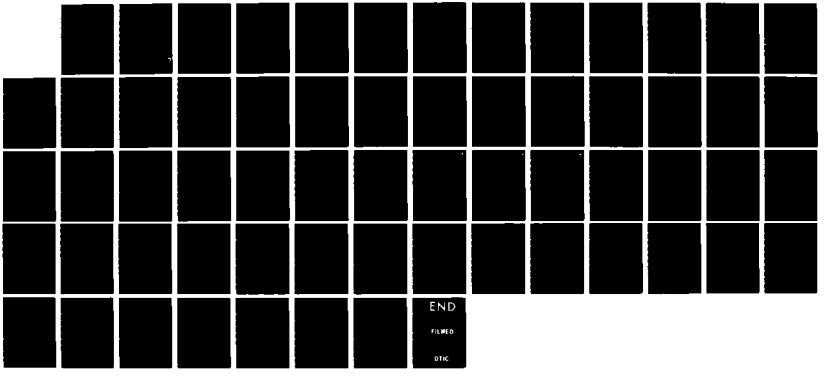
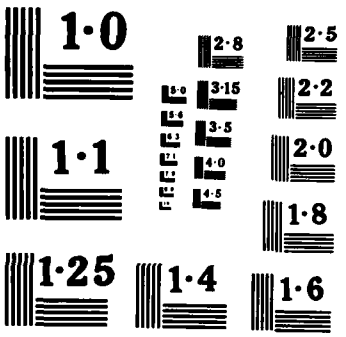
AD-A158 776 FAR-FIELD BOUNDARY CONDITIONS IN NUMERICAL SOLUTIONS OF 1/1
THE NAVIER-STOKES EQUATIONS(U) FLORIDA UNIV GAINESVILLE
P J MCKENNA 10 MAY 85 AFOSR-TR-85-0625 AFOSR-83-0330
UNCLASSIFIED F/G 20/4 NL






NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

AFOSR-TR- 85-0625

# Far-Field Boundary Conditions in Numerical Solutions of the Navier-Stokes Equations

AD-A158 776

P.J. McKenna

OTIC FILE COPY

Final Report AFOSR Grant 83-0330

DTIC
ELECTE
AUG 2 9 1985
D
G

Approved for public release;
distribution unlimited.

85 8 21 013

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

| | |
|---|---|
| 1a. REPORT SECURITY CLASSIFICATION: Unclassified AD-A158 776 | 1b. RESTRICTIVE MARKINGS: |
| 2a. SECURITY CLASSIFICATION AUTHORITY: --- | 3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: N/A | |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S): | 5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0625 |
| 6a. NAME OF PERFORMING ORGANIZATION: University of Florida; 6b. OFFICE SYMBOL (If applicable): | 7a. NAME OF MONITORING ORGANIZATION: AFOSR |
| 6c. ADDRESS (City, State and ZIP Code): Gainesville, FL 32611 | 7b. ADDRESS (City, State and ZIP Code): Bldg. 410 Bolling AFB, D.C. 20332-6448 |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR-83-0330 |
| 8c. ADDRESS (City, State and ZIP Code): Bldg. 410 Bolling AFB, D.C. 20332-6448 | 10. SOURCE OF FUNDING NOS. — PROGRAM ELEMENT NO.: 61102F; PROJECT NO.: 2304; TASK NO.: A3; WORK UNIT NO.: |
| 11. TITLE (Include Security Classification): Far Field Boundary Conditions in Numerical Solutions of the Navier-Stokes Equations | |
| 12. PERSONAL AUTHOR(S): P. J. McKenna | |
| 13a. TYPE OF REPORT: Final; 13b. TIME COVERED: FROM 24 Oct 84 TO 1 Apr 85; 14. DATE OF REPORT (Yr., Mo., Day): 10 May 1985; 15. PAGE COUNT: 59 | |
| 16. SUPPLEMENTARY NOTATION: | |

| 17. COSATI CODES | | | 18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) |
|---|---|---|---|
| FIELD | GROUP | SUB. GR. | artificail boundary conditions, Navier-Stokes equations |
| XX | XXXXXXXXX | XXX | |
| | | | |

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The purpose of this project was to investigate the artificial boundary conditions which must be imposed at the boundary of a numerical grid. Numerical experiments were performed which evaluated various boundary conditions of this type. Of particular interest was the question of whether linearization around the flow at infinity provides an adequate choice of boundary condition for the fully nonlinear equations. Inflow-outflow conditions as well as several choices of periodic boundary conditions were considered. The inflow-outflow conditions worked reasonably well for two dimensional problems. For the periodic conditions, results were mixed. Concerns about over specification when using pure periodic conditions proved to be groundless. However, the method failed when attempting to use periodicity in the two sideways traveling periodic waves.

| | |
|---|---|
| 20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☒ DTIC USERS ☐ | 21. ABSTRACT SECURITY CLASSIFICATION: Unclassified |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL: John P. Thomas, Capt., USAF | 22b TELEPHONE NUMBER (Include Area Code): (202)767-5026; 22c. OFFICE SYMBOL: NM |

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

## I INTRODUCTION

With the advent of high speed computers, new approaches to the solution of engineering problems have arisen. One such field is Computational Fluid Dynamics, in which numerical techniques such as finite differences are used to numerically integrate partial differential equations governing the physical phenomena. One problem with this approach is that on the boundary of the numerical grid, artificial boundary conditions must be imposed. Usually, these conditions are based on a linearization around the flow at infinity, and are thus non-physical. In this paper, we perform a series of numerical experiments, in which we evaluate various boundary conditions of this type, and investigate whether the linear model is an accurate representation of the fully non-linear equations.

The flow of fluid around obstacles in two dimensions is described by the compressible Navier-Stokes equations

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{1}$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix} \qquad E = \begin{bmatrix} \rho u \\ \rho u^2 - \sigma_{xx} \\ \rho uv - \tau_{xy} \\ \rho ue - u\sigma_{xx} - v\tau_{xy} - \dot{q}_x \end{bmatrix}$$

$$F = \begin{bmatrix} \rho v \\ \rho uv - z_{xy} \\ \rho v^2 - \sigma_{yy} \\ \rho ve - \sigma_{yy} - uz_{yx} - \dot{q}_y \end{bmatrix}$$

$$\sigma_{xx} = -\rho - \left(\frac{2}{3}\right) \mu \vec{\nabla} \cdot \vec{u} + 2\mu \frac{\partial u}{\partial x}$$

$$z_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = -\rho - \left(\frac{2}{3}\right) \mu \vec{\nabla} \cdot \vec{u} + 2\mu \frac{\partial u}{\partial y}$$

$$\dot{q}_x = k \frac{\partial T}{\partial x} \qquad \dot{q}_y = k \frac{\partial T}{\partial y}$$

$$p = \rho RT \qquad \mu = \mu(T)$$

$$e = c_v T + (u^2 + v^2)/2.$$

Accession For
NTIS GRA&I
DTIC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Dist | Avail and/or Special
A/1

DTIC COPY INSPECTED

Here, the four variables $\rho, u, v, e$ represent the physical quantities of density, x- and y- components of velocity, and internal energy.

This nonlinear system of mixed parabolic-hyperbolic type in two space dimensions and time, with four independent variables must be solved in an exterior region in $R^2$. The geometry will depend on the particular physical situation that one is attempting to model.

The situation we shall be interested in occurs in modelling

flight conditions, in which the conditions at infinity are prescribed with a large u-velocity and v-velocity zero and prescibed $\rho_\infty$ and $e_\infty$. Fluid flows around an obstacle in x-y space.

Usually, this equation is solved numerically using a finite difference scheme of the Lax-Wendroff type, such as the MacCormack ADE method. Since these calculations can only be made on a finite grid of points in xy space, an artificial far-field boundary is created. This boundary ought be sufficiently far away from the object around which the fluid is flowing so that local phenomena are not neglected by omitting part of the region of fluid flow. On the other hand, the farther away the region is, the more grid-points need to be included and thus the more expensive and time-consuming the computations become.

One then has the problem of deciding what effect this new boundary has on the solution of the problem. Because of the viscosity terms in (1) and the additional artificial viscosity introduced by the finite difference schemes, some boundary conditions must be imposed.

As we shall show in this report on numerical experiments, considerable care must be exercised in the choice of the boundary conditions. If one is interested in steady state flow, then one starts off with an initial approximation, and hopes that the errors in the numerical solution propagate out of the region as transitory disturbances in the physical variables. One then expects to converge to the steady state flow.

We have previously shown that the incorrect choice of boundary conditions can give rise to some of the following phenomena: i) reflecting boundary conditions, in which the disturbances in the physical variables represented by the difference between the steady state flow and the initial conditions are not allowed to exit through the far-field boundary but instead continue echoing within the grid, and giving rise to spurious oscillatory solutions; ii) under-specified boundary conditions, in which large errors are introduced before convergence takes place.

In this paper, we first discuss the theory for simple linear hyperbolic systems in one and two space dimension. We then analyze several computational experiments in the light of this theory in two space dimensions.

## II A REVIEW OF THE LINEAR CASE

The method used in the calculations which are the subject of this paper is the MacCormack alternating direction explicit

scheme. [6] [15]. This is a multistep efficient scheme which reduces in the linear case to the Lax-Wendroff scheme. For a diagonal N x N matrix A, this scheme approximates the equation $U_t + AU_x = 0$ by

$$(2) \qquad \begin{cases} U_j^{n+1} = U_j^n - A\frac{\Delta t}{\Delta x}(U_{j+1}^n - U_{j-1}^n) \cdot \\ \\ \qquad + A^2(\frac{\Delta t}{\Delta x})^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n) \end{cases}$$

As usual, $\{j_{1<j<J}\}$ represents the space step and n represents the time step. If we are considering the equation

$$U_t + AU_x = 0$$

on the region $\{(x,t),\ 0<x<1,\ t>0\}$ then the analytic solution is determined by the initial conditions and boundary conditions at x = 0 and x = 1. If the first k eigenvalues are positive and the remaining N - k are negative then the quantities $u_1 \text{---} u_k$ must be prescribed at x = 0 and $u_{k+1} \text{---} u_N$ must be prescribed at x = 1. Thus if $W_I = (u_1, u_2, \text{---} u_k, 0\ 0 \text{---} 0)$ and $W_{II} = (0, 0, \text{---} 0, u_{k+1} \text{---} u_N)$ then for well-posedness, the boundary conditions must be

$$W_I = f(t) + B_0 W_{II} \qquad \text{at } x = 0$$
$$W_{II} = g(t) + B_1 W_I \qquad \text{at } x = 1.$$

This gives a total of N boundary conditions. If either kx(n-k) matrix $B_1$ and the (n-k)xk matrix $B_0$ are non-zero, then the boundaries are reflecting, that is a wave in $W_I$ travelling left to right will be reflected as a wave in $W_{II}$ running right to left.

If a boundary is supposed to be non-physical, then it should not be reflecting, since the reflections would depend on the location of the artificial or numerical boundary.

Now let us consider the difference scheme of (2). If the grid points are given by $\{x_j\}_0^J$ with $x = 0$ and $x_J = 1$, then it is clear from (2) that 2N boundary conditions are required. Thus we must prescribe boundary conditions in such a way as to least affect the closeness of the numerical to the analytic solution.

This situation has been the subject of several papers. In [3], Gustaffson and Kreiss point out the danger of over-specification. By this is meant that all $u_i$ are specified at both endpoints. This might be tempting because one might argue that if the boundary onditions $u_i$---$u_k$ are given constants at $x = 0$, then eventually these same values will be assumed by these variables at $x = 1$. However, in [4] it is pointed out that convergence may or may not occur, depending on whether the number of grid points is odd or even.

A method which works well, as pointed out in [3], is to impose

$$\frac{\partial u_i}{\partial x}(1,t) = 0 \qquad 1 < i < k.$$

$$\frac{\partial u_i}{\partial x}(0,t) = 0 \qquad k+1 < i < n.$$

or in the numerical scheme

$$u_{i,J} = u_{i,J-1} \qquad 1 < i < k.$$

$$u_i,0 = u_{i,1} \qquad k+1 < i < n.$$

This introduces small errors at the outflow but these errors do not propagate upstream. This is proved analytically by Parter [10].

In [12], many different numerical boundary conditions are given. The conclusion is that upwind differencing at the point of outflow

$$u_J^{n+1} = u_j^n + \lambda_i(\frac{\Delta t}{\Delta x})(u_{J-1}-u_J)$$

is most accurate, although it converges with the same speed as the previously discussed $u_x = 0$.

The most serious error which one could make would be to prescribe conditions at the wrong end. In other words, since $u_1$ is right running, this would involve prescribing $u_1$ at $x = 1$ and imposing $u_{1x} = 1$ at $x = 0$. This would result in convergence to a

steady state which depends on the initial conditions.

## III THE NAVIER-STOKES EQUATIONS AND CHARACTERISTIC VARIABLES:

We now begin our discussion of the equations of gas dynamics. We will neglect viscosity for the purposes of this analysis. We will assume that the flow is one-dimensional and subsonic and that the deviations from free-stream solutions are small. This will allow us to neglect second order terms.

There are many forms of this equation, but the one most suitable for the present discussion is

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)\,u & \gamma - 1 \\ (\gamma-1)4^3 - \frac{\gamma e u}{\rho} & \frac{\gamma e}{\rho} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix}$$

and

$$U = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}$$

or in terms of physical variable

$$\frac{\partial \hat{U}}{\partial t} + \hat{A}\,\frac{\partial \hat{U}}{\partial x} = 0 \qquad (3)$$

where

$$\hat{A} = M^{-1} A M$$

and

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -u/p & 1/p & u \\ (\frac{\gamma-1}{2})u^2 & (1-\gamma)u & (\gamma-1) \end{pmatrix} \quad \hat{U} = \begin{bmatrix} \rho \\ u \\ \rho \end{bmatrix}$$

Here we make the key assumption that deviations from the free stream are going to be sufficiently small that we can treat the entries in the matrix A as being approximately constant (at least locally). Denote these frozen variables by 0-subscript. We then make the substitution

$$(4) \qquad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/c_0^2 \\ 0 & 1 & 1/\rho_0 c_0 \\ 0 & -1 & 1/\rho_0 c_0 \end{pmatrix} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}$$

and when this is substituted into ( ) we obtain

$$(5) \qquad \frac{\partial W_1}{\partial t} + u_0 \frac{\partial W_1}{\partial x} = 0 \qquad W_1 = \rho - \frac{1}{c_0^2} p$$

$$\frac{\partial W_2}{\partial t} + (u_0+c_0) \frac{\partial W_2}{\partial W} = 0 \qquad W_2 = u + \frac{1}{\rho_0 c_0} p$$

$$\frac{\partial W_3}{\partial t} + (u_0-c_0) \frac{\partial W_3}{\partial x} = 0 \qquad W_3 = -u + \frac{1}{\rho_0 c_0} p$$

Notice now how this breaks down into two separate cases. On the one hand, if flow is supersonic then all wave motion is in the left to right direction. In this case all analytic boundary conditions should be prescribed at the left hand side and only numerical boundary conditions prescribed at the right hand side.

Since the substitution (4) is equivalent to

$$\rho = K_1 + (\rho_0/2c_0)(K_2+K_3)$$
$$u = 1/2(K_2-K_3)$$
$$p = \rho_0 c_0/2(K_2+K_3)$$

it follows that prescribing all physical variables at the inflow and prescribing $\partial\rho/\partial x = \partial u/\partial x = \partial p/\partial x = 0$ at the outflow is legitimate in terms of analytical and numerical requirements in the supersonic case.

However, we must now consider the case of subsonic flow. In this case the situation is completely different. Here, two of the variables $W_1$ and $W_2$ go left to right with velocities $u_0$ and $u_0 + c_0$ respectively, whereas one of the variables runs right to left with velocity $c_0-u_0$. The variables $W_2$ and $W_3$ have no clear physical significance, yet it is only in considering these variables that the full wave structure of the equations (5) or (6) can be understood. Thus, one would be led to predict, for small deviations from free stream conditions, that the best boundary conditions would be, for an interval (0,L)

$$(6) \qquad W_1(0,t) = K_1 \qquad \frac{dW_1}{dx}(L,t) = 0$$

$$W_2(0,t) = K_2 \qquad \frac{dW_2}{dx}(L,t) = 0$$

$$\frac{dW_3}{dx}(0,t) = 0 \qquad W_3(L,t) = K_3$$

Note the curious aspect of these boundary conditions. In order to prescribe the numerical values $K_1$ and $K_2$, we need to know accurately all three physical variables at some distance to the left. However, only the two combinations $K_1$ and $K_2$ are prescribed. This can be summarized by saying that while we have used all three pieces of information upstream, we have done so in such a way that one degree of freedom remains, thus allowing the waves in $W_3$ to exit without reflections.

On the basis of the linearized model, various other combinations would be well-posed. For example, it is possible to prescribe $K_3$ in terms of either $K_1$ or $K_2$ at the outflow $x = L$. Thus at the outflow one may prescribe

$$W_2(L,t) = F_3(t) + c_1 W_1(L,t) + c_2 W_2(L,t)$$

For example if $c_1 = 0$, $c_2 = 1$, then this amounts to putting

$$(7) \qquad u(L,t) = 1/2\, F_3$$

(i.e., we prescribe velocity at the outflow).

Alternatively, we might take $c_1 = 0$, $c_2 = -1$ and we would get

$$(8) \qquad p(L,t) = ((\rho_0 c_0)/2)F_3$$

(i.e. we prescribe pressure at the outflow).

Many other combinations are possible, but as remarked in section II, all these will cause errors in the initial data to be reflected back into the medium as waves running from right to left. For example, we would predict that an error in $W_3$ would be reflected back as an error in $W_1$ if we use boundary condition (8). As we shall see, this is exactly what happens.

At the inflow end, we may prescribe $W_1$ and $W_2$ in terms of $W_3$. Thus the following boundary conditions are well posed;

$$(9)(a) \qquad W_1(0,t) = F_1 + c_1 W_3(0,t)$$

$$(b) \qquad W_2(0,t) = F_2 + c_2 W_3(0,t)$$

For example, choosing $c_2 = +1$ in (9b) corresponds to

$$u(0,t) = (1/2)F_2$$

(i.e. prescribing u at the inflow) and $c_2 = -1$ corresponds to

$$p(0,t) = (\rho_0 c_0/2)F_2$$

(i.e. prescribing p). One can prescribe the combination $(u,\rho)$ by first choosing $c_2 = 1$ (thereby prescribing u) and then choosing $c_1 = \rho_0/c_0$, thereby prescribing p in terms of a given F, and a

## VI TWO DIMENSIONAL RESULTS

Having now understood the phenomena which can occur when calculations are made with one space dimension, we now consider the more complicated situation of two space dimensions.

In this situation, we shall solve the Navier Stokes equation (1), again by the standard MacCormack A.D.E. scheme on a 20 x 20 grid, which is a very simplified model of a wind tunnel. We shall continue to consider flow close to the free stream flow ($M_\infty = 0.5$) previously studied in the one dimensional case. The physical values are given earlier. Figure 8 shows the geometry of the situation. The fluid is flowing in at the top right corner of the grid and flowing out at the bottom left.

We have three distinct types of boundaries to consider. We have the inflow and outflow boundaries (as before) and in addition, two sidewall boundaries, where the fluid is flowing parallel to the boundary.

We shall continue to impose one dimensional boundary conditions of the type given in the first section on the inflow and outflow, along with the additional condition $v = 0$. This says the fluid flow is one dimensional at the inflow and outflow, and could be physically reasonable.

When we come to the sidewall conditions we must undertake another one-dimensional analysis. Thus, we assume all variables are constant in the x-direction and variation only takes place in the y-direction.

