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SATELLITE DRAG PERTURBATIONS IN NONSINGULAR VARIABLES
PRODUCED BY AN OBLA (U) NAVAL SURFACE WEAPONS CENTER
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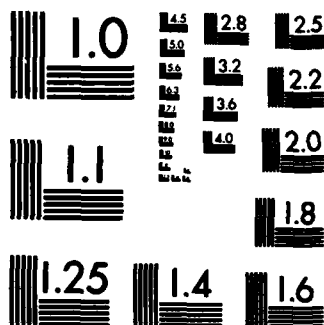
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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NSWC TR 84-353	2. GOVT ACCESSION NO. AD-A157869	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SATELLITE DRAG PERTURBATIONS IN NONSINGULAR VARIABLES PRODUCED BY AN OBLATE DIURNAL ATMOSPHERE I. THEORY		5. TYPE OF REPORT & PERIOD COVERED FINAL
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. D. PARKS		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS NAVAL SURFACE WEAPONS CENTER (CODE K13) DAHLGREN, VA 22448		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 35159B; 5K12CS
11. CONTROLLING OFFICE NAME AND ADDRESS DEFENSE MAPPING AGENCY WASHINGTON, DC 20305		12. REPORT DATE OCTOBER 1984
		13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Satellite Lagrange planetary equations Drag perturbations Atmosphere Oblate gravitation		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Analytic expressions are derived that describe the atmospheric drag induced decay rates for the semimajor axis and eccentricity for orbits of small eccentricity. The effects of both the oblate earth figure and diurnal density variations are included in the development. Equations describing the effects of geopotential perturbations are also summarized.		

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FOREWORD

Since the number of artificial earth satellites requiring operational flight control is continually increasing, more emphasis has been placed upon the development of rapid and accurate ephemeris prediction techniques. This study has been conducted under the auspices of the Defense Mapping Agency in order to develop a formalism that can be adapted to aid in the satisfaction of operational control requirements associated with low-altitude earth satellites. This report was reviewed and approved by Dr. R. J. Anderle and Mr. R. W. Hill.

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1. INTRODUCTION

There has in recent years been a renewed interest in satellite operational control using rapid and accurate prediction techniques. Of especial interest are satellites in low earth orbits that experience atmospheric drag disturbances. Early studies concerning the effects of atmospheric drag perturbations upon artificial satellites of small eccentricity were performed by Cook and King, ^{1,2} Zee, ³ Otterman and Lichtenfeld, ⁴ and Lane. ⁵ More recent analyses have been developed by Liu and Alford, ⁶ and Santora. ^{7,8} The latter author's works represent one of the more realistic approaches to the determination of drag perturbations on a decaying satellite. ⁹ There the oblateness and diurnal characteristics of the atmosphere have been combined with the effects induced by the oblate earth mass gravitational perturbations to provide a "unified" analytic theory describing both the drag and gravitational influences exerted upon a low altitude earth satellite.

Santora's theory is unified in the sense that the oblate diurnal atmospheric drag perturbations have been combined with those produced by the terrestrial gravity field. It is not a completely unified theory for low altitude satellite motion because a significantly different development must be used to treat orbits with $0 \leq e \leq .01$ than is used for orbits with $.01 < e \leq .20$. A completely unified theory is presented here that not only combines the oblate gravitational effects with those produced by an oblate diurnal atmosphere, but also eliminates the eccentricity dichotomy that exists in Santora's approach.

The following sections of this report discuss in detail the development of this unified theory. The nonsingular element set and the associated Lagrange planetary equations used in the development of the theory; a detailed derivation of the drag decay rates; and an overview of the geopotential perturbations are presented in the following sections. Appendixes are also provided that contain mathematical relationships used in the development of the decay rate expressions.

2. EQUATIONS OF MOTION IN NONSINGULAR VARIABLES

The eccentricity dichotomy that exists in the Santora theory can be removed by redeveloping the theory in terms of the following element set:

$$\left. \begin{aligned} a &= \text{semimajor axis} \\ \lambda &= M + \omega + \Omega \\ \xi &= e \cos \tilde{\omega} \quad (\tilde{\omega} = \omega + \Omega) \\ \eta &= e \sin \tilde{\omega} \\ P &= \sin (1/2) \cos \Omega \\ Q &= \sin (1/2) \sin \Omega \end{aligned} \right\} \quad (2.1)$$

where a , e , 1 , ω , Ω , and M are the usual Keplerian elements. This element set is nonsingular for $1 \neq \pi$ and $e < 1$.

The Lagrange planetary equations for the nonsingular element set defined in Equations 2.1 are:

$$\left. \begin{aligned} \dot{a} &= \frac{2}{na} R_\lambda \\ \dot{\lambda} &= n - \frac{2}{na} R_a + \frac{\gamma}{2na^2} (\xi R_\xi + \eta R_\eta) + \frac{1}{2na^2 \gamma} (PR_P + QR_Q) \\ \dot{\xi} &= -\frac{\gamma}{na^2(1+\gamma)} \xi R_\lambda - \frac{\gamma}{na^2} R_\eta - \frac{1}{2na^2 \gamma} \eta (PR_P + QR_Q) \\ \dot{\eta} &= -\frac{\gamma}{na^2(1+\gamma)} \eta R_\lambda + \frac{\gamma}{na^2} R_\xi + \frac{1}{2na^2 \gamma} \xi (PR_P + QR_Q) \\ \dot{P} &= -\frac{1}{2na^2 \gamma} PR_\lambda - \frac{1}{4na^2 \gamma} R_Q + \frac{1}{2na^2 \gamma} P(\eta R_\xi - \xi R_\eta) \\ \dot{Q} &= -\frac{1}{2na^2 \gamma} QR_\lambda + \frac{1}{4na^2 \gamma} R_P + \frac{1}{2na^2 \gamma} Q(\eta R_\xi - \xi R_\eta) \end{aligned} \right\} \quad (2.2)$$

