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RESIDUES OF INTEGRALS WITH THREE- DIMENSIONAL MULTIPOLE SINGULARITIES, WITH APPLICATION TO THE LAGALLY THEOREM

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I. INTRODUCTION

An important mathematical relation in the theory of three-dimensional irrotational flow is the Gauss-Green transformation between volume and surface integrals. A very useful result of this transformation, Green's third formula, requires, in its derivation, the evaluation of the limit of a singular integral over the surface of a sphere as the radius of the sphere approaches zero. Since, in this case, the singularity is due to the potential of a source at the center of the sphere, its limit may be called the residue of a source. Similarly, limits of integrals over the surface of a sphere of vanishingly small radius, with higher-order derivatives of the source potential in the integrand, will be called residues of multipoles. The latter occur in the derivation of the Lagally theorem for the force and moment acting on a body moving in an irrotational flow when multipoles are present in the hydrodynamic singularity system within the body (see Landweber, 1967 and Landweber and Miloh, 1980, 1981).

In contrast to the very simple derivation of the residue occurring in Green's third formula, the evaluation of the multipole residues was a challenging application of the theory of spherical harmonics. The derivations of a set of multipole residues, which were required but not included in the aforementioned references, will be presented here.

This work is dedicated to my long-time friend and colleague, Chia-Shun Yih, with whom I worked 30 years ago on the generalization of the Lagally theorem to unsteady flows (1956). Support by the Office of Naval Research under Contract N00014-83-K-0136 (NR-062-183) is gratefully acknowledged.

II. FORMULATION AND TABULATION OF RESULTS

The velocity potential ϕ_p of a multipole of strength M and of order $q = \alpha + \beta + \gamma$, situated at a point $P(x_s, y_s, z_s)$, with position vector $\bar{r}_p$, of a rectangular Cartesian coordinate system (x, y, z) will be expressed as

$$\phi_P = -M D_S^q \left(\frac{1}{R} \right), \quad D_S^q = \frac{\partial^q}{\partial x_s^\alpha \partial y_s^\beta \partial z_s^\gamma}, \quad R^2 = (x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2 \quad (1)$$

Let S denote the surface of a sphere of radius c with center at the point $O(x_O, y_O, z_O)$ and position vector $\bar{r}_O$ such that $c > b = |\bar{r}_P - \bar{r}_O|$. It is necessary to displace the singularity from the origin in order to ensure that the coordinates (x, y, z) of a point on S are independent of (x_s, y_s, z_s) . Put

$$\bar{\rho} = \bar{r} - \bar{r}_O = \rho \bar{n}, \quad \bar{\rho}_P = \bar{r}_P - \bar{r}_O = b \bar{n}_P \quad (2)$$

where $\bar{n}_P$ is the unit vector in the direction from O to P and $\bar{n}$ the unit vector from O to the point (x, y, z) .

The integrals of interest involve a pair of vectors $\bar{u}_P$ and $\bar{v}$ where $\bar{u}_P = \nabla \phi_P$, $\bar{v} = \nabla \Psi$, and Ψ is regular harmonic within the sphere and on S . Thus both $\bar{u}_P$ and $\bar{v}$ are irrotational and solenoidal within the sphere and on S , except for the multipole singularity of ϕ_P at P . Results for the residues are given in Table 1. Their derivations will be presented in the following sections.

We observe that, when $q = 0$, the first integral in Table 1 reduces to the residue of Green's third formula,

$$\lim_{c \rightarrow 0} \int_S \Psi \bar{n} \cdot \nabla (1/R) dS = -4\pi M \Psi(x_s, y_s, z_s) \quad (3)$$

The second integral is a corollary of the first since each of its components is of the type of the first. This shows that the second integral does not require the conditions $\nabla \times \bar{v} = 0$ and $\nabla \cdot \bar{v} = 0$. It is included in the list of integrals because pairs of the three integrals, (5), (6) and (7), yield the residues for the three vector-product combinations of $\bar{u}_P$, $\bar{v}$ and $\bar{n}$, viz:

Table 1. Multipole Residues

$$\lim_{c \rightarrow 0} \int_S \bar{n} \cdot \bar{u}_P \psi dS = \frac{4\pi M(q+1)}{2q+1} D_S^q(\psi)_P^* \quad (4)$$

$$\lim_{c \rightarrow 0} \int_S \bar{n} \cdot \bar{u}_P \bar{v} dS = \frac{4\pi M(q+1)}{2q+1} D_S^q(\bar{v})_P \quad (5)$$

$$\lim_{c \rightarrow 0} \int_S \bar{u}_P \cdot \bar{v} \bar{n} dS = \frac{4\pi M}{(2q+1)(2q+3)} D_S^q(\bar{v})_P \quad (6)$$

$$\lim_{c \rightarrow 0} \int_S \bar{v} \cdot \bar{n} \bar{u}_P dS = \frac{4\pi M(q+1)}{2q+3} D_S^q(\bar{v})_P \quad (7)$$

$$\lim_{c \rightarrow 0} \int_S \bar{n} \cdot \bar{u}_P \rho x v dS = \frac{4\pi M(q+1)}{2q+1} [D_S^q(\bar{r} \times \bar{v})_P - \bar{r}_P \times D_S^q(\bar{v})_P] \quad (8)$$

$$\lim_{c \rightarrow 0} \int_S \bar{v} \cdot \bar{n} \rho x u_P dS = \frac{4\pi Mq}{2q+1} [D_S^q(\bar{r} \times \bar{v})_P - \bar{r}_P \times D_S^q(\bar{v})_P] \quad (9)$$

$$\lim_{c \rightarrow 0} \int_S \Phi_P \bar{v} \cdot \bar{n} dS = -\frac{4\pi Mq}{2q+1} D_S^q(\psi)_P \quad (10)$$

$$\lim_{c \rightarrow 0} \int_S \Phi_P \bar{n} dS = -\frac{4\pi}{3} M \delta_{1q} D_S^q \bar{r}_P \quad (11)$$

$$\lim_{c \rightarrow 0} \int_S \Phi_P \bar{r} \times \bar{n} dS = -\frac{4\pi}{3} M \delta_{1q} \bar{r}_P \times D_S^q \bar{r}_P \quad (12)$$

* ()_P indicates that the quantity between the parentheses is evaluated at the point P(x_S, y_S, z_S).

$$\lim_{c \rightarrow 0} \int_S \overline{nx}(\overline{vxu}_p) dS = \frac{8\pi M(q+1)}{(2q+1)(2q+3)} D_S^q(\overline{v})_P \quad (13)$$

$$\lim_{c \rightarrow 0} \int_S \overline{vx}(\overline{u}_p \overline{xn}) dS = \frac{4\pi Mq}{2q+1} D_S^q(\overline{v})_P \quad (14)$$

$$\lim_{c \rightarrow 0} \int_S \overline{u}_p \overline{x}(\overline{nxv}) dS = - \frac{4\pi M(q+2)}{2q+3} D_S^q(\overline{v})_P \quad (15)$$

III. SOME LEMMAS

It will facilitate the derivations if we first collect some useful spherical-harmonic properties. Put $\xi = x - x_0$, $\eta = y - y_0$, $\zeta = z - z_0$, so that $\overline{\rho}$ has the components (ξ, η, ζ) . We shall also employ spherical coordinates (ρ, μ, ϕ) , $\mu = \cos \theta$, with the origin at the point 0 and the polar axis along $\overline{OP}$.

a) The scalar Ψ and the vector $\overline{v}(\xi, \eta, \zeta)$ may be expanded in the forms

$$\Psi = \sum_{m=0}^{\infty} \Psi_m(\xi, \eta, \zeta), \quad \Psi_m = \rho^m S_m(\mu, \phi) \quad (16a)$$

$$\overline{v} = \sum_{m=0}^{\infty} \overline{v}_m(\xi, \eta, \zeta), \quad \overline{v}_m = \rho^m \overline{T}_m(\mu, \phi) \quad (16b)$$

where Ψ_m and $\overline{v}_m$ are solid spherical harmonics, homogeneous of degree m in ξ, η, ζ , and S_m and $\overline{T}_m$ are the corresponding surface spherical harmonics of order m .