This leads to the set of equations

$$(6) \qquad V_t + AV_y = 0$$

where

$$\vec{V} = \begin{bmatrix} p \\ u \\ v \\ e \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & u & 0 \\ \frac{u^2}{2}(\gamma-1) & -u(\gamma-1) & 0 & \gamma - 1 \\ 0 & 0 & \frac{e}{e}\gamma - \frac{u^2}{2}(\gamma-1) & 0 \end{bmatrix}$$

Freezing the coefficients of A in (6), we substitute

$$(7) \qquad T_1 = \rho + \frac{p}{c_0^2}$$

$$T_2 = v + \frac{1}{\rho_0 c_0} p$$

$$T_3 = -v + \frac{1}{\rho_0 c_0} p$$

$$T_4 = u$$

Equation (6) is then transformed to

(8) $$T_{1t} = 0$$

$$T_{2t} + c_0 T_{2y} = 0$$

$$T_{3t} - c_0 T_{3y} = 0$$

$$T_{4t} = 0$$

Thus, if we only consider deviations transverse to the free stream flow, we have on the basis of the linearized model, four non-physical variable $T_1$, $T_2$, $T_3$, $T_4$, two of which are u and entropy. $T_1$ move with zero velocity in the y direction, one of which, $T_2$, moves with speed $c_0$ in the positive x direction, and one of which moves with speed $c_0$ in the negative x direction. Thus one dimensional theory predicts that at the sidewall, we should impose

| $y = 0$ | $y = L$ |
|---|---|
| $T_{1y} = \text{const}$ | $T_{1y} = 0$ |
| $T_{3y} = 0$ | $T_3 = \text{const}$ |
| $T_{4y} = 0$ | $T_{4y} = 0$ |

This, together with the inflow and outflow conditions gives the following set of boundary conditions

Inflow

$$v_1 = v_\infty$$

$$p_1 = \frac{1}{2}\,[p_2 + \rho_0 c_0 (k_3 - u_2)]$$

$$u_1 = \frac{1}{2}\,[k_3 - \frac{p_2}{\rho_0 c_0} + u_2]$$

$$\rho_1 = k_1 + \frac{\rho_0}{2c_0}\,[\frac{p_2}{\rho_0 c_0} + k_3 - u_2]$$

where the free stream values are specified in the characteristic variables $k_1 = \rho_\infty - p_\infty/c_0^2$, $k_3 = U_\infty + p_\infty/\rho_0 c_0$ and where the zero-subscript refers to frozen variables

Outflow

$$v_n = v_{n-1}$$

$$u_n = \frac{1}{2}\,[u_{n-1} + p_{n-1}/(\rho_0 c_0) - k_4$$

$$p_n = \frac{\rho_0 c_0}{2}\,[k_4 + u_{n-1} + p_{n-1}/(\rho_0 c_0)]$$

$$\rho_n = \rho_{n-1} - \frac{p_{n-1}}{c_0^2} + \frac{\rho_0}{2c_0}\,[k_u + u_{n-1} + \frac{p_{n-1}}{\rho_0 c_0}]$$

Top wall

$$p_j = \frac{\rho_0 c_0}{2} \left[ v_{j-1} + \frac{p_{j-1}}{\rho_0 c_0} - k_3 \right]$$

$$v_j = 1/2 k_3 + 1/2 (v_{j-1} + p_{j-1}/\rho_0 c_0)$$

$$\rho_j = \frac{\rho_0}{2 c_0} \left( v_{j-1} + \frac{p_{j-1}}{\rho_0 c_0} - k_3 \right) + \rho_{j-1} - \frac{1}{c^2} p_{j-1}$$

$$u_j = u_\infty$$

Bottom wall

$$u_1 = u_\infty$$

$$v_1 = 1/2 [v_2 + k_2 - p_2/(\rho_0 c_0)]$$

$$p_1 = \frac{\rho_0 c_0}{2} [-v_2 + k_2 + p_2/(\rho_0 c_0)]$$

$$\rho_1 = \rho_2 + (\rho_0/2c_0) [-V_2 + k_2 - p_2/\rho_0 c_0]$$

where the free stream values are used to specify the characteristic combinations

$$W_2 = v_\infty + \frac{p_\infty}{\rho_0 c_0} \qquad W_3 = v_\infty - p_\infty/\rho_0 c_0 .$$

The purpose of the present grant was two-fold: first to expand the previous work to situations where the flow was at an angle $\alpha$ to the artificial boundary, but is primarily steady-state, and second, to see how the artificial boundary conditions apply in <u>period</u> situations where convergence to free stream is not expected.

The first objective is clearly desirable, simply to save in additional computing time. The requirement that the grid not be rectangular with boundaries either parallel to or perpendicular to the free stream at infinity clearly can be used to remove a large number of points from the grid. The entire boundary can be chosen approximately equidistant from the object whose flow characteristics are being studied.

The second objective was chosen because of its importance in modelling flow in the compressors of turbines. If one is attempting to design turbine compressors, one must have knowledge of how the flow in the area between the stators and the rotors is behaving. This is almost impossible to measure.

On the other hand, to accurately model the entire inside of the compressor requires a computational capability far exceeding the largest of today's computers. Thus one models only the flow past two blades of the rotor, and uses periodicity to extend the calculations to the entire rotor. The flow coming in past the

stator is regarded as a periodic inflow term.

The following pages summarize the results of the research on these two projects.

## Section I Flow at an angle of incidence $\alpha$.

The previously outlined boundary conditions were used. Computations were done on the basis of the inflow or outflow depending on the flow perpendicular to the surface. For computational simplicity, we chose a rectangular grid, but choose $u_\infty = V_\infty = 300$ This results in a 45° angle of incidence or exit. We impose an initial 10% disturbance in $W_2$, the right running characteristic variable and observe what happens as it exits. Recall that this disturbance runs at an angle of 45° to the flow. The following plots for $W_2$ at various iteration numbers show it exiting without any surprising phenomena, similar to the one-dimensional case. By the sixty-fifth iteration, the flow has almost converged to steady state. There appear to be some small reflections (as shown in the last plot) in $W_3$, but nothing to worry about.

Thus we can conclude that the inflow-outflow boundary conditions work reasonably well. Subject to a rotation of coordinates, they can work on any geometry - the only decision is whether we are at an inflow or outflow point.

II. The case of periodic BCs

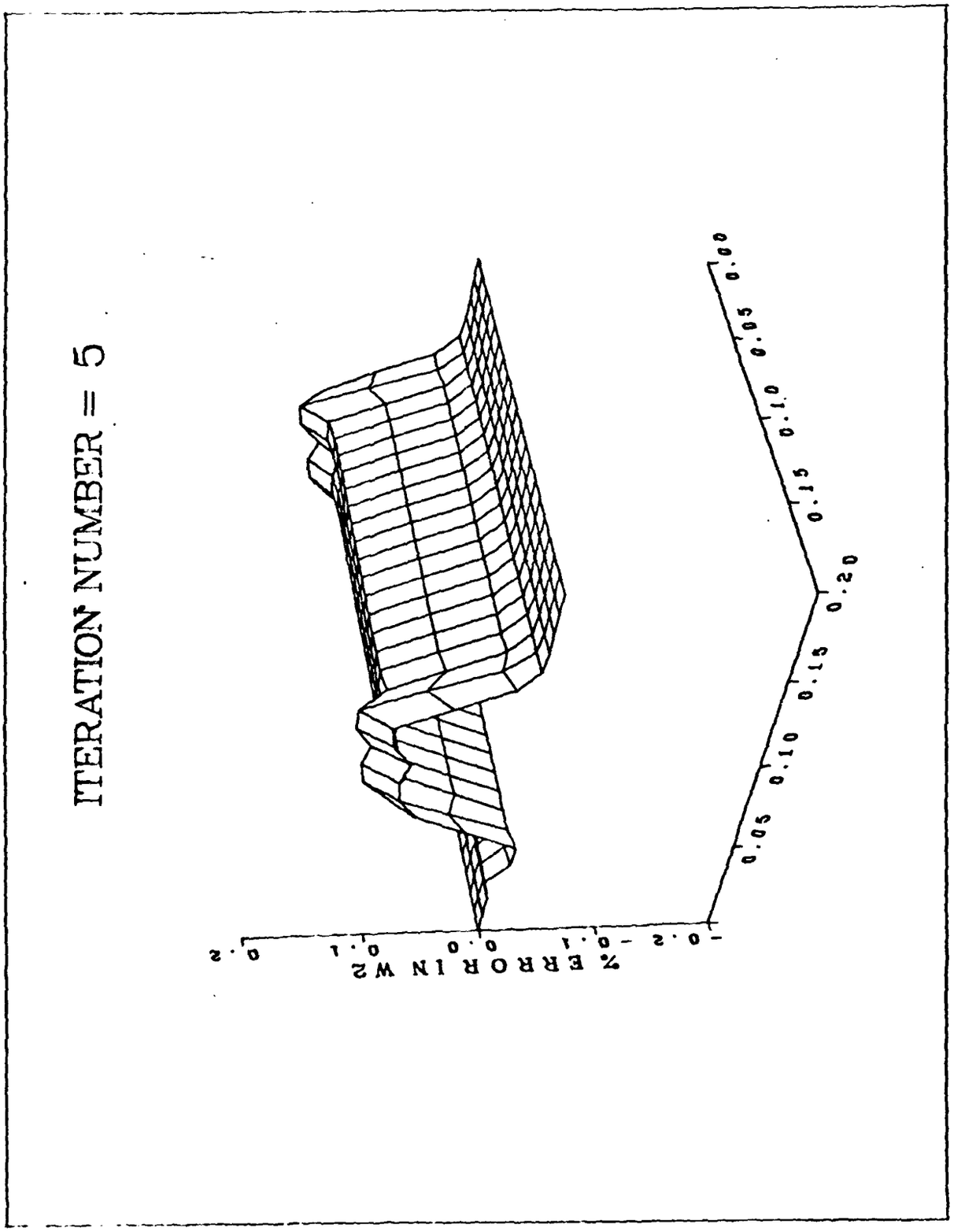
We now consider some experiments on the case of periodic boundary conditions. We study flow parallel to the rectangular grid, perpendicular to the inflow and parallel to the side-walls.

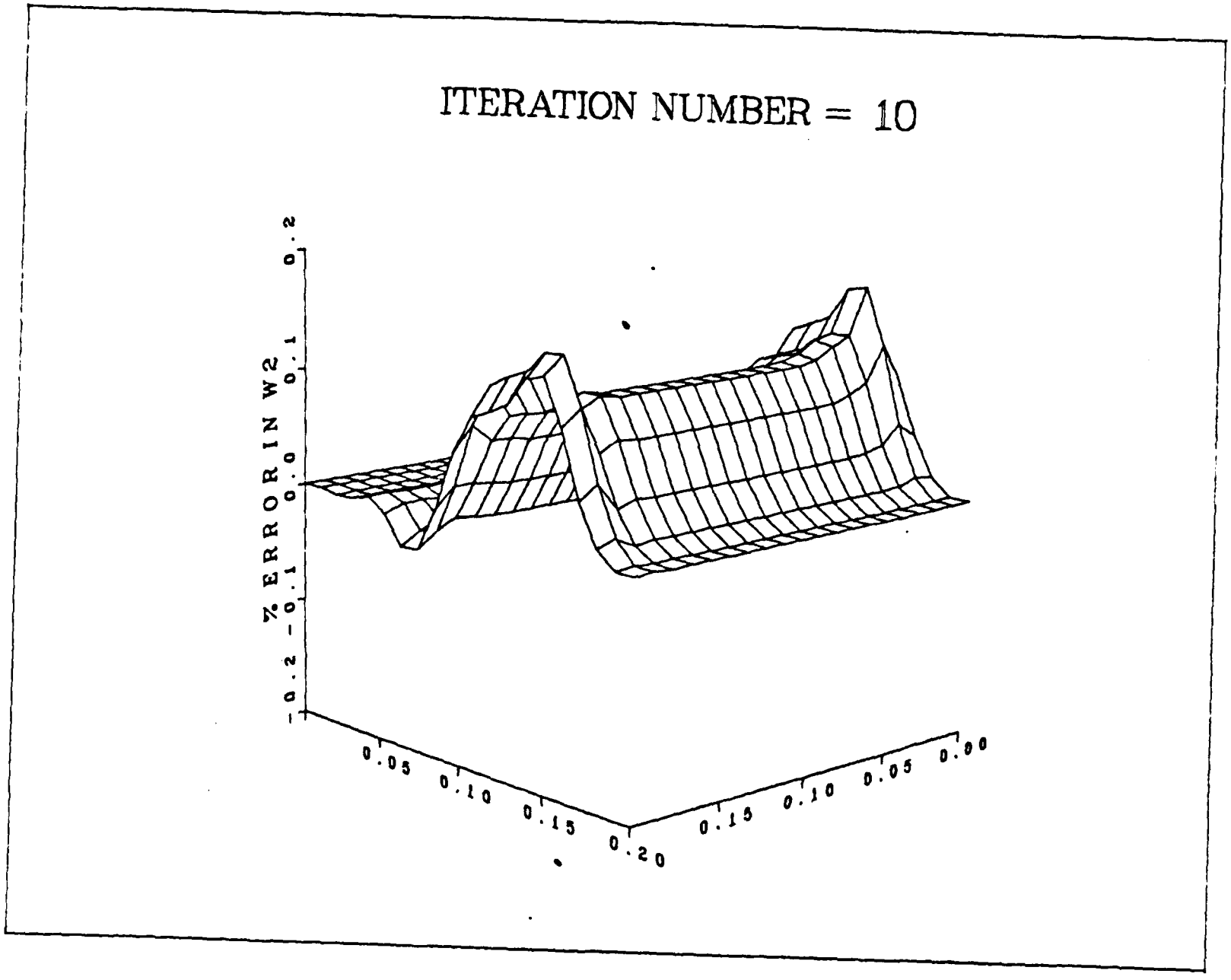
The most obvious boundary condition is pure periodic. $S_1 = S_{n-1}$, $S_n = S_2$ for all physical variables S. This, it was feared might lead to over specification, but these fear appeared groundless. A sight-traveling wave would exit to the right, and reappear at the left in a predictable and correct manner.

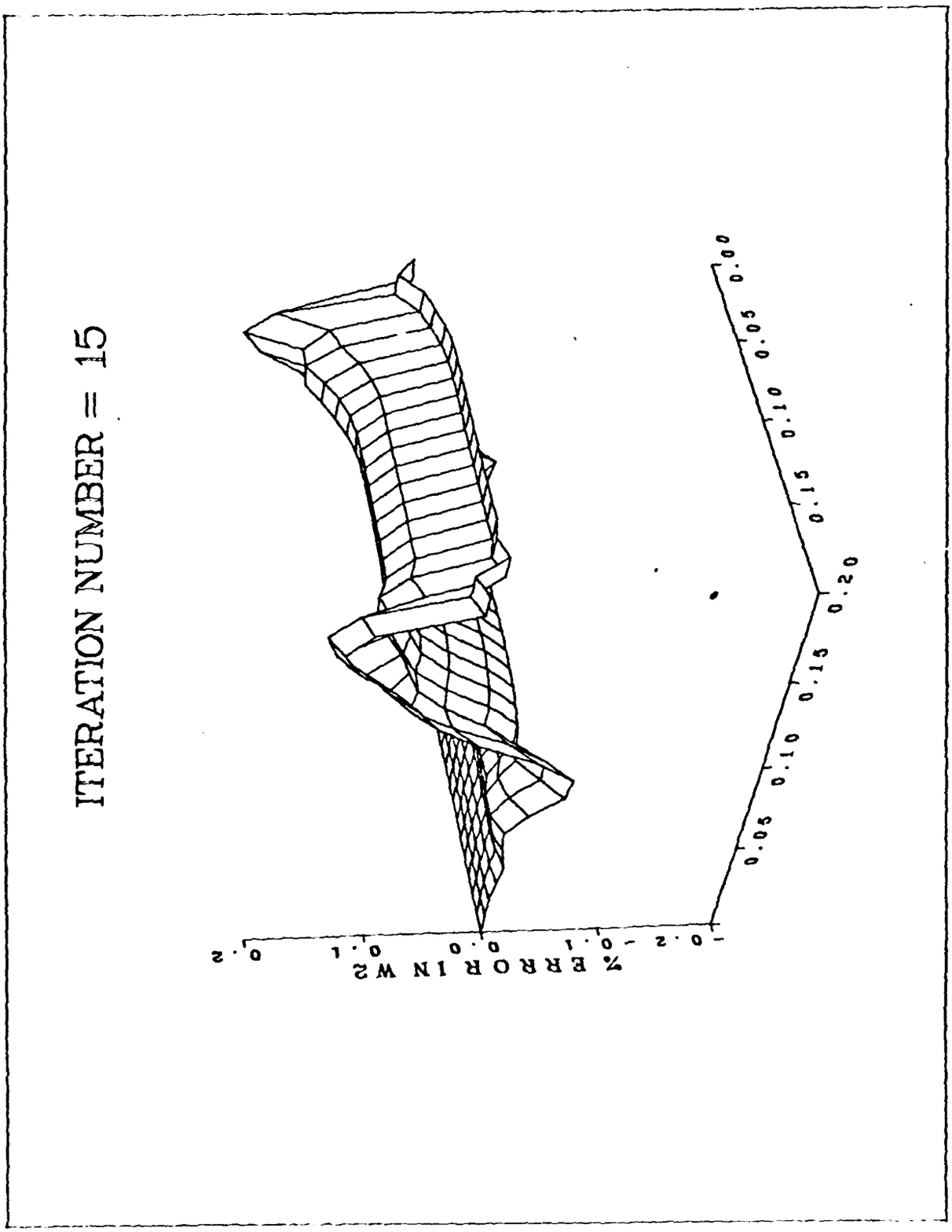
The next objective was to see if this could be done by only using periodicity in the two sideways traveling periodic waves. The reason for this was to minimize storage problems for the more complicated turbine problem already discussed.

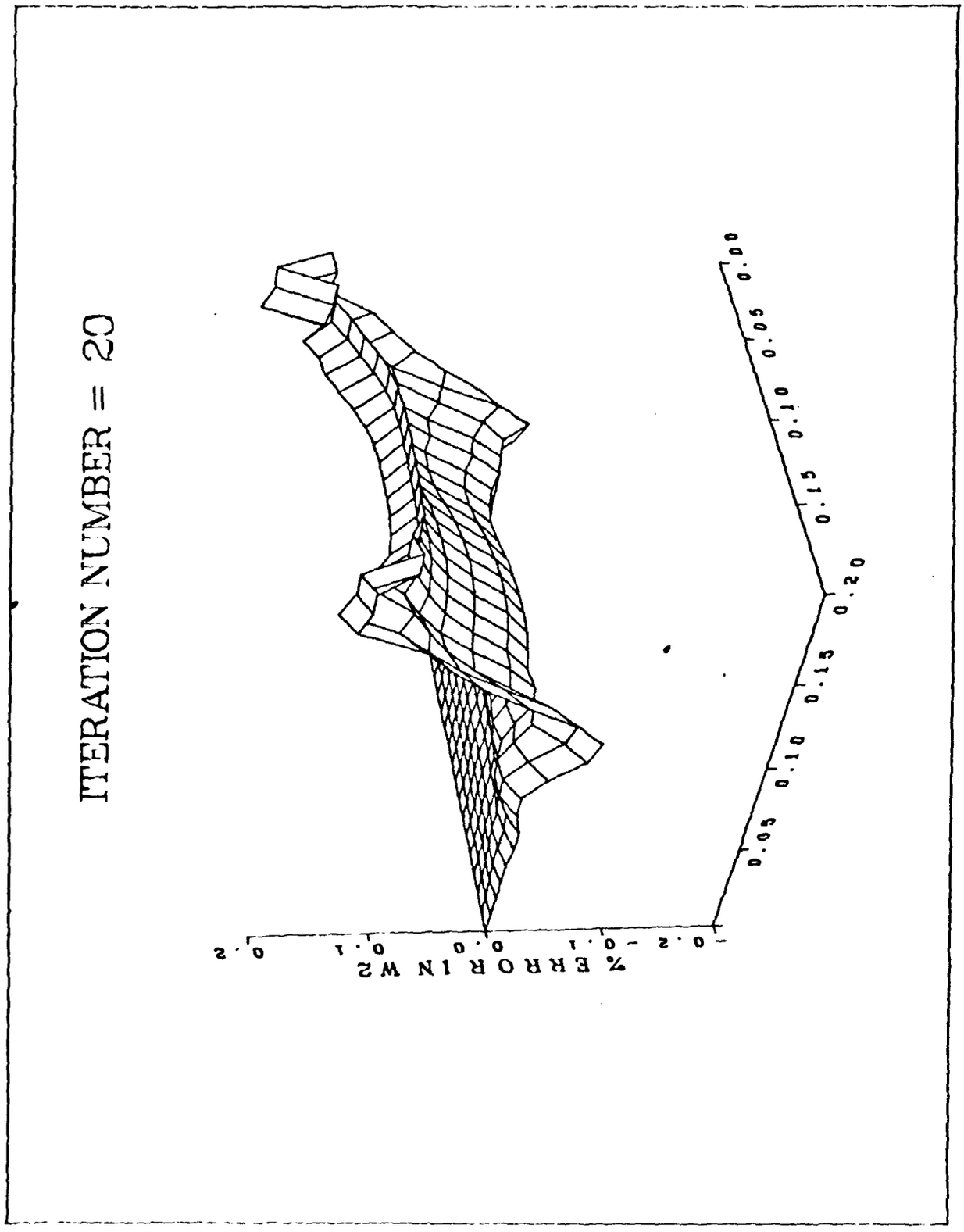
Curiously, this method failed. Furthermore, the failure seems intrinsic to the whole idea, as we shall demonstrate.