where n is the mean motion and

$$\gamma = \sqrt{1 - e^2} \quad (2.3)$$

For close-earth artificial satellites perturbed by the earth's oblateness and atmospheric resistance, the partials of the disturbing function are of the form

$$R_{\alpha} = R_{\alpha}^D + R_{\alpha}^G \quad (2.4)$$

where α represents any of the elements of Equations 2.1, and the superscripts D and G designate the perturbing terms due to atmospheric drag and the earth's oblateness, respectively. To be more explicit

$$R_{\alpha}^D = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \alpha} \quad (2.5)$$

and

$$R_{\alpha}^G = \frac{\partial R}{\partial \alpha} \quad (2.6)$$

where $\vec{F}$ is the nonconservative drag force, $\vec{r}$ is the satellite radius vector, and R is the gravitational disturbing function.

3. DRAG DECAY RATES FOR NONSINGULAR ELEMENTS IN AN OBLATE DIURNAL ATMOSPHERE

Analytic expressions for drag decay rates in an oblate diurnal atmosphere using nonsingular variables are developed in this section. Since the principal drag effects on a close-earth satellite are perturbations in a and e , it is necessary only to consider expressions for the drag induced changes in a , ξ , and η .

An analytic form for the atmospheric density can be obtained by combining the oblate atmosphere used by Cook, King-Hele, and Walker¹⁰ with that of an atmosphere with diurnal variation discussed by Cook and King-Hele.¹ The resulting form for the density is given by

$$\rho = \rho_0(1 + F \cos \phi) \exp[-\beta(r - \sigma)] \quad (3.1)$$

where ρ_0 is an average atmospheric density defined by

$$\rho_0 = \frac{1}{2}(\rho_{\max} + \rho_{\min}) \quad (3.2)$$

F is a density amplitude factor given by

$$F = \frac{\rho_{\max} - \rho_{\min}}{\rho_{\max} + \rho_{\min}} \quad (3.3)$$

$$\beta = H^{-1} \quad (3.4)$$

$$\sigma = r_p \left\{ \frac{1 - \epsilon \sin^2 i \sin^2 u}{1 - \epsilon \sin^2 i \sin^2 \omega} \right\} \quad (3.5)$$

and

$$\cos \phi = A \left[\frac{\cos E - e}{1 - e \cos E} \right] + B \left[\frac{(1 - e^2)^{\frac{1}{2}} \sin E}{1 - e \cos E} \right] \quad (3.6)$$

In the last four equations $\rho_{\max}$ and $\rho_{\min}$ are the maximum daytime and minimum nighttime densities, respectively, at the osculating perigee altitude h_p ; H is the density scale height; ϵ is the earth's ellipticity; r_p is the osculating perigee radius; E is the eccentric anomaly; $u = \theta + \omega$, where θ is the true anomaly; and

$$A = \sin \delta_B \sin i \sin \omega + \cos \delta_B \left\{ \cos (\Omega - \alpha_B) \cos \omega - \cos i \sin (\Omega - \alpha_B) \sin \omega \right\} \quad (3.7)$$

and

$$B = \sin \delta_B \sin i \cos \omega - \cos \delta_B \left\{ \cos (\Omega - \alpha_B) \sin \omega + \cos i \sin (\Omega - \alpha_B) \cos \omega \right\} \quad (3.8)$$

The declination and right ascension of the center of the diurnal bulge are designated by δ_B and α_B , respectively, in the last two equations.

Equation 3.5 may be expanded to first order in i to give

$$\sigma = r_p \left[1 + \frac{1}{2} \epsilon \sin^2 i (\cos 2u - \cos 2\omega) \right] \quad (3.9)$$

Similarly, Equation 3.6 may be expanded to first order in e to give

$$\cos \phi = A(\cos E - e + e \cos^2 E) + B(\sin E + e \sin E \cos E) \quad (3.10)$$

Substituting Equations 3.9 and 3.10 into Equation 3.1 and using the relation

$$r = a(1 - e \cos E) \quad (3.11)$$

allows the following first-order expression to be written for the atmospheric density:

$$\begin{aligned} \rho = \rho_0 [1 + FA(\cos E - e + e \cos^2 E) + FB(\sin E \\ + e \sin E \cos E)] \exp \left\{ -\beta a e (1 - \cos E) \right. \\ \left. + c \cos 2u - c \cos 2\omega \right\} \end{aligned} \quad (3.12)$$

where

$$c = \frac{1}{2} \epsilon \beta r_p \sin^2 i \quad (3.13)$$

This quantity may be treated as a small parameter of the same order of magnitude as the eccentricity so that the following expansion may be used:

$$\exp \{ c \cos 2u \} = 1 + c \cos 2u + \frac{1}{2} c^2 \cos^2 2u \quad (3.14)$$

The changes in the elements a , ξ , and η over one orbital revolution due to atmospheric drag deceleration are given by (see Appendix B)

$$\Delta a = - \left(\frac{C_D S}{m} \right) \delta a^2 \int_0^{2\pi} \rho \frac{(1 + e \cos E)^{3/2}}{(1 - e \cos E)^{1/2}} dE \quad (3.15)$$

$$\Delta \xi = - \left(\frac{C_D S}{m} \right) \delta a \gamma \int_0^{2\pi} \rho \left(\frac{1 + e \cos E}{1 - e \cos E} \right)^{1/2} (\gamma \cos \tilde{\omega} \cos E - \sin \tilde{\omega} \sin E) dE \quad (3.16)$$

and

$$\Delta \eta = - \left(\frac{C_D S}{m} \right) \delta a \gamma \int_0^{2\pi} \rho \left(\frac{1 + e \cos E}{1 - e \cos E} \right)^{1/2} (\gamma \sin \tilde{\omega} \cos E + \cos \tilde{\omega} \sin E) dE \quad (3.17)$$

where $\left(\frac{C_D S}{m} \right)$ is the inverse ballistic coefficient for the satellite, and

$$\delta = \left(1 - \frac{r_p}{v_p} \Lambda \omega_e \cos i \right)^2 \quad (3.18)$$

In the last expression v_p is the speed of the satellite at perigee and Λ is the ratio of atmospheric angular rotation rate to that of the earth, designated by ω_e .