b) Since Ψ_m and $\overline{v}_m$ are homogeneous in ξ, η, ζ , they satisfy the Euler relations

$$\overline{\rho} \cdot \nabla \Psi_m = m \Psi_m, \quad \overline{\rho} \cdot \nabla \overline{v}_m = m \overline{v}_m \quad (17)$$

c) When the irrotational vector $\bar{v}$ is regular harmonic, the quantities $\overline{\rho x v}$ and $\overline{\rho \cdot v}$ are also regular harmonic.

Proof: In index notation with the summation convention, we have

$$\begin{aligned} \frac{\partial^2}{\partial \xi_\ell \partial \xi_\ell} \epsilon_{ijk} \xi_j v_k &= \epsilon_{ijk} \left(\xi_j \frac{\partial^2 v_k}{\partial \xi_\ell \partial \xi_\ell} + 2 \frac{\partial v_k}{\partial \xi_\ell} \delta_{j\ell} \right) = 2 \epsilon_{ijk} \frac{\partial v_k}{\partial \xi_j} = 0 \\ \frac{\partial^2}{\partial \xi_\ell \partial \xi_\ell} \xi_i v_i &= \frac{\partial^2 v_i}{\partial \xi_\ell \partial \xi_\ell} + 2 \frac{\partial v_i}{\partial \xi_\ell} \delta_{i\ell} = 2 \frac{\partial v_i}{\partial \xi_i} = 0 \end{aligned}$$

where ϵ_{ijk} is the permutation tensor, and δ_{ij} is the kronecker delta function.

d) The orthogonality relation between Legendre polynomials $P_n(\mu)$ and a surface spherical harmonic S_m or $\bar{T}_m$ shows that the integral of their product over the unit sphere is proportional to the value of S_m or $\bar{T}_m$ when $\mu = 1$; (see MacRobert, 1945). Since S_m or $\bar{T}_m$ is homogeneous of degree m in ξ, η, ζ , and the point P lies on the polar axis, this result may be expressed in terms of $(\psi)_{mP} = \rho_P^m (S)_{mP}$ or $(\bar{v})_{mP} = \rho_P^m (\bar{T})_{mP}$,

$$b^m \int_{-1}^1 \int_0^{2\pi} P_n(\mu) \left\{ \begin{matrix} S \\ \bar{T} \end{matrix} \right\}_m d\mu d\phi = \frac{4\pi \delta_{mn}}{2n+1} \left\{ \begin{matrix} (\psi) \\ (\bar{v}) \end{matrix} \right\}_{mP} \quad (18)$$

e)

$$b^{n-1} \int_{-1}^1 \int_0^{2\pi} P_n^1(\mu) e^{i\phi} S_m(\mu, \phi) d\mu d\phi = \frac{4\pi \delta_{mn}}{2n+1} \left(\frac{\partial \psi}{\partial y} + i \frac{\partial \psi}{\partial z} \right)_P \quad (19)$$

where $P_n^1(\mu)$ is the associated Legendre function of degree n and order 1, $i = \sqrt{-1}$, and the (x, y, z) -coordinate system has been taken so that the x -axis is parallel to the polar axis of the spherical coordinate system.

Proof: $S_m(\mu, \phi)$ may be expanded in the series

$$S_m = A P_m^1(\mu) + P_n^1(\mu) (B \cos \phi + C \sin \phi) + \dots$$

Then, by orthogonality, we have,

$$\int_{-1}^1 \int_0^{2\pi} P_n^1(\mu) e^{i\phi} S_m d\mu d\phi = \frac{2\pi n(n+1)}{2n+1} \delta_{mn} (B+iC) \quad (20)$$