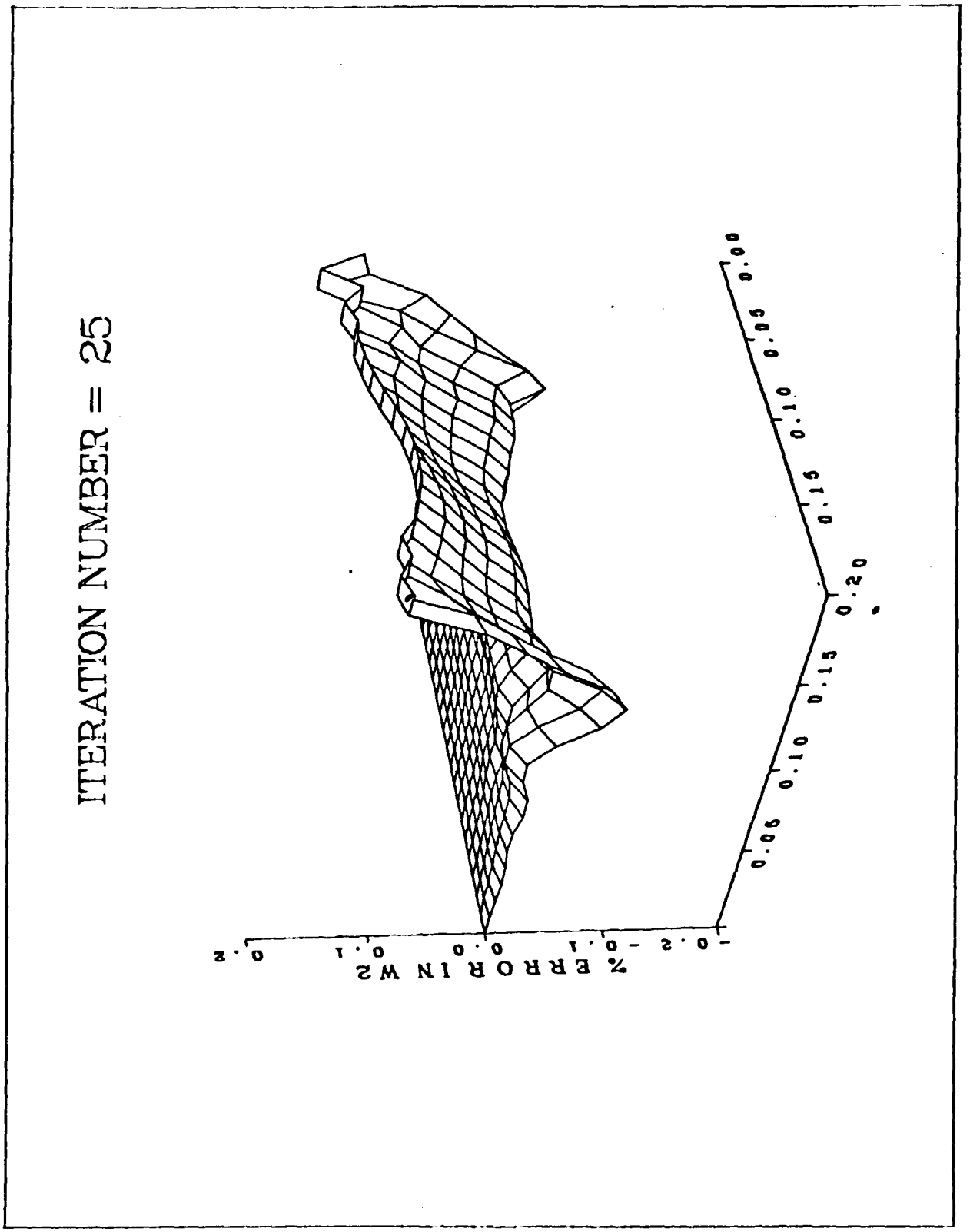
First the $W_2$ wave appears in iteration 5 and proceeds to exit. Periodicity requires that it reappear, and this can be seen in iteration 20. However, already, a sharp spike (reminiscent of over specification is appearing in $W_3$ by iteration 20. As far as iteration 40 $W_2$ continues to progress from right to left, but at iteration 55, sharp spikes begin to develop in $W_2$ and $W_3$. This seems to suggest overspecification at the corners, but as far as we know, this is not the problem. Shortly afterwards the program crashes. The relevant portion of the program for the boundary condition implementation is included.

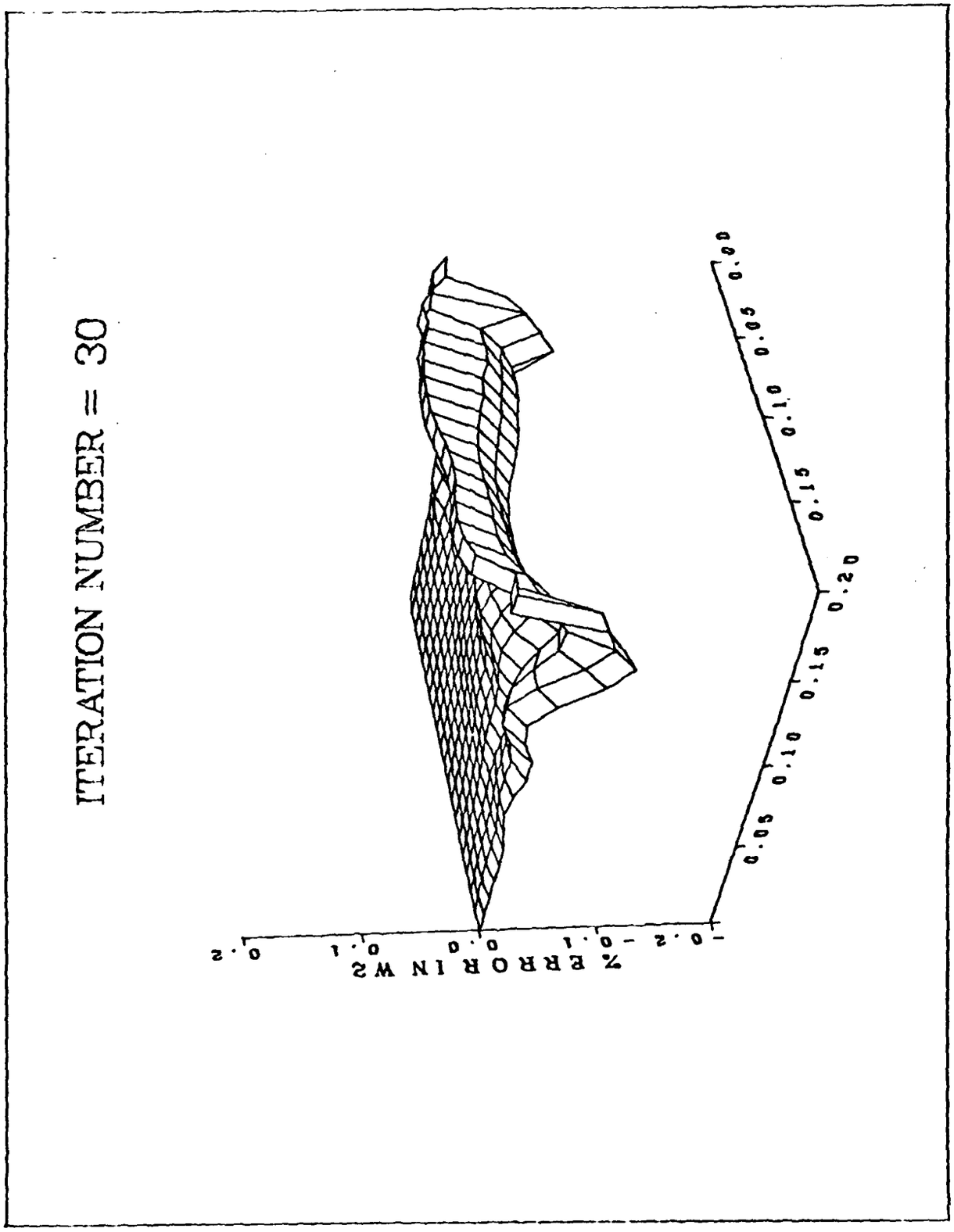

ITERATION NUMBER = 5
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.15
0.10
0.05

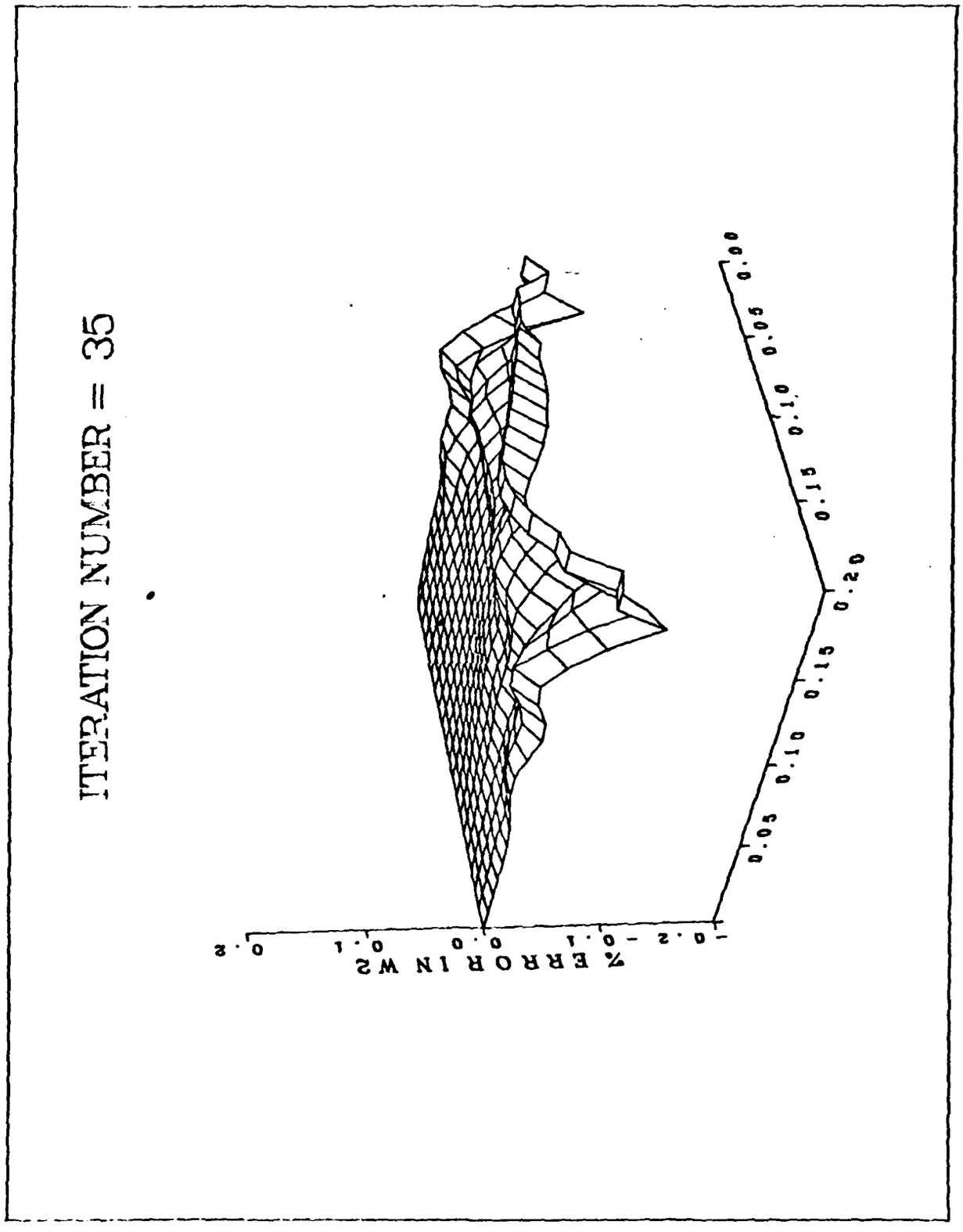

ITERATION NUMBER = 10
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

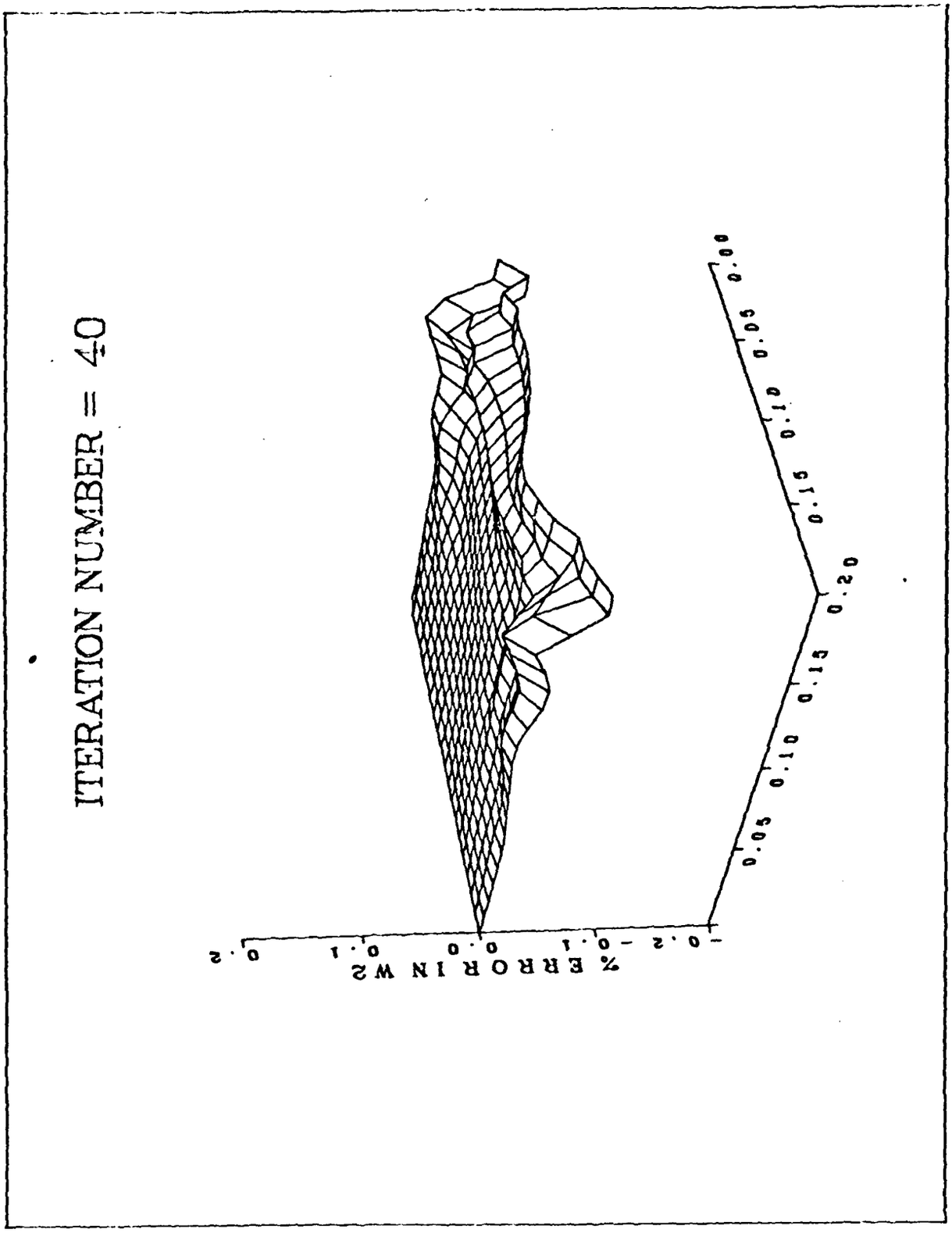

ITERATION NUMBER = 15
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.15
0.10
0.05

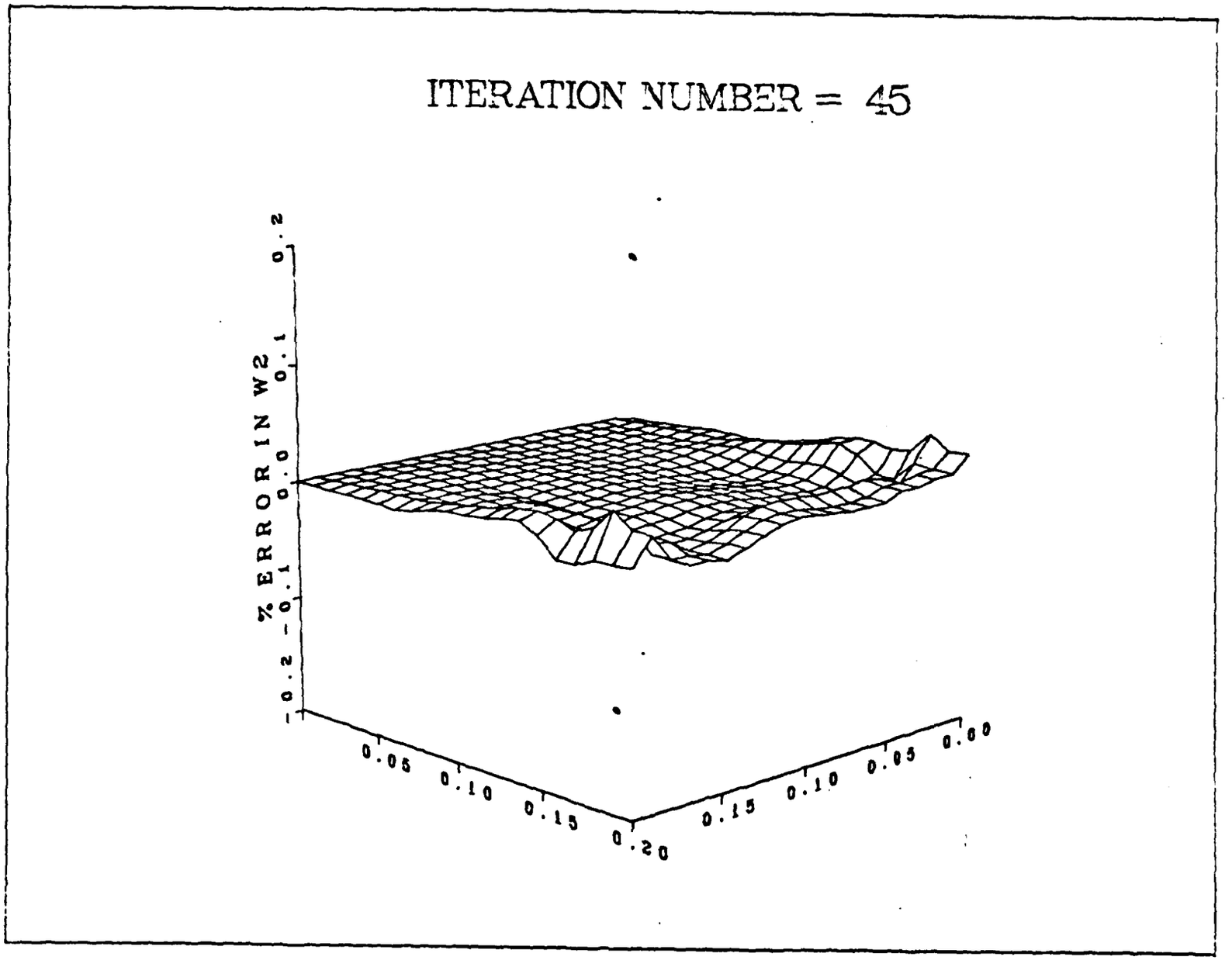

ITERATION NUMBER = 20
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.05
0.10
0.15
0.20