Using the relations

$$\cos \theta = \frac{\cos E - e}{1 - e \cos E} \quad (3.19)$$

and

$$\sin \theta = \frac{(1 - e^2)^{1/2} \sin E}{1 - e \cos E} \quad (3.20)$$

to eliminate θ to first order in e ; applying Equations 3.12 and 3.14; and expanding the resulting integrands of Equations 3.15 - 3.17 yields:

$$\begin{aligned} \Delta a = & - \left(\frac{C_D S}{m} \right) \delta \rho_0 a^2 \exp[-\beta a e - c \cos 2\omega] \int_0^{2\pi} \left\{ 1 + F_A \cos E \right. \\ & + e F_A \cos^2 E - e F_A + F_B \sin E + e F_B \sin E \cos E \left. \right\} \cdot \left\{ 1 \right. \\ & \left. + c \cos 2(\omega + E) - 2c e \sin 2(\omega + E) + \frac{1}{4} c^2 \right\} dE \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} c^2 \cos 4(\omega + E) - c^2 e \sin 4(\omega + E) \sin E \} \cdot \{1 \\
& + 2e \cos E \} \exp [\beta a e \cos E] dE \quad (3.21)
\end{aligned}$$

$$\begin{aligned}
\Delta \xi = & - \left(\frac{C_D S}{m} \right) \delta \rho_0 a \gamma \exp[-\beta a e - c \cos 2\omega] \int_0^{2\pi} \{ \text{SAME} \} \cdot \{ \text{SAME} \} \cdot \{ i \\
& + e \cos E \} \cdot \{ \gamma \cos \tilde{\omega} \cos E - \sin \tilde{\omega} \sin E \} \exp[\beta a e \cos E] dE \quad (3.22)
\end{aligned}$$

and

$$\begin{aligned}
\Delta \eta = & - \left(\frac{C_D S}{m} \right) \delta \rho_0 a \gamma \exp[-\beta a e - c \cos 2\omega] \int_0^{2\pi} \{ \text{SAME} \} \cdot \{ \text{SAME} \} \cdot \{ 1 \\
& + e \cos E \} \cdot \{ \gamma \sin \tilde{\omega} \cos E + \cos \tilde{\omega} \sin E \} \exp[\beta a e \cos E] dE \quad (3.23)
\end{aligned}$$

When the integrands of the last three equations are multiplied, the results contain trigonometric terms that are expressible as functions of $\cos(nE)$, $n = 0, 1, 2, \dots, 7$. This permits them to be written in terms of the integral representation of the Bessel function of the first kind and imaginary argument defined by

$$I_n(\beta a e) = \frac{1}{2\pi} \int_0^{2\pi} \cos(nE) \exp(\beta a e \cos E) dE \quad (3.24)$$

The changes in a , ξ , and η over one orbital revolution due to atmospheric drag deceleration then become

