 NUMERICAL EXAMPLES

This section presents the results of two example problems which illustrate the analysis of MDOF systems with random stiffness and damping. In both examples the random portion of any element of the global stiffness matrix,  $[K]$ , was assumed to be perfectly correlated with the random portion of the corresponding element in the damping matrix.

This assumption of independent terms has a serious shortcoming in that a rigid body displacement of any element will contribute to the variances of the displacements.

The random properties of each element of the stiffness and damping matrices were assumed independent of all other entries of these matrices.

These assumptions were made for reasons of computational convenience only. When one is modeling simple structures such as homogeneous bars and beams where the random properties are assumed highly correlated along their length, these assumptions may not be valid.

However, in the examples below the elements are considered to be condensed representations of many beams, columns and girders. For example, if one were to model a large building the axial and flexural properties of several bays and stories may be combined and represented by a single frame element. Thus a large structure may be modeled in an economical manner. Such subassembly modeling is common for large structures including aircraft, ships, and dams as well as buildings.

### EXAMPLE 1 - Deflection of a Non-Prismatic Cantilever Beam.

The structure considered here is shown in Figure 3-3. All dimensions are unitless. The structure possesses six degrees of freedom. However, the discussion here will concern the midpoint and tip deflection. The mean and standard deviation of the tip deflection are plotted in Figure 3-4. The mean and standard deviation of the midpoint deflection are plotted in Figure 3-5.

The correlation coefficient between the two is plotted in Figure 3-6.

The mean and standard deviations for this problem follow the same basic trend as that observed for the SDOF examples, namely that the standard deviation curve has the same basic shape as the mean curve. However, the standard deviation curves do not lag behind the mean curves as in the SDOF cases. Furthermore the standard deviation does not display the secondary peaks which were indicated in the SDOF case.

The ratio between the midpoint and tip deflections appear to be about a factor of  $1/8$ . The standard deviation at the tip is approximately  $1/3$  the value of the mean. The standard deviation at the midpoint on the other hand is approximately  $1/2$  the mean. The coefficient of variation of the stiffness and damping were  $0.15$ . Since more elements contributed to the stiffness at the midpoint, and the stiffness of the elements is independent, it is not surprising that the midpoint has a larger coefficient of variation.

The coefficient of variation at the tip, however, is still approximately double, ( $1/3$  vs  $0.15$ ) what would be expected based on the results of the SDOF computations. However the assumption of independence of terms within the stiffness matrix, means only positive terms, specifically those of Equation (3-102) enter the computations. If the terms