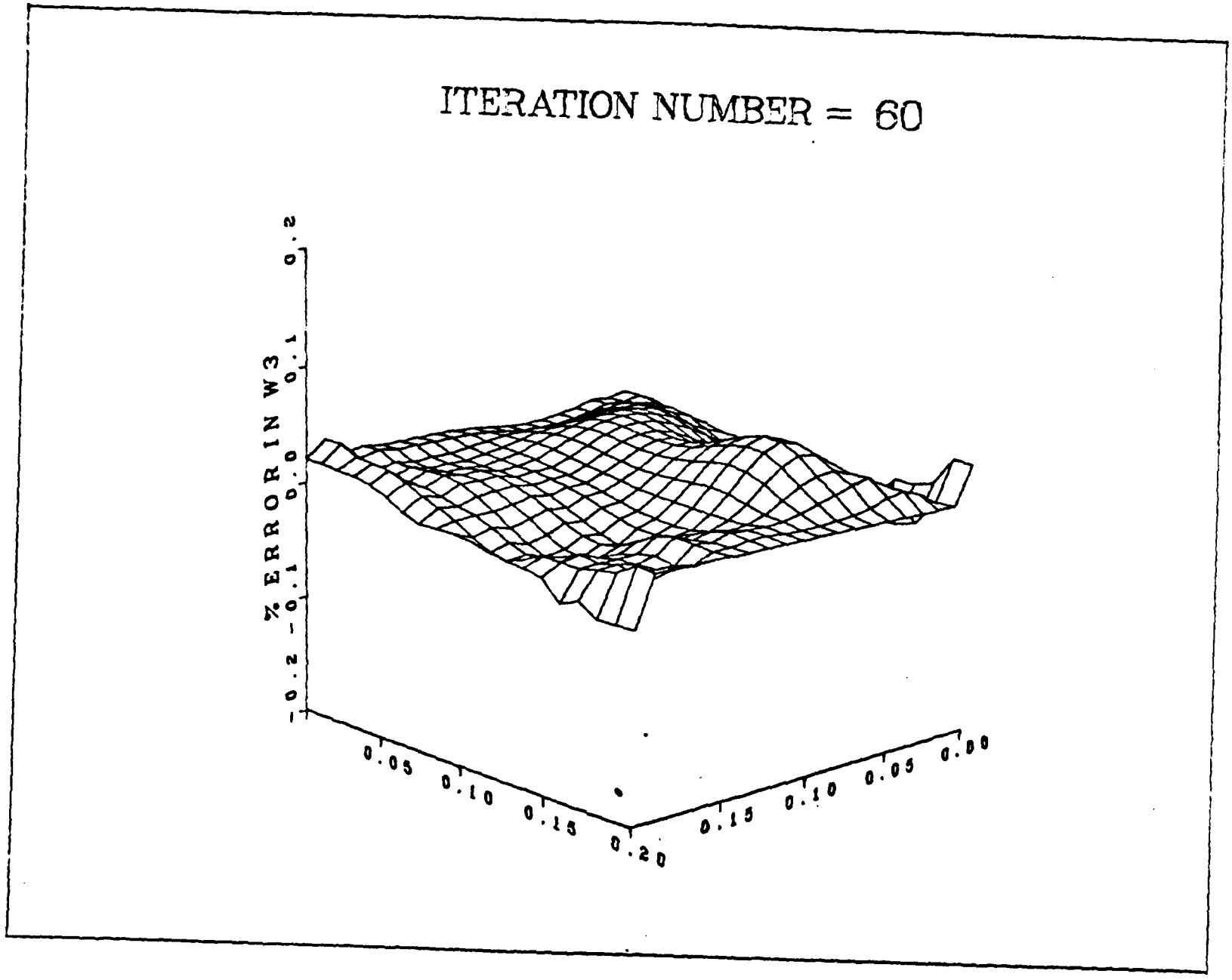

ITERATION NUMBER = 25
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20

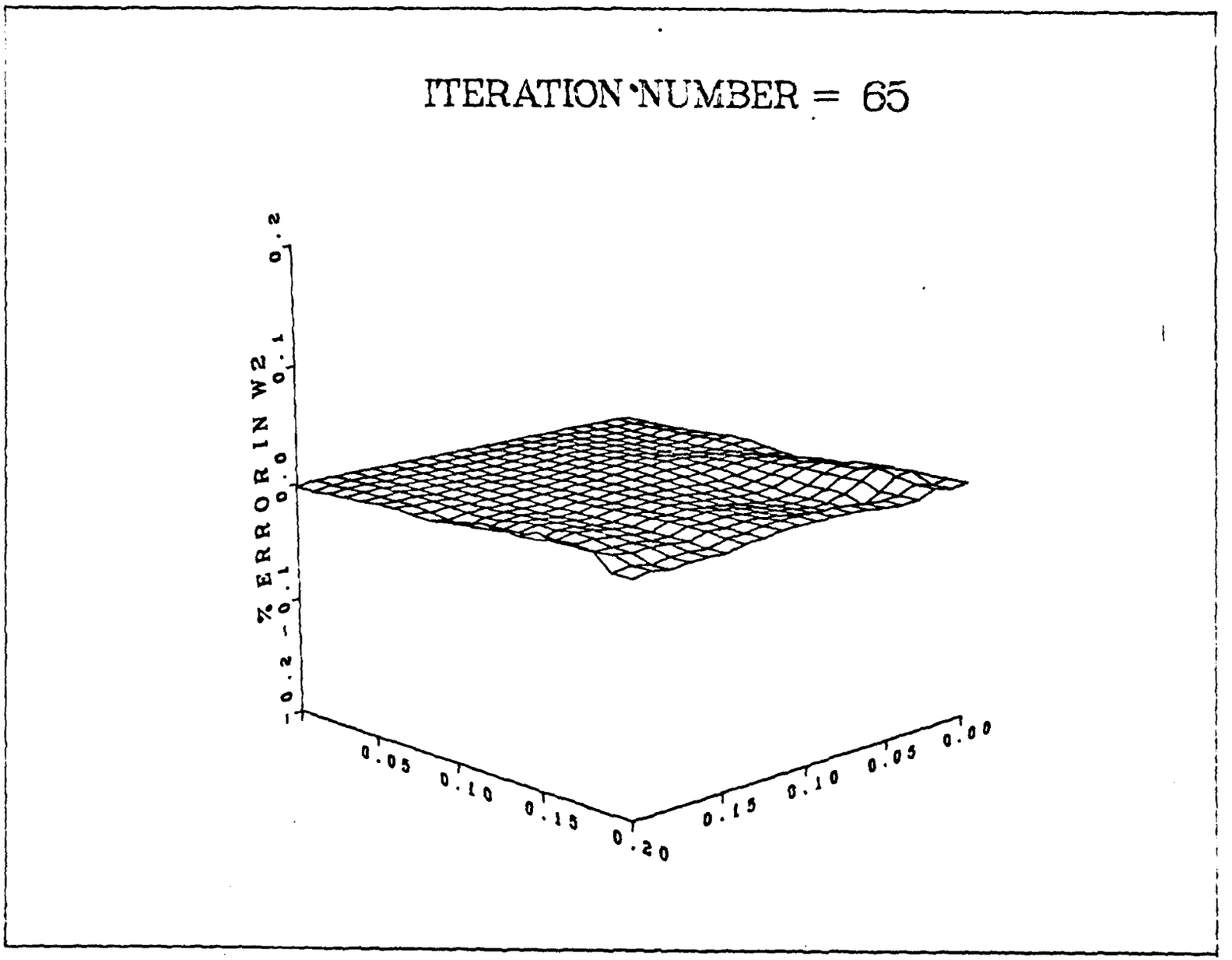

ITERATION NUMBER = 30
% ERROR IN WS
2.0
1.0
0.0
1.0
2.0
0.00
0.05
0.10
0.15
0.20
0.15
0.10
0.05


ITERATION NUMBER = 35
% ERROR IN WS
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.05
0.10
0.15
0.20


ITERATION NUMBER = 40
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.15
0.10
0.05


ITERATION NUMBER = 45
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00


ITERATION NUMBER = 60
% ERROR IN W3
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00


ITERATION NUMBER = 65
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

IMPLEMENTATION

```
      SUBROUTINE BC
      COMMON   DETA,DZETA,CV,RC,PR,PRT,GAMMA,EP(20,20,1),DTL(20),
     1 GAMM1,CFL,BETA,UINF,RHOINF,CINF,TINF,TW,DT,CX,CY,L,JBL,JBU,
     2 KT1,KT2,JL,KL,JLM,KLM,ILE,ISMTHX,ISMTHY,PI,PINF,YRL
     3 AK1,AK2,AK3,AK4,AW1,AW2,AW3,AW4,IDEV,XRL
      COMMON TW  ETX(20,20,1),ETY(20,20,1),ZTX(20,20,1),ZTY(20,20,1)
      COMMON  DEP  RHO(20,20,1),RHOU(20,20,1),RHOV(20,20,1),
     1 RHOE(20,20,1)
      COMMON  DEPP  RHOP(20,20,1),RHOUP(20,20,1),RHOVP(20,20,1),
     1 RHOEP(20,20,1)
      COMMON  DOF  X(20,20,1),Y(20,20,1)
      COMMON  DIP  P(20,20),PP(20,20),W31(20),W2JL(20)
C
      RUINF=RHOINF*UINF
      REINF=RHOINF*(CV*TINF+0.5*UINF**2)
C
      IF(L.EQ.2) GO TO 10
C
C                 ***************
C                 PREDICTOR SWEEP
C                 ***************
C
      DO 15 J=1,JL
      DO 15 K=1,KL
      PP(K,J)=(RC/CV)*(RHOEP(K,J,1)-(((RHOUP(K,J,1)**2)+
     1 (RHOVP(K,J,1)**2))/(2.0*RHOP(K,J,1))))
   15 CONTINUE
C
      DO 25 K=1,KL
      W31(K)=PP(K,2)/(RHOINF*CINF)-RHOVP(K,2,1)/RHOP(K,2,1)
      W2JL(K)=PP(K,JLM)/(RHOINF*CINF)+RHOVP(K,JLM,1)/RHOP(K,JLM,1)
   25 CONTINUE
C
C
C
C     TRANSVERSE BOUNDARY CONDITIONS
C
C        TOP FAR FIELD  **********
C
C
      DO 1 K=1,KL
      RHOUP(K,JL,1)=RHOUP(K,JLM,1)
      RHOVP(K,JL,1)=0.5*RHOP(K,JLM,1)*(-W31(K)+RHOVP(K,JLM,1)/RHOP(K,JLM
     1,1)+PP(K,JLM)/(RHOINF*CINF))
      PP(K,JL)=0.5*RHOINF*CINF*(W31(K)+RHOVP(K,JLM,1)/RHOP(K,JLM,1)+
     1 PP(K,JL)/(RHOINF*CINF))
      RHOP(K,JL,1)=RHOP(K,JLM,1)+(PP(K,JL)-PP(K,JLM))/(CINF**2)
      RHOEP(K,JL,1)=(PP(K,JL)*CV)/RC+((RHOUP(K,JL,1)**2)+(RHOVP(K,JL,1)*
     1*2))/(RHOP(K,JL,1)*2.0)
C
    1 CONTINUE
C
C
C
C    ******  BOTTOM WALL  ************
C
      DO 2 K=1,KL
      RHOUP(K,1,1)=RHOUP(K,2,1)
      RHOVP(K,1,1)=0.5*RHOP(K,2,1)*(-RHOVP(K,2,1)/RHOP(K,2,1)+W2JL(K)
     1 -PP(K,2)/(RHOINF*CINF))
      PP(K,1)=0.5*RHOINF*CINF*(RHOVP(K,2,1)/RHOP(K,2,1)+W2JL(K)+
     1 PP(K,2)/(RHOINF*CINF))
```

Reproduced from best available copy.

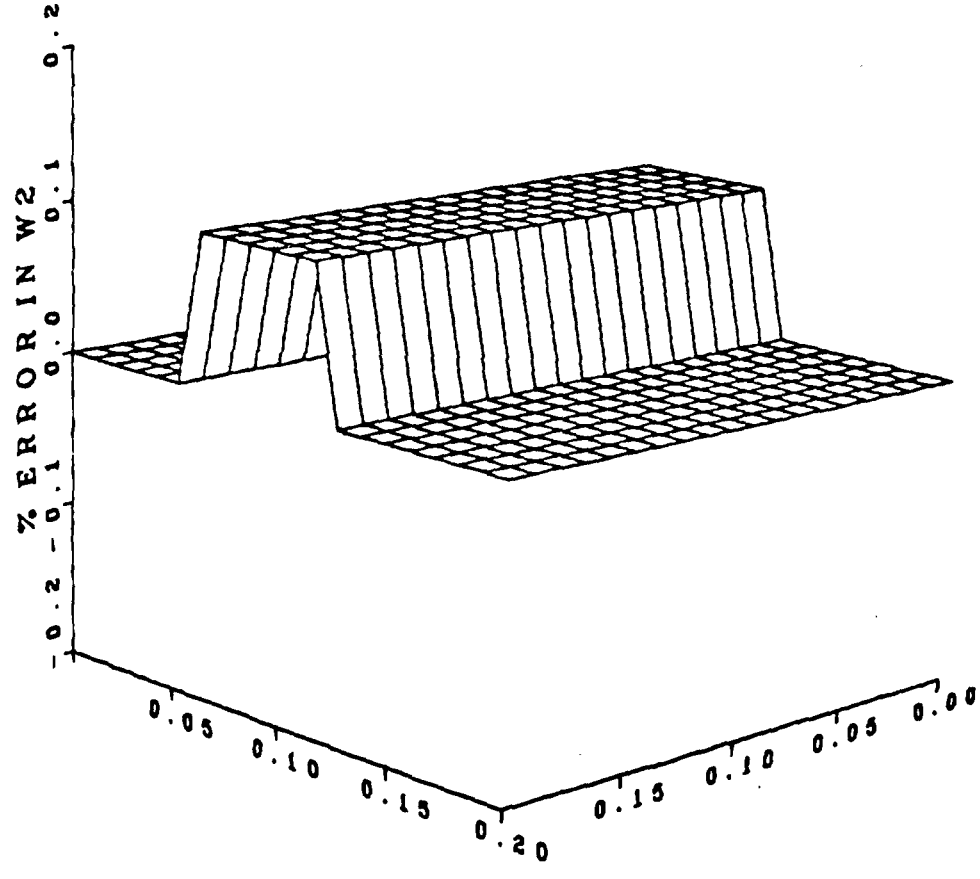
```
      RHOP(K,1,1)=RHOP(K,2,1)+(PP(K,1)-PP(K,2))/(CINF**2)
      RHOEP(K,1,1)=(PP(K,1)*CV)/RC+((RHOUP(K,1,1)**2)+(RHOVP(K,1,1)*
     1*2))/(RHOP(K,1,1)*2.0)
  2   CONTINUE
C
C
C   UPSTREAM & DOWNSTREAM BOUNDARY CONDITIONS
C
C     ******** UPSTREAM **********
C
      DO 3 J=1,JL
      RHOP(1,J,1)=AK1+(RHOINF/(2.*CINF))*((PP(2,J)/(RHOINF*CINF))+AK3
     1 -(RHOUP(2,J,1)/RHOP(2,J,1)))
      RHOUP(1,J,1)=RHOP(1,J,1)*0.5*(AK3-(PP(2,J)/(RHOINF*CINF))+
     1 (RHOUP(2,J,1)/RHOP(2,J,1)))
      RHOVP(1,J,1)=RHOVP(2,J,1)
      PP(1,J)=0.5*(PP(2,J)+RHOINF*CINF*(AK3-(RHOUP(2,J,1)/RHOP(2,J,1))))
      RHOEP(1,J,1)=(PP(1,J)*CV)/RC+((RHOUP(1,J,1)**2)+(RHOVP(1,J,1)**2))
     1 /(RHOP(1,J,1)*2.0)
C
C
C     ********** DOWNSTREAM **********
C
      RHOP(KL,J,1)=RHOP(KLM,J,1)-PP(KLM,J)/(CINF**2)+(RHOINF/(2.*CINF))*
     1 (AK4+RHOUP(KLM,J,1)/RHOP(KLM,J,1)+(PP(KLM,J)/(RHOINF*CINF)))
      RHOUP(KL,J,1)=RHOUP(KLM,J,1)
      RHOVP(KL,J,1)=RHOVP(KL,J,1)*(RHOUP(KLM,J,1)/RHOP(KLM,J,1)-AK4
     1 +(PP(KLM,J)/(RHOINF*CINF)))
      PP(KL,J)=0.5*(PP(KLM,J)+RHOINF*CINF*(AK4+(RHOUP(KLM,J,1)/RHOP(KLM,
     1 J,1))))
      RHOEP(KL,J,1)=(PP(KL,J)*CV)/RC+((RHOUP(KL,J,1)**2)+(RHOVP(KL,J,1)*
     1 *2))/(RHOP(KL,J,1)*2.0)
  3   CONTINUE
C
C
      GO TO 100
C
C             ********************************
C                  CORRECTOR SWEEP
C             ********************************
C
C
  10  CONTINUE
      DO 16 J=1,JL
      DO 16 K=1,KL
      P(K,J)=(RC/CV)*(RHOE(K,J,1)-(((RHOU(K,J,1)**2)+(RHOV(K,J,1)**2))/
     1 (2.0*RHO(K,J,1))))
  16  CONTINUE
C
      DO 26 K=1,KL
      U31(K)=P(K,2)/(RHOINF*CINF)-RHOV(K,2,1)/RHO(K,2,1)
      U2JL(K)=P(K,JLM)/(RHOINF*CINF)+RHOV(K,JLM,1)/RHO(K,JLM,1)
  26  CONTINUE
C
C     TRANSVERSE BOUNDARY CONDITIONS
C
C       *****  TOP FAR FIELD   ******
C
      DO 11 K=1,KL
      RHOU(K,JL,1)=RHOU(K,JLM,1)
      RHOV(K,JL,1)=0.5*RHO(K,JLM,1)*(-U31(K)+RHOV(K,JLM,1)/RHO(K,JLM
     1,1)+P(K,JLM)/(RHOINF*CINF))
      P(K,JL)=0.5*RHOINF*CINF*(U31(K)+RHOV(K,JLM,1)/RHO(K,JLM,1)+
```