$$\begin{aligned}
\Delta a = & - 2\pi \left(\frac{C_D S}{m} \right) \delta \rho_0 a^2 \exp[-\beta a e - c \cos 2\omega] \left\{ I_0 + 2eI_1 + FA[I_1 \right. \\
& + \frac{1}{2} e(I_0 + 3I_2)] + c \left\{ I_2 + 2eI_3 + \frac{1}{2} FA[I_1 + I_3 \right. \\
& + \frac{1}{2} e(I_0 + 2I_2 + 5I_4)] \} \cos 2\omega - \frac{1}{2} cFB \{ I_1 - I_3 \\
& + \frac{1}{2} e(I_0 + 4I_2 - 5I_4) \} \sin 2\omega + \frac{1}{4} c^2 \{ I_0 + 2eI_1 \\
& + FA[I_1 + \frac{1}{2} e(I_0 + 3I_2)] \} + \frac{1}{4} c^2 \{ I_4 - e(I_3 - 3I_5)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \text{FA} [I_3 + I_5 - \frac{1}{2} e(I_2 - 2I_4 - 7I_6)] \left\{ \cos 4\omega \right. \\
& \left. - \frac{1}{8} c^2 \text{FB} [I_3 - I_5 - \frac{1}{2} e(I_2 - 8I_4 + 7I_6)] \sin 4\omega \right\}
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\Delta \xi = & -2\pi \left(\frac{C_D S}{m} \right) \delta \rho_0 a \exp[-\beta a e - c \cos 2\omega] \left\{ \left(1 + \frac{1}{2} \text{FA} e \right) I_1 \right. \\
& + \frac{1}{2} (e + \text{FA}) (I_0 + I_2) + \frac{1}{2} \text{FA} e I_3 \left. \left(1 + \frac{c^2}{4} \right) \cos \tilde{\omega} \right. \\
& - \frac{1}{2} \text{FB} [I_0 - I_2 + e(I_1 - I_3)] \left. \left(1 + \frac{c^2}{4} \right) \sin \tilde{\omega} \right. \\
& + \left\{ \frac{c}{2} [I_1 + I_3 + \text{FA} e (I_3 + I_5) - \frac{1}{2} e(I_0 - 2I_2 - 3I_4) \right. \\
& + \frac{1}{2} \text{FA} (I_0 + 2I_2 + I_4)] \cos \tilde{\omega} + \frac{1}{4} \text{FB} c [I_0 - 2I_2 + I_4 \\
& + 2e(I_1 - 2I_3 + I_5)] \sin \tilde{\omega} \left. \right\} \cos 2\omega - \frac{1}{4} c \left\{ \text{FB} [I_0 \right. \\
& - I_4 + 2e(I_1 - I_5)] \cos \tilde{\omega} + [e(I_0 - 4I_2 + 3I_4) \\
& + 2(I_3 - I_1) - \text{FA} (I_0 - I_4 + 2e(I_3 - I_5))] \sin \tilde{\omega} \left. \right\} \sin 2\omega \\
& + \frac{c^2}{16} \left\{ [2(I_3 + I_5) + e(5I_6 + 2I_4 - 3I_2) + \text{FA} (I_6 + 2I_4 \right. \\
& + I_2 + e(3I_7 + 3I_5 - I_3 - I_1))] \cos \tilde{\omega} + \text{FB} [I_6 - 2I_4 \\
& + I_2 + e(3I_7 - 7I_5 + 5I_3 - I_1)] \sin \tilde{\omega} \left. \right\} \cos 4\omega \\
& + \frac{c^2}{16} \left\{ \text{FB} [I_6 - I_2 + e(3I_7 - I_5 - 3I_3 + I_1)] \cos \tilde{\omega} \right. \\
& + [2(I_3 - I_5) - e(5I_6 - 8I_4 + 3I_2) - \text{FA} (I_6 - I_2 \\
& + e(3I_7 - 3I_5 - I_3 + I_1))] \sin \tilde{\omega} \left. \right\} \sin 4\omega \left. \right\}
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
\Delta\eta = -2\pi \left(\frac{C_D S}{m} \right) \delta\rho_0 a \exp[-\beta a e - c \cos 2\omega] & \left\{ \frac{1}{2} FB \left(1 + \frac{c^2}{4} \right) [I_0 - I_2 \right. \\
& + e(I_1 - I_3)] \cos \tilde{\omega} + \left[\left(1 + \frac{1}{2} eFA \right) I_1 + \frac{1}{2} (e + FA) (I_0 \right. \\
& + I_2) + \frac{1}{2} eFA I_3 \left. \right] \left(1 + \frac{c^2}{4} \right) \sin \tilde{\omega} - \left\{ \frac{c}{4} FB [I_0 - 2I_2 + I_4 \right. \\
& + 2e(I_1 - 2I_3 + I_5)] \cos \tilde{\omega} - \frac{c}{2} [I_1 + I_3 + eFA(I_3 + I_5) \\
& + e(-\frac{1}{2} I_0 + I_2 + \frac{3}{2} I_4) + FA(\frac{1}{2} I_0 + I_2 + \frac{1}{2} I_4)] \sin \tilde{\omega} \left. \right\} \cos 2\omega \\
& + \frac{c}{4} \left\{ [e(I_0 - 4I_2 + 3I_4) - 2(I_1 - I_3) - FA(I_0 - I_4 + 2e(I_3 \right. \\
& - I_5))] \cos \tilde{\omega} - FB[I_0 - I_4 - 2e(I_1 - I_5)] \sin \tilde{\omega} \left. \right\} \sin 2\omega \\
& - \frac{c^2}{16} \left\{ FB[I_6 - 2I_4 + I_2 + e(3I_7 - 7I_5 + 5I_3 - I_1)] \cos \tilde{\omega} \right. \\
& - [2(I_3 + I_5) + e(5I_6 + 2I_4 - 3I_2) + FA(I_6 + 2I_4 + I_2 \\
& + e(3I_7 + 3I_5 - I_3 - I_1))] \sin \tilde{\omega} \left. \right\} \cos 4\omega - \frac{c^2}{16} \left\{ [2(I_3 - I_5) \right. \\
& - e(5I_6 - 8I_4 + 3I_2) - FA(I_6 - I_2 + e(3I_7 - 3I_5 - I_3 \\
& + I_1))] \cos \tilde{\omega} - FB[I_6 - I_2 + e(3I_7 - I_5 - 3I_3 \\
& + I_1)] \sin \tilde{\omega} \left. \right\} \sin 4\omega \left. \right\} \quad (3.27)
\end{aligned}$$

It should be noted that γ does not appear in the last two equations since it has been replaced with its first-order expansion equivalent of unity. The changes in eccentricity and orbital period per revolution can be found from

$$\Delta e = (\Delta \xi^2 + \Delta \eta^2)^{1/2} \quad (3.28)$$

and

$$\Delta\tau = \frac{3}{2} \frac{\tau}{a} \Delta a \quad (3.29)$$

respectively, where τ is the orbital period.

4. GEOPOTENTIAL PERTURBATIONS IN NONSINGULAR ELEMENTS

In order to evaluate the drag-induced changes in a , ξ , and η given by Equations 3.25 through 3.27 on a rev-by-rev basis, it is necessary to propagate the gravitationally perturbed six-member nonsingular element set to the time of interest. This subsection is devoted to a development of the gravitational disturbing function R of Equation 2.6.