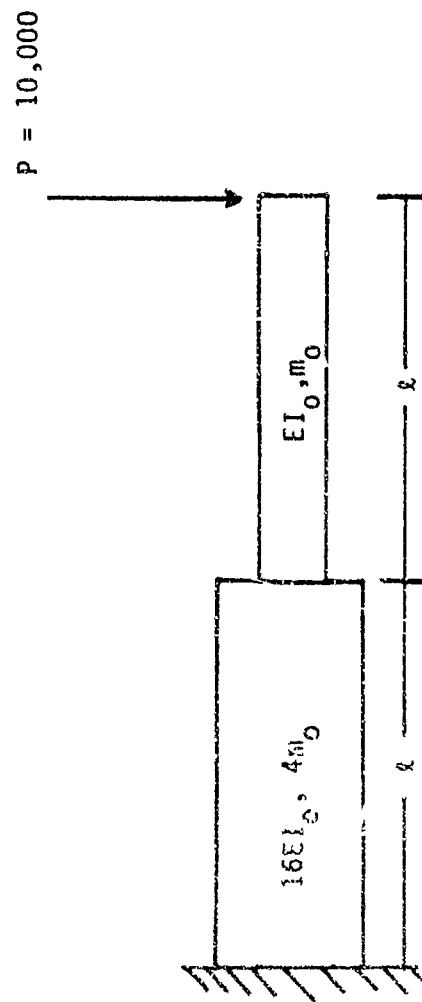


Figure 3-3 Cantilever Beam of Example 1

$$l = 20 \quad EI_0 = 3.867 \times 10^7 \quad m_0 = 2.938 \times 10^{-3}$$

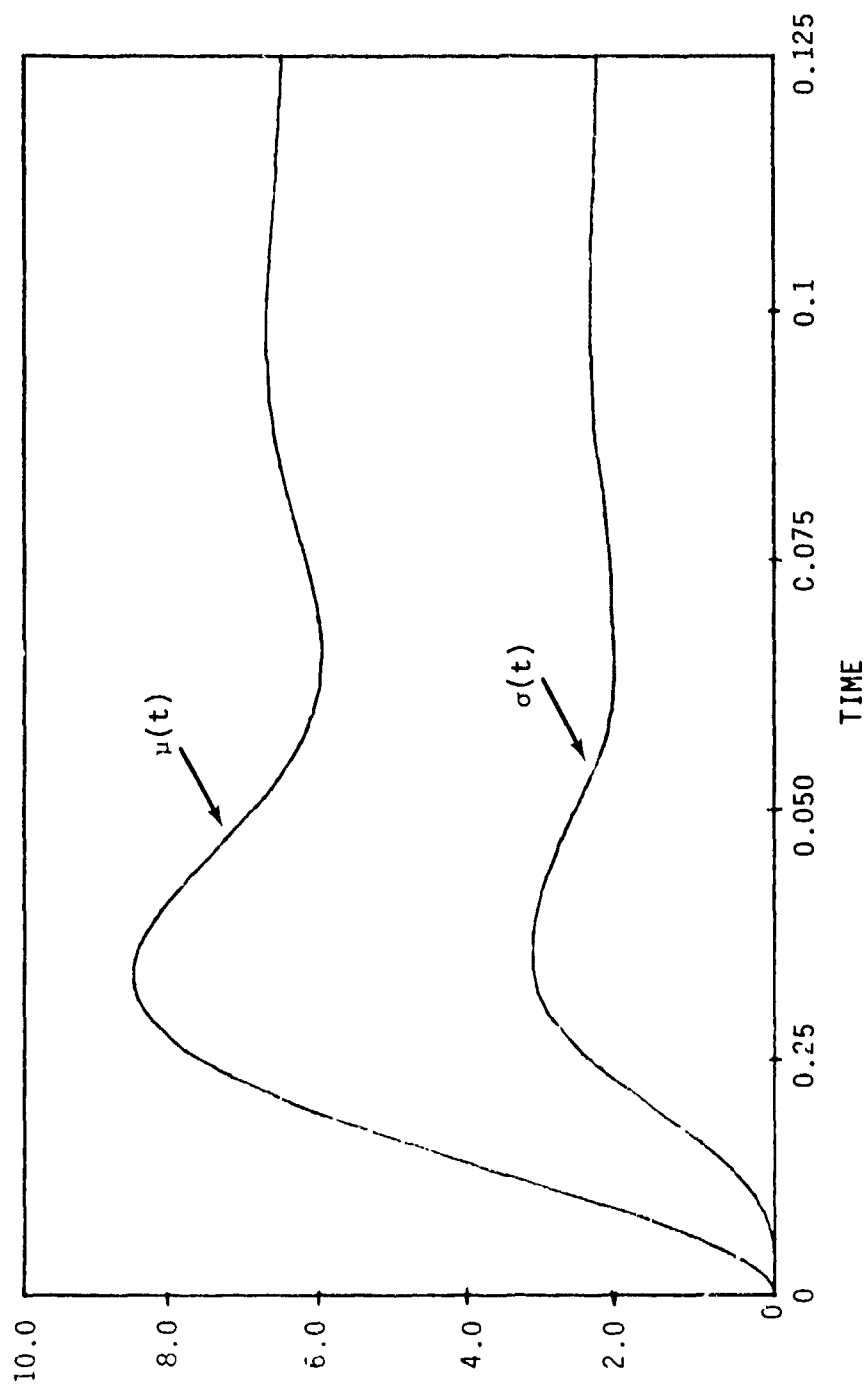


Figure 3-4  $\mu(t)$  and  $\sigma(t)$  for Tip Deflection, Example 1

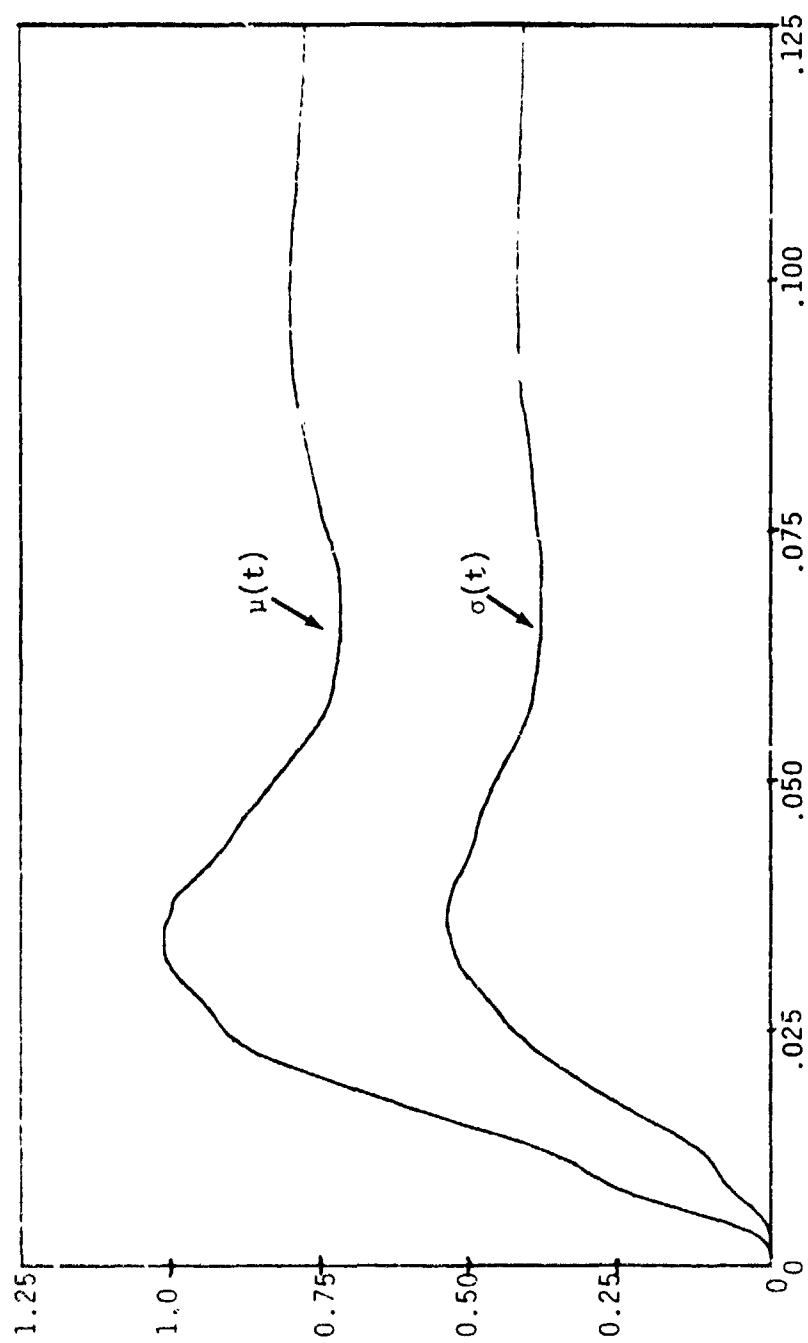


Figure 3-5  $u(t)$  and  $\sigma(t)$  for Midpoint Deflection, Example 1

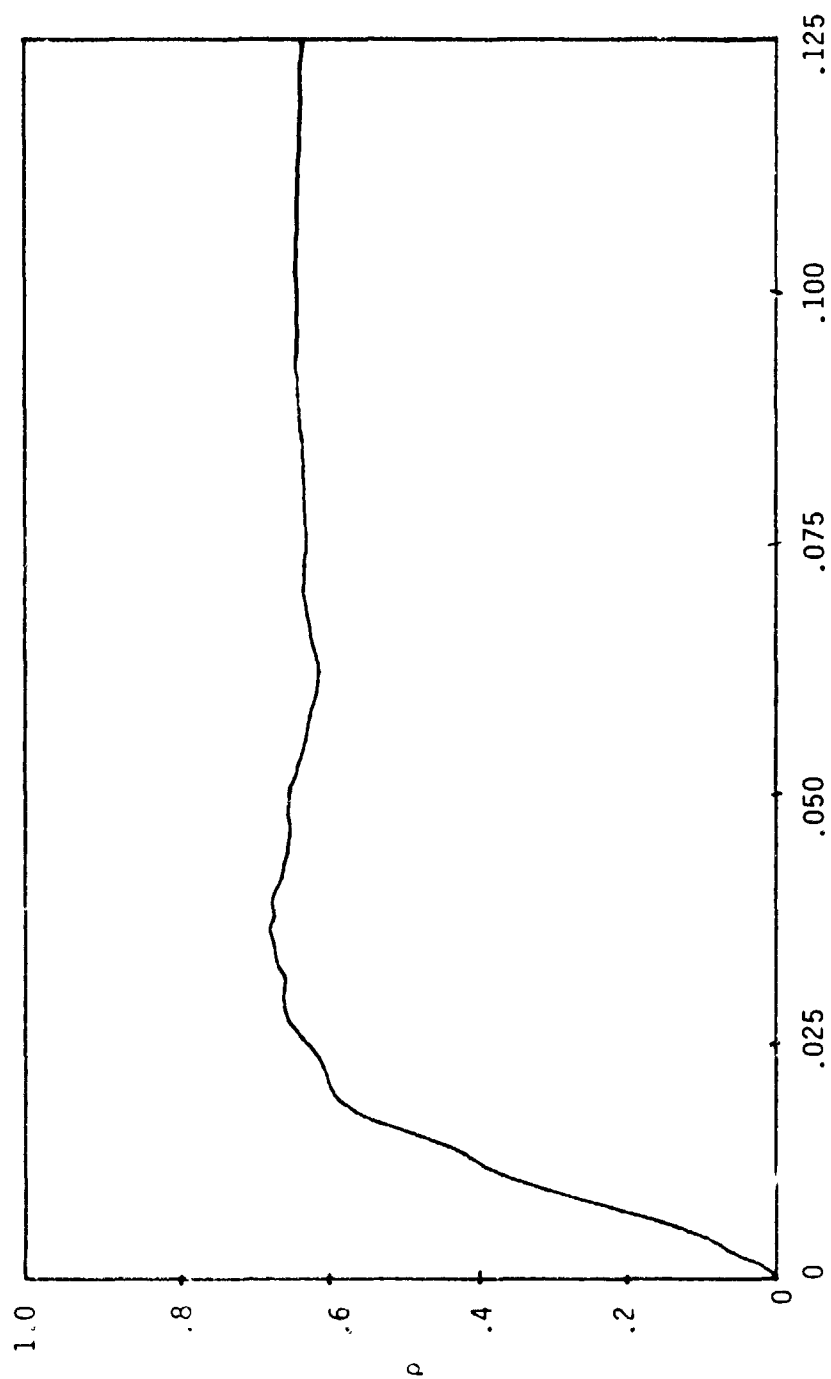


Figure 3-6 Correlation Coefficient Between Deflections at Tip and Midpoint

of the stiffness matrix were to be considered perfectly correlated then negative terms would enter the computation as well, reducing the overall variance. The SDOF approximation to this same problem would in effect assume perfect correlation between the random variables, thus the results would not be comparable.

The correlation between the tip and midpoint deflections approaches a constant value of 0.615. The correlation remains positive since the deflections always are in phase, and tied to one another in that the deflected shape of the structure is well defined for this particular problem.

## EXAMPLE 2 - Axial Response of a Prismatic Bar.

The structure of interest for this problem is shown in Figure 3-7. All dimensions are unitless. The response parameters of interest were the midpoint and tip deflections. Two loading conditions were investigated, the first being a loading applied suddenly to the midpoint and the second being a loading applied suddenly to the end. The loading in both cases had a value of .1.

For the case of the loading at the midpoint the mean and standard deviation for the deflection at the tip are plotted in Figure 3-8. The mean and standard deviation for the midpoint are plotted in Figure 3-9, and the correlation between them is plotted in Figure 3-10. The same information for loading at the tip is plotted in Figures 3-11, 3-12, and 3-13.

Examination of Figures 3-8 and 3-9 indicate that the standard deviation of the tip deflection is significantly higher than that at the midpoint even though the deflection are nearing their final values. The increase in the tip deflection's standard deviation is due to the assumption of independent entries of the stiffness matrix. Hence while the end element has undergone only a rigid body deformation once the oscillations are damped out this has still contributed to the final standard deviations.

The resulting means are oscillating about a point which is approximately 7% greater than their deterministic static deflection. This is due to the impact of the uncertainties on the mean response.

The correlation coefficient for both cases is approaching a constant value of 0.5. The standard deviations for the case of the end

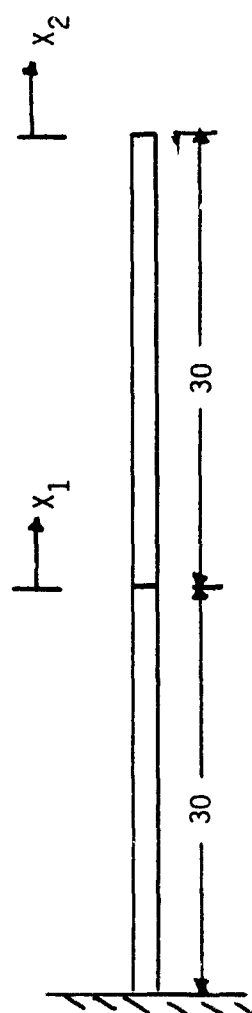


Figure 3-7 Homogeneous Bar of Examples 2 and 3  
 $AE = 60$

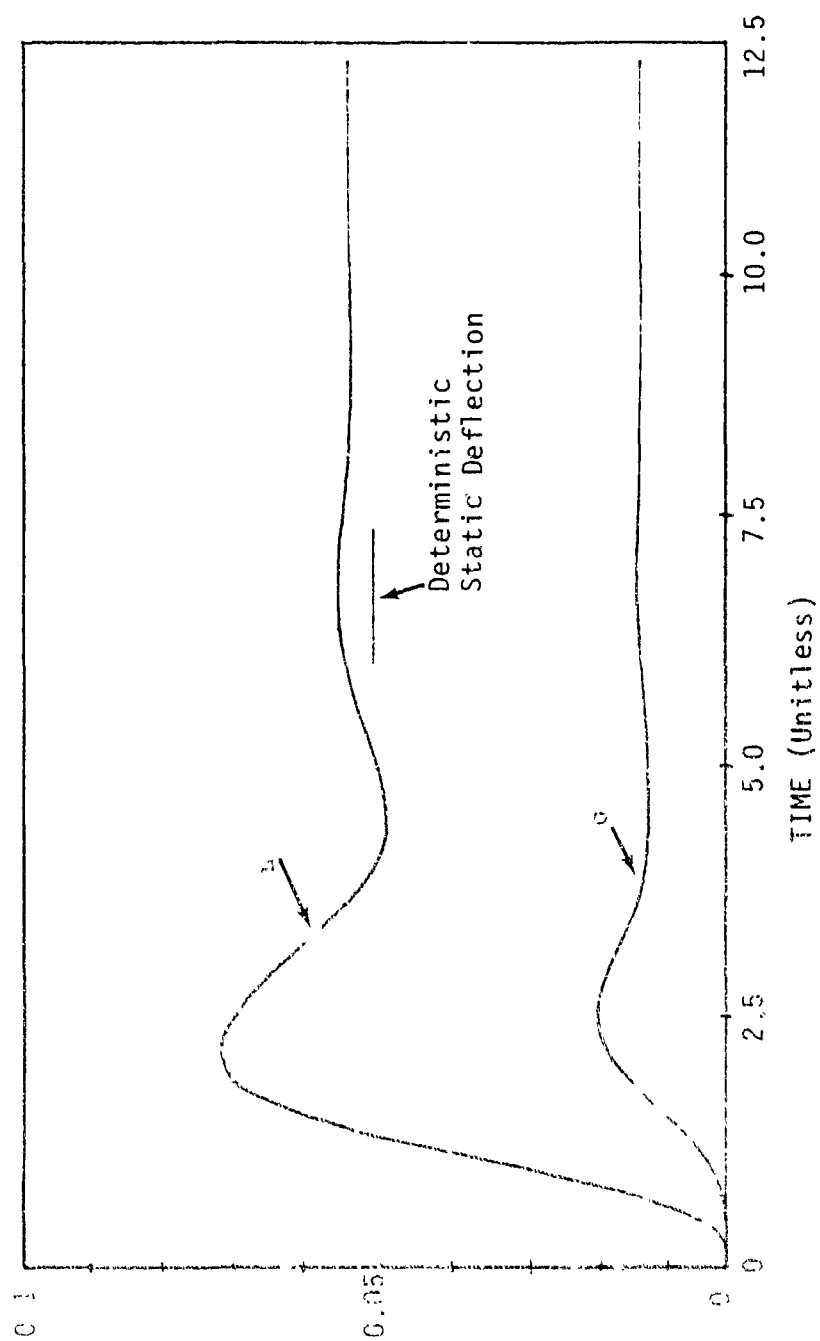


Figure 3-8 Tip Deflection for Bar with Loading Applied at Midpoint

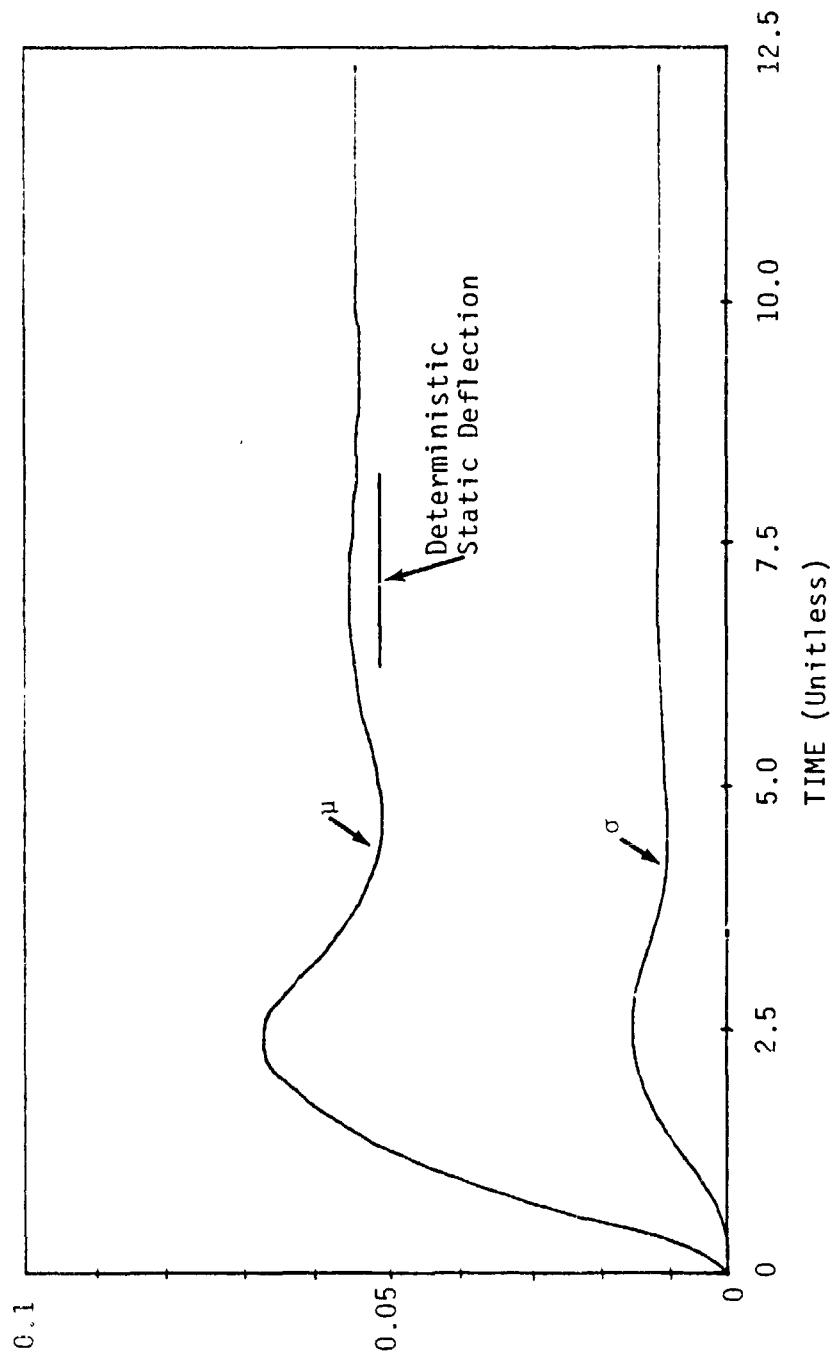


Figure 3-9  $\mu$  and  $\sigma$  for Midpoint Deflection of Bar with Loading Applied at Midpoint