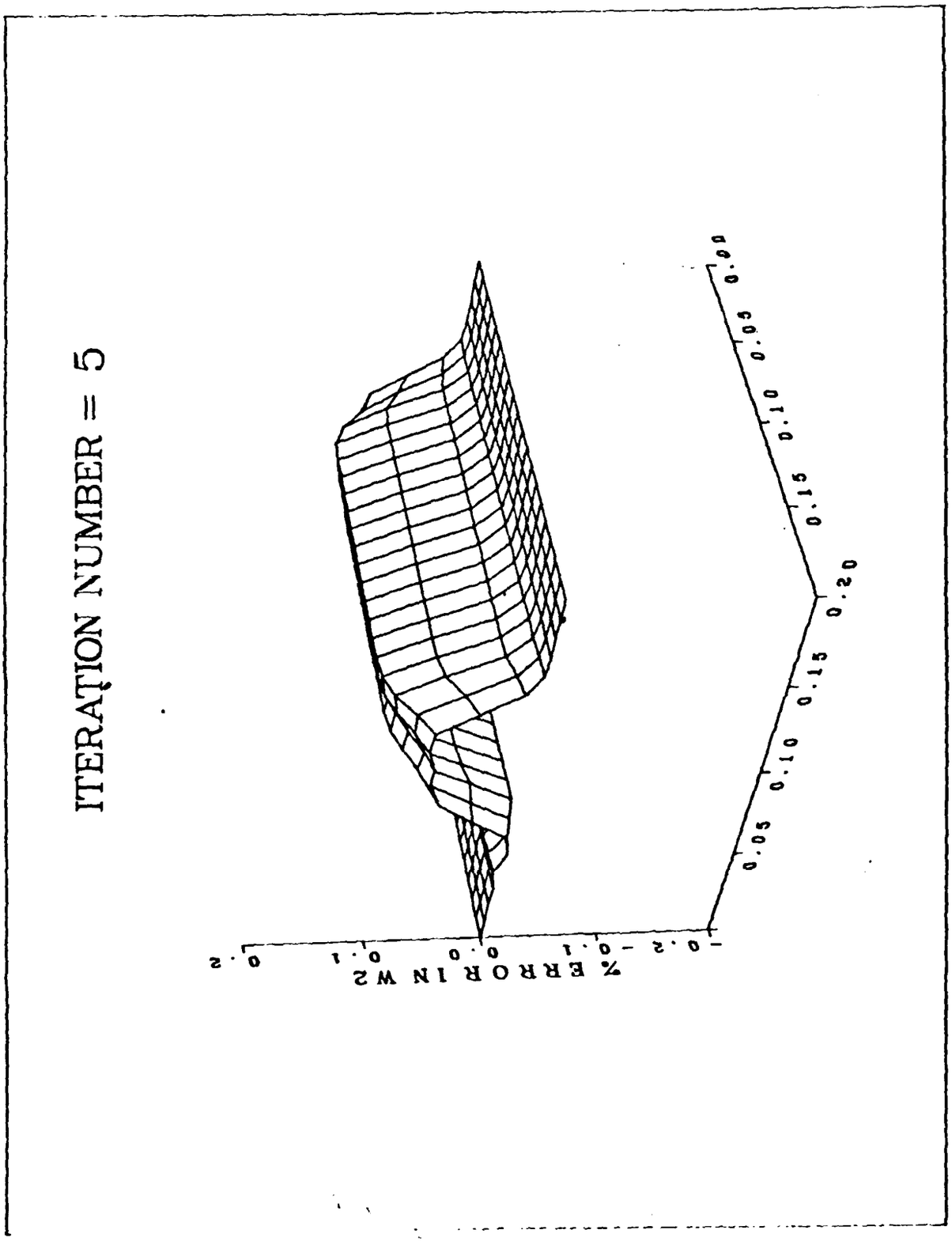
Reproduced from best available copy.

```
     1 P(K,JL)/(RHOINF*CINF))
      RHO(K,JL,1)=RHO(K,JLM,1)+(P(K,JL)-P(K,JLM))/(CINF**2)
      RHOE(K,JL,1)=(P(K,JL)*CV)/RC+((RHOU(K,JL,1)**2)+(RHOV(K,JL,1)*
     1*2)/(RHO(K,JL,1)*2.0)

 11   CONTINUE


      ******  BOTTOM WALL  ********

      DO 12 K=1,KL
      RHOU(K,1,1)=RHOUP(K,2,1)
      RHOV(K,1,1)=0.5*RHO(K,2,1)*(-RHOV(K,2,1)/RHO(K,2,1)+U2JL(K)
     1 -P(K,2)/(RHOINF*CINF))
      P(K,1)=0.5*RHOINF*CINF*(RHOV(K,2,1)/RHO(K,2,1)+U2JL(K)+
     1 P(K,2)/(RHOINF*CINF))
      RHO(K,1,1)=RHO(K,2,1)+(P(K,1)-P(K,2))/(CINF**2)
      RHOE(K,1,1)=(P(K,1)*CV)/RC+((RHOU(K,1,1)**2)+(RHOV(K,1,1)*
     1*2)/(RHO(K,1,1)*2.0)
 12   CONTINUE
C
C
C     UPSTREAM & DOWNSTREAM BOUNDARY CONDITIONS
C
C     *******  UPSTREAM  ********
C
      DO 13 J=1,JL
      RHO(1,J,1)=AK1+(RHOINF/(2.*CINF))*((P(2,J)/(RHOINF*CINF))+AK3
     1 -(RHOU(2,J,1)/RHO(2,J,1)))
      RHOU(1,J,1)=RHO(1,J,1)*0.5*(AK3-(P(2,J)/(RHOINF*CINF))
     1 +(RHOU(2,J,1)/RHO(2,J,1)))
      RHOV(1,J,1)=RHOV(2,J,1)
      P(1,J)=0.5*(P(2,J)+RHOINF*CINF*(AK3-(RHOU(2,J,1)/RHO(2,J,1))))
      RHOE(1,J,1)=(P(1,J)*CV)/RC+((RHOU(1,J,1)**2)+(RHOV(1,J,1)**2))
     1 /(RHO(1,J,1)*2.0)
C
C
C
C     ******  DOWNSTREAM  ***********
C
      RHO(KL,J,1)=RHO(KLM,J,1)-P(KLM,J)/(CINF**2)+(RHOINF/(2.*CINF))*
     1 (AK4+RHOU(KLM,J,1)/RHO(KLM,J,1)+(P(KLM,J)/(RHOINF*CINF)))
      RHOV(KL,J,1)=RHOV(KLM,J,1)
      RHOU(KL,J,1)=0.5*RHO(KL,J,1)*(RHOU(KLM,J,1)/RHO(KLM,J,1)-AK4
     1 +(P(KLM,J)/(RHOINF*CINF)))
      P(KL,J)=0.5*(P(KLM,J)+RHOINF*CINF*(AK4+(RHOU(KLM,J,1)/RHO(KLM,
     1 J,1))))
      RHOE(KL,J,1)=(P(KL,J)*CV)/RC+((RHOU(KL,J,1)**2)+(RHOV(KL,J,1)*
     1 *2))/(RHO(KL,J,1)*2.0)
 13   CONTINUE
C
C
100   RETURN
      END
      SUBROUTINE TRANS
      COMMON / / DETA,DZETA,CV,RC,PR,PRT,GAMMA,EP(20,20,1),DTL(20),
     1 GAMM1,CFL,BETA,UINF,RHOINF,CINF,TINF,TW,DT,CX,CY,L,JBL,JBU,
     2 KT1,KT2,JL,KL,JLM,KLM,ILE,ISMTHX,ISMTHY,PI,PINF,YRL,
```

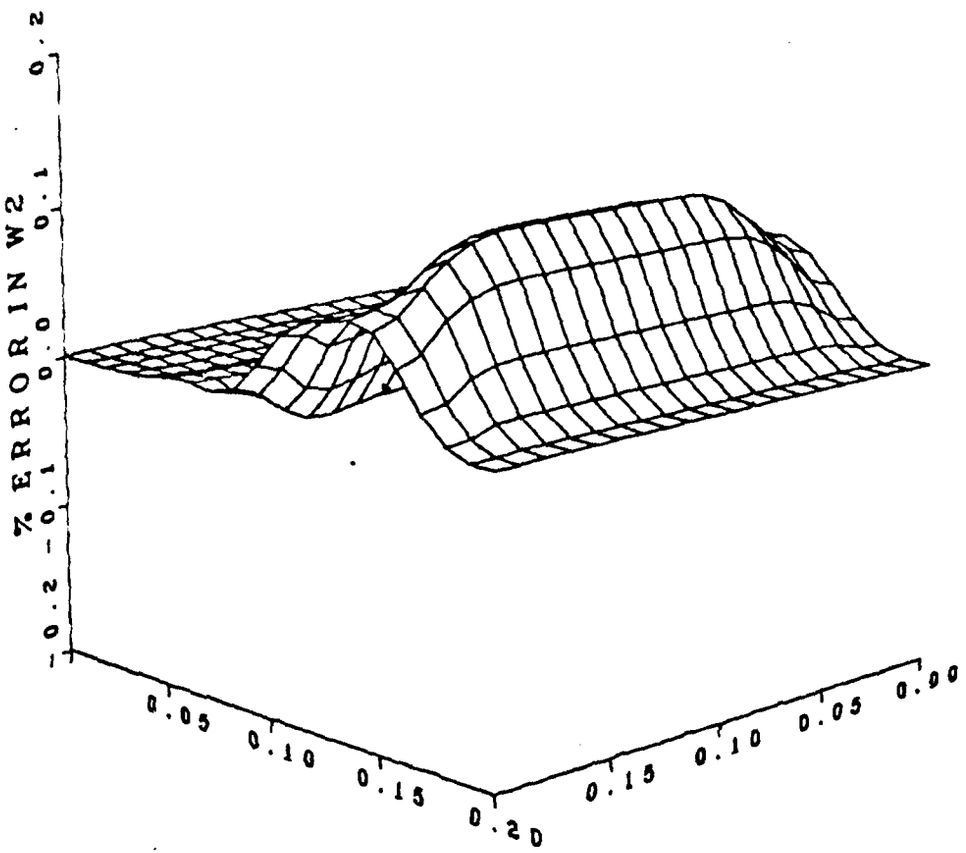
Reproduced from best available copy.

ITERATION NUMBER = 1

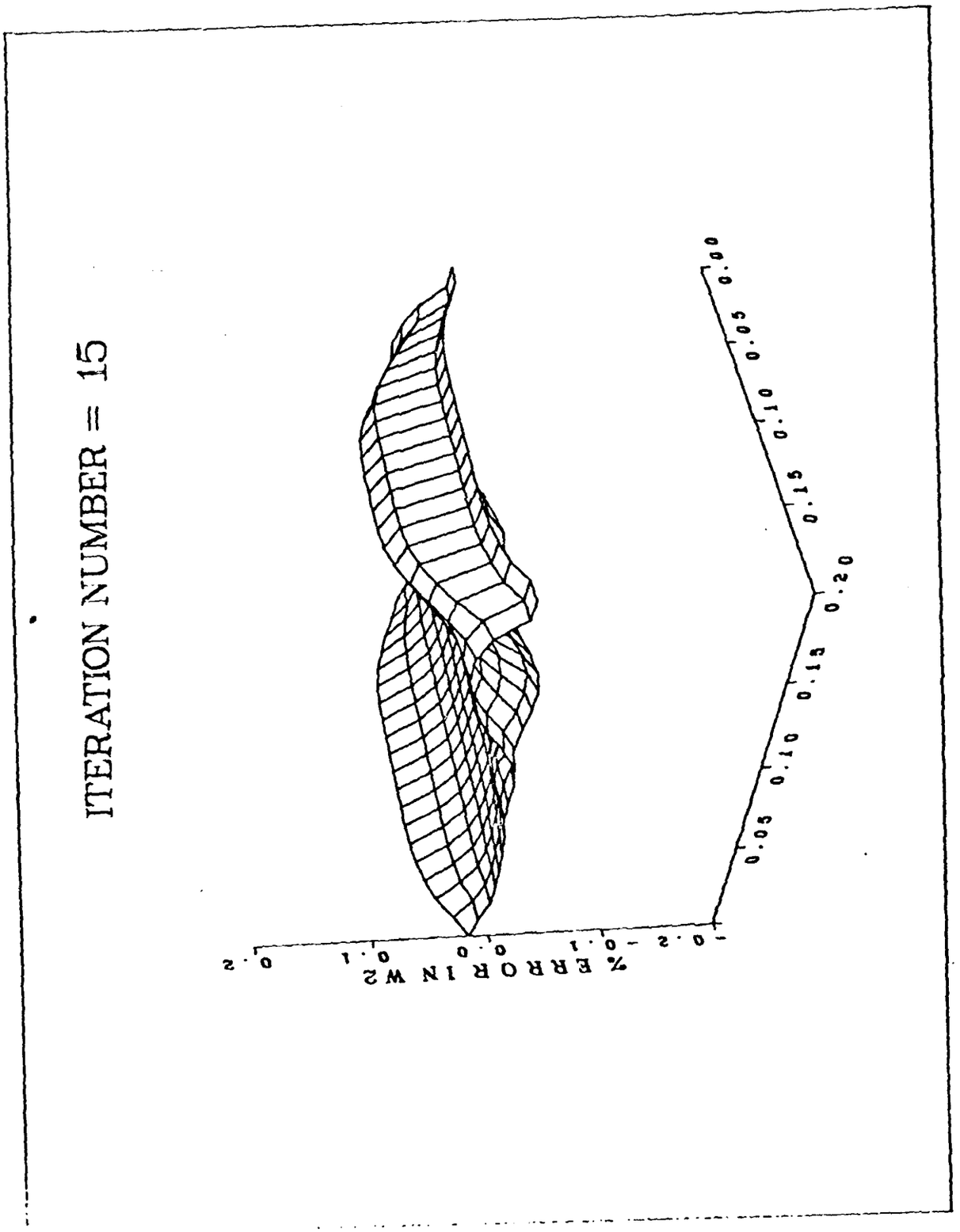


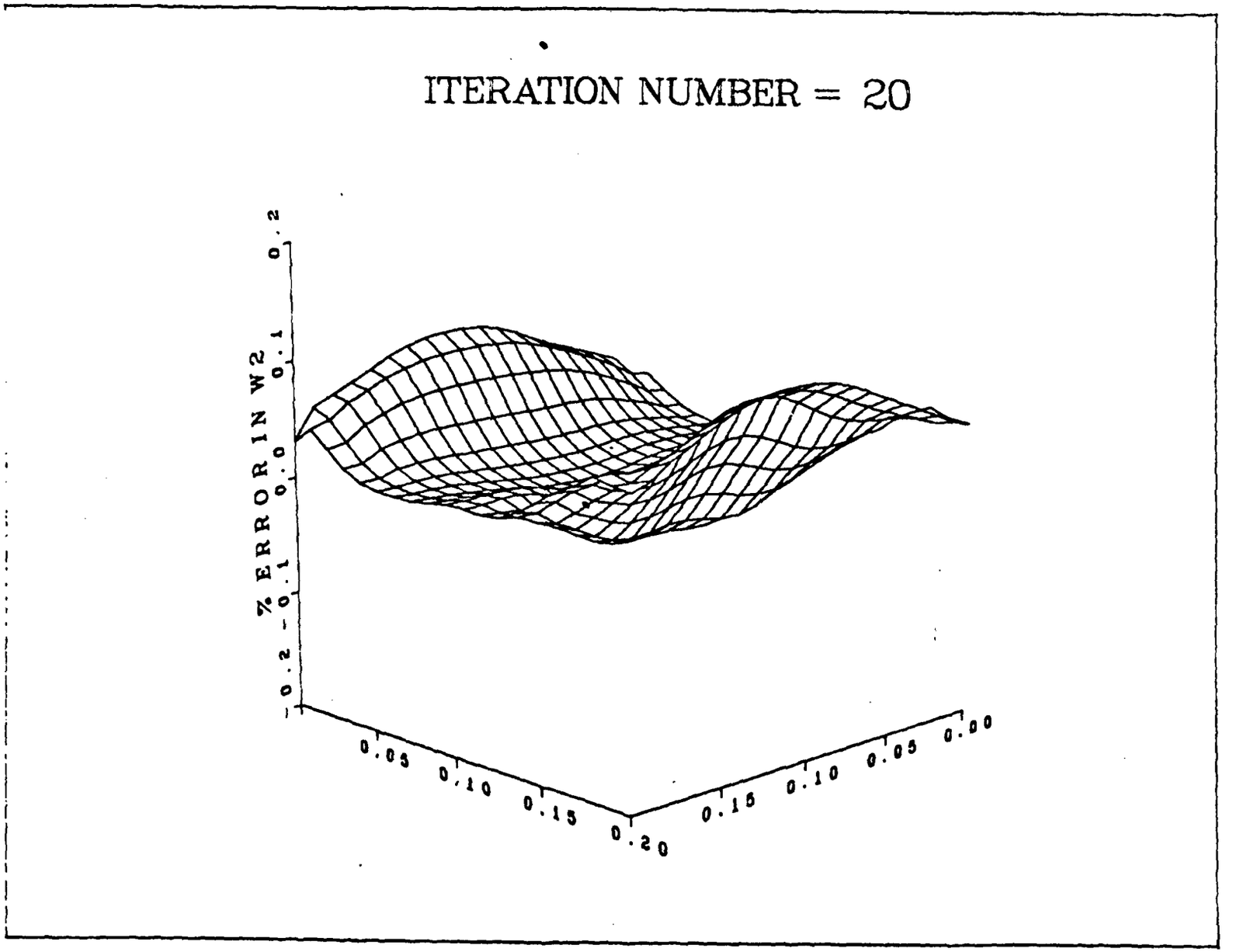

ITERATION NUMBER = 5
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.05
0.10
0.15
0.20