As will be discussed in subsequent studies, computations performed using this development will be done in terms of the mean state. Thus, it is convenient, yet completely general, to divide the gravitational disturbing function into short-periodic and secular-long-periodic parts:

$$R = R_{SLP} + R_{SP} \quad (4.1)$$

where the subscripts "SLP" and "SP" denote the secular-long-periodic and short-periodic parts, respectively, and are given by

$$R_{SLP} = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=2p-\ell} R_{\ell mpq} \quad (4.2)$$

and

$$R_{SP} = \sum_{\ell \geq 2} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q \neq 2p-\ell} R_{\ell mpq} \quad (4.3)$$

The $(\ell mpq)^{th}$ terms in the disturbing function summations are¹¹

$$\begin{aligned}
R_{\ell mpq} = & \left(\frac{\mu a_e^\ell}{a^{\ell+1}} \right) J_{\ell mp}(c) K_{\ell pq}(\gamma) \left\{ \text{I}_{\ell mpq} (A_{\ell m} \cos \theta_{\ell mpq} \right. \\
& + B_{\ell m} \sin \theta_{\ell mpq}) + \text{II}_{\ell mpq} (A_{\ell m} \sin \theta_{\ell mpq} \\
& \left. - B_{\ell m} \cos \theta_{\ell mpq}) \right\}
\end{aligned} \tag{4.4}$$

where a_e is the mean equatorial radius of the earth, μ is the earth's gravitational constant

$$A_{\ell m} = \begin{cases} C_{\ell m}, & \ell - m \text{ even} \\ -S_{\ell m}, & \ell - m \text{ odd} \end{cases} \tag{4.5}$$

$$B_{\ell m} = \begin{cases} S_{\ell m}, & \ell - m \text{ even} \\ C_{\ell m}, & \ell - m \text{ odd} \end{cases} \tag{4.6}$$

and

$$\theta_{\ell mpq} = (\ell - 2p + q)\lambda - m\theta \tag{4.7}$$

Here θ is the Greenwich sidereal time.

The $J_{\ell mp}$ and $K_{\ell pq}$ functions in Equation 4.4 are the inclination and eccentricity functions given by

$$\begin{aligned}
J_{\ell mp}(c) = & (-1)^k \frac{(\ell + m)!}{2^\ell p! (\ell - p)!} \sum_{j=j_1}^{j_2} \left[(-1)^j \binom{2\ell-2p}{j} \binom{2p}{\ell-m-j} c^{2\ell-\alpha-2j} \right. \\
& \left. \cdot (1 - c^2)^{j+(\alpha-|\alpha|)/2} \right]
\end{aligned} \tag{4.8}$$

$$K_{\ell pq}(\gamma) = (-1)^{|q|} 2^{\ell} (1 + \gamma)^{-\ell - |q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^k \left[\frac{(-1)^r}{r!t!} \binom{2p-2\ell}{|q|+k-r} \right. \\ \left. \cdot \binom{-2p}{k-t} \left(\frac{\ell-2p+q}{2} \right)^{r+t} (1 + \gamma)^{r+t-k} (1 - \gamma)^k \right], \quad (\text{for } q > 0) \quad (4.9)$$

and

$$K_{\ell pq}(\gamma) = (-1)^{|q|} 2^{\ell} (1 + \gamma)^{-\ell - |q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^k \left[\frac{(-1)^t}{r!t!} \binom{-2p}{|q|+k-r} \right. \\ \left. \cdot \binom{2p-2\ell}{k-t} \left(\frac{\ell-2p+q}{2} \right)^{r+t} (1 + \gamma)^{r+t-k} (1 - \gamma)^k \right], \quad (\text{for } q < 0) \quad (4.10)$$

where

$$c = \cos(1/2)$$

$$k = \text{integral part of } \left[\frac{\ell-m}{2} \right]$$

$$j_1 = \max(0, -\alpha)$$

$$j_2 = \min(2\ell - 2p, \ell - m)$$

$$\alpha = m - \ell + 2p$$

The $\mathbb{R}_{\ell mpq}$ and $\mathbb{I}_{\ell mpq}$ functions in Equation 4.4 are given by

$$\mathbb{R}_{\ell mpq} = \sum_{n=0}^k \sum_{u=u_1}^{u_2} (-1)^{n+u} \delta_u \binom{|q|}{u} \binom{|\alpha|}{2n-u} \xi^{|q|-u} \eta^u p^{|\alpha|-2n+u} q^{2n-u} \quad (4.11)$$

and

$$\mathbb{I}_{\ell mpq} = \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \delta_u \binom{|q|}{u} \binom{|\alpha|}{2n+1-u} \xi^{|q|-u} \eta^{u_P} |\alpha|^{-2n-1+u} Q^{2n-u+1} \quad (4.12)$$

where

$$k = \left\lfloor \frac{|q| + |\alpha|}{2} \right\rfloor, \quad k' = \left\lfloor \frac{|q| + |\alpha| - 1}{2} \right\rfloor$$

$$u_1 = \max(0, 2n - |\alpha|), \quad u_2 = \min(2n, |q|)$$

$$u_1' = \max(0, 2n + 1 - |\alpha|), \quad u_2' = \min(2n + 1, |q|)$$

$$\delta_u = 1, \text{ if } q, \alpha \text{ are both positive or negative}$$

$$\delta_u = (-1)^u, \text{ if } q \text{ or } \alpha \text{ is negative}$$

The partial derivatives of the inclination and eccentricity functions that are required for use in the Lagrange planetary equations are

$$\left. \begin{aligned} \frac{\partial J_{\ell mp}}{\partial P} &= -2P \frac{\partial J_{\ell mp}}{\partial c} \\ \frac{\partial J_{\ell mp}}{\partial Q} &= -2Q \frac{\partial J_{\ell mp}}{\partial c} \\ \frac{\partial K_{\ell pq}}{\partial \xi} &= -\frac{\xi}{\gamma} \frac{\partial K_{\ell pq}}{\partial \gamma} \\ \frac{\partial K_{\ell pq}}{\partial \eta} &= -\frac{\eta}{\gamma} \frac{\partial K_{\ell pq}}{\partial \gamma} \end{aligned} \right\} \quad (4.13)$$