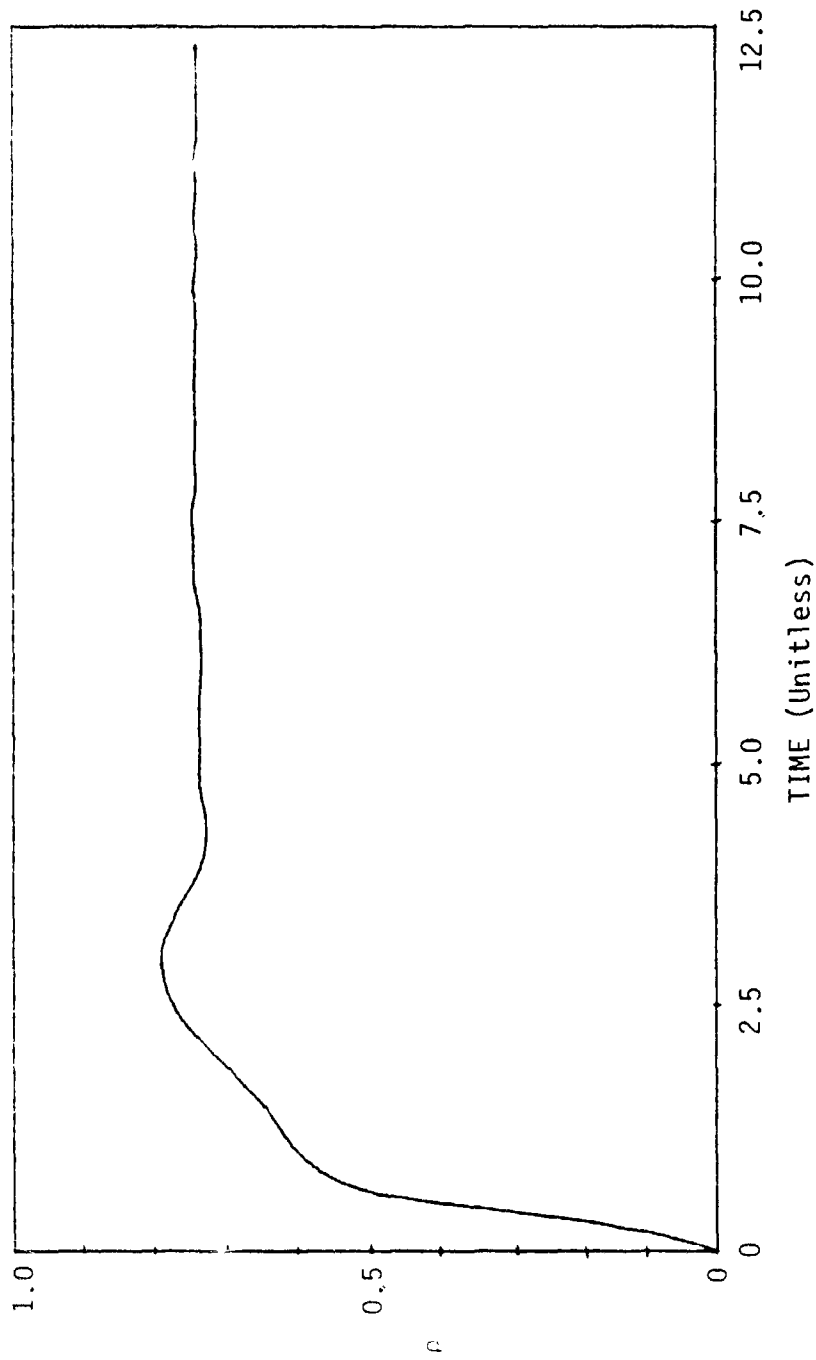


Figure 3-10 Correlation Between the Midpoint and Tip Deflections for a Bar With Loading Applied at the Midpoint

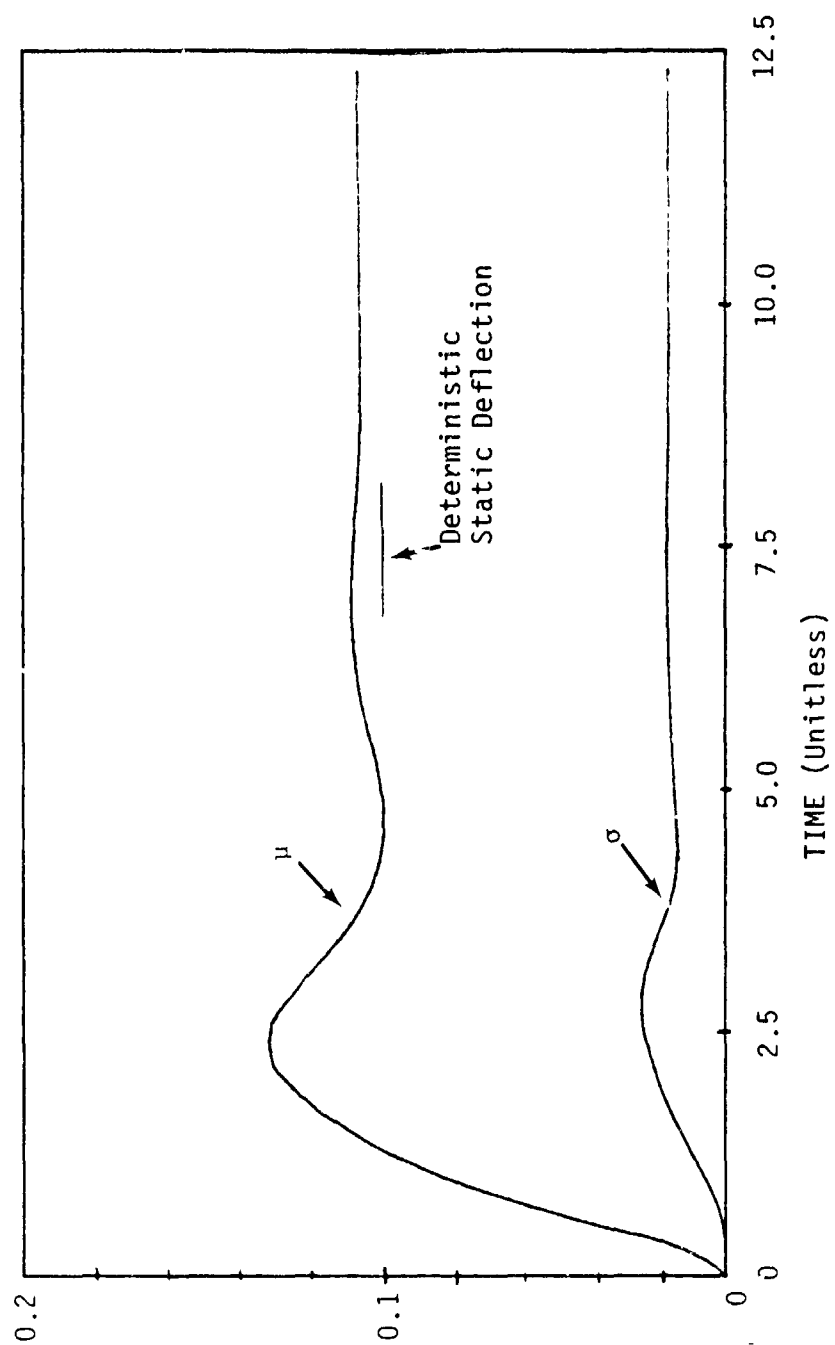


Figure 3-11  $\mu$  and  $\sigma$  of Tip Deflection of Bar With Loading Applied at the End

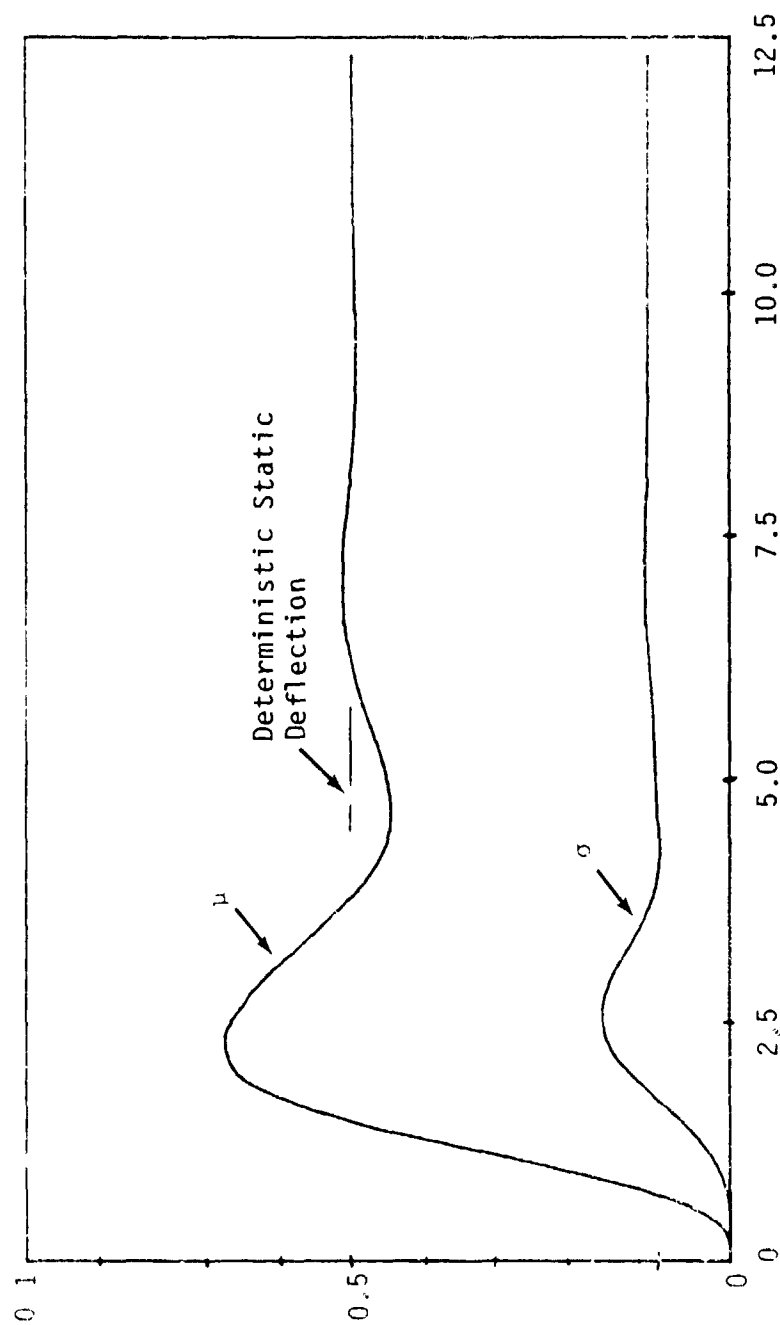


Figure 3-12  $\mu$  and  $\sigma$  for Midpoint Deflection of Bar With Loading Applied at the End