In terms of unit vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ in the directions of increasing values of (ρ, θ, ϕ) , the gradient of Ψ_m at $(1, 0, \phi)$ is

$$(\nabla \Psi_m)_{1,0,\phi} = \bar{e}_1 A + \frac{1}{2} m(m+1) [\bar{e}_2 (B \cos \phi + C \sin \phi) + \bar{e}_3 (-B \sin \phi + C \cos \phi)]$$

Then, taking components of the gradient in the y- and z-directions, we obtain

$$\left(\frac{\partial \Psi_m}{\partial y}\right)_{1,0,\phi} = \frac{1}{2} m(m+1) B, \quad \left(\frac{\partial \Psi_m}{\partial z}\right)_{1,0,\phi} = \frac{1}{2} m(m+1) C$$

which, substituted for B and C in (20), yields (19), the result we wished to prove.

$$f) \quad \lim_{c \rightarrow 0} D_S^q \left\{ \frac{\Psi}{v} \right\} = D_S^q \left\{ \frac{\Psi_m}{v_m} \right\} \delta_{qm} \quad (21)$$

since the terms of the expansion (16a,b) with $m < q$ vanish under q differentiations, and those with $m > q$ vanish as $c \rightarrow 0$.

g) Two different expansions of the velocity field of a multipole $\bar{u}_p$ will be useful. We have

$$\bar{u}_p = M D_S^q (\bar{R}/R^3), \quad \bar{R} = \bar{\rho} - \bar{\rho}_p \quad (22)$$

The expansion of $1/R$ then gives

$$\frac{\bar{R}}{R} = \nabla_S \frac{1}{R} = \nabla_S \sum_{n=1}^{\infty} \frac{b^n}{\rho^{n+1}} P_n(\mu), \quad \mu = \bar{n} \cdot \bar{n}_p, \quad \bar{n}_p = \bar{\rho}_p / \rho_p \quad (23a)$$

and

$$\frac{\bar{R}}{R} = \sum_{n=1}^{\infty} \frac{b^{n-1}}{\rho} [\dot{P}_n(\mu) \bar{n} - \dot{P}_{n-1}(\mu) \bar{n}_p] \quad (23b)$$

since

$$\begin{aligned}\nabla_s (b^n P_n) &= nb^{n-1} P_n \bar{n} + b^n \dot{P}_n \nabla_s \left(\frac{\bar{\rho}_P}{b} \right) \cdot \bar{n} \\ &= b^{n-1} [nP_n \bar{n} + \dot{P}_n (\bar{n} - \mu \bar{n}_P)]\end{aligned}$$

and (see MacRobert, 1947)

$$\mu \dot{P}_n(\mu) - n P_n(\mu) = \dot{P}_{n-1}(\mu) \quad (24)$$

Here the dot indicates differentiation with respect to the argument. We shall also require the relation

$$\dot{P}_n(\mu) - \mu \dot{P}_{n-1}(\mu) = n P_{n-1}(\mu) \quad (25)$$

IV. DERIVATIONS OF RESIDUES

The residues in Table 1 will be derived in the order shown, except that we must first verify (15) in order to derive (6).

A. Derivation of (4) and (5). Substituting (16a) for Ψ , (22) and (23b) for $\bar{u}_P$, putting

$$dS = -c^2 d\mu d\phi$$

and applying (25), we obtain

$$\int_S \bar{n} \cdot \bar{u}_P \Psi dS = MD_S^Q \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_{-1}^1 \int_0^{2\pi} c^{m-n+1} b^{n-1} n P_{n-1}(\mu) S_m d\mu d\phi$$

Orthogonality of the surface spherical harmonics requires $n=m+1$, so that the right number of this equation becomes

$$MD_S^q \sum_{m=0}^{\infty} (m+1)b^m \int_{-1}^1 \int_0^{2\pi} P_m(\mu) S_m d\mu d\phi$$

or, by (18),

$$\lim_{c \rightarrow 0} M \sum_{m=0}^{\infty} \frac{4\pi(m+1)}{2m+1} D_S^q(\Psi_m)_P = \frac{4\pi M(q+1)}{2q+1} D_S^q(\Psi)_P$$

since the terms of the expansion of Ψ in (16a) vanish under q differentiations, and those with $m > q$ vanish as $c \rightarrow 0$. This is the result given in (4). As has already been remarked, the residue (5) is an immediate consequence of (4).