and

where

$$\frac{\partial J_{\ell mp}}{\partial c} = (-1)^k \frac{(\ell+m)!}{2^\ell p! (\ell-p)!} \sum_{j=j_1}^{j_2} \left[(-1)^j \binom{2\ell-2p}{j} \binom{2p}{\ell-m-j} \right. \\ \left. \cdot \left\{ c^{2\ell-\alpha-2j-1} s^{\alpha-|\alpha|} \left[(2\ell-|\alpha|) s^{2j} - (2j+\alpha-|\alpha|) s^{2j-2} \right] \right\} \right] \quad (4.14)$$

$$s = \sin (1/2)$$

$$\frac{\partial K_{\ell pq}}{\partial \gamma} = \frac{(-\ell-|q|)}{1+\gamma} K_{\ell pq} + (-1)^{|q|} 2^\ell (1+\gamma)^{-\ell-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^k \\ \cdot \left[\frac{(-1)^r}{r!t!} \binom{2p-2\ell}{|q|+k-r} \binom{-2p}{k-t} \left(\frac{\ell-2p+q}{2} \right)^{r+t} (1+\gamma)^{r+t-k-1} [(r+t-k)] \right. \\ \left. \cdot (1-\gamma)^k - k(1+\gamma)(1-\gamma)^{k-1} \right], \quad (q > 0) \quad (4.15)$$

and

$$\frac{\partial K_{\ell pq}}{\partial \gamma} = \frac{(-\ell-|q|)}{1+\gamma} K_{\ell pq} + (-1)^{|q|} 2^\ell (1+\gamma)^{-\ell-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^k \\ \cdot \left[\frac{(-1)^t}{r!t!} \binom{-2p}{|q|+k-r} \binom{2p-2\ell}{k-t} \left(\frac{\ell-2p+q}{2} \right)^{r+t} (1+\gamma)^{r+t-k-1} \right. \\ \left. \cdot [(r+t-k)(1-\gamma)^k - k(1+\gamma)(1-\gamma)^{k-1}] \right], \quad (q < 0) \quad (4.16)$$

The partial derivatives of the $\mathbb{R}$ and $\mathbb{I}$ functions are as follows:

$$\left. \begin{aligned}
 \frac{\partial \mathbb{R}_{\ell m p q}}{\partial \xi} &= |q| \mathbb{R}_{\ell m p q'} ; \quad \frac{\partial \mathbb{R}_{\ell m p q}}{\partial \eta} = -q \mathbb{I}_{\ell m p q'} \\
 \frac{\partial \mathbb{I}_{\ell m p q}}{\partial \xi} &= |q| \mathbb{I}_{\ell m p q'} ; \quad \frac{\partial \mathbb{I}_{\ell m p q}}{\partial \eta} = q \mathbb{R}_{\ell m p q'} \\
 \frac{\partial \mathbb{R}_{\ell m p q}}{\partial P} &= |\alpha| \mathbb{R}_{\ell m' p q} ; \quad \frac{\partial \mathbb{R}_{\ell m p q}}{\partial Q} = -\alpha \mathbb{I}_{\ell m' p q} \\
 \frac{\partial \mathbb{I}_{\ell m p q}}{\partial P} &= |\alpha| \mathbb{I}_{\ell m' p q} ; \quad \frac{\partial \mathbb{I}_{\ell m p q}}{\partial Q} = \alpha \mathbb{R}_{\ell m' p q}
 \end{aligned} \right\} \quad (4.17)$$

where

$$q' = \begin{cases} q - 1, & (q > 0) \\ q + 1, & (q < 0) \end{cases}$$

and

$$m' = \begin{cases} m - 1, & (\alpha > 0) \\ m + 1, & (\alpha < 0) \end{cases}$$

The nonsingular element set of Equations 2.1 can be decomposed into the standard Keplerian element set through application of the following relations:

$$\left. \begin{aligned}
 e &= (\xi^2 + \eta^2)^{1/2} \\
 i &= 2 \sin^{-1} [(P^2 + Q^2)^{1/2}] \\
 \Omega &= \tan^{-1} (Q/P) \\
 \omega &= \tan^{-1} (\eta/\xi) - \Omega \\
 \text{and} \\
 M &= \lambda - (\omega + \Omega)
 \end{aligned} \right\} \quad (4.18)$$

In closing it should be mentioned that the osculating perigee radius r_p is required to compute the decay rates of Equations 3.25 - 3.27. This quantity is given by

$$r_p = r_{SLP} + r_{SP} \quad (4.19)$$

evaluated at $\theta = 0$. The r_{SLP} portion is computed using the a and e obtained from the Lagrange planetary equations when the gravitational disturbing function R is represented by R_{SLP} only. The short periodic variation of the satellite radius is

$$\begin{aligned}
 r_{SP} = \frac{1}{2} C_{20} \left(\frac{a_e^2}{p} \right) \left(1 - \frac{3}{2} \sin^2 i \right) & \left[1 + \frac{e}{1 + \sqrt{1 - e^2}} \cos \theta \right. \\
 & \left. + \frac{2}{\sqrt{1 - e^2}} \left(\frac{r}{a} \right) \right] - \frac{1}{4} C_{20} \left(\frac{a_e^2}{p} \right) \sin^2 i \cos (2\omega + 2\theta)
 \end{aligned} \quad (4.20)$$

where p is the orbital semi-latus rectum.