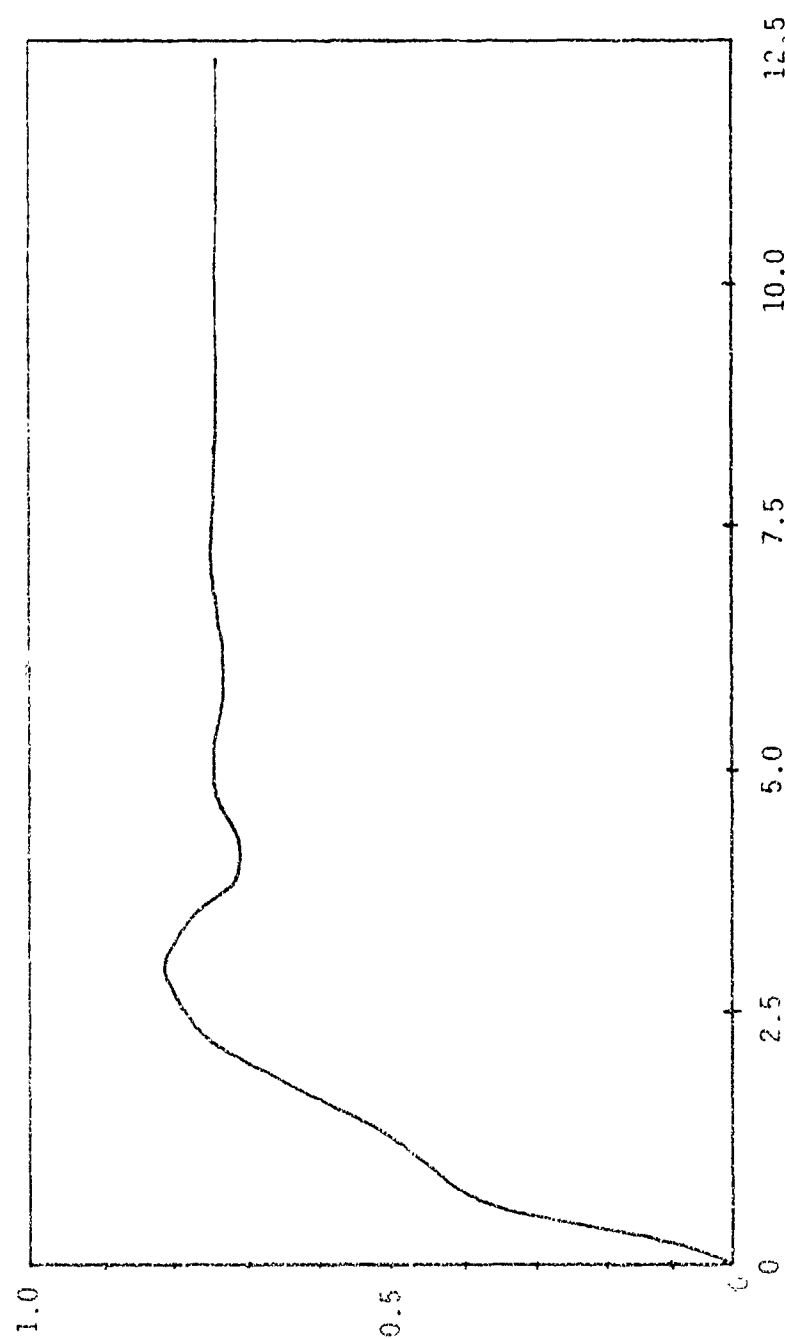


Figure 3-13 Correlation Between the Midpoint and End Deflections for a Bar With the Loading Applied at the End

loading are higher than those for the midpoint loading. This is due to larger deflections on the second half of the bar for the case of an end loading. The mean value of the midpoint deflection is also greater for the case of an end loading than the case of a midpoint loading, but by only a few percent. This is due once again to the effect of the randomness of the system parameters on the mean values.

## CHAPTER 4

### PASSAGE PROBABILITIES FOR NONSTATIONARY RANDOM PROCESSES

#### 4.1 INTRODUCTION

This chapter presents a new method for estimating the probability of passage beyond a barrier level,  $b$ , by the nonstationary random process  $X(t)$ .

When the barrier  $b$  represents a predetermined failure level then the probability of passing beyond this barrier is identical to the probability of failure.

The method of computing the passage probabilities for the nonstationary random process,  $X(t)$ , presented in this chapter is a new extension of existing methods used for stationary processes. The extension involves the application of conditional probabilities and the assumption that the process  $X(t)$  appears stationary during short time intervals.

The calculation of passage probabilities for stationary processes is discussed in Section 4.2. The extension of this to nonstationary processes is presented in Section 4.3 with the computational algorithm for implementing the new method presented in Section 4.4. Numerical examples of the application of the technique are provided in Section 4.5.

#### 4.2 PASSAGE PROBABILITIES FOR STATIONARY RANDOM PROCESSES

The methodology for approximating the passage probabilities for stationary random processes has been well developed by Rice (21) and others. The method developed by Rice is summarized below.

If one considers the random process  $X(t)$  to be weakly stationary then its mean,  $\mu_X(t)$ , and variance,  $\sigma_X^2(t)$ , are constant values for all values of  $t$ . Furthermore the process defining its derivative with regard to  $t$ ,  $\dot{X}(t)$ , is independent of  $X(t)$ .

The characteristic of interest of  $X(t)$  is the collection of its random peaks which fall above a predetermined level,  $b$ , where  $b > \mu_X$ . Let  $N(b, \tau)$  be a counting process associated with the crossing of the barrier level  $b$ . Thus every crossing of  $b$  by  $X(t)$  is counted by  $N(b, t)$ . Specifically it is assumed that  $N(b, t)$  is a Poisson counting process.

The events counted by a Poisson counting process, in this case the crossings of  $b$  by  $X(t)$ , are assumed to have the following properties:

- a) The events occur independently, hence knowing an event has just occurred provides no information regarding the probability of when the next event will occur,
- b) The probability of one event occurring in a time interval  $dt$  is  $\lambda_b(t)dt$  where  $dt \ll 1$ , and  $\lambda(t)_b$  is the mean crossing rate. When  $X(t)$  is stationary  $\lambda_b(t) = \lambda_b$  a constant.
- c) The probability of more than one event occurring in a time interval  $dt$  is negligible.

Assumption (a) above is arbitrary for structural response since the realizations of the process  $X(t)$  are not independent. When  $X(t)$  is at or near a barrier level then  $X(t+dt)$  will be as well for  $dt \ll 1$ . This effect is called "clumping" and is addressed by Vanmarke (4) and others. However, this effect was neglected by Rice and will be neglected in the following sections as well.

The Poisson process,  $N(b,t)$ , has the following probability function,

$$P_N(j,t) = \exp(-\lambda_b t) \frac{(\lambda_b t)^j}{j!} \quad (4-1)$$

where  $j$  is the number of events counted, that is the number of times  $X(t)$  crosses the barrier level  $b$ .

The probability of not crossing the barrier in the interval  $(0,t)$  is provided by Equation (4-1) with  $j=0$ . This yields,

$$P_N(0,t) = \exp(-\lambda_b t) \quad (4-2)$$

Let  $P_N(F,t)$  be the probability of crossing the barrier level at least once in the increment  $(0,t)$ , then,

$$P_N(F,t) = 1 - \exp(-\lambda_b t) \quad (4-3)$$

All that now remains is to compute the mean crossing rate,  $\lambda_b$ . The mean crossing rate at any time  $(t)$ ,  $\lambda_b(t)$ , is given by,

$$\lambda_b(t) = \int_{-\infty}^{\infty} |\dot{x}| p_{X\dot{X}}(b, \dot{x}, t) d\dot{x} \quad (4-4)$$

where  $p_{X\dot{X}}(b, x, t)$  is the joint probability density function of  $X(t)$  and  $\dot{X}(t)$  evaluated at  $X(t) = b$ . Since the random processes  $X(t)$  and  $\dot{X}(t)$  have been assumed stationary,  $\lambda_b(t) = \lambda_b$ , and Equation (4-4) may be reduced to,

$$\lambda_b = \int_{-\infty}^{\infty} |\dot{x}| p_{X\dot{X}}(b, \dot{x}) d\dot{x}. \quad (4-5)$$

When  $X(t)$ ,  $\dot{X}(t)$  are independent, mean zero, normal random processes their joint density is given by,

$$P_{s_{X\dot{X}}}(x, \dot{x}) = \frac{1}{2\pi \sigma_X \sigma_{\dot{X}}} \exp \left( -\frac{x^2}{2\sigma_X^2} - \frac{\dot{x}^2}{2\sigma_{\dot{X}}^2} \right) \quad (4-6)$$

Substituting this expression into Equation (4-5) yields,

$$\lambda_b = \frac{1}{\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \exp \left( -\frac{b^2}{2\sigma_X^2} \right) \quad (4-7)$$

Equation (4-7) now defines the mean crossing rate of a barrier  $b$  by the stationary mean zero random process  $X(t)$ . However, this rate reflects crossing the barrier from both above and below and since the engineer is only concerned with crossings from below the crossing rate must be modified. Fortunately this is straightforward since for a stationary random process all crossings from below must be paired with a crossing from above. Hence the mean crossing rate from below,  $\lambda_b^+$  is merely half the total mean crossing rate,

$$\lambda_b^+ = \frac{1}{2\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \exp \left( -\frac{b^2}{2\sigma_X^2} \right) \quad (4-8)$$

The resulting probability of crossing the barrier level  $b$  from below within the increment  $(0, t)$  is then

$$P_N^+(F, t) = 1 - \exp(-\lambda_b^+ t) \quad (4-9)$$

These results may now be expanded to include the present case of interest, the nonstationary response of a structure with random stiffness and damping.

### 4.3 THE NONSTATIONARY CASE

A new method for computing the probability of having crossed a barrier,  $b$ , at least once in the interval  $(0, t)$  for a nonstationary process will now be presented. The barrier  $b$  is greater than the largest values of  $\mu_X(t)$ . When  $\mu_X(t)$  is greater than  $b$ , the probability of passage is assumed to be 1.

The method is formulated under the following assumptions,

- a) The nonstationary random process  $X(t)$  and  $\dot{X}(t)$  are continuous. Hence their values may not instantaneously jump at any time. This is completely appropriate for processes defining structural displacement and velocity response under most loadings of interest, but excludes processes such as Brownian motion and quite possibly buckling responses.
- b) The means and variances of  $X(t)$  and  $\dot{X}(t)$  vary only slightly over increments of time,  $\Delta t$ , which are small in comparison to the response time of the structure.
- c)  $X(t)$  and  $\dot{X}(t)$  are correlated normal random processes whose means,  $\mu_X(t)$ ,  $\mu_{\dot{X}}(t)$ , standard deviations,  $\sigma_X(t)$ ,  $\sigma_{\dot{X}}(t)$  and correlation coefficient  $r(t)$  are known.

Such a function obviously does not allow the direct use of Equation (4-9) to calculate the probability of passage beyond a barrier,  $b$ . Equation (4-4) will still provide the mean crossing rate at any time  $t$ . However, consider the behavior of  $X(t)$  over a short period of time. For example the interval  $(\tau, \tau + \Delta t)$ . Then as  $\Delta t$  becomes small,

$$\mu_X(\tau) \approx \mu_X(\tau + \Delta t) \quad (4-10)$$

$$\text{and} \quad \sigma_X(\tau) \approx \sigma_X(\tau + \Delta t) \quad (4-11)$$

Considering the approximations of Equations (4-10) and (4-11) the mean and standard deviation of  $X(t)$  within the increment  $(\tau, \tau+\Delta t)$  may be approximated by,

$$\mu_X(t) \approx \mu_X(\tau) \quad \tau \leq t \leq \tau+\Delta t \quad (4-12)$$

$$\sigma_X(t) \approx \sigma_X(\tau) \quad \tau \leq t \leq \tau+\Delta t \quad (4-13)$$

The values for  $\mu_X(\tau+\Delta t)$  and  $\sigma_X(\tau+\Delta t)$  could have been used for the right hand sides of Equations (4-12) and (4-13). However, this would only result in the final results being shifted to the left on the time axis by an amount  $\Delta t$ .

The similar expressions for the process  $\dot{X}(t)$  are,

$$\mu_{\dot{X}}(t) \approx \mu_{\dot{X}}(\tau) \quad \tau \leq t \leq \tau+\Delta t \quad (4-14)$$

and

$$\sigma_{\dot{X}}(t) \approx \sigma_{\dot{X}}(\tau) \quad \tau \leq t \leq \tau+\Delta t \quad (4-15)$$

The approximations of Equations (4-12) through (4-15) will have the largest error when the response begins, that is as the structure moves away from the deterministic zero start condition. However, since the probability of exceeding a barrier in a time  $\Delta t$  when starting from a zero position is approximately zero, the errors occurring at early times will in effect be "laundered out" by the computations.