B. Derivation of (6) and (15). Substituting (16b) for $\bar{v}$, and (22) and (23a) for $\bar{u}_P$, we obtain

$$\int_S \bar{u}_P x(\bar{n}x\bar{v}) dS = MD_S^q \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c^{m-n+1} b^n \nabla_S x \int_{-1}^1 \int_0^{2\pi} P_n(\mu) \bar{n}x\bar{T}_m d\mu d\phi$$

According to Lemma c, $\bar{n}x\bar{T}_m$ is a surface spherical harmonic of order $m+1$, and hence, by orthogonality, the integral vanishes unless $n=m+1$. Then, by (18),

$$\begin{aligned} \lim_{c \rightarrow 0} \int_S \bar{u}_P x(\bar{n}x\bar{v}) dS &= MD_S^q \nabla_S x \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} (\bar{\rho}x\bar{v}_{n-1})_P \\ &= \frac{4\pi M}{2q+3} D_S^q [\nabla_S x(\bar{\rho}x\bar{v}_q)]_P \end{aligned}$$

since the differential operator $D_S^q \nabla_S$ is of order $q+1$. But

$$\nabla_S x(\bar{\rho}x\bar{v})_P = (\bar{v}_q \cdot \nabla_S \bar{\rho} + \bar{\rho} \nabla_S \cdot \bar{v}_q - \bar{\rho} \cdot \nabla_S \bar{v}_q - \bar{v}_q \nabla_S \cdot \bar{\rho})_P = - (q+2)$$

since

$$\overline{\mathbf{v}}_q \cdot \overline{\nabla} \overline{\rho} = \overline{\mathbf{v}}_q, \quad \nabla \cdot \overline{\mathbf{v}}_q = 0, \quad \nabla \cdot \overline{\rho} = 3$$

and, by (17),

$$\overline{\rho} \cdot \overline{\nabla} \overline{\mathbf{v}}_q = q \overline{\mathbf{v}}_q$$

Then, by (21),

$$\lim_{c \rightarrow 0} \int_S \overline{\mathbf{u}}_P \cdot \overline{\mathbf{x}} (\overline{\mathbf{n}} \times \overline{\mathbf{v}}) dS = - \frac{4\pi M(q+2)}{2q+3} D_S^q(\overline{\mathbf{v}}_q)_P = - \frac{4\pi M(q+2)}{2q+3} D_S^q(\overline{\mathbf{v}})_P$$

as we wished to show.

The result in (6) can now be obtained as the sum of the residues in (5) and (15), since

$$\overline{\mathbf{u}}_P \cdot \overline{\mathbf{v}} \overline{\mathbf{n}} = \overline{\mathbf{u}}_P \cdot \overline{\mathbf{x}} (\overline{\mathbf{n}} \times \overline{\mathbf{v}}) + \overline{\mathbf{n}} \cdot \overline{\mathbf{u}}_P \overline{\mathbf{v}}$$

C. Derivation of (7). By the same procedure as in the previous cases, the integral in (7) is first expressed as a double summation and then reduced to the single term

$$\lim_{c \rightarrow 0} \int_S \overline{\mathbf{v}} \cdot \overline{\mathbf{n}} \overline{\mathbf{u}}_P dS = \frac{4\pi M}{2q+3} D_S^q \nabla (\overline{\rho} \cdot \overline{\mathbf{v}}_q)_P$$

since, by Lemma c, $\overline{\mathbf{T}}_m \cdot \overline{\mathbf{n}}$ is a surface spherical harmonic. Also put $\overline{\mathbf{v}}_q = \nabla \Psi_{q+1}$ where Ψ_{q+1} is a spherical harmonic of degree $q+1$. Then we have, by (17),

$$\nabla_S (\overline{\rho} \cdot \overline{\mathbf{v}}_q)_P = \nabla_S (\overline{\rho} \cdot \nabla \Psi_{q+1})_P = (q+1) \nabla_S (\Psi_{q+1})_P = (q+1) (\overline{\mathbf{v}}_q)_P$$