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APPENDIX A

POSITION PARTIAL DERIVATIVES WITH RESPECT TO ORBITAL ELEMENTS

Relationships that are required for the transformation of R_{α}^D into a more usable form are collected in this appendix. Let x_j , $j = 1, 2, 3$, represent the inertial Cartesian coordinates of the satellite position vector $\vec{r}$. Then

$$\frac{\partial x_j}{\partial a} = \frac{x_j}{a} \quad (A.1)$$

$$\frac{\partial x_1}{\partial \lambda} = \frac{a}{\gamma} \left[\frac{1}{r} \frac{\partial x_1}{\partial u} - e(\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i) \right] \quad (A.2)$$

$$\frac{\partial x_2}{\partial \lambda} = \frac{a}{\gamma} \left[\frac{1}{r} \frac{\partial x_2}{\partial u} - e(\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i) \right] \quad (A.3)$$

$$\frac{\partial x_3}{\partial \lambda} = \frac{a}{\gamma} \left[\frac{1}{r} \frac{\partial x_3}{\partial u} + e \cos \omega \sin i \right] \quad (A.4)$$

$$\frac{\partial x_1}{\partial u} = -r[\sin u \cos \Omega + \cos u \sin \Omega \cos i] \quad (A.5)$$

$$\frac{\partial x_2}{\partial u} = -r[\sin u \sin \Omega - \cos u \cos \Omega \cos i] \quad (A.6)$$

$$\frac{\partial x_3}{\partial u} = r \cos u \sin i \quad (A.7)$$

$$\frac{\partial x_j}{\partial \omega} = \frac{\partial x_j}{\partial u} - \frac{\partial x_j}{\partial \lambda} \quad (A.8)$$

$$\frac{\partial x_j}{\partial \Omega} = (-1)^j \frac{\partial (x_1 x_2)}{\partial x_j} - \frac{\partial x_j}{\partial u} \quad (A.9)$$

$$\frac{\partial x_1}{\partial e} = \frac{\sin \theta}{\gamma} \frac{\partial x_1}{\partial u} - a(\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i) \quad (A.10)$$

$$\frac{\partial x_2}{\partial e} = \frac{\sin \theta}{\gamma^2} \frac{\partial x_2}{\partial u} - a(\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i) \quad (A.11)$$

$$\frac{\partial x_3}{\partial e} = \frac{\sin \theta}{\gamma^2} \frac{\partial x_3}{\partial u} - a \sin \omega \sin i \quad (A.12)$$

$$\frac{\partial x_1}{\partial i} = r \sin u \sin \Omega \sin i \quad (A.13)$$

$$\frac{\partial x_2}{\partial i} = -r \sin u \cos \Omega \sin i \quad (A.14)$$

$$\frac{\partial x_3}{\partial i} = r \sin u \cos i \quad (A.15)$$

$$\frac{\partial x_j}{\partial \xi} = \frac{\partial x_j}{\partial e} \cos \tilde{\omega} - \frac{\partial x_j}{\partial \tilde{\omega}} \frac{1}{e} \sin \tilde{\omega} \quad (A.16)$$

$$\frac{\partial x_j}{\partial \eta} = \frac{\partial x_j}{\partial e} \sin \tilde{\omega} + \frac{\partial x_j}{\partial \tilde{\omega}} \frac{1}{e} \cos \tilde{\omega} \quad (A.17)$$

$$\frac{\partial x_j}{\partial P} = 2 \frac{\partial x_j}{\partial i} \frac{\cos \Omega}{\cos (i/2)} - \frac{\partial x_j}{\partial \Omega} \frac{\sin \Omega}{\sin (i/2)} \quad (A.18)$$

and

$$\frac{\partial x_j}{\partial Q} = 2 \frac{\partial x_j}{\partial i} \frac{\sin \Omega}{\cos (i/2)} + \frac{\partial x_j}{\partial \Omega} \frac{\cos \Omega}{\sin (i/2)} \quad (A.19)$$

In the above expressions u is the true argument of latitude defined by

$$u = \omega + \theta \quad (A.20)$$

APPENDIX B

A DERIVATION OF THE INTEGRAL EXPRESSIONS FOR Δa , $\Delta \xi$, $\Delta \eta$

In order to obtain integral expressions for Δa , $\Delta \xi$, and $\Delta \eta$, it is necessary to develop the Gauss planetary equations from the Lagrange planetary equations of section 2 for the case when

$$R_\alpha = R_\alpha^D = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \alpha} \quad (\text{B.1})$$

This is accomplished by first resolving the disturbing force $\vec{F}$ into three components U, V, and W, where W is perpendicular to the plane of the orbit (positive toward the north pole); V is in the plane of the orbit at right angles to the radius vector (positive in the direction of motion); and U is along the radius vector (positive in the positive r direction). The required transformation is:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \cos u \cos \Omega & -\sin u \cos \Omega & \sin \Omega \sin i \\ -\sin u \sin \Omega \cos i & -\cos u \sin \Omega \cos i & \\ \cos u \sin \Omega & -\sin u \sin \Omega & -\cos \Omega \sin i \\ +\sin u \cos \Omega \cos i & +\cos u \cos \Omega \cos i & \\ \sin u \sin i & \cos u \sin i & \cos i \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad (\text{B.2})$$

Applying Equations B.2 along with the relationships of Appendix A to Equation B.1 gives the following results for R_α^D :

$$R_a^D = \left(\frac{r}{a}\right)U \quad (\text{B.3})$$

$$R_\lambda^D = \left(\frac{ae}{\gamma} \sin \theta\right)U + \left(\frac{a^2 \gamma}{r}\right)V \quad (\text{B.4})$$

$$R_{\xi}^D = a \left(\frac{\sin \theta \sin \tilde{\omega}}{\gamma} - \cos \tilde{\omega} \cos \theta \right) U + \left[a \left(\frac{\cos \theta \sin \tilde{\omega}}{\gamma} + \cos \tilde{\omega} \sin \theta \right) - A \right] V \quad (B.5)$$