Consider the random processes  $X(t)$  and  $\dot{X}(t)$ , with the properties outlined above. The mean upcrossing rate of a barrier  $b$  at an instant of time,  $\tau$ , is then

$$\lambda_D^+(\tau) = \int_{-\infty}^{\infty} \dot{x} P_{X\dot{X},\tau}(b, \dot{x}) d\dot{x} \quad (4-16)$$

where  $P_{\dot{X}\dot{X},\tau}(b,\dot{x})dx$  is,

$$P_{\dot{X}\dot{X},\tau}(b,x) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}\rho}} \exp \left\{ -\frac{1}{2\rho^2} \left[ \left( \frac{b-\mu_X}{\sigma_X} \right)^2 - 2r \left( \frac{b-\mu_X}{\sigma_X} \right) \left( \frac{\dot{x}-\mu_{\dot{X}}}{\sigma_{\dot{X}}} \right) + \left( \frac{\dot{x}-\mu_{\dot{X}}}{\sigma_{\dot{X}}} \right)^2 \right] \right\} \quad (4-17a)$$

and  $\mu_X = \mu_X(\tau) \quad (4-17b)$

$$\mu_{\dot{X}} = \mu_{\dot{X}}(\tau) \quad (4-17c)$$

$$\sigma_X = \sigma_X(\tau) \quad (4-17d)$$

$$\sigma_{\dot{X}} = \sigma_{\dot{X}}(\tau) \quad (4-17e)$$

$$r = r(\tau) \quad (4-17f)$$

$$\rho = \sqrt{1-r^2} \quad (4-17g)$$

Letting,  $b' = \frac{b-\mu_X}{\rho\sigma_X} \quad (4-18)$

and,  $Z = \frac{\dot{x}-\mu_{\dot{X}}}{\sigma_{\dot{X}}} \quad (4-19)$

Equation (4-16) may be written,

$$\lambda_b^+(\tau) = \frac{1}{2\pi\sigma_X} \int_{\frac{-\mu_{\dot{X}}}{\pi\sigma_{\dot{X}}}}^{\infty} (\rho\sigma_X Z + \mu_{\dot{X}}) \exp \left\{ -\frac{1}{2} [Z^2 - 2rb'Z + b'^2] \right\} dZ \quad (4-20)$$

Completing the square of the exponential term in the integral yields,

$$\lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}(\rho^2 b'^2)\right\}}{2\pi\sigma_{\dot{\chi}}} \int_{\frac{-\mu_{\dot{\chi}}}{\rho\sigma_{\dot{\chi}}}}^{\infty} (\rho\sigma_{\dot{\chi}}Z + \mu_{\dot{\chi}}) \exp\left\{-\frac{1}{2}(Z-rb')^2\right\} dz \quad (4-21)$$

Now letting,

$$Z-rb' = S \quad (4-22)$$

and,

$$\frac{-\mu_{\dot{\chi}}}{\rho\sigma_{\dot{\chi}}} - rb' = S_L \quad (4-23)$$

where  $S_L$  is the new lower bound of integration yields,

$$\lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}(\rho^2 b'^2)\right\}}{2\pi\sigma_{\dot{\chi}}} \int_{S_L}^{\infty} (\rho\sigma_{\dot{\chi}}(rb'+s) + \mu_{\dot{\chi}}) \exp\left\{-\frac{1}{2}(S)^2\right\} dS \quad (4-24)$$

Expanding this expression to two integrals,

$$\begin{aligned} \lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}\rho^2 b'^2\right\}}{2\pi\sigma_{\dot{\chi}}} & \left[ (\rho\sigma_{\dot{\chi}}\rho b' + \mu_{\dot{\chi}}) \int_{S_L}^{\infty} \exp\left\{-\frac{1}{2}S^2\right\} dS \right. \\ & \left. + \rho\sigma_{\dot{\chi}} \int_{S_L}^{\infty} S \exp\left\{-\frac{1}{2}S^2\right\} dS \right] \quad (4-25) \end{aligned}$$

Upon solution this yields,

$$\begin{aligned} \lambda_b^+(\tau) = \frac{\exp\left\{-\frac{1}{2}\rho^2 b'^2\right\}}{2\pi\sigma_{\dot{\chi}}} & \left[ \rho\sigma_{\dot{\chi}} \exp\left\{-\frac{1}{2}S_L^2\right\} - S_L\rho\sigma_{\dot{\chi}} \right. \\ & \left. - S_L\rho\sigma_{\dot{\chi}} \int_{S_L}^{\infty} \exp\left\{-\frac{1}{2}S^2\right\} dS \right] \quad (4-26) \end{aligned}$$

The integral on the right hand side of Equation (4-24) may be evaluated in terms of the error function, erf(x) where,

$$\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{1}{2} y^2\right) dy \quad (4-27)$$

and,

$$\text{erf}(\infty) = \frac{1}{2} \quad (4-28)$$

Hence, Equation (4-24) becomes,

$$\lambda_b(\tau) = \frac{\rho\sigma_{\chi} \exp\left\{-\frac{1}{2} \rho^2 b'^2\right\}}{2\pi\sigma_{\chi}} \left[ \exp\left\{-\frac{1}{2} S_L^2\right\} - S_L \sqrt{2\pi} G \right] \quad (4-29a)$$

where,

$$G = \frac{1}{2} - \text{erf}(S_L) \quad 0 < S_L \quad (4-29b)$$

or,

$$G = \frac{1}{2} + \text{erf}(S_L) \quad S_L < 0 \quad (4-29c)$$

Now replacing  $b'$  with its definition, Equation (4-18), Equation (4-29a) becomes

$$\lambda_b(\tau) = \frac{\rho\sigma_{\chi} \exp\left\{-\frac{1}{2} \left(\frac{x-b}{\sigma_{\chi}}\right)^2\right\}}{2\pi\sigma_{\chi}} \left[ \exp\left\{-\frac{1}{2} S_L^2\right\} - S_L \sqrt{2\pi} G \right] \quad (4-30)$$

where  $S_L$  is defined by Equation (4-23) and  $G$  is defined by Equations (4-29b) and (4-29c). The value of erf(x) is available as a computer generated function.

Then the probability of crossing the barrier  $b$  at least once during the interval  $(\tau, \tau + \Delta t)$ ,  $P(b, \tau, \Delta t)$ , is

$$P(b, \tau, \Delta t) = 1 - \lambda_b^+(\tau) \Delta t \quad . \quad (4-31)$$

Having the probability of at least one crossing occurring in the time  $(0, t)$  may now be developed. Let  $t_n = n\Delta t$ , where  $n$  is a positive integer, and  $\Delta t$  is a small positive time. Let the event  $\alpha$  be the crossing of the barrier at least once in the interval  $(0, t_n)$ . Let the event  $\beta$  be the crossing of the barrier at least once in the interval  $(0, t_{n-1})$ .

The probability of  $\alpha$ ,  $P(\alpha)$  may be written as a the sum of probabilities conditioned on  $\beta$ . This results in,

$$P(\alpha) = P(\alpha/\beta)P(\beta) + P(\alpha/\beta^C)P(\beta^C) \quad (4-32)$$

where  $P(\alpha/\beta)$  is the probability that  $\alpha$  occurs given that  $\beta$  has occurred and  $P(\alpha/\beta^C)$  is the probability that  $\alpha$  occurs given  $\beta$  did not occur.

Since  $P(\alpha/\beta)$  is the probability of at least one crossing in the interval  $(0, t_n)$  given at least one crossing in  $(0, t_{n-1})$ ,

$$P(\alpha/\beta) = 1 \quad (4-33)$$

And Equation (4-32) becomes,

$$P(\alpha) = P(\beta) + P(\alpha/\beta^C)P(\beta^C) \quad (4-34)$$

However since the only way for a passage to occur in the interval  $(0, t_n)$  given that none occurred in the interval  $(0, t_{n-1})$  is for the passage to occur in the interval  $(t_{n-1}, t_n)$ . This is given by Equation (4-31) evaluated for  $\tau = t_{n-1}$ . Hence

$$P(\alpha) = P(\beta) + (1 - \lambda_b^+(t_{n-1})\Delta t)P(\beta^C) \quad (4-35)$$

Letting  $P_{xb}(j)$  be the probability of  $X(t)$  achieving at least one crossing of the barrier  $b$  by the end of the  $j^{\text{th}}$  time step yields,

$$P(\alpha) = P_{xb}(n) \quad (4-36a)$$

$$P(\beta) = P_{xb}(n-1) \quad (4-36b)$$

$$P(\beta^C) = 1 - P(\beta) = 1 - P_{xb}(n-1) \quad (4-36c)$$

Substituting the above into Equation (4-35) yields the recurrence relation,

$$P_{xb}(n) = P_{xb}(n-1) + (1 + \lambda_b^+(t_{n-1})\Delta t)(1 - P_{xb}(n-1)) \quad (4-37)$$

This expression may now be evaluated in a step-by-step manner to yield the probability of  $X(t)$  crossing a barrier  $b$  by the time  $t_n$ .

The computational algorithm for implementing Equation (4-37) is provided in Section 4.4 and some numerical examples are provided and discussed in Section 4.4.

#### 4.4 THE COMPUTATIONAL ALGORITHM

The following algorithm was developed to evaluate Equation (4-37) to compute the probability of passage beyond a barrier  $b$ . The algorithm is implemented in the code RSDOF. A listing of RSDOF is included in Appendix A.

- 1) Let  $n=0$ , set barrier level of interest
- 2) Let  $n=n+1$ ,  $t_n=n\Delta t$
- 3) Get the values for  $\sigma_X(t_n)$ ,  $\sigma_{\dot{X}}(t_n)$ ,  $\mu_X(t_n)$ ,  $\mu_{\dot{X}}(t_n)$ ,  $r(t_n)$  as computed in Chapter 2.
- 4) Evaluate  $B$  of Equation (4-20c) to determine which expression for  $\lambda_b^+$  to use
- 5) If  $B < 0$  use  $\lambda_b^+(t_n)$  as defined in Equation (4-30)
- 6) If  $B > 0$  use  $\lambda_b^+(t_n)$  as defined in Equation (4-29)
- 7) Compute the probability of crossing a barrier  $b$  during this time step by Equation (4-31).
- 8) Compute the probability of crossing the barrier in the interval  $(0, t_n)$  from Equation (4-37)
- 9) Go to Step 2 until problem complete.

The results obtained by implementing the above algorithm are discussed in the following sections.

#### 4.5 NUMERICAL EXAMPLES

This section presents the results of five calculations using the method developed in the previous sections of this chapter.

The first example consisted of three calculations. The probability of having achieved at least one crossing of a barrier was computed for Cases 7, 8, and 9 of Example 2 in Chapter 2. These cases had the parameters shown in Table 2-2. The barrier level was 0.36, which is 90% of the peak response for the undamped system. The only variation between cases was the correlation between the random stiffness and damping.

The resulting passage probabilities are plotted in Figure 4-1. The probabilities rapidly reach a value that is close to their final value as the response reaches its first peak. The probabilities increase slightly with each additional peak but are asymptotically approaching a final value.

The probability of passage for the case of a negative correlation between the stiffness and damping is the largest of the three cases followed by the case of independent stiffness and damping, with positively correlated stiffness and damping yielding the smallest probability of passage.

The second example is the probability of the system defined in Example 3 of Chapter 2 exceeding the peak level obtained by the deterministic calculation.

The resulting passage probability is plotted in Figure 4-2. These values are for the system with 8% damping and a coefficient of variation for the stiffness of 0.15, the values used in Chapter 2.