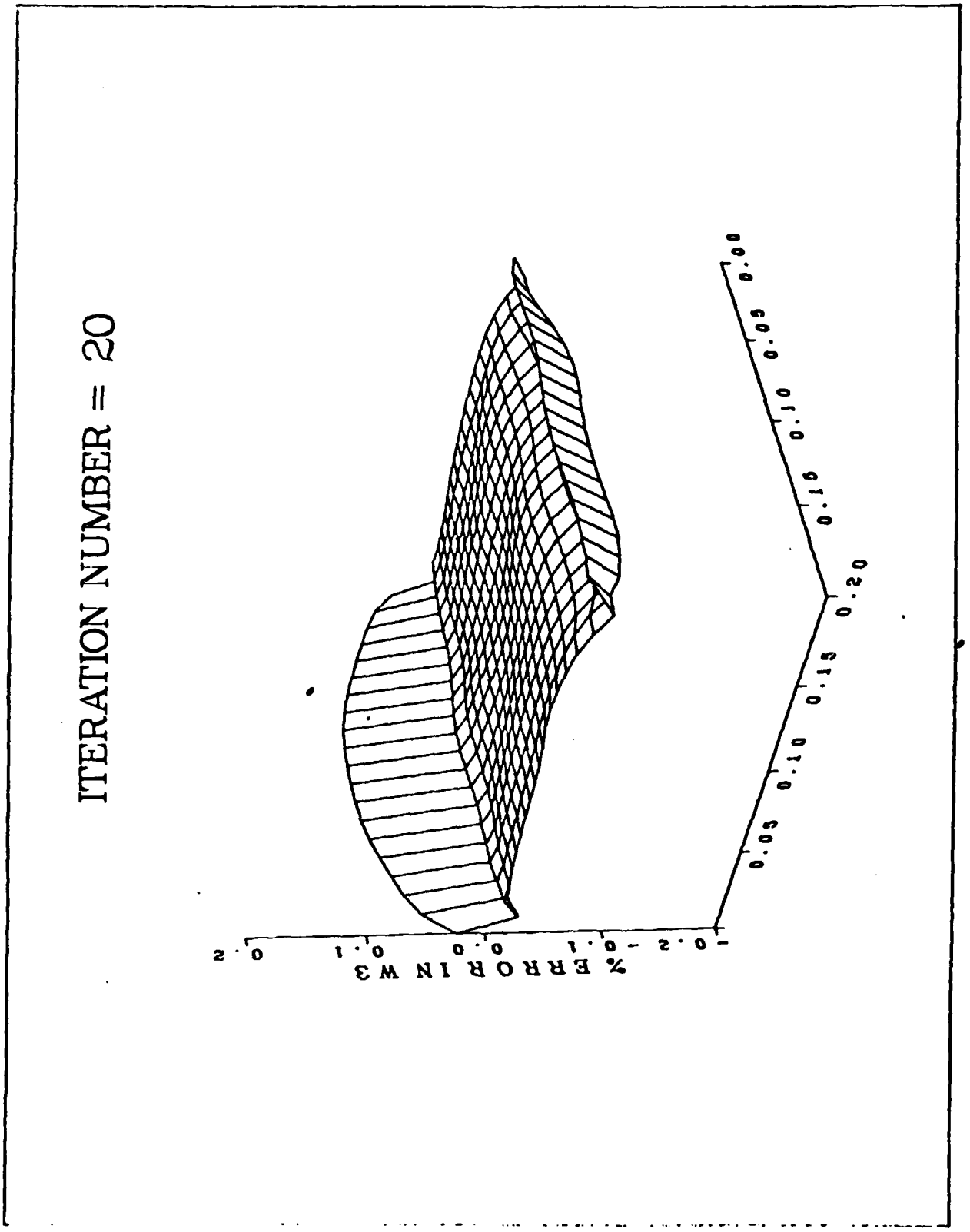
ITERATION NUMBER = 10



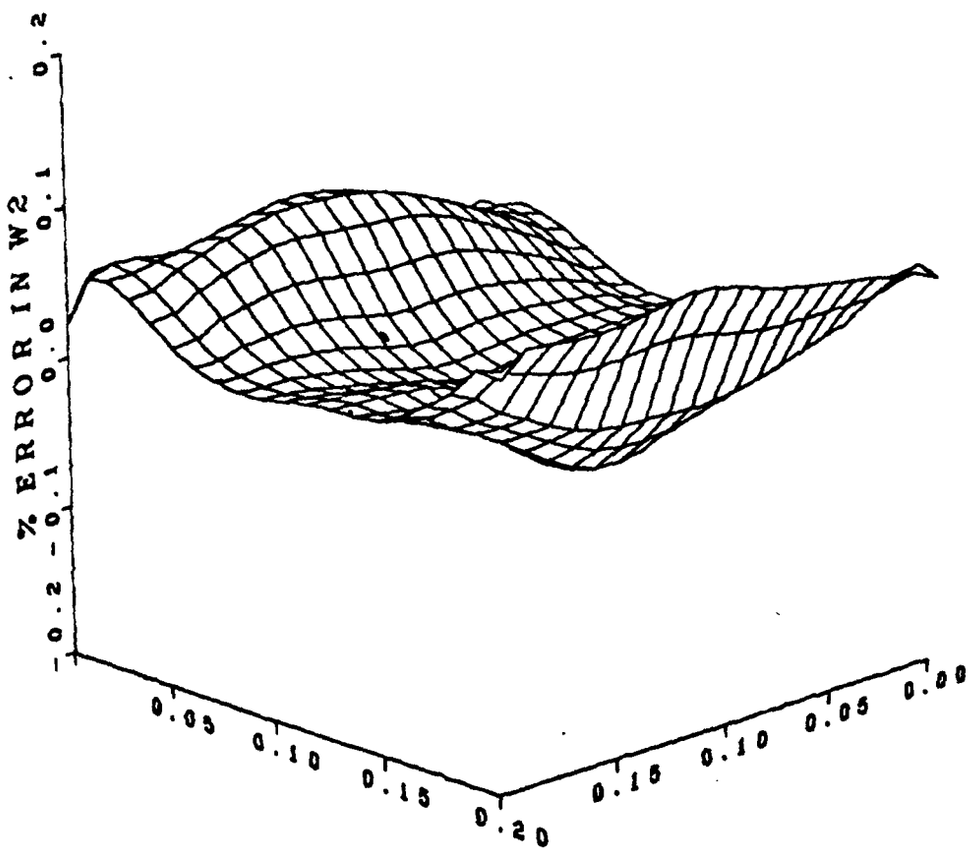
P1

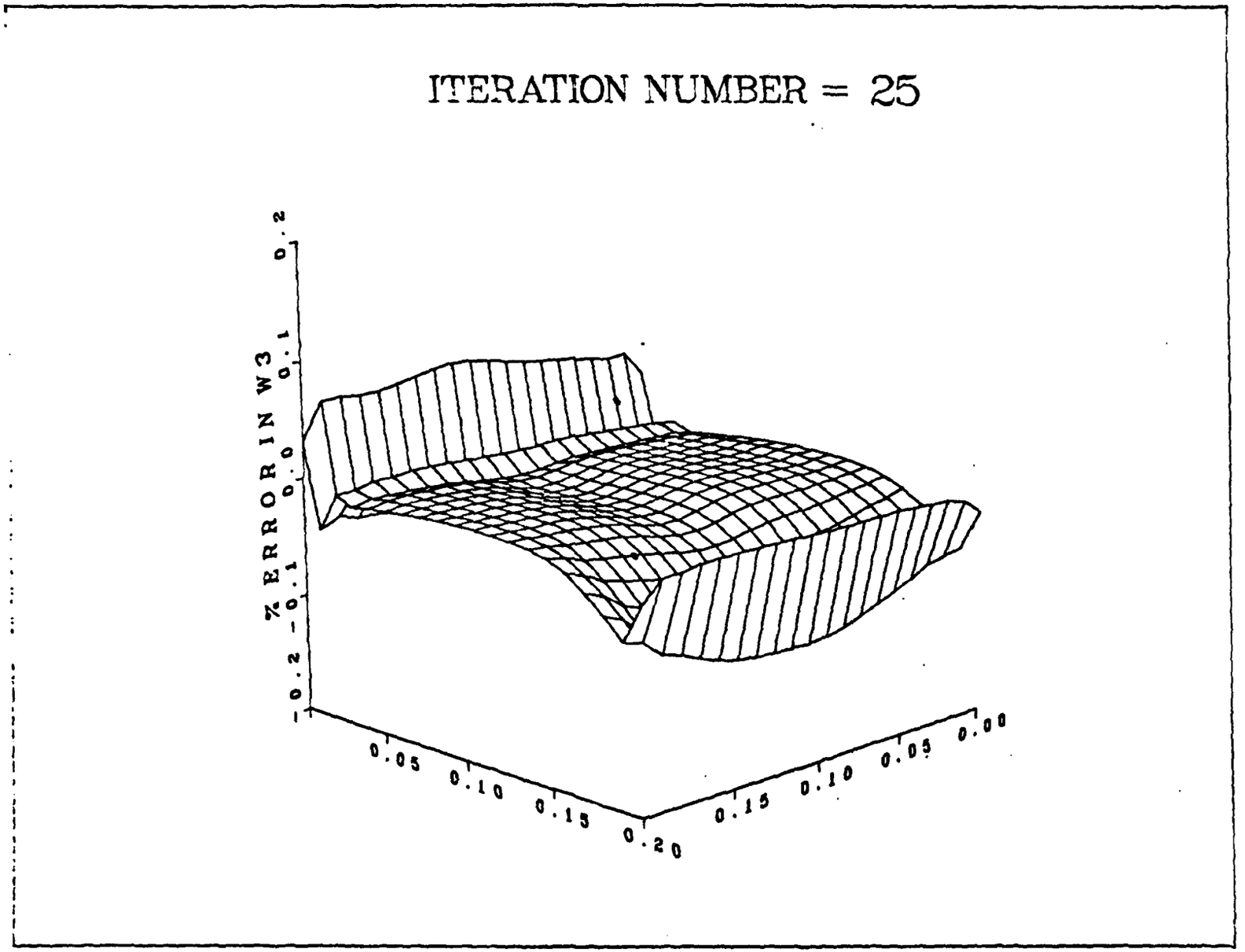

ITERATION NUMBER = 15
% ERROR IN W2
2.0
1.0
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.15
0.10
0.05

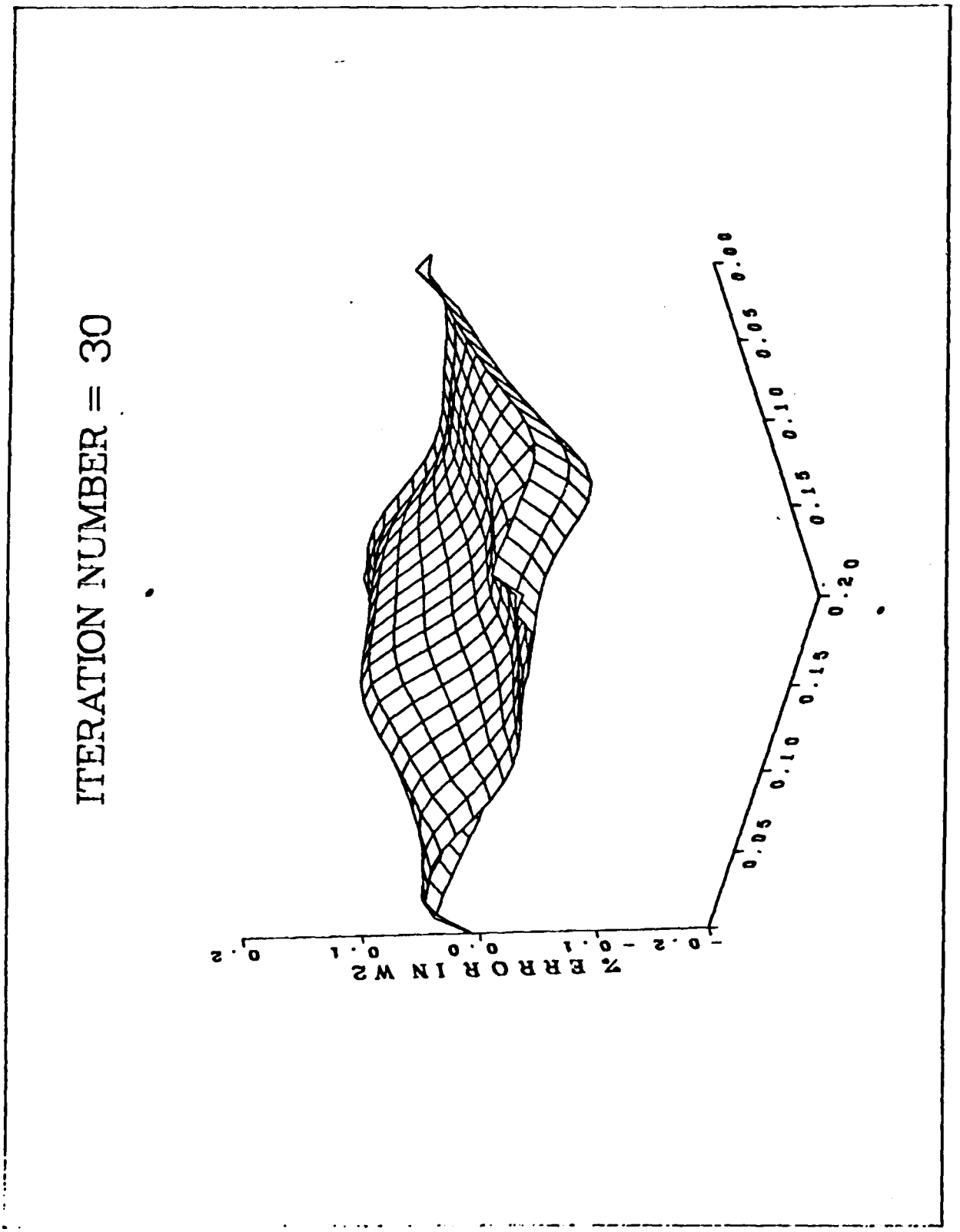

ITERATION NUMBER = 20
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

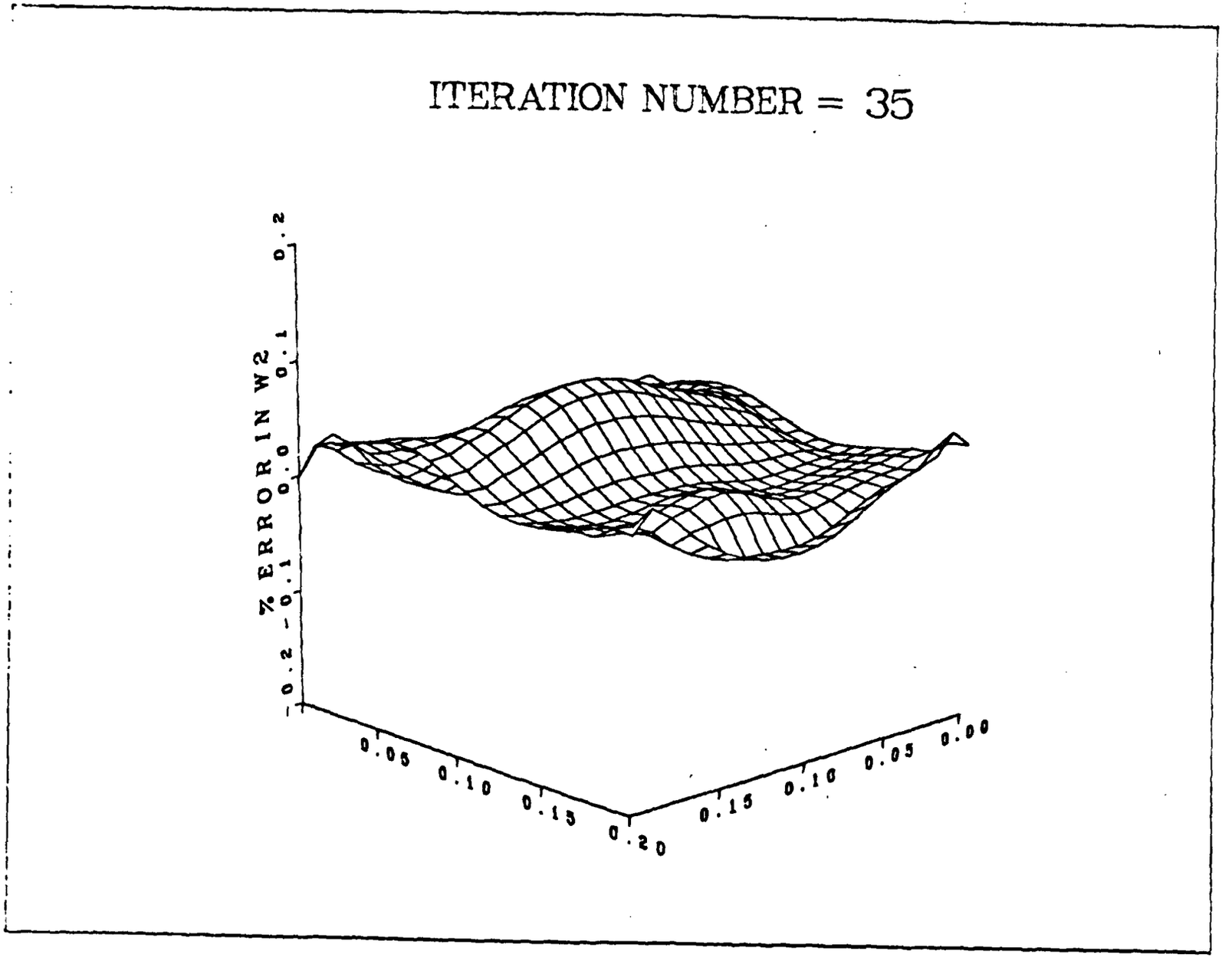

ITERATION NUMBER = 20
% ERROR IN W3
2.0
1.0
0.0
-1.0
-2.0
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