Hence, by (21),

$$\lim_{c \rightarrow 0} \int_S \overline{\mathbf{v}} \cdot \overline{\mathbf{n}} \overline{\mathbf{u}}_P dS = \frac{4\pi M(q+1)}{2q+3} D_S^q(\overline{\mathbf{v}}_q)_P = \frac{4\pi M(q+1)}{2q+3} D_S^q(\overline{\mathbf{v}})_P$$

as was stated.

D. Derivation of (8). Since $\overline{\rho x v}$ is regular harmonic (Lemma c) and $\overline{\rho} = \overline{r} - \overline{r}_0$, we obtain from (5)

$$\begin{aligned} \lim_{c \rightarrow 0} \int_S \overline{n} \cdot \overline{u}_P \overline{\rho x v} dS &= \lim_{c \rightarrow 0} \frac{4\pi Mq}{2q+1} D_S^q [(\overline{r} - \overline{r}_0) x \overline{v}]_P \\ &= \frac{4\pi Mq}{2q+1} [D_S^q (\overline{r} x \overline{v})_P - \overline{r}_P x D_S^q (\overline{v})_P] \end{aligned}$$

in agreement with (8).

E. Derivation of (9). As in the derivation of (7), we have

$$\overline{\rho} \cdot \overline{v} = \overline{\rho} \cdot \nabla \Psi = \sum_{m=0}^{\infty} \overline{\rho} \cdot \nabla \Psi_m = \sum_{m=0}^{\infty} m \Psi_m = \sum_{m=0}^{\infty} m \rho^m S_m \quad (26)$$

Hence, by (22) and (23b), since $\overline{\rho} = \overline{\rho n}$, the integral in (9) becomes

$$I_9 = \int_S \overline{v} \cdot \overline{n} \overline{\rho x u}_P dS = M D_S^q \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{m b^{n-1} \overline{n}}{c^{n-m-1}} x \int_{-1}^1 \int_0^{2\pi} \overline{n} \cdot \overline{p}_{n-1}(\mu) S_m d\mu d\phi$$

We have

$$\overline{n} = \mu \overline{n}_P + \sin \theta (\overline{j} \cos \phi + \overline{k} \sin \phi) \quad (27)$$

where $\overline{n}_P$, $\overline{j}$ and $\overline{k}$ are an orthogonal set of unit vectors. Hence, applying (19) and putting $b \overline{n}_P = \overline{\rho}_P$, we obtain

$$\overline{\rho}_P x (\overline{v}_{m-1})_P I_9 = M D_S^q \sum_m \frac{4\pi m}{2m+1} \overline{\rho}_P x (\overline{v}_{m-1})_P = \frac{4\pi Mq}{2q+1} D_S^q (\overline{\rho x v})_P$$

and then, since $\overline{\rho}_P = \overline{r}_P - \overline{r}_0$,

$$\lim_{c \rightarrow 0} I_9 = \frac{4\pi Mq}{2q+1} [D_S^q (\overline{r x v})_P - \overline{r}_P x D_S^q (\overline{v})_P]$$

in agreement with (9).

F. Derivation of (10). The desired result is immediately obtained by writing

$$\phi_P = -MD_S^Q \sum_{n=0}^{\infty} \frac{b^n}{c^{n+1}} P_n(\mu) \quad (28)$$

and applying (26), (18), and (21).

G. Derivation of (11). Since $\bar{n}$ is also a surface spherical harmonic, application of (27) and (28) yields, by orthogonality,

$$\int_S \phi_P \bar{n} dS = -\frac{4\pi M}{3} \delta_{1q}^D \delta_{sq} (b \bar{n}_P) = -\frac{4\pi M}{3} \delta_{1q}^D \delta_{sq} \bar{r}_P$$

since $b \bar{n}_P = \bar{\rho}_P = \bar{r}_P - \bar{r}_0$. This agrees with (11).