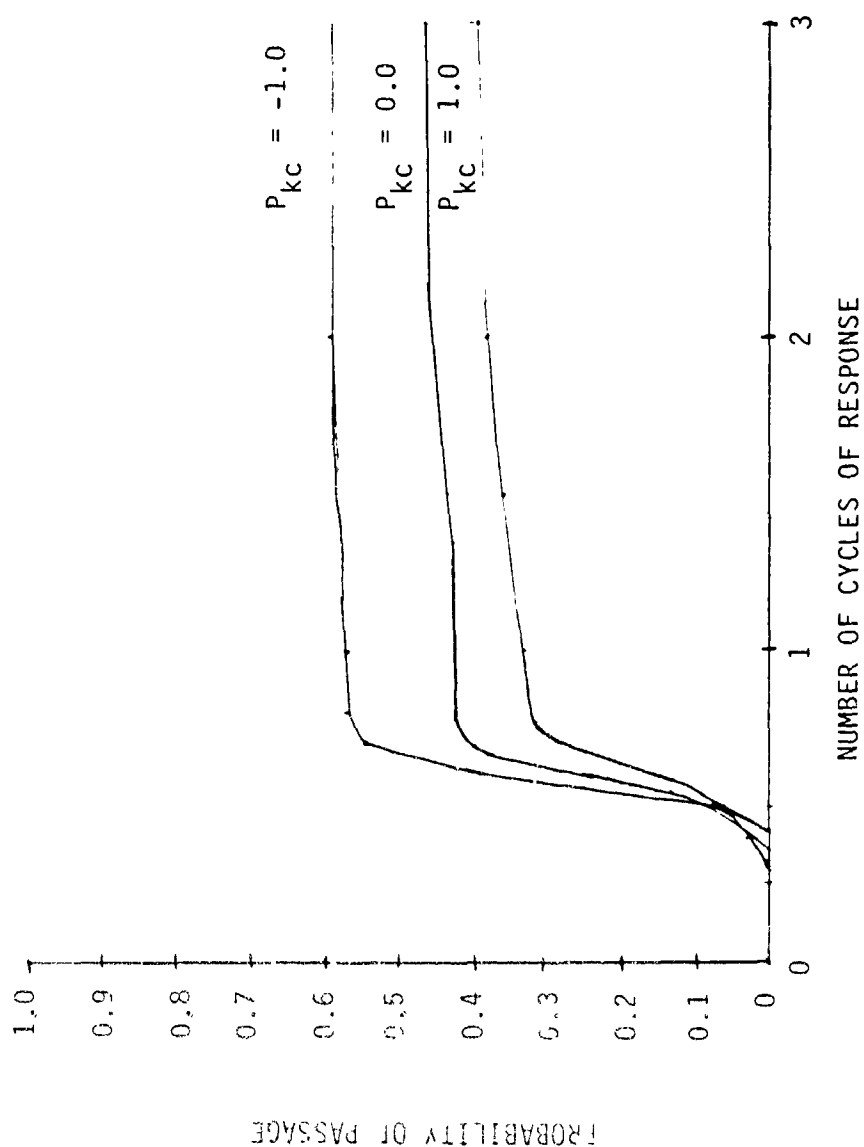


Figure 4-1 Probability of Passage for 3 Systems With Random Stiffness and Damping

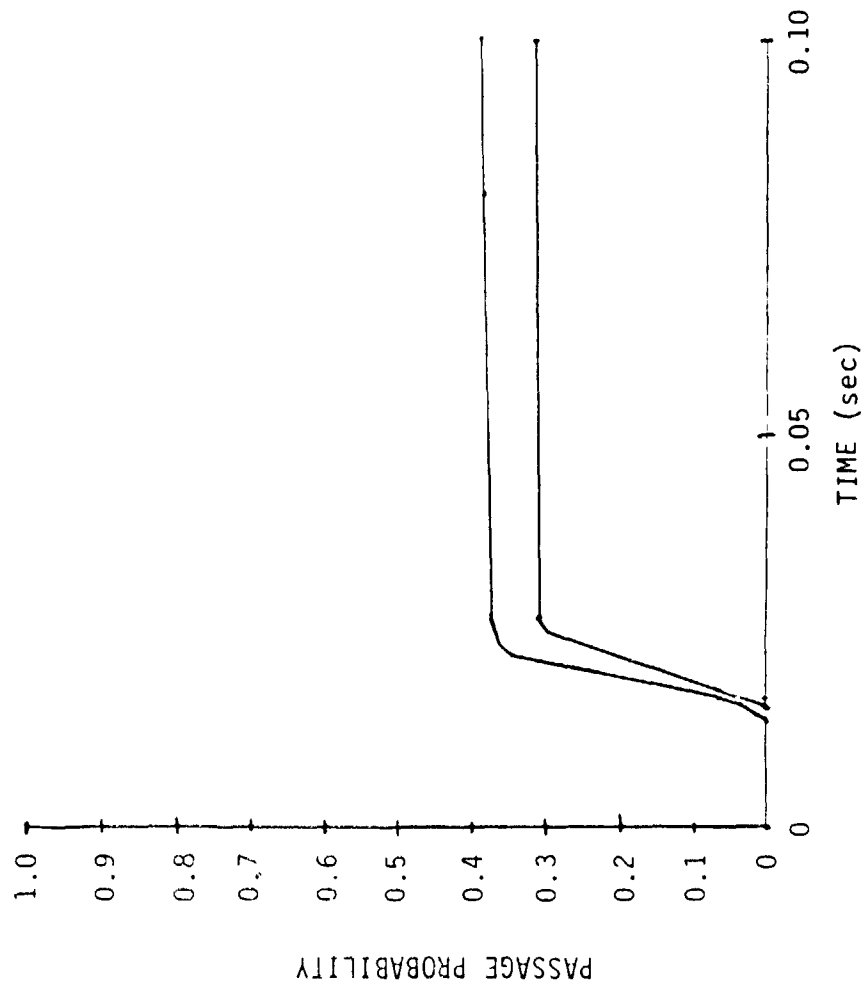


Figure 4-2 Passage Probabilities for Blast Excited Beam

Hence there is almost a 40% chance of this structure failing when failure is defined as exceeding the peak determinist response. The second calculation computed the probability of exceeding the value of 0.2884", which is the deflection resulting from twice the peak load when it is applied as a static load,

$$0.2884" = \frac{2 \times 300,000}{2.08 \times 10^6} \quad (4-38)$$

The results of this computation are plotted in Figure 4-3. The shape of the curve is similar to that of Figure 4-2; however, the resulting probability of passage is now only about .32. Hence an increase in the barrier level of less than 6% decreased the passage probability by 20%.

## CHAPTER 5

### SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

#### 5.1 SUMMARY

A technique to explicitly account for the randomness of structural stiffness and damping in the computation of the structural response was developed. The technique is suitable for use in cases of both single and multiple degrees of freedom.

The technique was applied to several problems in Chapters 2 and 3 to demonstrate the influence of these underlying uncertainties on the response of typical structures.

The results of some of these computations were used to compute the probability of the response exceeding a given value using a new method developed in Chapter 4.

The methods of Chapters 2 and 3 are limited to linear elastic structures, however they may be readily extended to cases of nonlinearity or plastic response.

The method for computing the passage probabilities developed in Chapter 4 may be applied to any continuous nonstationary random process which satisfies the assumptions made in Chapter 4.

#### 5.2 CONCLUSIONS

The randomness of a systems stiffness and damping will effect the mean and standard deviation of that systems response. The uncertainties will have only minimal influence when the system response is mean zero. This is to be expected since the systems stiffness and damping only enter the problem when the system is displaced.

The relative influence of the system stiffness appears constant for all non-zero deflections. The mean at any time may be approximated by,

$$\mu_x = X_D \cdot (1 + \text{Cov}_K^2) \quad (5-1)$$

where  $\mu_x$  is the mean of  $X$ ,  $X_D$  is the deterministic value and  $\text{cov}_K$  is the coefficient of variation for the stiffness.

The relative influence of the damping depends on the level of damping, and has no effect on the static deflections. For lightly damped systems,  $\epsilon \leq 0.15$ , the influence of the randomness of the damping may probably be ignored for most problems.

The results of Chapter 4 indicate that the randomness in the stiffness must be accounted for when a high degree of survivability is required. A blast excited slab had a significant probability of exceeding the theoretical maximum value obtained for the deterministic system.

### 5.3 RECOMMENDATIONS

The problem of MDOF system response needs more work, specifically an efficient method of accounting for the correlation of the random variables within the stiffness and damping matrices. This may be easier in a plane strain or plane stress computation where the system is a continuous medium.

A method for accounting for system uncertainties in cases where the response is not mean zero should also be developed. This includes such important cases as the response to earthquake, and turbulence.

APPENDIX A  
PROGRAM RSDOF

```

10 C*****
11 C*
12 C* PROGRAM RSDOF
13 C* THIS PROGRAM COMPUTES THE FIRST AND SECOND
14 C* MOMENTS OF THE RESPONSE OF A LINEAR ELASTIC
15 C* SDOF SYSTEM WITH RANDOM STIFFNESS AND DAMPING
16 C*
17 C* THE LOADING MAY BE DETERMINISTIC OR RANDOM
18 C*
19 C*****
80 DOUBLE PRECISION KM,M,PSI,KV,TD,RHO
90 DOUBLE PRECISION UN,UN1,UN2,CVN2,EDN,EDN1,EDN2
100 DOUBLE PRECISION ZN,EPDN1,C,W1,W2,TP,DELTA,ALPHA
110 DOUBLE PRECISION MB,MB2,ALPB,T,CK,A1,A2,A12,A22
120 DOUBLE PRECISION EKN,EKN1,EKN2,DUM,UN,EDNN1,U21,V
130 DOUBLE PRECISION P,VP,VPN
140 DOUBLE PRECISION CV,VK,VC,KRCR,ECDN,ECDN1,ECDN2,VN,VH1
150 DOUBLE PRECISION W3,W4,W5,W6,W7,EXDNF,W8,T1,ERR
160 DOUBLE PRECISION SIGV,SIG,MUA,PPFP,PPFT,B,BAR,EX
170 DOUBLE PRECISION R,R2,BP,BZ,TEST,SIGV2
180 DOUBLE PRECISION ZZ
190 DOUBLE PRECISION SL,SL2,T2,B2
200 C
210 C READ AND WRITE PROBLEM PARAMETERS
220 10 CONTINUE
230 READ(5,*) KM,M,PSI,KV,CV,RHO,TD,N,IN
240 READ(5,*) BAR
250 WRITE(6,110) KM,M,PSI,KV,CV,RHO,N
260 WRITE(6,115) BAR
270 110 FORMAT(1H , 'K=' , F15.4 , / , 1H , 'M=' , F10.4 , / , 1H , 'PSI=' , F5.3 , / ,
280 11H , 'KV=' , F5.3 , / , 1H , 'CV=' , F5.3 , / , 'RHO=' , F5.3 , / , 1H ,
290 2'N=' , I4)
300 115 FORMAT(1X , 'BARRIER LEVEL IS=' , F10.6)

```

310 C  
320 C INITIALIZE VARIABLES

330 UN=0.0D0  
340 UN1=0.0P2  
350 UN2=0.0D0  
360 VN=0.0D0  
370 EDN=0.0D0  
380 EDN1=0.0D0  
390 EDN2=0.0D0  
400 ECDN=0.0D0  
410 ECDN1=0.0D0  
420 ECDN2=0.0D0  
430 EKDN=0.0D0  
440 EKDN1=0.0D0  
450 EKDN2=0.0D0  
460 P=0.0D0  
470 VP=0.0D0  
480 EDNN1=0.0D0  
490 PFP=0.0D0  
500 PFPT=0.0D0  
510 MUA=0.0D0

520 C  
530 C PERFORM ONE TIME CALCULATIONS

540 C=KM\*M  
550 C=DSQRT(C)  
560 C=2.0D0\*PSI\*C  
570 VC=CV\*CV\*C\*C  
580 VK=KM\*KM\*KV\*KV  
590 KRCR=RHO\*KM\*C\*CV\*KV  
600 W1=KM/M  
610 W1=DSQRT(W1)  
620 W2=1.0D0-PSI\*PSI  
630 W2=DSQRT(W2)  
640 W2=W2\*W1

```

650 TP=6.28318D0/W2
660 TD=TD*TP
670 DELT=TP/N
680 NT=TD/DELT+1
690 PT=TP/10.
700 N1=PT/DELT+1
710 N2=N1
720 C
730 C SCALE FOR TIME EFFECTS
740 C
750 MB=M/(DELT*DELT)
760 T=0.D0
770 C=C/DELT
780 A1=2.D0*MB-C
790 A2=C-MB-KM
800 A12=A1*A1
810 A22=A2*A2
820 MB2=MB*MB
830 MB2=1.0D0/MB2
840 C
850 C START CALCULATION AND WRITE TITLES
860 C
870 WRITE(6,120) DELT
880 120 FORMAT(1H,'TIMESTEP= ',F10.8,'/ ',
890 11X,'TIME',11X,'DISPL',12X,'VEL',10X,'SIG',10X,'PFP')
900 DO 20 I=1,NT
910 UM1=UN
920 UN=UN1
930 UN1=UN2
940 UN2=0.D0
950 VM1=VN
960 VN=(UN1-UN)/DELT
970 EDN=EDN1
980 EDN1=EDN2
990 EDN2=0.D0