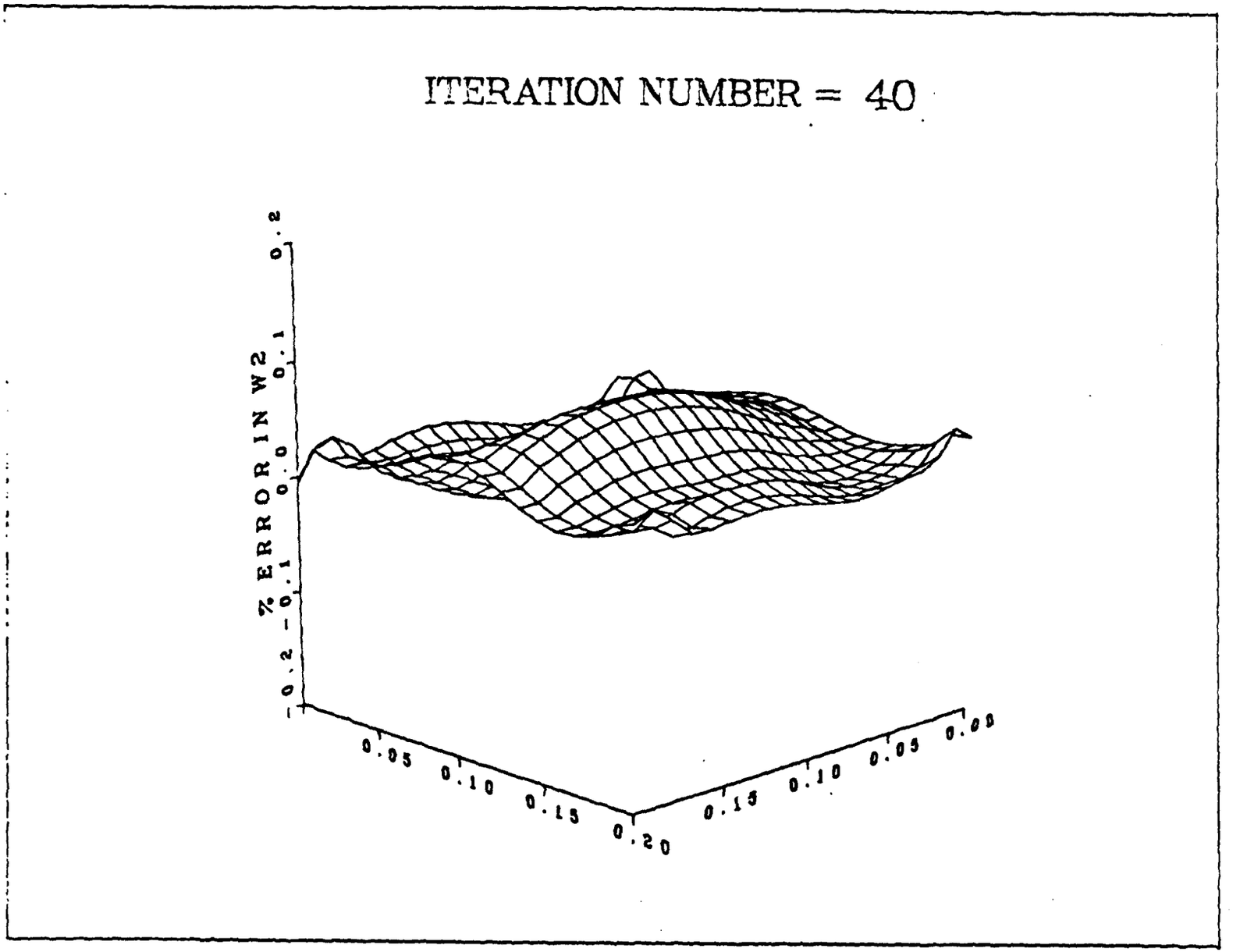
ITERATION NUMBER = 25

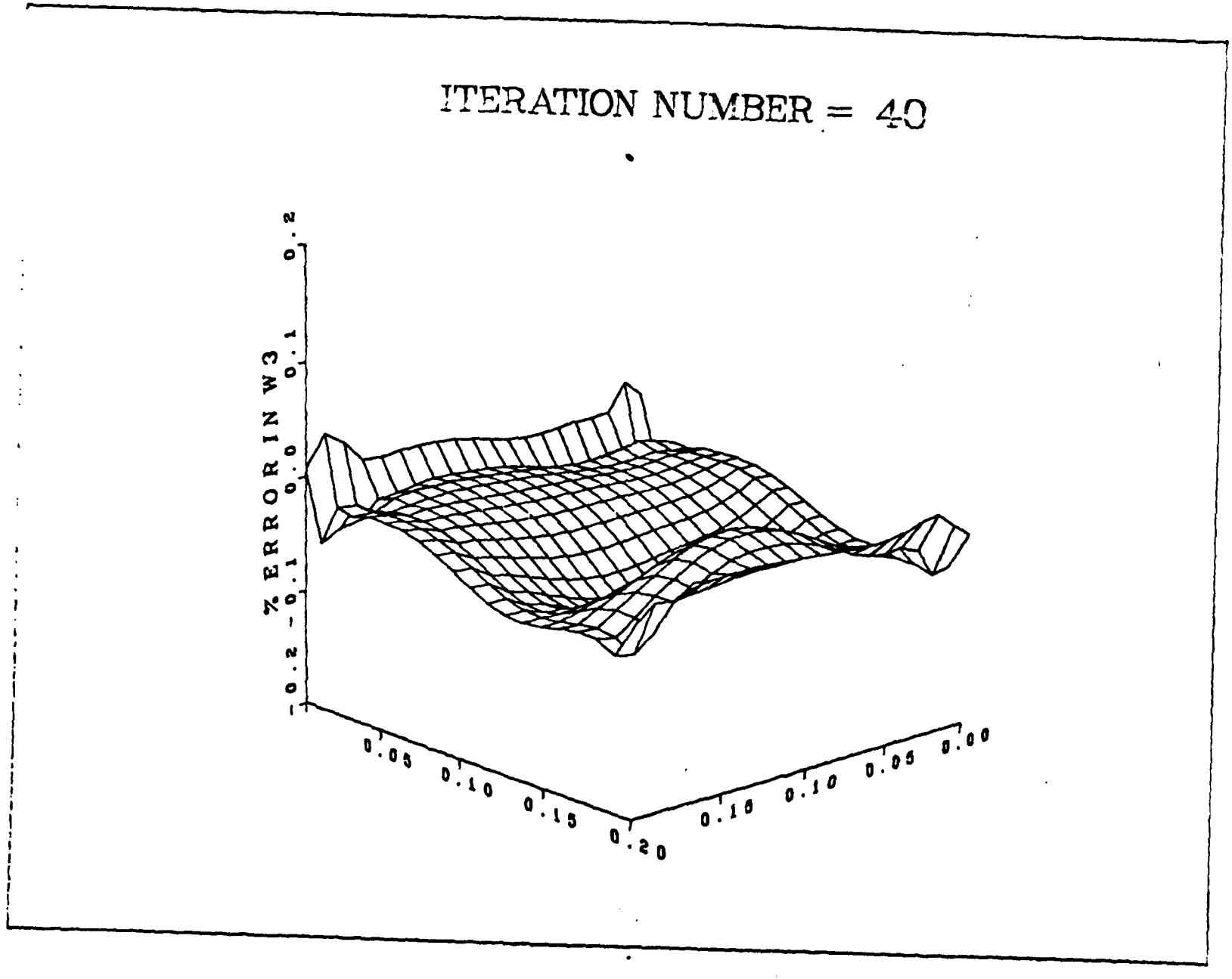


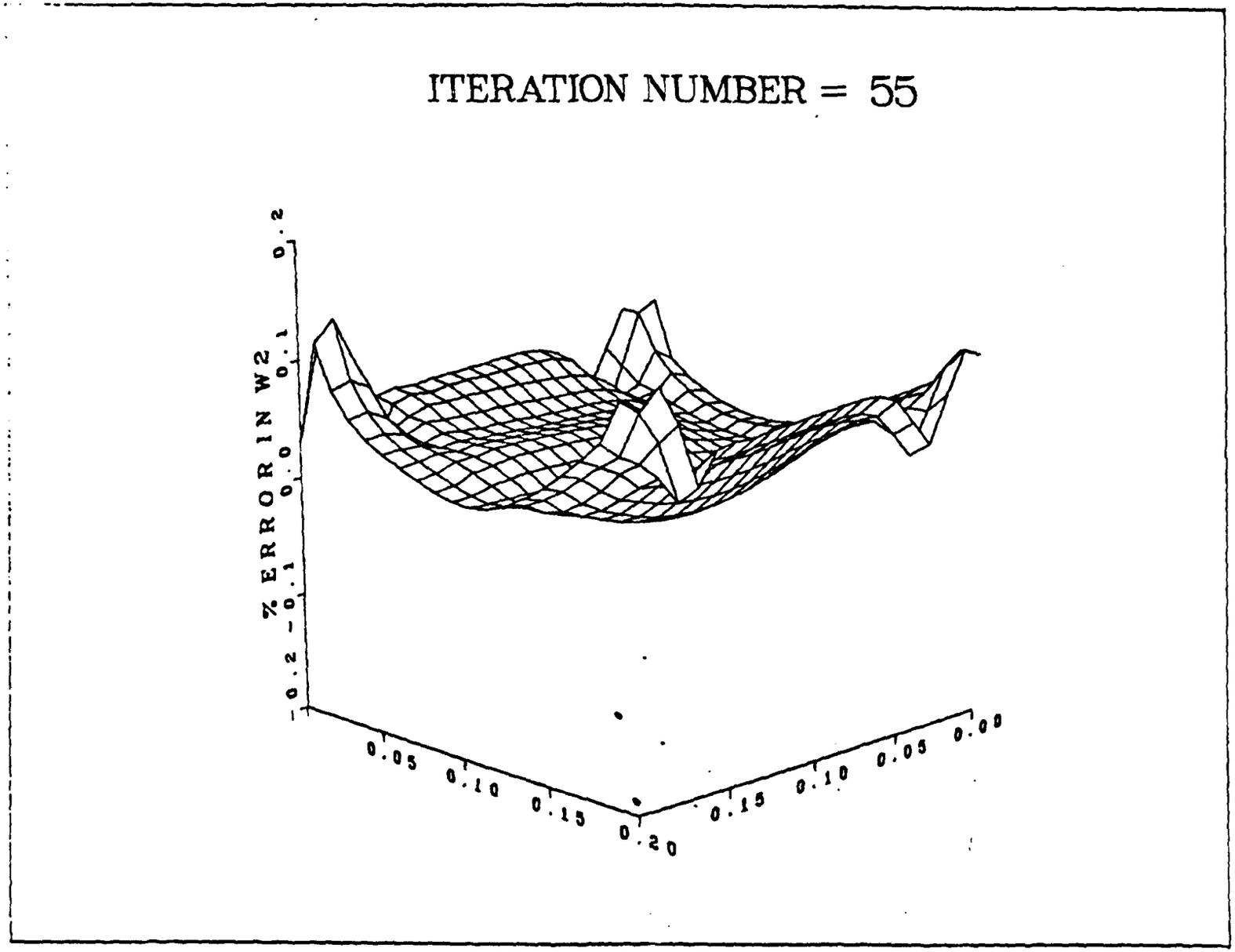

ITERATION NUMBER = 25
% ERROR IN W3
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00


ITERATION NUMBER = 30
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.00
0.05
0.10
0.15
0.20
0.05
0.10
0.15
0.20

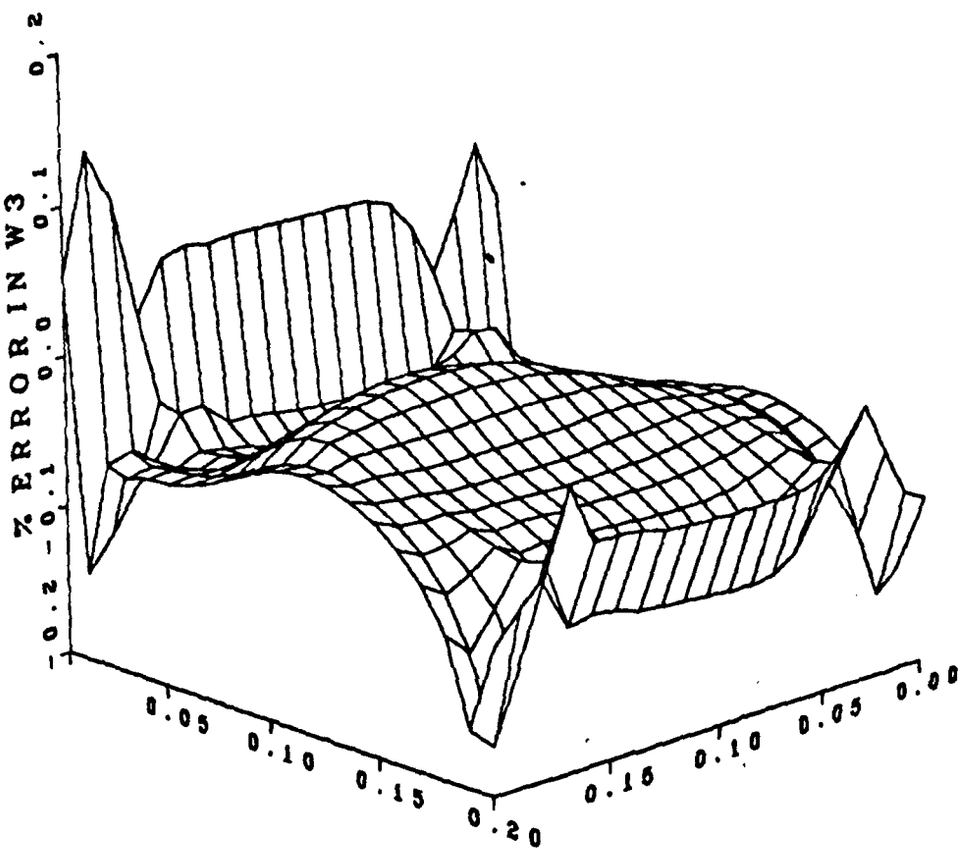

ITERATION NUMBER = 35
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

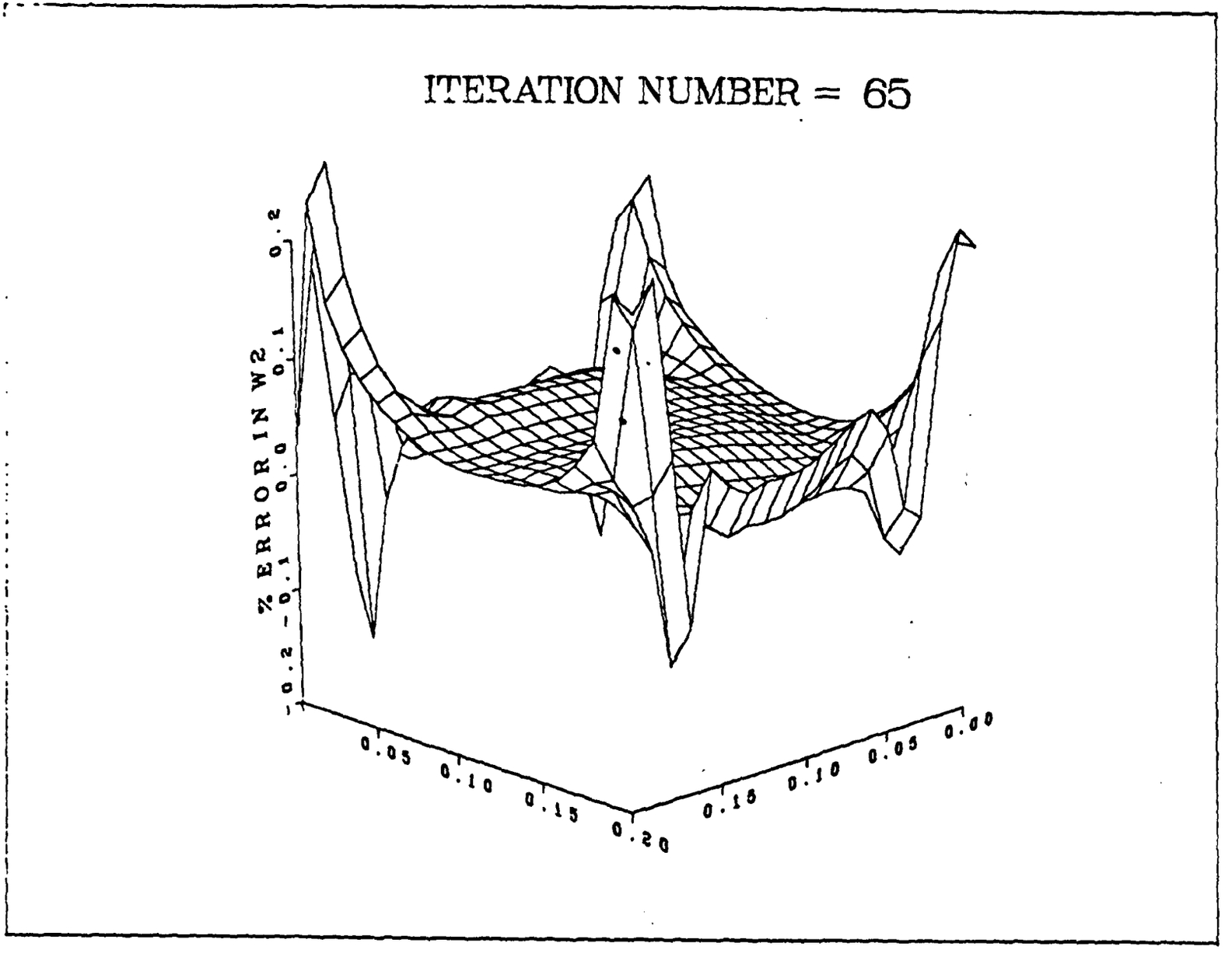

ITERATION NUMBER = 40
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

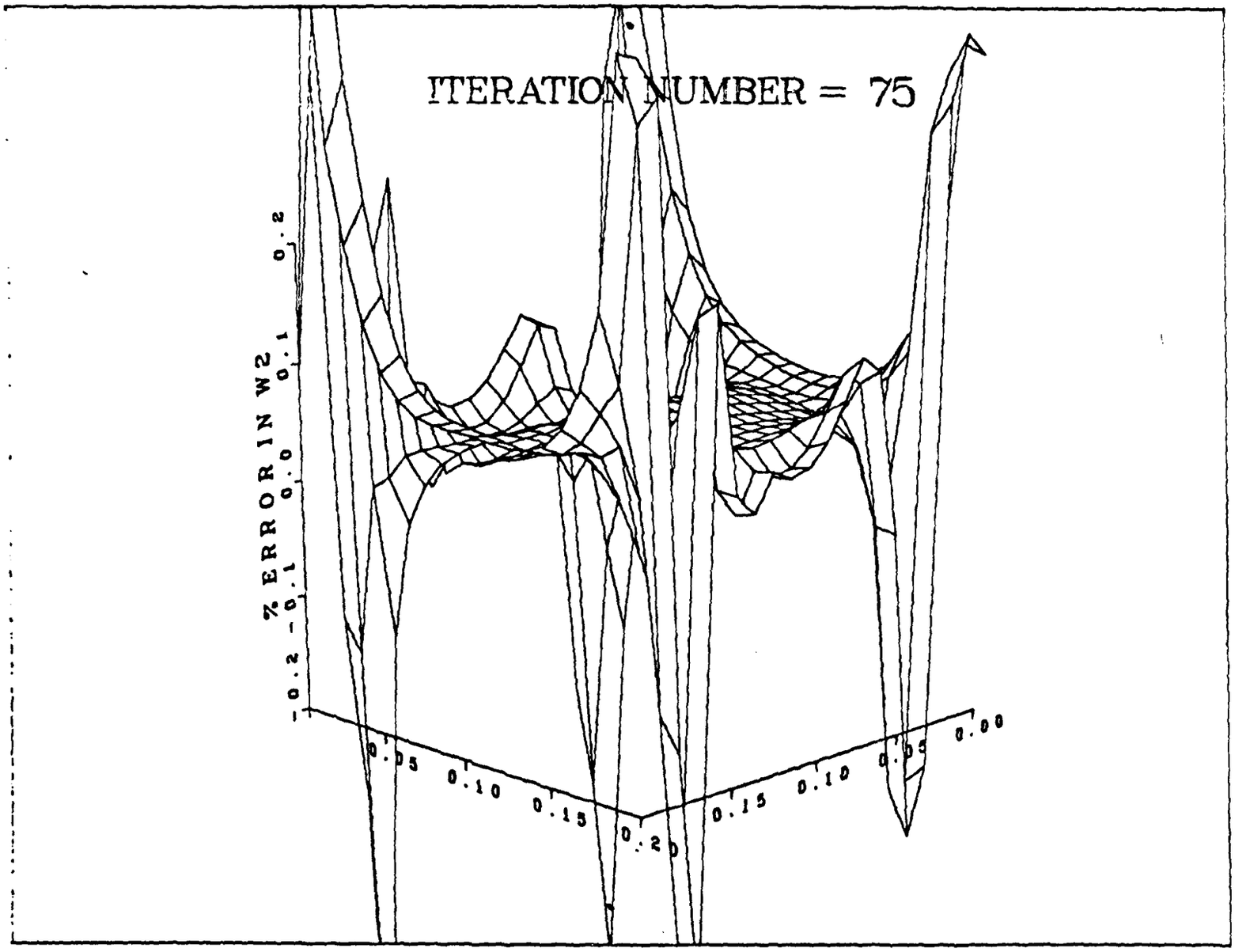

ITERATION NUMBER = 40
% ERROR IN W3
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00


ITERATION NUMBER = 55
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

ITERATION NUMBER = 55




ITERATION NUMBER = 65
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00


ITERATION NUMBER = 75
% ERROR IN W2
0.2
0.1
0.0
-0.1
-0.2
0.05
0.10
0.15
0.20
0.15
0.10
0.05
0.00

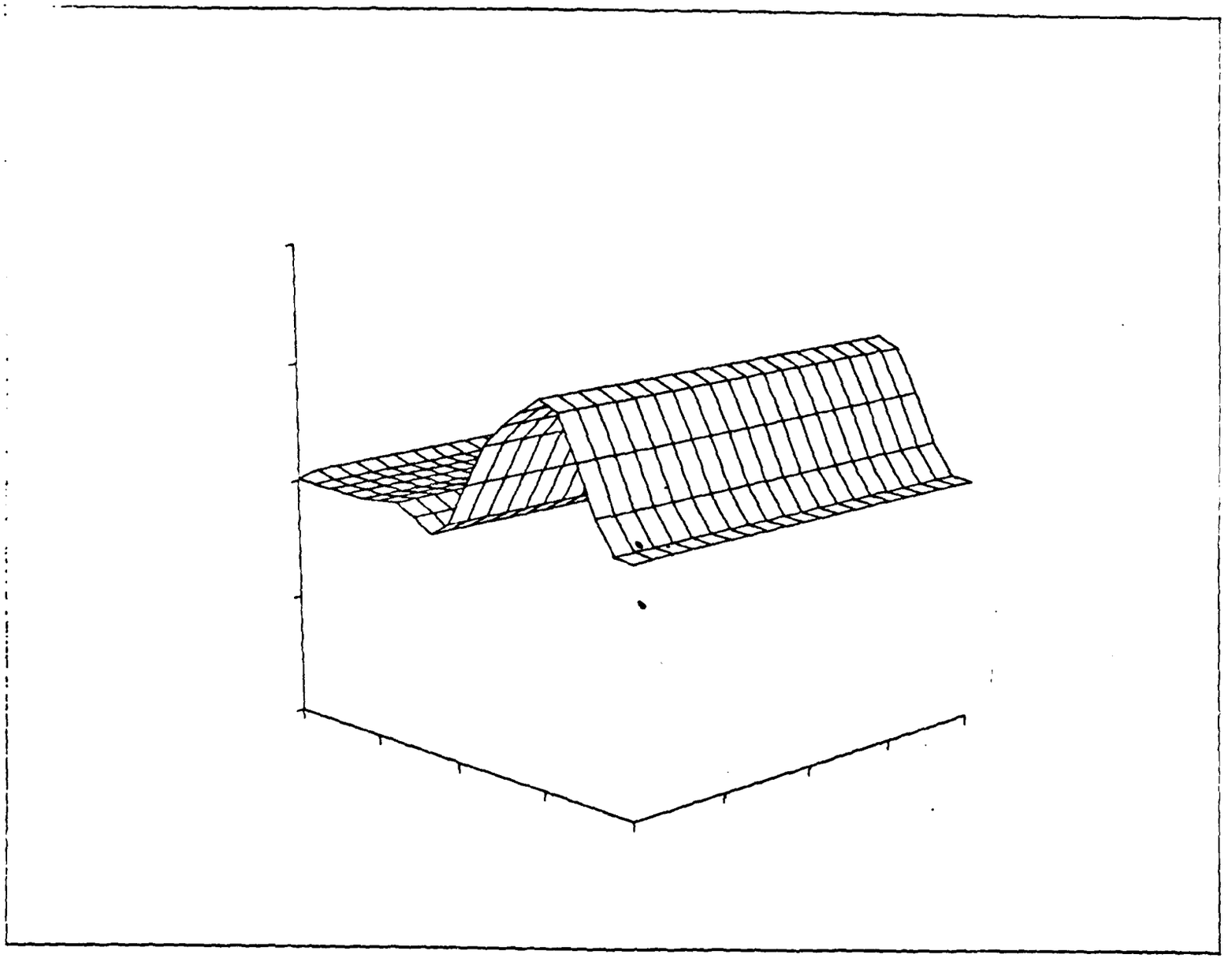
Finally, we attempted to run periodic characteristic B.C.s on a one-dimensional basis. We felt that since the problem manifested itself at the corners, if we removed the corners the problem should go away. We next show what happened in this situation.