H. Derivation of (12). Substituting $\bar{r} = \bar{r}_0 + c \bar{n}$ into (12) and applying (11), we obtain

$$\begin{aligned} \lim_{c \rightarrow 0} \int_S \phi_P \bar{r} \bar{n} dS &= \lim_{c \rightarrow 0} \bar{r}_0 \times \int_S \phi_P \bar{n} dS \\ &= -\frac{4\pi}{3} M \delta_{1q}^D \bar{r}_P \times \delta_{sq}^D \bar{r}_P \end{aligned}$$

in agreement with (12).

V. EXAMPLE

Force on a Stationary Sphere in a Steady, Slightly Nonuniform Flow

A sphere of radius c is introduced into an irrotational flow of an inviscid fluid. We shall employ a rectangular, Cartesian coordinate system (x, y, z) , and a spherical coordinate system (R, μ, ϕ) , $\mu = \cos \theta$, with

polar axis along the x-axis and both with origin at the center of the sphere, so that

$$x = R \cos \theta, y = R \sin \theta \cos \phi, z = R \sin \theta \sin \phi \quad (29)$$

The velocity potential of the undisturbed flow will be written as

$$\begin{aligned} \Psi = UR\mu + R^2 S_2(\mu, \phi), \quad S_2 = A_0 P_2(\mu) + P_2^1(\mu)(A_1 \cos \phi + B_1 \sin \phi) \\ + P_2^2(\mu)(A_2 \cos 2\phi + B_2 \sin 2\phi) \end{aligned} \quad (30)$$

This corresponds to the most general harmonic flow quadratic in x, y, z . The x-axis has been taken in the direction of the uniform stream, represented by the first term. Its expression in terms of x, y, z is

$$\begin{aligned} \Psi = Ux + \frac{1}{2} A_0 (2x^2 - y^2 - z^2) + 3A_1 xy + 3B_1 xz + 3A_2 (y^2 - z^2) + 3B_2 yz \\ = Ux + A_0 x^2 + (3A_2 - \frac{1}{2}A_0)y^2 - (3A_2 + \frac{1}{2}A_0)z^2 + 3B_2 yz + 3B_1 xz + 3A_1 xy \end{aligned} \quad (31)$$

The velocity field corresponding to Ψ is then

$$\begin{aligned} \bar{v} = \nabla \Psi = \bar{i}(U + 2A_0 x + 3A_1 y + 3B_1 z) + \bar{j}[3A_1 x + (6A_2 - A_0)y + 3B_2 z] \\ + \bar{k}[3B_1 x + 3B_2 y - (6A_2 + A_0)z] \end{aligned} \quad (32)$$

where $\bar{i}, \bar{j}, \bar{k}$ are the unit vectors in the x, y, z directions.

The disturbance potential Φ which satisfies the boundary condition

$$\frac{\partial}{\partial R} (\Phi + \Psi) \Big|_{R=c} = 0 \quad (33)$$

is

$$\phi = \frac{Uc}{2R} \mu + \frac{2c}{3R} S_2(\mu, \phi) \quad (34)$$

Each of the terms of (34) can be expressed in terms of multipoles situated at the center of the sphere by applying the formula given by Hobson (1931),

$$\frac{1}{R^{n+1}} P_n^m(\mu) e^{im\phi} = \frac{(-1)^n}{(n-m)!} \frac{\partial^{n-m}}{\partial x^{n-m}} \left(\frac{\partial}{\partial y} + i \frac{\partial}{\partial z} \right)^m \frac{1}{R} \quad (35)$$

which gives

$$\begin{aligned} \frac{\mu}{R^2} &= - \frac{\partial}{\partial x} \frac{1}{R}, \quad \frac{P_2^1(\mu)}{R^3} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \frac{1}{R} \\ \frac{P_2^1(\mu) \cos \phi}{R^3} &= \frac{\partial^2}{\partial x \partial y} \frac{1}{R}, \quad \frac{P_2^1(\mu) \sin \phi}{R^3} = \frac{\partial^2}{\partial x \partial z} \frac{1}{R} \\ \frac{P_2^2(\mu) \cos 2\phi}{R^3} &= \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \frac{1}{R}, \quad \frac{P_2^2(\mu) \sin 2\phi}{R^3} = 2 \frac{\partial^2}{\partial y \partial z} \frac{1}{R} \end{aligned} \quad (36)$$