```

```

1000 EKD1=EKD1
1010 EKD1=EKD2
1020 EKD2=0.D0
1030 ECD1=ECD1
1040 ECD1=ECD2
1050 ECD2=0.D0
1060 EDN1=(-UM1*EKD1-VM1*ECD1+A1*EDN+A2*EDN1)/MB
1070 CALL LOAD(P,T,VP,DELT)
1080 UN2=(P-((ECD1-ECD1)/DELT)-EKD1+A1*UN1+A2*UN)/MB
1090 EDN2=MB2*(VP+VK*UN*UN+VC*VN*VN+2.D0*UN*A2*EKD1-2.D0*VN*A1*ECD1
1100 1 -2.D0*VN*A2*EKD1+A12*EDN1+2.D0*A1*A2*EDN1+A22*EDN)
1110 2 -2.D0*VN*A2*EKD1+A12*EDN1+2.D0*A1*A2*EDN1+A22*EDN)
1120 EKDN2=(-VK*UN-KRCR*VN+A1*EKDN1+A2*EKDN)/MB
1121 C*****
1122 C*
1123 C* THIS BLOCK OF CODE COMPUTES THE PASSAGE
1124 C* PROBABILITY OF A BARRIER LEVEL SPECIFIED BY THE
1125 C* USER.
1126 C*****
1130 IF(EDN.LE.0.0) GOTO 11
1140 SIGV2=(EDN+EDN1-2*EDN1)/(DELT*DELT)
1150 IF(SIGV2.LT.0.0) SIGV2=0.0
1160 SIGV=DSQRT(SIGV2)
1170 SIG=DSQRT(EDN)
1180 IF(SIGV.EQ.0.0.OR.SIG.EQ.0.0) GOTO 11
1190 R=(EDN1-EDN)/(DELT*SIG*SIGV)
1200 IF(R.GT.1.) R=1.
1210 IF(R.LT.-1.) R=-1.
1220 IF(R.EQ.1.0) R=0.999
1230 IF(R.EQ.-1.0) R=-0.999
1240 R2=1-R*R
1250 R2=DSQRT(R2)
1260 B=(BAR-UN)/(SIG*R2)
1270 IF(B.LT.0.0) PFP=1.0
1280 SL=VN/(R2*SIGV)-R*B

```

```

1290      B2=B*B*R2/R2/2.D0
1300      IF(B2.GT.112.5) GOTO 11
1310      R2=DEXP(B2)
1320      T1=R2*SIGV
1330      SL2=SL*SL/2.D0
1340      T2=-T1*SL*2.50628
1350      B2=1.D0/(6.283185D0*SIG*B2)
1360      IF(SL.GT.0.0) GOTO 13
1370      IF(SL2.GT. 115) GOTO 15
1380      SL=-SL
1390      MUA=B2*(T1/DEXP(SL2)+T2*(.5+DERF(SL)))
1400      GOTO 12
1410
1420      MUA=B2*T2
1430      GOTO 12
1440
1450      IF(SL2.GT.112.5) GOTO 11
1460      MUA=B2*(T1/DEXP(SL2)+T2*(.5-DERF(SL)))
1470      GOTO 12
1480
1490      MUA=0.D0
1500      MUA=MUA*DELT
1510      PFPT=1.D0-DEXP(-MUA)
1520      PFP=PFPT+(1.D0-PFP)*PFPT
1530      IF(N2.NE.N1) GOTO 40
1540      WRITE(6,150) T,UN,VN,SIG,PFP
1550      R2=0
1560      N2=N2+1
1570      T=T+DELT
1580      20 CONTINUE
1590      150 FORMAT(F10.6,4X,F10.6,4X,F10.6,4X,F10.6,4X,F10.8)
1600      IF(1N.NE.0) GOTO 10
1610      STOP
1620      END

```

SUBROUTINE LOAD(P,T,VP,DELT)  
DOUBLE PRECISION P,T,VP,DELT,T1  
P=100  
VP=0.0  
RETURN  
END

1620  
1630  
1650  
1660  
1670  
1710

**APPENDIX B**  
**PROGRAM RMDOF**

```

1 0.4#####
2 C*
3 C* PROGRAM RND0F
4 C*
5 C* THIS PROGRAM USES A PLANE FRAME FINITE ELEMENT
6 C* CODE TO EVALUATE THE FIRST AND SECOND MOMENTS OF
7 C* THE RESPONSE OF A MULTI-DEGREE-OF-FREEDOM SYSTEM *
8 1.*
9 C* THE LOADING MAY BE DETERMINISTIC OR RANDOM *
10 C#####
60 C
70 C INITIALIZE AND MAKE DECLARATIONS ETC
80 C
90 DOUBLE PRECISION RNODE(7,2),K(21,21),VK(21,21),M(21),MI(21)
100 DOUBLE PRECISION A1(21,21),A2(21,21),UM1(21),UN(21),UN1(21)
110 DOUBLE PRECISION UN2(21),VM1(21),VN(21),EKN(21),EKN1(21)
120 DOUBLE PRECISION EDN(21,21),EDN1(21,21),EDN2(21,21)
130 DOUBLE PRECISION EKN2(21),ZN(21),ZH1(21)
140 DOUBLE PRECISION W1(21,21),W2(21,21)
150 DOUBLE PRECISION W3(21,21),W4(21,21),W5(21,21),P(21)
160 DOUBLE PRECISION EQN(21,21)
170 DOUBLE PRECISION EDNN1(21,21),DIAG(21),AE,COVA,EI,COVE,RHO,Z1
180 DOUBLE PRECISION OMEG1,Z2,OMEG2,BETA,ALPHA,DELT,S1,S2,T1,T2
190 DOUBLE PRECISION SM
200 COMMON/RND/RNODE
210 COMMON/GLB/K,M,VK,N
220 DO 1 I=1,21
230 UN1(I)=0.D0
240 UN2(I)=0.D0
250 VN(I)=0.D0
260 ZN(I)=0.D0
270 EKN1(I)=0.D0
280 EKN2(I)=0.D0
290 P(I)=0.D0

```

```

300      DO 1 J=1,21
310        A1(I,J)=0.D0
320        A2(I,J)=0.D0
330        K(21,21)=0.D0
340        EDNN1(I,J)=0.D0
350        EDN(I,J)=0.D0
360        EDN1(I,J)=0.D0
370        EQN(I,J)=0.D0
380        W1(I,J)=0.D0
390        W2(I,J)=0.D0
400        W3(I,J)=0.D0
410        W4(I,J)=0.D0
420        W5(I,J)=0.D0
430        VK(I,J)=0.D0
440      1 CONTINUE
450      C
460      C
470      C
480      C
490      C
500      C
510      C
520      READ(4,120) NN,NEL,NRST
530      N=3*NN
540      120 FORMAT(4I4)
550      C
560      C
570      C
580      WRITE(7,140) NN
590      140 FORMAT(1H , 'TOTAL NUMBER OF NODES=',I2,/,1H , 'NODE',
600      1 5X, 'X',I2X, 'Y')
610      DO 10 I=1,NN
620        READ(4,130)(RNODE(I,J),J=1,2)
630        WRITE(7,150) I,(RNODE(I,J),J=1,2)
640      10 CONTINUE
650      150 FORMAT(15,5X,E10.3,5X,E10.3)

```

```

660      130 FORMAT(2F10.3)
670 C
680 C
690 C      READ IN ELEMENT DATA AND COMPUTE STIFFNESS AND MASS MATRICES
700 C
710 C
720      DO 20 LELM=1,NEL
730          READ(4,180) N1,N2,AE,COVA,EI,COVE,RHO
740          CALL ELEMEN(N1,N2,AE,COVA,EI,COVE,RHO,LELM)
750      20 CONTINUE
760      180 FORMAT(I2,I2,F10.3,F5.3,F10.3,F5.3,F10.3)
770 C
780 C      READ DAMPING PARAMETERS
790 C
800          READ(4,190) OMEG1,Z1,OMEG2,Z2
810      190 FORMAT(4F10.3)
820          ALPHA=2.DO*(OMEG2**2-OMEG1**2)/(OMEG2**2.DO-OMEG1**2.DO)
830          BETA=2.DO*OMEG1**21-ALPHA*OMEG1**2.DO
840          WRITE(7,191) ALPHA,BETA
850      191 FORMAT(1X,'ALPHA=',E10.4,5X,'BETA=',E10.4)
860 C
870 C      READ TIMESTEP AND PROBLEM DURATION
880 C
890 C
900          READ(4,200) DELT,TD
910      200 FORMAT(2F10.4)
920          NT=TD/DELT
930          S1=1.DO-ALPHA/DELT
940          S2=ALPHA/DELT
950          WRITE(7,192) S1,S2
960      192 FORMAT(1X,'S1=',E10.4,5X,'S2=',E10.4)
970 C
980      DO 55 I=1,N
990          M(I)=M(I)/(DELT**2.DO)
1000          MI(I)=1.DO/M(I)
1010      55 CONTINUE

```

```

1020 C
1030 C COMPUTE THE MATRICES USED IN THE FINITE DIFFERENCE SCHEME
1040 C
1050 DO 70 I=1,N
1060 DO 80 J=1,N
1070 A1(I,J)=-S2*K(I,J)
1080 A2(I,J)=-S1*K(I,J)
1090 CONTINUE
1100 A1(I,I)=A1(I,I)+(2.D0-DELTA*BETA)*M(I)
1110 A2(I,I)=A2(I,I)-(1.D0-BETA*DELTA)*M(I)
1120 CONTINUE
1130 C IMPOSE CONSTRAINT CONDITIONS
1140 C
1150 DO 50 IL=1,NRST
1160 READ(4,210) NODE, IDOF
1170 FORMAT(2I4)
1180 IDOF=3*NODE-3+IDOF
1190 DO 60 J=1,N
1200 A1(J, IDOF)=0.D0
1210 A1( IDOF, J)=0.D0
1220 VK(J, IDOF)=0.D0
1230 VK( IDOF, J)=0.D0
1240 A2(J, IDOF)=0.D0
1250 A2( IDOF, J)=0.D0
1260 CONTINUE
1270 MI( IDOF)=0.D0
1280 CONTINUE
1290 C
1300 C START TIME STEP EVALUATION
1310 C
1320 NP=20
1330 TIME=Q,D0
1340 C
1350 C
1360 C
1370 C
1380 C
1390 C
1400 C
1410 C
1420 C
1430 C
1440 C
1450 C
1460 C
1470 C

```

```

1480      DO 90 IIL=1,NT
1490 C
1500 C   UPDATE ARRAYS
1510 C
1520      CALL LOAD(TIME,P,EQN,N)
1530      DO 201 I=1,N
1540          UN1(I)=UN(I)
1550          UN(I)=UN1(I)
1560          UN1(I)=UN2(I)
1570          UN2(I)=0.DO
1580          VM1(I)=VN(I)
1590          ZM1(I)=ZN(I)
1600          VN(I)=(UN1(I)-UN(I))/DELT
1610          ZN(I)=UN(I)+ALPHA*VN(I)
1620          EKDN(I)=EKDN1(I)
1630          EKDN1(I)=EKDN2(I)
1640      DO 250 J=1,N
1650          EDN(I,J)=EDN1(I,J)
1660          EDN1(I,J)=EDN2(I,J)
1670          W1(I,J)=0.DO
1680          W2(I,J)=0.DO
1690          W3(I,J)=0.DO
1700          W4(I,J)=0.DO
1710          W5(I,J)=0.DO
1720      CONTINUE
1730      CONTINUE
1740      DO 260 I=1,N
1750          EKDN2(I)=M1(I)*(-VK(I,I)*ZN(I)+A1(I,I)*EKDN1(I)
1760              +A2(I,I)*EKDN(I))
1770      CONTINUE
1780      CALL MAMA(A1,EDN,W1,N)
1790      CALL MAMA(EDN1,A2,W2,N)
1800      DO 280 I=1,N
1810          DO 279 J=1,N
1820              FIMN1(I,J)=M1(I,I)*W1(I,I)+W2(I,I)