$$R_{\eta}^D = -a \left(\frac{\sin \theta \cos \tilde{\omega}}{\gamma} + \sin \tilde{\omega} \cos \theta \right) U + \left[a \left(\sin \tilde{\omega} \sin \theta - \frac{\cos \tilde{\omega} \cos \theta}{\gamma} \right) - B \right] V \quad (B.6)$$

$$R_P^D = 2r \left[QV + 2 \left(\frac{P \sin u + Q \cos u \cos^2 (1/2)}{\sin i} \right) W \right] \quad (B.7)$$

and

$$R_Q^D = -2r \left[PV - 2 \left(\frac{Q \sin u - P \cos u \cos^2 (1/2)}{\sin i} \right) W \right] \quad (B.8)$$

where

$$A = -r \left[\frac{\sin \theta \cos \tilde{\omega}}{\gamma^2} - \left(1 - \frac{a}{\gamma r} \right) \frac{\sin \tilde{\omega}}{e} \right] \quad (B.9)$$

$$B = -r \left[\frac{\sin \theta \sin \tilde{\omega}}{\gamma^2} + \left(1 - \frac{a}{\gamma r} \right) \frac{\cos \tilde{\omega}}{e} \right] \quad (B.10)$$

$$\gamma = \sqrt{1 - e^2} \quad (B.11)$$

and

$$u = \theta + \omega \quad (B.12)$$

The results of Equations B.3 - B.12 can be applied to Equations 2.2 to provide the U, V, W representation of the Gauss planetary equations. Since only the retarding force that is opposite the direction of motion is considered here, it is convenient to transform the equations of motion for a, ξ , and η from

the U, V, W representation to the T, N, W representation, where T is the component along the velocity vector; N is normal to T (positive toward the interior of the ellipse); and W is normal to the orbital plane as before. The required transformation is:

$$\begin{bmatrix} V \\ U \end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix} (1 + e \cos \theta) & e \sin \theta \\ e \sin \theta & - (1 + e \cos \theta) \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix} \quad (\text{B.13})$$

where

$$\sigma = \sqrt{1 + e^2 + 2e \cos \theta} \quad (\text{B.14})$$

Applying this transformation to the U, V, W representation for $\dot{a}$, $\dot{\xi}$, and $\dot{\eta}$ and retaining only the T component yields:

$$\dot{a} = \frac{2T}{\eta\gamma} \sigma \quad (\text{B.15})$$

$$\begin{aligned} \dot{\xi} = \frac{\gamma T}{na\sigma} [2 \cos (\tilde{\omega} + \theta) + e(\sin (\tilde{\omega} + \theta) \sin \theta \\ + \cos (\tilde{\omega} + \theta) \cos \theta + \cos \tilde{\omega})] \end{aligned} \quad (\text{B.16})$$

and

$$\begin{aligned} \dot{\eta} = \frac{\gamma T}{na\sigma} [2 \sin (\tilde{\omega} + \theta) + e(\sin (\tilde{\omega} + \theta) \cos \theta \\ - \cos (\tilde{\omega} + \theta) \sin \theta + \sin \tilde{\omega})] \end{aligned} \quad (\text{B.17})$$

The retarding force due to atmospheric drag resistance is

$$T = - \frac{1}{2} \left(\frac{C_D S}{m} \right) \delta \rho v^2 \quad (\text{B.18})$$

where δ is given by Equation 3.18 and ρ is the atmospheric density. Substituting Equation B.18 into Equations B.15 - B.17; using Equations 3.19 - 3.20; and applying the relations

$$v^2 = \left(\frac{\mu}{a}\right) \left(\frac{1 + e \cos E}{1 - e \cos E}\right) \quad (\text{B.19})$$

and

$$\frac{d}{dt} = \left(\frac{n}{1 - e \cos E}\right) \frac{d}{dE} \quad (\text{B.20})$$

allows the following to be written

$$\frac{da}{dE} = - \left(\frac{C_D S}{m}\right) \delta a^2 \rho \frac{(1 + e \cos E)^{3/2}}{(1 - e \cos E)^{1/2}} \quad (\text{B.21})$$

$$\frac{d\xi}{dE} = - \left(\frac{C_D S}{m}\right) \delta a \gamma \rho \left(\frac{1 + e \cos E}{1 - e \cos E}\right)^{1/2} (\gamma \cos \tilde{\omega} \cos E - \sin \tilde{\omega} \sin E) \quad (\text{B.22})$$

and

$$\frac{d\eta}{dE} = - \left(\frac{C_D S}{m}\right) \delta a \gamma \rho \left(\frac{1 + e \cos E}{1 - e \cos E}\right)^{1/2} (\gamma \sin \tilde{\omega} \cos E + \cos \tilde{\omega} \sin E) \quad (\text{B.23})$$

The changes in these elements over one orbital revolution are obtained by integrating the last three equations over 2π radians of eccentric anomaly to give the final results

$$\Delta a = - \left(\frac{C_D S}{m}\right) \delta a^2 \int_0^{2\pi} \rho \frac{(1 + e \cos E)^{3/2}}{(1 - e \cos E)^{1/2}} dE \quad (\text{B.24})$$

$$\begin{aligned} \Delta \xi = & - \left(\frac{C_D S}{m}\right) \delta a \gamma \int_0^{2\pi} \rho \left(\frac{1 + e \cos E}{1 - e \cos E}\right)^{1/2} (\gamma \cos \tilde{\omega} \cos E \\ & - \sin \tilde{\omega} \sin E) dE \end{aligned} \quad (\text{B.25})$$

and

$$\Delta\eta = -\left(\frac{C_D S}{m}\right) \delta a \gamma \int_0^{2\pi} \rho \left(\frac{1 + e \cos E}{1 - e \cos E} \right)^{1/2} (\gamma \sin \tilde{\omega} \cos E + \cos \tilde{\omega} \sin E) dE \quad (\text{B.26})$$

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