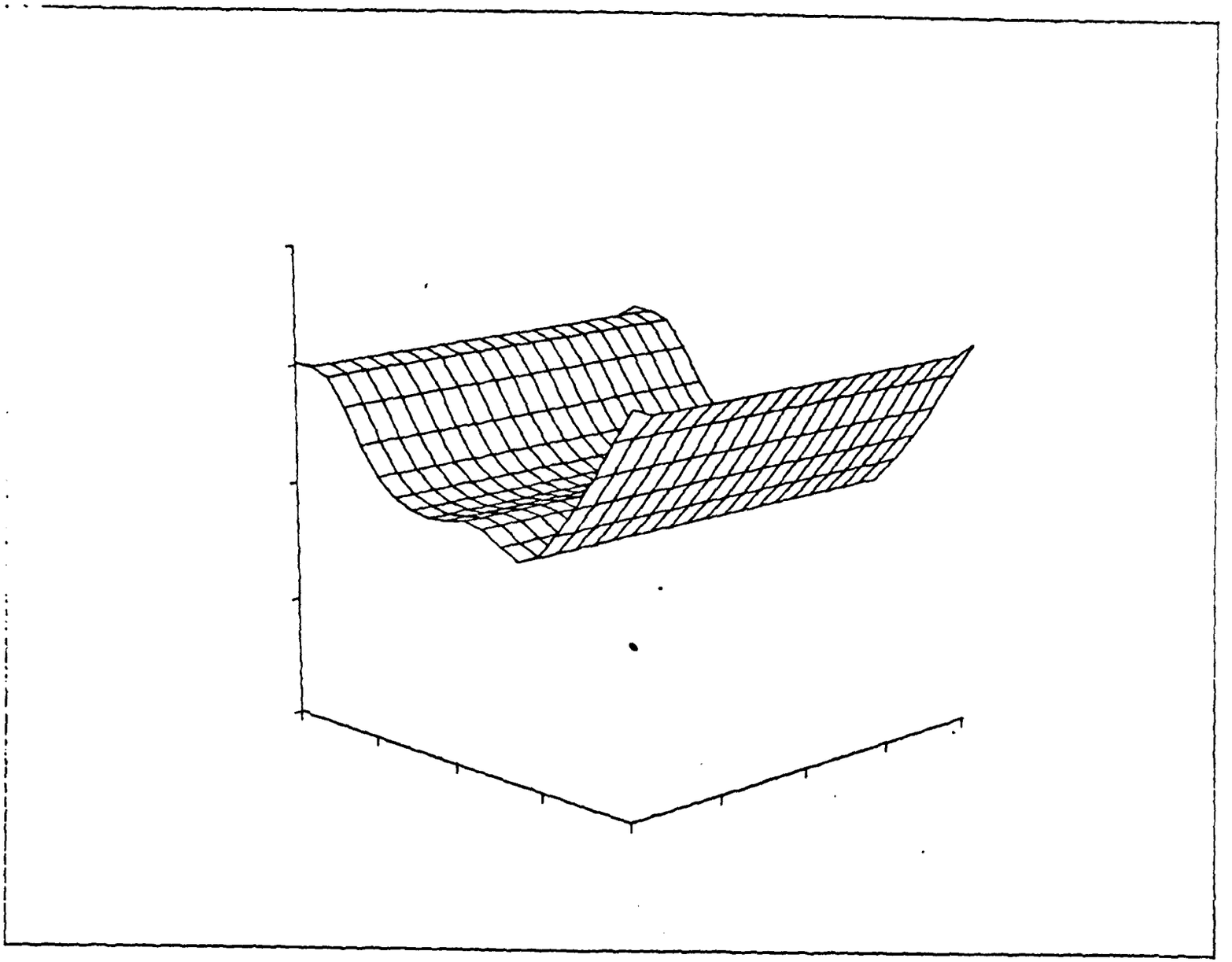
First we see the one-dimensional wave in $W_2$ moving to the right, exiting and reappearing by the iteration 20. However, sharp spikes appear in $W_3$ by this time. $W_2$ appears to behave well until iteration 85, but either reflection or some non-linear instability lies caused $W_3$ to grow catastrophically. Clearly this represents failure of the boundary conditions, even in the one-dimensional case.

Various minor variations were tried, but all produced the same basic phenomena.

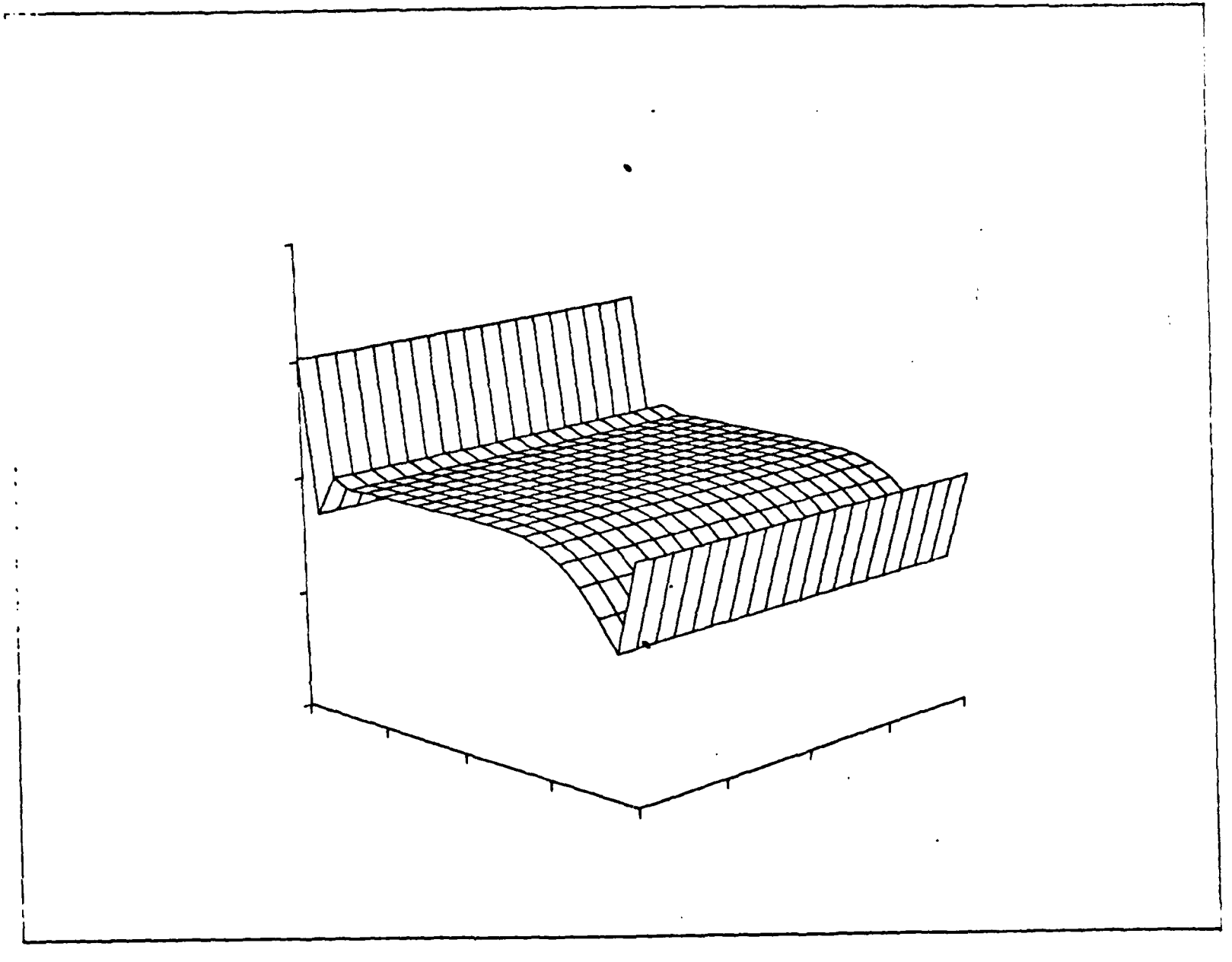
$\boxed{\text{W2}}$ 10 iter

PERMCK4

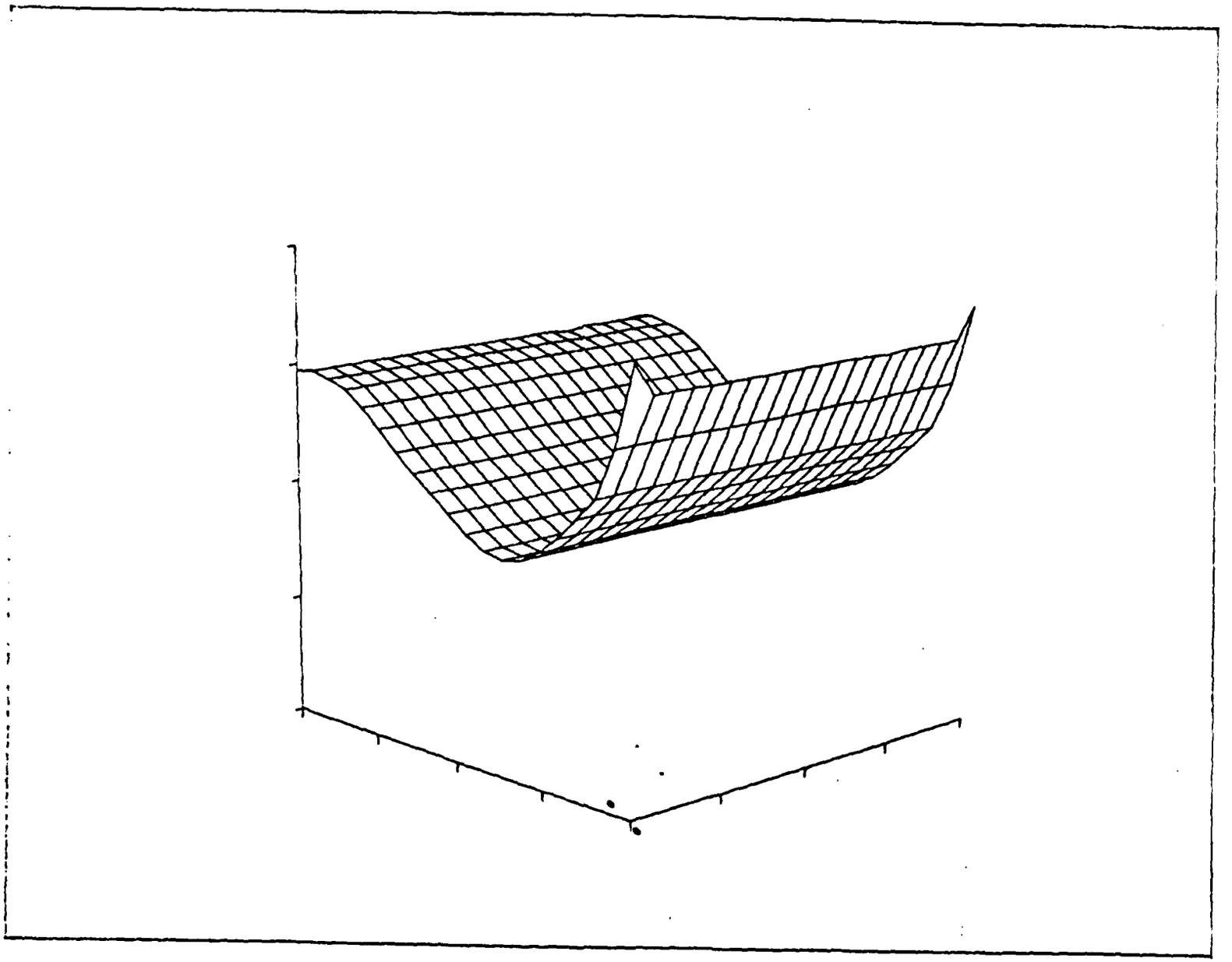




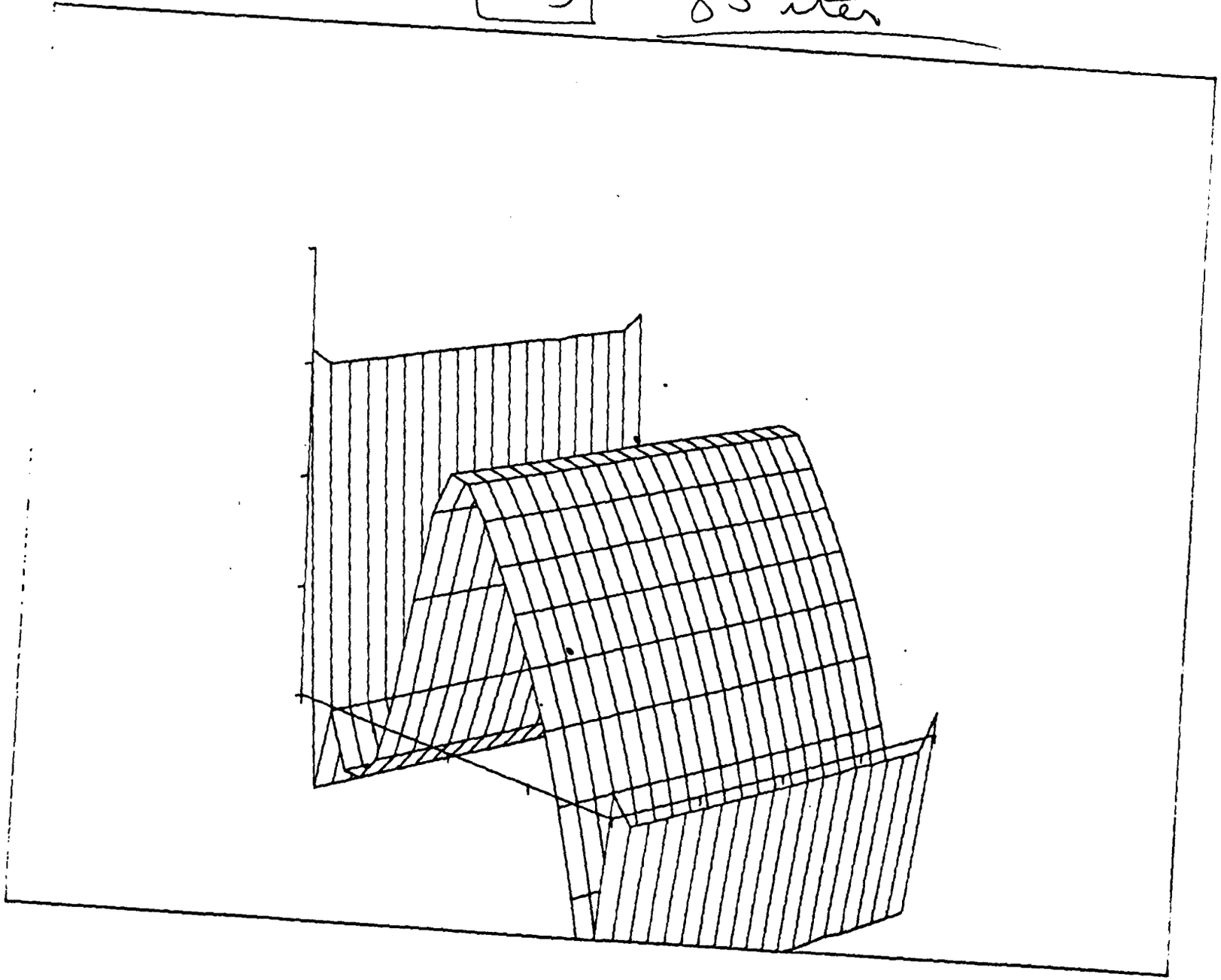
W3 20 iter



W 2 85 ter



W3 85 ster



## Conclusion

Non-reflecting characteristic boundary conditions work well so long as the flow is basically returning to steady state. This was seen in the first series of numerical experiments.

However, at this stage of development they cause strange instabilities which prevent efficient modelling of periodic phenomena. The reasons for this are not understood at the present.

## BIBLIOGRAPHY


[1] Engquist, B., and Majda, A., "Absorbing Boundary Conditions for the Numerical Simulation of Waves", Math Comp. Vol. 31, pp 629-651, (1977).

[2] Gottlieb, D., and Turkel, E., "Boundary Conditions for Multi-Step Finite Difference Methods for Time Dependent Equations", J. Comp. Phys. Vol. 26, pp 181-196 (1978).

[3] Gustaffson, B., and Kriess, H.O., "Boundary Conditions for Time Dependent Problems with an Artifical Boundary", Jour. Comp. Phys. Vol. 30, pp 333-351 (1979).

[4] Gustaffson, B., Kreiss, H.O., and Sundstrom, A., "Stability Theory of Difference Approximations for Mixed Initial Boundary Value Problems II", Math. Comp., Vol. 26, pp 649-686 (1972).

[5] Kreiss, H.O., "Stability Theory for Difference Approximations of Mixed Initial Boundary Value Problems L", Math. Comp., Vol. 22, pp 703-714 (1968).

[6] MacCormack, R.W., "Numerical Solutions of the Interactions of a Shock Wave with Laminar Boundary Layer", Lecture Notes in Physics, Vol. 19, Springer Verlag (1976).

[7] McKenna P.J., Graham, J.E., and Hankey, W.L., The role of far-field boundary conditions in numerical solutions of the Navier-Stokes equations, AFWAL Tech. Report TR. 81.

[8] Moretti, G., Comment on "Stability Aspects of Divergent Subsonic Flow", AIAA Jour., Vol. 19, No. 5., p 669 (1981).

[9] Oliger, J., and Sundstrom, A., Theoretical and practical aspects of some initial boundary value problems in fluid dynamics, SIAM J. of Appld. Math., 35 (1978) pp. 419-446.

[10] Parter, S.V., "Stability, Convergence and Pseudo-Stability of Finite Difference Equations for an Over-Determined Problems", Numer. Math, Vol. 4, pp 277-292 (1962).

[11] Rudy, D.H., and Strikwerda, J.S., "Boundary Conditions for Subsonic Compressible Navier-Stokes Equations", Computers and Fluids, Vol. 9, pp 327-338 (1981).

[12] Rudy, D.H., and Strikwerda, J.C., "A Non-reflecting Outflow Boundary Condition for Subsonic Navier-Stokes Calculations", Jour. Compu. Phys. Vol. 36, pp 55-70 (1980).

[13] Schlichting, H., "Boundary Layer Theory". Seventh Edition, McGraw Hill N.Y. (1979).

[14] Shang, J.S., Oscillatory compressible flow round a cylinder, Proc. AIAA 20th Aerospace Meeting, Jan 11-14 (1982). Orlando, Florida.

[15] Shang, J.S., "Numerical Simulation of Wing Fuselage Interference", AIAA 19th Aerospace Sciences Meeting, AIAA-81-0048, Jan 12-15, 1981.

[16] Steger, J.L., Pulliam. T.H., and Chima, R.V., "An Implicity Finite Difference Code for Inviscid and Viscous Cascade Flow", AIAA Preprint-80-1427, AIAAA 13th Fluid and Plama Dynamics Conference, July 14-16, 1980.

[17] Yee, H.C., "Numerical Approximation of Boundary Conditions with Application to Inviscid Equation of Gas Dynamics", NASA Technical Memorandum 81265, March 1981.

[18] Yee, H.C., R.M., Beam, and Warming, R.F., "Stable Boundary Approximations for a Class of Implicit Schemes for the One-Dimensional Inviscid Equation of Gas Dynamics", AIAA Paper 81-1009, presented at Palo Alto, CA, June 1981.

END

FILMED

10-85

DTIC