```

```

1830      CONTINUE
1840      EDNN1(I,I)=MI(I)*(-EKDN(I)*ZN(I))+EDNN1(I,I)
1850      CONTINUE
1860 C
1870 C
1880 C      COMPUTE MEAN RESPONSE UN2
1890 C
1900      DO 290 I=1,N
1910          T1=0.00
1920          T2=0.00
1930          DO 300 J=1,N
1940              T1=T1+A1(I,J)*UN1(J)
1950              T2=T2+A2(I,J)*UN(J)
1960          CONTINUE
1970          UN2(I)=MI(I)*(P(I)-S1*EKDN(I)-S2*EKDN1(I)+T1+T2)
1980      CONTINUE
1990 C
2000 C      COMPUTE DIAG VECTOR
2010 C
2020      DO 310 I=1,N
2030          DIAG(I)=0.00
2040          DO 310 J=1,N
2050              DIAG(I)=DIAG(I)+VK(I,J)*(ZN(J)*ZN(J))
2060          CONTINUE
2070      101 FORMAT(1X,6(E8.2,2X))
2080 C
2090 C      COMPUTE THE EXPECTED VALUE OF K*U*D
2100 C
2110      DO 320 I=1,N
2120          DO 319 J=1,N
2130              W1(I,J)=0.00
2140              W2(I,J)=0.00
2150          CONTINUE
2160          W1(I,I)=EKDN1(I)*ZN(I)
2170          W2(I,I)=EKDN(I)*ZN(I)

```

```

2180      CONTINUE
2190      CALL MAMA(W1,A1,W1,N)
2200      CALL MAMA(W2,A2,W2,N)
2210      CALL MAMA(A1,EDN1,W3,N)
2220      CALL MAMA(A2,EDN,W4,N)
2230      CALL MAMA(W3,A1,W3,N)
2240      CALL MAMA(W4,A2,W4,N)
2250      CALL MAMA(A1,EDNN1,W5,N)
2260      CALL MAMA(W5,A2,W5,N)
2270      DO 330 I=1,N
2280          DO 340 J=1,N
2290              EDN2(I,J)=0.D0
2300              EDN2(I,J)=MI(I)*MI(J)*(EQN(I,J)-W1(I,J)-W2(J,I)
1          -W1(J,I)-W2(I,J)+W3(I,J)+W4(I,J)+W5(I,J)+W5(J,I))
2310
2320      CONTINUE
2330      EDN2(I,I)=MI(I)*MI(I)*DIAG(I)+EDN2(I,I)
2340      CONTINUE
2350      DO 335 I=1,N
2360          DO 335 J=1,I
2370              SM=(EDN2(I,J)+EDN2(J,I))/2.D0
2380              EDN2(I,J)=SM
2390              EDN2(J,I)=SM
2400      CONTINUE
2410 C
2420 C WRITE OUT DESIRED RESULTS
2430 IF(NP.NE. 20) GOTO 92
2440 NP=0
2450 WRITE(7,91) TIME,UN(4),UN(7),EDN(4,4),EDN(4,7),EDN(7,7)
2530 91 FORMAT(6(E9.3,2X))
2540 92 CONTINUE
2550 NP=NP+1
2560 C
2570 TIME=TIME+DELT
2580 90 CONTINUE
2590 END

```

```

2770 C
2780 C
2790 C
2800 C
2810 C
2820 C
2830 C
2840 C
2850 C
2860 C
2870 C
2880 C
2890 C
2900 C
2910 C
2911 C
2912 C
2920 C
2930 C
2940 C
2950 C
2960 C
2970 C
2971 C
2972 C
2973 C
2980 C
2990 C
3000 C
3010 C
3011 C
3012 C
3013 C
3020 C

      SUBROUTINE ELEMEN(N1,N2,AE,COVA,EI,COVE,RHO,LELI)

      THIS SUBROUTINE COMPUTES THE ELEMENT STIFFNESS AND MASS
      MATRICES,THE VARIANCE OF THE STIFFNESS MATRIX AND SUMS
      THE GLOBAL MATRICES.

      COMMON BLOCKS AND TYPE DECLARATIONS

      DOUBLE PRECISION KE(6,6),ME(6,6),K(21,21),M(21),VKE(6,6)
      DOUBLE PRECISION VK(21,21),W1(6,6),AE,COVE,EI,COVI
      DOUBLE PRECISION L,L2,L3,RHO,T(6,6),TT(6,6),RNODE(7,2)
      DOUBLE PRECISION X1,X2,Y1,Y2,XDIF,YDIF,SNB,CSB,C1,C2
      DOUBLE PRECISION W2(6,6),W3(6,6)
      DOUBLE PRECISION W4(6,6),W5(6,6),W6(6,6)
      DOUBLE PRECISION P,S,G,VP,VS,VG
      INTEGER CON(6)

      COMMON/GLB/K,M,VK,N
      COMMON/RND/RNODE
      DO 1 I=1,6
        DO 1 J=1,6
          W1(I,J)=0.D0
          W2(I,J)=0.D0
          W3(I,J)=0.D0
          T(I,J)=0.D0
          VKE(I,J)=0.D0
          ME(I,J)=0.D0
          KE(I,J)=0.D0
          W4(I,J)=0.D0
          W5(I,J)=0.D0
          W6(I,J)=0.D0
        1 CONTINUE

```

```

3030      IROT=0
3040 C
3050 C      FORM CONNECTIVITY VECTOR
3060 C
3070      CON(1)=3*N1-2
3080      CON(2)=3*N1-1
3090      CON(3)=3*N1
3100      CON(4)=3*N2-2
3110      CON(5)=3*N2-1
3120      CON(6)=3*N2
3130      X1=RNODE(N1,1)
3140      Y1=RNODE(N1,2)
3150      X2=RNODE(N2,1)
3160      Y2=RNODE(N2,2)
3170      XDIF=X2-X1
3180      YDIF=Y2-Y1
3190      L2=XDIF*XDIF+YDIF*YDIF
3200      L=USQRT(L2)
3210      L3=L*L2
3220 C
3230 C      CHECK TO SEE IF ROTATION IS REQUIRED AND SET FLAG
3240 C
3250 C      IF(YDIF.NE. 0.0) IROT=1
3260 C
3270 C      COMPUTE AXIAL TERMS
3280 C
3290      C1=AE/L
3300      VC1=C1*C1*COVA*COVA
3310      KE(1,1)=C1
3320      KE(1,4)=-C1
3330      KE(4,4)=C1
3340      VKE(1,1)=VC1
3350      VKE(1,4)=VC1
3360      VKE(4,4)=VC1

```

```

3370 C
3380 C AXIAL MASS
3390 C
3400 C2=RHO*L/2.D0
3410 ME(1,1)=C2
3420 ME(4,4)=C2
3430 C
3440 C ASSEMBLE BENDING PORTIONS
3450 C
3460 P=12.D0*EI/L3
3470 S=6.D0*EI/L2
3480 G=4.D0*EI/L
3490 VP=P*P*COVE*COVE
3500 VS=S*S*COVE*COVE
3510 VG=G*G*COVE*COVE
3520 KE(2,2)=P
3530 KE(2,3)=S
3540 KE(2,5)=-P
3550 KE(2,6)=S
3560 KE(3,3)=G
3570 KE(3,5)=-S
3580 KE(3,6)=G/2.D0
3590 KE(5,5)=P
3600 KE(5,6)=-S
3610 KE(6,6)=G
3620 C
3630 C VARIANCE TERMS
3640 C
3650 VKE(2,2)=VP
3660 VKE(2,3)=VS
3670 VKE(2,5)=VP
3680 VKE(2,6)=VS
3690 VKE(3,3)=VG
3700 VKE(3,5)=VS

```

```

3710      VKE(3,6)=VG/4.D0
3720      VKE(5,5)=VP
3730      VKE(5,6)=VS
3740      VKE(6,6)=VG
3750 C
3760 C      MASS TERMS
3770 C
3780      ME(2,2)=C2
3790      ME(3,3)=C2*L2/12.D0
3800      ME(5,5)=C2
3810      ME(6,6)=C2*L2/12.D0
3820 C
3830 C      FILL IN LOWER PORTION OF MATRICES
3840 C
3850      DO 10 I=1,6
3860      DO 10 J=1,I
3870          KE(I,J)=KE(J,I)
3880          VKE(I,J)=VKE(J,I)
3890      10 CONTINUE
3900 C
3910 C      PERFORM ROTATIONS IF NECESSARY
3920 C
3930      IF (IROT.NE.1) GOTO 100
3940      CSB=XDIF/L
3950      SNB=YDIF/L
3960      T(1,1)=CSB
3970      T(1,2)=-SNB
3980      T(2,1)=SNB
3990      T(2,2)=CSB
4000      T(3,3)=1.D0
4010      T(4,4)=CSB
4020      T(4,5)=-SNB

```

```

4030 I(5,4)=SNB
4040 I(5,5)=CSB
4050 T(6,6)=1.D0
4060 DO 50 I=1,6
4065 DO 50 J=1,6
4070 DO 50 N3=1,6
4075 W1(I,J)=W1(I,J)+T(N3,I)*KE(N3,J)
4080 W2(I,J)=W2(I,J)+T(N3,I)*ME(N3,J)
4085 W3(I,J)=W3(I,J)+T(N3,I)*VKE(N3,J)
4090 50 CONTINUE
4095 DO 60 I=1,6
4100 DO 60 J=1,6
4105 DO 60 N3=1,6
4110 W4(I,J)=W4(I,J)+W1(I,N3)*T(N3,J)
4115 W5(I,J)=W5(I,J)+W2(I,N3)*T(N3,J)
4120 W6(I,J)=W6(I,J)+W3(I,N3)*T(N3,J)
4125 60 CONTINUE
4130 DO 70 I=1,6
4135 DO 70 J=1,6
4140 KE(I,J)=W4(I,J)
4145 ME(I,J)=W5(I,J)
4146 VKE(I,J)=W6(I,J)
4155 70 CONTINUE
4200 100 CONTINUE
4210 C
4220 C ASSEMBLE GLOBAL MATRICES
4240 DO 110 I=1,6
4250 II=CON(I)
4260 H(II)=H(II)+ME(I,I)
4270 DO 110 J=1,6
4280 JI=CON(J)
4290 K(II,JI)=K(II,JI)+KE(I,J)
4300 VK(II,JI)=VK(II,JI)+VKE(I,J)
4310 110 CONTINUE

```

```

4320 WRITE(7,15) LELM
4330 15 FORMAT(1X,'STIFFNESS MATRIX ELEMENT',I2)
4340 DO 5 I=1,6
4350 WRITE(7,17)(KE(I,J),J=1,6)
4360 5 CONTINUE
4370 WRITE(7,17)(ME(I,I),I=1,6)
4380 17 FORMAT(1X,6(E9.3,2X))
4390 18 FORMAT(1X,'MASS MATRIX',/,6(E9.3,2X))
4400 RETURN
4410 END
4411 C THIS SUBROUTINE PERFORMS THE OPERATION
4412 C A*B=C. WHERE A,B,AND C ARE N*N MATRICES
4420 SUBROUTINE MAMA(A,B,C,N)
4430 DOUBLE PRECISION A(21,21),B(21,21),C(21,21),D(21,21)
4440 DO 20 I=1,N
4450 DO 20 J=1,N
4460 D(I,J)=0.D0
4470 20 CONTINUE
4480 DO 30 I=1,N
4490 DO 30 J=1,N
4500 DO 30 K=1,N
4510 D(I,J)=A(I,K)*B(K,J)+D(I,J)
4520 30 CONTINUE
4530 DO 40 I=1,N
4540 DO 40 J=1,N
4550 C(I,J)=D(I,J)
4560 40 CONTINUE
4570 RETURN
4580 END

```

```
4590      SUBROUTINE LOAD(T,P,EQN,N)
4600      DOUBLE PRECISION P(21),EQN(21,21),T
4610 C
4620 C      DEFINE FURTHER VARIABLES HERE
4630 C
4640 C
4650 C      USER DEFINED LOADING FUNCTIONS
4660 C
4720      RETURN
4730      END
OFF
13:31:35 03/31/85 60237
CONNECTED 00:11:41
CPU TIME 0.0
```

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