The force on the sphere can now be obtained from the Lagally formula (Landweber, 1967)

$$\overline{F} = - 4\pi\rho \sum_s M_s (D_s^q \overline{V})_0 \quad (37)$$

Here, by (1) and (36), the successive operators D_s^q are

$$D_s^1 = \frac{\partial}{\partial x}, \quad D_s^2 = \left(\frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial x \partial z}, \dots, \frac{\partial^2}{\partial y \partial z} \right)$$

Since $\bar{v}$, given in (32), is linear in x, y, z , the second derivatives D_s^2 do not contribute to $\bar{F}$, and hence $\bar{F}$ becomes

$$\begin{aligned} \bar{F} &= -2\pi\rho U c^3 \left(\frac{\partial \bar{v}}{\partial x} \right)_0 \\ &= -2\pi\rho U c^3 (2i\bar{A}_0 + 3j\bar{A}_1 + 3k\bar{B}_1) \end{aligned} \quad (38)$$

We observe that $\frac{1}{2} U c^3$ is the doublet strength at the center of the sphere in a uniform stream. The result in (38) is the well-known Lagally expression for the force on a doublet in a nonuniform stream, but, unexpectedly, with the doublet strength unaffected by the quadratic nonuniformity.

The foregoing procedure can be readily extended to nonuniform flows of higher order. If the velocity potential contains a homogeneous harmonic of degree n , its contribution $\bar{v}_{n-1}$ to the velocity would be of degree $n-1$. Then

$$(D_s^q \bar{v}_{n-1})_0 = \delta_{q,n-1} (D_s^{n-1} \bar{v}_{n-1})_0 \quad (39)$$

since the terms with $q < n-1$ vanish at the origin. The force (37) would then become

$$\bar{F} = -4\pi\rho \sum_{n=1}^{\infty} M_n (D_s^{n-1} \bar{v}_{n-1})_0 \quad (40)$$

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Appendix

An Alternative Procedure for Deriving the Multipole Integrals

The residues given in Table 1 may also be derived by applying a theorem due to Maxwell for the surface integral of the product of two harmonics of the same degree; see Hobson (1931, p. 157). This alternative procedure has the advantage that the singularity is located at the center of the sphere. It will be illustrated by rederiving residue (4).

Associated with the multipole potential (1) of degree $-(q+1)$ we have the regular homogeneous harmonic of degree q ,

$$\Omega_q = R^{2q+1} \Phi_P \quad (40)$$

Then

$$\bar{u}_P = \nabla \Phi_P = \frac{1}{R^{2q+1}} [\nabla \Omega_q - (2q+1) \Omega_q \frac{\bar{n}}{R}] \quad (41)$$

The integral in (4) then becomes

$$\begin{aligned} \int_S \bar{n} \cdot \bar{u}_P \Psi dS &= \frac{1}{c^{2q+1}} \int_S [\bar{n} \cdot \nabla \Omega_q - \frac{1}{c} (2q+1) \Omega_q] \Psi dS \\ &= - \frac{q+1}{c^{2q+2}} \int_S \Omega_q \Psi dS = - \frac{q+1}{c^{2q+2}} \int_S \Omega_q \Psi_q dS \end{aligned} \quad (42)$$

by (17) and (16a), and the orthogonality of the harmonics. The Maxwell formula then gives

$$\int_S \Omega_q \Psi_q dS = - \frac{4\pi M}{2q+1} c^{2q+2} D_S^q \Psi_q \quad (43)$$

in which, in the limit as $c \rightarrow 0$, Ψ_q may be replaced by Ψ , by (21). Substitution of (43) into (42) then gives the residue (4).

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